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Positive Solution to a Nonlinear Elliptic Problem

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Abstract Let L be a second order elliptic operator with smooth coefficients satisfying $L1 = 0$ defined in a domain Ω that is Greenian for L. Under fairly general hypotheses on the function φ, we solve the following problem:

$$\begin{cases} Lu + \varphi(\cdot, u) = 0, & \text{in the sense of distributions in } \Omega; \\ u > 0, & \text{in } \Omega \text{ ;} \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Keywords Nonlinear elliptic problems · Regular domain · Greenian domain · Green function

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1 Introduction

Let L be a second order elliptic operator with smooth coefficients satisfying $L1 = 0$ defined in a domain Ω that is Greenian for L.[1] In particular, boundedness of Ω is not assumed here.

[1] See the definition in Section 2.

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We study existence, uniqueness and regularity of solutions to the following problem

$$\begin{cases} Lu + \varphi(\cdot, u) = 0, & \text{in the sense of distributions in } \Omega; \\ u > h, & \text{in } \Omega; \\ u = h, & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

with h being an L-harmonic function. L is applied to u in the sense of distributions, $u = h$ on $\partial\Omega$ means that $\lim\limits_{x \to \partial\Omega} (u - h)(x) = 0$ and $\varphi : \Omega \times]0, \infty[\to [0, \infty[$ is measurable and satisfies some appropriate hypotheses detailed below. A particular case is obtained when $h = 0$ i.e.

$$\begin{cases} Lu + \varphi(\cdot, u) = 0, & \text{in the sense of distributions in } \Omega; \\ u > 0, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Both problems have attracted a lot of attention for $L = \Delta$ the Laplace operator in $\mathbb{R}^d$, in particular the second one. Much less has been done for a general L.

Boundary value problems such as

$$\begin{cases} \Delta u + \varphi(\cdot, u) = 0, & \text{in the sense of distributions in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

with various φ arise in a large number of mathematical models in physics, mechanics, chemistry and astronomy. In particular, they describe population dynamics, chemical reactions and morphogenesis. Therefore, positive solutions are often of main interest. Moreover, solutions to Eq. 1.3 can be interpreted as stationary solutions to the associated parabolic problem. When $\varphi(x, u) = g(x)u^\gamma$, Eq. 1.3 is known as the generalized Emden-Fowler equation [23] and has been extensively studied since the beginning of the 20th century. For a good overview we refer to [14] and [5].

Let G_Ω be the Green function for L in Ω. In this paper the function φ is required to satisfy the following hypotheses:

- (H_1) φ is continuous and nonincreasing with respect to the second variable.
- (H_2) $\forall c > 0$, $\varphi(\cdot, c) \in K_d^{loc}(\Omega)$ (see Eq. 2.1).
- (H_3) for every $c > 0$, the Green potential $G_\Omega(\varphi(\cdot, c))$ of $\varphi(\cdot, c)$ is finite at least on one point.
- (H_4) $G_D(\varphi(\cdot, c))(x) > 0$ for every $c > 0$, $x \in D$ where D is a regular bounded domain contained with its closure in Ω.

Notice that this allows the existence of a singularity at 0 of the type $\lim\limits_{s \to 0^+} \varphi(x, s) = \infty$ as well as some growth of $\varphi(\cdot, c)$ at the boundary of Ω.

The idea of taking such φ comes from [7, 18], where problems (1.1), (1.2) were studied for $L = \Delta$. Our aim here is to generalize the results of both papers. We do it in many ways, not only taking an arbitrary elliptic operator but also by weakening hypotheses on φ and Ω. The following main theorem says something new also in the case of $L = \Delta$.

Theorem 1 *L is a second order elliptic operator with smooth coefficients satisfying $L1 = 0$ defined in the domain Ω. We assume that Ω is a Greenian for L. Suppose $\varphi : \Omega \times]0, \infty[\to$ $[0, \infty[$ satisfies (H_1) - (H_4) and for every $c > 0$,*

$$\lim_{x \to \partial\Omega} G_\Omega(\varphi(\cdot, c))(x) = 0. \tag{1.4}$$

Then there is a unique continuous solution to Eq. 1.1. If additionally

$$\lim_{x\to\infty} G_\Omega(\varphi(\cdot, c))(x) = 0 \quad \text{for every } c > 0,$$

then $\lim_{x\to\infty}(u - h)(x) = 0$. *With further assumptions on regularity of* φ, *we get more regularity of* u *(see Theorem 19).*

Notice that if h has a well defined boundary value then so does u. In particular, in a Dirichlet domain Ω i.e. a bounded domain Ω satisfying *an exterior sphere condition*, for h just continuous on $\partial\Omega$, we obtain solution $u \in C(\bar{\Omega})$ under very mild assumptions both on regularity and growth of φ. Dirichlet domains are called here *regular*.[2]

To have a feeling of H_3, it is worth mentioning that Theorem 1 gives solution to the following problem for a uniformly elliptic operator L in a bounded domain Ω with $C^{1,1}$ boundary,

$$\begin{cases} Lu + \frac{g(x)}{d(x)^b} u^{-a} = 0, & \text{in the sense of distributions in } \Omega; \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

$0 \le b < 2$, $g \in L^\infty(\Omega)$ positive, $a > 0$, $d(x) = \operatorname{dist}(x, \partial\Omega)$. H_1 and H_2 are clearly satisfied. To check H_3 notice that $G_\Omega \le CG$, where G is the Green function for Δ in Ω [16] and to use the estimates for G proved in [26]. Then for a fixed x, there exist a constant $M > 0$ such that for every y outside a compact neighbourhood of x

$$G_\Omega(x, y)\frac{g(y)}{d(y)^b} \le Md(y)^{-b+1},$$

which is integrable. Problem (1.5) for $L = \Delta$ has been recently considered by Díaz, Hernández and Rakotoson in [5] in bounded domains without restrictions on the boundary. So we get here a partial generalization of their results to uniformly elliptic operators.

As far as we know, there is no result about existence of solutions to Eq. 1.1 for elliptic operators in unbounded Ω. There are some papers concerning (1.1) but the domain is always bounded and there are much stronger regularity assumptions on φ and $\partial\Omega$ [3, 14, 20, 22] although monotonicity of φ with respect to the second variable is not always required [14, 22]. The strength of our approach relies on a very mild regularity of φ as well as practically no assumptions on the domain except of being Greenian which is perfectly natural provided solution to Eq. 1.1, if it exists, is of the form

$$u = h + G_\Omega(\varphi(\cdot, u)).$$

However, we need to keep monotonicity with respect to the second variable which fits very well into the potential theoretical approach developed in [7] for the problem

$$\Delta u + p(x)\psi(u) = 0 \tag{1.6}$$

with $p \in L^\infty_{loc}(\Omega)$ positive, $\psi :]0, \infty[\to]0, \infty[$ continuous, nonincreasing. The method of El Mabrouk works perfectly here. In fact, a possible generalization of Eq. 1.6 to elliptic operators and p being in the Kato class is mentioned in [7]. Combining ideas both of [7] and [18], we do it here with weaker hypotheses on φ than in [7, 18].

Problems (1.1) and (1.2) for $L = \Delta$ have been recently considered under variety of hypotheses on φ. Monotonicity with respect to the second variable is crucial in [4, 6, 8, 12,

[2] See the definition of regular domain in Section 2.

17, 24] but it is not longer required in very recent papers [9, 11, 13, 17, 21, 25]. However, there are always more regularity assumptions on φ than in this work like Hölder continuity or the product form $\varphi(x,s) = p(x)\psi(s)$. Also, the problem is usually considered either in a bounded regular domain or in $\Omega = \mathbb{R}^d$. The approach via potential theory initiated in [7] and used here allows us to go beyond these restrictions.

Theorem 1 is a result of a few steps. First, we prove that for every regular bounded domain D such that $\bar{D} \subset \Omega$ and for every $f \in \mathcal{C}^+(\partial D)$ the problem:

$$\begin{cases} Lu + \varphi(\cdot, u) = 0, & \text{in the sense of distributions in } D; \\ u > 0, & \text{in } D; \\ u = f, & \text{on } \partial D, \end{cases} \tag{1.7}$$

has a unique solution $u \in \mathcal{C}^+(\bar{D})$. Moreover,

$$u(x) = H_D f(x) + \int_D G_D(x,y)\varphi(y, u(y))\, dy, \qquad \forall x \in D,$$

where $H_D f$ is the solution of the classical Dirichlet problem for L with boundary values f.

If for every $c > 0$, $\varphi(\cdot, c) \in L^\infty_{loc}(D)$, then $u \in \mathcal{C}^+(\bar{D}) \cap \mathcal{C}^1(D)$. Further, if $\varphi \in \mathcal{C}^\alpha_{loc}(D\times]0,\infty[)$, then $u \in \mathcal{C}^{2,\alpha}_{loc}(D) \cap \mathcal{C}(\bar{D})$. After that, in a Greenian domain, we establish one-to-one correspondence between nonnegative continuous solutions u of the equation

$$Lu + \varphi(\cdot, u) = 0 \text{ in the sense of distributions in } \Omega, \tag{1.8}$$

and nonnegative L-harmonic functions h:

$$u(x) = h(x) + \int_\Omega G_\Omega(x,y)\varphi(y, u(y))\, dy. \tag{1.9}$$

It turns out that the solution preserves the same regularity as in regular bounded domains under the same hypotheses.

The remainder of this paper is organized as follows. In Section 2, we introduce the some notations and tools needed for the sequel. Next in Section 3, we solve the problem (1.7) in a regular domain and we investigate the regularity of the solution. Then, in Section 4, we establish a 1-to-1 correspondence between nonnegative L-harmonic functions and nonnegative continuous solutions of Eq. 1.8 in Greenian domain and we also address their regularity. In Section 5, we focus on boundary conditions and we prove Theorem 1. Finally in Appendix, we recall, for readers convenience, some basic tools of the potential theory used here.

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2 Preliminaries

For every open set Ω of $\mathbb{R}^d$ with $(d \geq 3)$ let $\mathcal{B}(\Omega)(resp.\ \mathcal{C}(\Omega))$ be the set of all real valued Borel measurable (resp. continuous) functions on Ω. We are also going to consider $\mathcal{C}^1(\Omega)$ - the space of continuously differentiable functions on Ω, $\mathcal{C}^\infty_c(\Omega)$ - the space of infinitely differentiable functions on Ω with compact support, $\mathcal{C}^{2,\alpha}(\Omega)$ - the space of functions with the second derivative being α-Hölder continuous. Finally, for every set $\mathcal{F}$ of numerical functions, we denote by $\mathcal{F}^+$ the set of all functions in $\mathcal{F}$ which are nonnegative.

A bounded set D satisfying $\overline{D} \subset \Omega$ is called regular (for L) if each function $f \in \mathcal{C}(\partial D)$ admits a continuous extension $H_D f$ on $\overline{D}$ such that $H_D f$ is L-harmonic in D, in other words, the function $h = H_D f$ is the unique solution to the classical Dirichlet problem i.e.

$$\begin{cases} Lh = 0, & \text{in } D; \\ h = f, & \text{on } \partial D. \end{cases}$$

An open subset Ω of $\mathbb{R}^d$ is called Greenian set if Ω possesses a Green function (for L) which will be denoted by G_Ω i.e. for every $y \in \Omega$ $G_\Omega(\cdot, y)$ is a potential on Ω and we have $L(G_\Omega(\cdot, y)) = -\varepsilon_y$, in the sense of distributions, where ε_y denotes the Dirac measure at the point y.

In this paper, by a solution to a partial differential equation we shall mean a continuous solution in the sense of distributions. In particular, a solution to Eq. 1.8 in an open set $D \subset \Omega$ will be a function $u \in \mathcal{C}^+(D)$ such that $\varphi(\cdot, u)$ is locally integrable on D and for all $\psi \in \mathcal{C}_c^\infty(D)$ we have

$$\int_D uL^*(\psi) + \int_D \varphi(\cdot, u)\psi = 0.$$

A lower semi-continuous function is said to be L-superharmonic on a open set Ω if $Ls \leq 0$ in the sense of distributions. Every function v such that $-v$ is L-superharmonic on Ω will be called L-subharmonic on Ω.

Now we are going to recall basic properties of potentials of functions belonging to the Kato class.

Definition 2 (see e.g. [18]) A Borel measurable function ψ on Ω belongs to the Kato class $K_d(\Omega)$ if ψ satisfies

$$\lim_{\alpha \to 0} \sup_{x \in \Omega} \int_{\Omega \cap (|x-y| \leq \alpha)} \frac{|\psi(y)|}{|x-y|^{d-2}}\, dy = 0. \tag{2.1}$$

Proposition 3 (see e.g. [18]) *Let $\psi \in K_d(\Omega)$. Then for each $M > 0$, we have*

$$\int_{\Omega \cap (|y| \leq M)} |\psi(y)|\, dy < \infty.$$

In particular, if Ω is a bounded domain, then $\psi \in L^1(\Omega)$.

The following proposition was proved in [18] for the Green function corresponding to the Laplace operator in $\mathbb{R}^d$. But due to the estimate (2) the proof is the same.

Proposition 4 *[see* [18] *and* [16]*]*
Let D be a bounded regular domain in $\mathbb{R}^d$ $(d \geq 3)$ and $\psi \in K_d(D)$, then

$$\sup_{x \in D} \int_D \frac{|\psi(y)|}{|x-y|^{d-2}}\, dy < \infty,$$

and

$$G_D \psi \in \mathcal{C}_0(D).$$

Definition 5 A Borel measurable function ψ on Ω belongs to $K_d^{loc}(\Omega)$ if for every bounded subset D in Ω, $\psi \in K_d(D)$.

Proposition 6 *Let $p \in L^{\infty}_{loc}(\Omega)$ then $p \in K^{loc}_d(\Omega)$.*

Proof Let D a bounded domain satisfying $\bar{D} \subset \Omega$ then there exists a constant a_d depending only on the dimension d such that

$$0 \leq \int_{D\cap B(x,\alpha)} \frac{|p(y)|}{|x-y|^{d-2}}\, dy \leq a_d \sup_{y\in D} |p(y)| \frac{\alpha^2}{2},$$

for every $x \in D$ and $\alpha > 0$. We can deduce the result. □

Proposition 7 *Let Ω be a Greenian domain, $\phi \in K^{loc}_d(\Omega)$ and there exists $x_0 \in \Omega$ such that $G_\Omega\phi(x_0)$ is finite. Then $G_\Omega\phi \in \mathcal{C}(\Omega)$.*

Proof Let $x \in \Omega$ and D be a bounded regular domain such that $x_0 \in D$ and $\bar{D} \subset \Omega$.

$$\int_\Omega G_\Omega(x,y)\phi(y)\, dy = \int_{\Omega\cap D^c} G_\Omega(x,y)\phi(y)\, dy + \int_D G_\Omega(x,y)\phi(y)\, dy.$$

$G_\Omega(\cdot, y)$ is L-harmonic in D for every $y \in \Omega \cap D^c$, so using Harnack inequality (see Theorem 28) we can deduce that the first part is finite continuous in D.

Also

$$\int_D G_\Omega(x,y)\phi(y)\, dy = \int_D (G_\Omega(x,y) - G_D(x,y))\phi(y) + G_D(x,y)\phi(y)\, dy.$$

Or $x \mapsto \int_D G_D(x,y)\phi(y)\, dy$ is continuous on D by Proposition 4 and $G_\Omega(\cdot,y) - G_D(\cdot,y)$ is L-harmonic in D. We can deduce by Harnack inequality that $G_\Omega\phi$ is continuous in Ω. □

3 Solution of Eq. 1.7 in a Regular Domain

In this section, we solve the problem (1.7) in an arbitrary regular bounded set D and a given $f \in \mathcal{C}^+(\partial D)$:

Theorem 8 (Solution of Eq. 1.7 in a regular domain) *Let D be a bounded regular domain such that $\overline{D} \subset \Omega$ and let L be a second order elliptic operator with smooth coefficients satisfying $L1 \leq 0$. Suppose that $f \in \mathcal{C}^+(\partial D)$ non identically zero and $\varphi : \Omega\times]0,\infty[\to [0,\infty[$ is a measurable function satisfying $H_1 - H_2$. Then there exists a unique solution $u \in \mathcal{C}(\overline{D})$ of problem (1.7). Furthermore, we have:*

$$u(x) = H_D f(x) + \int_D G_D(x,y)\varphi(y,u(y))\, dy, \text{ for every } x \in D .$$

If in addition, φ satisfies (H_4), then the statement remains true for f being the zero function.

Before dealing with the proof, we start with a lemma that allows us to compare solutions to Eq. 1.1. For $L = \Delta$ this result is stated and proved in [7]. The proof goes along the same lines - only properties of L-superharmonic functions in the sense of abstract potential theory are used.

Lemma 9 (Comparison with values on the boundary) *Let Ω be a domain, $u, v \in \mathcal{C}^+(\Omega)$ such that $u, v > 0$, $Lu, Lv \in \mathcal{L}^1_{loc}(\Omega)$ and $\varphi : \Omega\times]0,\infty[\to [0,\infty[$ a decreasing function* *with respect to the second variable. If :*

$$\begin{cases} Lu + \varphi(\cdot, u) \leq Lv + \varphi(\cdot, v) & \textit{in the sense of distributions,} \\ \liminf\limits_{\substack{x \to y \\ y \in \partial\Omega}} (u - v)(x) \geq 0. \end{cases}$$

Then:

$$u - v \geq 0 \ in \ \Omega.^3$$

Proof Let $V = \{x \in \Omega, \ u(x) < v(x)\}$. V is open in Ω because u, v are continuous. Let suppose that V is nonempty. On V we have :

$$Lu - Lv \leq \varphi(\cdot, v) - \varphi(\cdot, u) \leq 0.$$

Furthermore,

$$\text{if } z \in \partial V \cap \partial\Omega \text{ then } \liminf_{x \to z}(u - v)(x) \geq 0,$$

$$\text{if } z \in \partial V \cap \Omega \text{ then } z \in V^c \cap \Omega \text{ and so } u(z) \geq v(z).$$

Therefore we can conclude:

$$\begin{cases} L(u - v) \leq 0, & \text{in the sense of distributions, in } V, \\ \liminf\limits_{x \to z}(u - v)(x) \geq 0 & \text{on } \partial V. \end{cases}$$

Hence by Lemma 42

$$u - v \geq 0 \text{ in } V,$$

and so V is empty. □

Now we are ready to prove the main theorem of this section.

Proof First, we suppose that: $\inf\limits_{x \in \partial D} f(x) = \alpha > 0$.

Let

$$\beta = \sup_{x \in \overline{D}} H_D f(x) + \sup_{x \in D} G_D(\varphi(\cdot, \alpha))(x) < \infty,^4$$

and

$$C = \{u \in \mathcal{C}(\overline{D}), \alpha \leq u \leq \beta\}.$$

In C we consider the topology of uniform convergence, C is nonempty bounded closed convex set. Also for every $u \in C$ and every $x \in \overline{D}$

$$\alpha \leq H_D f(x) + G_D(\varphi(\cdot, u))(x) \leq H_D f(x) + G_D(\varphi(\cdot, \alpha))(x) \leq \beta.$$

We consider

$$\begin{aligned} T : & C \to C \\ & u \mapsto H_D f + G_D(\varphi(\cdot, u)). \end{aligned}$$

T is well defined. Indeed, $H_D f$ is a continuous function on $\overline{D}$, $\varphi(\cdot, \alpha) \in K_d(D)$ hence for every $u \in C$, $\varphi(\cdot, u) \in K_d(D)$. Consequently $G_D(\varphi(\cdot, u)) \in \mathcal{C}_0(D)$. Additionally, $T(C)$ is contained in C.

[3] If Ω is unbounded, then writing $x \to \partial\Omega$ we include also the case $|x| \to +\infty$.

[4] In the first case we need $\alpha > 0$ in order that $\varphi(\cdot, \alpha)$ be well defined.

Using the Schauder theorem we will prove that T has a fixed point in C. We start with equicontinuity of the set $T(C)$. Let $\epsilon > 0$. First, observe that $H_D f$ is uniformly continuous in $\overline{D}$. Therefore, only equicontinuity of $\{G_D(\varphi(\cdot, u)), u \in C\}$ remains to be proved.

If $x \in \partial D$, then $G_D(\varphi(\cdot, u))(x) = 0$ for every $u \in C$. In addition, $G_D(\varphi(\cdot, \alpha)) \in \mathcal{C}_0(D)$, hence there exists a neighbourhood $\mathcal{V}_x$ of x such that for every $y \in \mathcal{V}_x \cap D$ and $u \in C$

$$|G_D(\varphi(\cdot, u))(y) - G_D(\varphi(\cdot, u))(x)| \leq G_D(\varphi(\cdot, \alpha))(y) \leq \epsilon.$$

Now we consider points x in D. Let $\gamma > 0$ be such that

$$\sup_{x \in D} \int_{D \cap (|x-y| \leq 2\gamma)} \frac{\varphi(y, \alpha)}{|x - y|^{d-2}}\, dy \leq \epsilon.$$

Let $\Delta = \{(x, x) : x \in D\}$. The map $(x, y) \mapsto G_D(x, y)$ is uniformly continuous on every compact set $\subset \{D \times \overline{D} \backslash \Delta\}$ (see Proposition 35), hence it exists ν such that if for every $x, x' \in \overline{B(x, \gamma/2)}$ such that $|x - x'| \leq \nu$ and $y \in D \cap B(x, \gamma)^c$,

$$|G_D(x, y) - G_D(x', y)| < \epsilon,$$

which implies for every $u \in C$ we have

$$\begin{aligned}
|Tu(x) - Tu(x')| \leq{} & |H_D f(x) - H_D f(x')| \\
& + \int_{D \cap B(x,\gamma)^c} |G_D(x, y) - G_D(x', y)| \varphi(y, u(y))\, dy \\
& + |\int_{B(x,\gamma)} (G_D(x, y) - G_D(x', y))\varphi(y, u(y))\, dy| \\
\leq{} & |H_D f(x) - H_D f(x')| + \epsilon \int_D \varphi(y.\alpha)\, dy \\
& + |\int_{B(x,\gamma)} (G_D(x, y) - G_D(x', y))\varphi(y, u(y))\, dy|
\end{aligned}$$

In addition: $|\int_{B(x,\gamma)} \varphi(y, u(y))(G_D(x, y) - G_D(x', y))\, dy|$

$$\begin{aligned}
& \leq \int_{B(x,\gamma)} \varphi(y, u(y))(G_D(x, y) + G_D(x', y))\, dy \\
& \leq k\Big(\int_{B(x,\gamma)} \frac{\varphi(y, u(y))}{|x - y|^{d-2}}\, dy + \int_{B(x',2\gamma)} \frac{\varphi(y, u(y))}{|x' - y|^{d-2}}\, dy\Big) \\
& \leq 2k \sup_{x \in D} \int_{B(x,2\gamma)} \frac{\varphi(y, \alpha)}{|x - y|^{d-2}}\, dy \\
& \leq 2k\epsilon.
\end{aligned}$$

Therefore, given ϵ, we can choose ν sufficiently small such that for all $u \in C$

$$|Tu(x) - Tu(x')| \leq \epsilon.$$

Secondly, T is continuous. Indeed, Let u_n tend to $u \in C$. Then

$$|Tu_n(x) - Tu(x)| \leq \int_D G_D(x, y)|\varphi(y, u_n(y)) - \varphi(y, u(y))|\, dy,$$

the function inside the integral tends to zero and it is dominated by

$$2k \frac{\varphi(y, \alpha)}{|x - y|^{d-2}}\, dy \in L^1(D).$$

By the dominated convergence theorem, we can conclude the pointwise convergence, by equicontinuity we can deduce the uniform convergence.

Now in view of the Schauder theorem, there is a fixed point $u \in \mathcal{C}(\overline{D})$ of T i.e.

$$u = H_D f + G_D(\varphi(\cdot, u)).$$

Moreover, $G_D(\varphi(\cdot, u)) \in C_0(D)$ and $0 \le \varphi(\cdot, u) \le \varphi(\cdot, \alpha) \in L^1(D)$ then by Lemma 38

$$L(G_D(\varphi(\cdot, u))) = -\varphi(\cdot, u) \text{ in the sense of distributions.}$$

Secondly, let f be a nontrivial nonnegative continuous function on ∂D such that $\inf\limits_{x\in\partial D} f(x) = 0$

Let

$$f_k = f + \frac{1}{k} \ge \frac{1}{k} \qquad , k \in \mathbb{N}^*$$

and let u_k be the solution to Eq. 1.7 with the boundary value f_k.

Then

$$u_k = H_D f_k + G_D(\varphi(\cdot, u_k)). \tag{3.1}$$

In addition,

$$\begin{cases} L(u_k) + \varphi(\cdot, u_k) = L(u_{k+1}) + \varphi(\cdot, u_{k+1}) = 0, & \text{in } D; \\ u_{k+1} = f_{k+1} \le f_k = u_k, & \text{on } \partial D, \end{cases}$$

with

$$0 \le -L(u_k) = \varphi(y, u_k(y)) \le \varphi(y, \frac{1}{k}) \in L^1(D).$$

So by Lemma 9 we get

$$0 \le u_{k+1} \le u_k \quad in\ D. \tag{3.2}$$

We denote:

$$u(x) = \lim_{k\to+\infty} u_k(x), \text{ for } x \in D.$$

Now we turn to prove that u is continuous in D. On one hand, (u_n) is a decreasing sequence of continuous function, so the limit u is upper semi-continuous. On the other hand, $(u_n - H_D f_n)$ is an increasing sequence of continuous functions, so the limit $u - H_D f$ is lower semi-continuous. Since $H_D f$ is continuous, we can conclude that u is continuous too. Moreover,

$$H_D f \le H_D f_n \le u_n,$$

then

$$H_D f \le u,$$

however if f is nontrivial nonnegative then

$$H_D f > 0, \text{ in } D,$$

which implies that

$$u > 0, \text{ in } D.$$

Following this $\lim\limits_{n\to+\infty} \varphi(\cdot, u_n) = \varphi(\cdot, u) < \infty$, then by monotone convergence theorem

$$u = H_D f + G_D(\varphi(\cdot, u)). \tag{3.3}$$

Further, for every compact set K in D, there exists $\eta > 0$ such that $u(x) > \eta$ for $x \in K$. Therefore, $\varphi(\cdot, u) \le \varphi(\cdot, \eta) \in L^1(K)$. Also, $G_D(\varphi(\cdot, u)) = u - H_D f$ is continuous, by Corollary 39 we may conclude

$$L(G_D(\varphi(\cdot, u))) = -\varphi(\cdot, u), \text{ in the sense of distributions in } D.$$

Now we turn our attention to the boundary conditions. Since $H_D f \leq u \ \ in \ D$ then

$$f(y) = \liminf_{\substack{x \to y \\ y \in \partial D}} f(x) \leq \liminf_{\substack{x \to y \\ y \in \partial D}} u(x).$$

On the other hand

$$\limsup_{\substack{x \to y \\ y \in \partial D}} u(x) \leq \limsup_{\substack{x \to y \\ y \in \partial D}} u_n(x) = f(y) + \frac{1}{n}.$$

Hence

$$\lim_{\substack{x \to y \\ y \in \partial D}} u(x) = f(y).$$

Third, we suppose that $f \equiv 0$ on ∂D then $H_D f \equiv 0$ too as so it is not evident that $u = \lim_{n \to +\infty} u_n > 0$ in D. For this particular case we need H_4. First, by Eqs. 3.1 and 3.2

$$0 \leq u_n(x) - H_D f_n(x) \leq u(x) - H_D f(x) \leq u(x), \qquad x \in D, n \in \mathbb{N}^*.$$

Let's suppose that there exists $x_0 \in D$ such that $u(x_0) = 0$ then $u_n(x_0) - H_D f_n(x_0) = 0$. In addition

$$\begin{cases} L(u_n - H_D f_n) = -\varphi(\cdot, u_n) \leq 0, & \text{in } D; \\ (u_n - H_D f_n)(x_0) = 0, & \end{cases}$$

But, a nonconstant L-superharmonic function cannot take a negative minimum inside D, so $u_n - H_D f_n = 0$ which implies that

$$G_D(\varphi(\cdot, H_D f_n))(x_0) = 0,$$

however $H_D f_n \leq \frac{1}{n}$ hence

$$G_D(\varphi(\cdot, \frac{1}{n}))(x_0) = 0.$$

This is a contradiction with H_4 and hence $u > 0$ in D.

Uniqueness of the solution follows immediately from Lemma 9. □

Remark 10 Notice that in view of Proposition 4, H_3 is always satisfied in D. However, H_4 is needed in order that φ can take zero on $\Omega \times]0, +\infty[$. The above theorem generalizes both Lemma 4.3 in [7] and Corollary 2 in [18].

A careful observation of the proof of the above theorem allows us to conclude the following upper bound for the solution which depends only on the boundary value and the size of the potential, not on the domain itself i.e. as far as $D \subset D_0$ the bound depends only on f, φ and G_{D_0}.

Corollary 11 *Under the same hypothesis as in the Theorem 8 the unique solution $u \in \mathcal{C}(\overline{D})$ of problem* (1.7) *satisfies*

$$||u||_\infty \leq \inf_k \Big(||f + \frac{1}{k}||_\infty + ||G_D(\varphi(\cdot, \frac{1}{k}))||_\infty \Big). \tag{3.4}$$

Proof By Theorem 8 the solution u is the limit of a decreasing sequence (u_k) and $u_k = H_D f_k + G_D(\varphi(\cdot, u_k))$. Clearly,

$$||u_k||_\infty \leq ||f_k||_\infty + ||G_D(\varphi(\cdot, \frac{1}{k}))||_\infty.$$

Hence (3.4) follows. □

With further hypotheses on φ we may conclude more regularity of u.

Theorem 12 *Suppose that the assumptions of Theorem 8 are satisfied and additionally that for every* $c > 0$*,* $\varphi(\cdot, c) \in L^{\infty}_{loc}(D)$*, then the unique solution* u *of problem* (1.7) *belongs to* $\mathcal{C}^{+}(\overline{D}) \cap \mathcal{C}^{1}(D)$*. Furthermore, if* $\varphi \in \mathcal{C}^{\alpha}_{loc}(D\times]0, \infty[)$ *then* $u \in \mathcal{C}^{2,\alpha}_{loc}(D)$ $\cap\mathcal{C}(\bar{D})$*.*

Remark 13 Regularity of solutions is not mentioned in [7] and in [18] it is proved only for the case where the boundary value is a constant function.

Proof Let $x_0 \in D$ and $r > 0$ be such that $\overline{B(x_0, r)} \subset D$. Let $f_0 = u_{/\partial B(x_0,r)}$ be continuous and strictly positive. We denote $B = B(x_0, r)$. By uniqueness, u is also given by:

$$u = H_B f_0 + G_B(\varphi(\cdot, u)), \qquad \text{in } B,$$

$H_B f_0$ being smooth. Since L is uniformly elliptic on B, there is $\eta > 0$ such that for $y \in B$[5]

$$|\frac{\partial}{\partial x_i} G_B(x_0, y)| \leq \frac{\eta}{|x_0 - y|^{d-1}}.$$

Moreover, $\varphi(\cdot, u)$ is bounded in B. Therefore,

$$\frac{\partial G_B(\varphi(\cdot, u))}{\partial x_i}(x_0) = \int_D \frac{\partial G_B(x_0, y)}{\partial x_i} \varphi(y, u(y))\, dy$$

and we may deduce that u is differentiable on x_0 and then in all D.

Now, if $\varphi \in \mathcal{C}^{\alpha}_{loc}(D\times]0, \infty[)$ then we first deduce that $u \in \mathcal{C}^{1}(D)$ and then we take D_1 to be a regular bounded domain with $C^{1,1}$ boundary such that $\bar{D_1} \subset D$. We denote u_L the solution of problem (1.7) so, $\varphi(\cdot, u_L) \in \mathcal{C}^{\alpha}(D_1)$ bounded in D_1. Then by what has been said above, $\varphi(\cdot, u_L) \in \mathcal{C}^{\alpha}(D_1)$ and $u_L|_{\partial D_1} \in C(\partial D_1)$. We denote $\psi = u_{L/\partial D_1}$ which is continuous on ∂D_1. Hence by [[10] p. 101] we deduce that $u_L \in \mathcal{C}^{2,\alpha}_{loc}(D)$ $\cap\mathcal{C}(\bar{D})$. □

4 Solution in a Greenian Domain without Boundary Condition

In this section, we establish one-to-one correspondence between nonnegative L-harmonic functions and nonnegative continuous solutions of the Eq. 1.8 in an arbitrary Greenian domain:

Theorem 14 *Let* Ω *be a Greenian domain, L a second order elliptic operator with smooth coefficients satisfying* $L1 = 0$ *and let* $\varphi : \Omega\times]0, \infty[\mapsto [0, \infty[$ *be a measurable function*

[5] See [19] for more details.

satisfying H_1-H_4. Then there is a one-to-one correspondence between nonnegative continuous solutions of Eq. 1.8 and nonnegative L-harmonic functions in Ω Furthermore:

$$u - G_\Omega(\varphi(\cdot, u)) = h,$$

u is a minimal solution satisfying $u > h$ and h is a maximal L-harmonic function dominated by u .

Before we prove the main result, we need two lemmas analogous to Lemmas 5.1 and 5.2 in [7].

Lemma 15 *Let $\varphi : \Omega\times]0, \infty[\to [0, \infty[$ a measurable function satisfying H_1-H_2, $u_i \in \mathcal{C}^+(\Omega)$, $u_i > 0$, $h_i \in \mathcal{C}^+(\Omega)$ such that*

$$h_i = u_i - G_\Omega(\varphi(\cdot, u_i)), \quad 1 \le i \le 2.$$

If $h_1 - h_2$ is L-superharmonic positive function in Ω then:

$$u_1 - u_2 \ge 0.$$

Proof We are going to apply Proposition 44. Let

$$K = \{x \in \Omega, (u_1 - u_2)(x) \ge 0\}.$$

By assumption, K is closed and non empty. Let

$$v = \varphi(\cdot, u_1) - \varphi(\cdot, u_2).$$

Then

$$h_1 - h_2 + G_\Omega(v^+) = u_1 - u_2 + G_\Omega(v^-),$$

with $t^+ = max(t, 0)$ and $t^- = max(-t, 0)$. It is clear that $v^+ \in L^1_{loc}(\Omega)$, because

$$0 \le v^+ \le \varphi(\cdot, u_1) + \varphi(\cdot, u_2) \in L^1_{loc}(\Omega).$$

Also $G_\Omega(v^+) \in L^1_{loc}(\Omega)$, because

$$0 \le G_\Omega(v^+) \le G_\Omega(\varphi(\cdot, u_1)) + G_\Omega(\varphi(\cdot, u_2)),$$

the latter being continuous. So by Proposition 45

$$L(G_\Omega(v^+))= - v^+, \text{ in the sense of distributions in } \Omega.$$

Therefore, $h_1 - h_2 + G_\Omega(v^+)$ is a L-superharmonic positive function in Ω so it's lower semi-continuous on $\overline{\Omega - K}$. In addition, $G_\Omega(v^-)$ is a potential L-harmonic in $\Omega \setminus K$, because v^- is supported in K and

$$L(G_\Omega(v^-))= - v^-, \text{ in the sense of distributions in } \Omega.$$

Furthermore, it's clear that $G_\Omega(v^-)$ is lower semi-continuous. Also, $G_\Omega(v^-) = G_\Omega(\frac{v-|v|}{2})$ is upper semi-continuous because $G_\Omega(v)$ is continuous and $G_\Omega(-|v|)$ is upper semi-continuous.

Finally,

$$h_1 - h_2 + G_\Omega(v^+) \ge G_\Omega(v^-),$$

on the boundary of K.

We can conclude by Proposition 44

$$h_1 - h_2 + G_\Omega(v^+) \ge G_\Omega(v^-),$$

holds everywhere which implies that $u_1 - u_2 \ge 0$ in Ω. □

Lemma 16 *Let* $(H_1) - (H_2)$ *be satisfied and* (u_n) *be an increasing sequence of positive continuous solutions of Eq. 1.8 in* Ω. *Then,* $u = \sup u_n$ *is either identically* $+\infty$ *or it is a continuous solution in* Ω.

Proof Suppose that $\lim_{n\to+\infty} u_n = u$ is not identically $+\infty$ in Ω. Then there exists x_0 such that $\lim_{n\to+\infty} u_n(x_0) = u(x_0) < \infty$. Let D, D' be regular bounded domains of Ω, such that $x_0 \in D'$ and $\overline{D'} \subset D$. By Theorem 8

$$u_n(x) = H_D u_n(x) + \int_D G_D(x, y)\varphi(\cdot, u_n(y))\, dy, \text{ in } D.$$

Hence,

$$H_D u_n(x_0) \le u(x_0)$$

so by Harnack inequality there exists $c > 0$ such that for every $n \in \mathbb{N}$ and $x \in D'$

$$H_D u_n(x) \le c u(x_0),$$

then $H_D u$ is a positive L-harmonic function in D'. Also

$$0 < H_D u \le u, \text{ in } D'.$$

Consequently, by the monotone convergence theorem, we get:

$$u(x) = H_D u(x) + \int_D G_D(x, y)\varphi(\cdot, u(y))\, dy \text{ in } D,$$

which is finite at least on one point x_0. Also, one one hand, u is a limit of an increasing sequence of continuous functions so it's lower semi-continuous and on the other hand, $u - H_D u$ is the limit of $u_n - H_D u_n = G_D(\varphi(\cdot, u_n))$ which is a decreasing sequence of continuous functions then $u - H_D u$ is upper-semi-continuous. Since $H_D u$ is continuous we can conclude the continuity of u as well as $G_D(\varphi(\cdot, u))$.

In addition, u_1 is a continuous, positive function on $\bar{D}$ which compact, so $\alpha = \inf_D u_1 > 0$. Therefore:

$$\varphi(\cdot, u) \le \varphi(\cdot, u_1) \le \varphi(\cdot, \alpha) \in L^1(D).$$

Whence, by Proposition Eq 38:

$$L(G_D(\varphi(\cdot, u))) = -\varphi(\cdot, u), \text{ in the sense of distributions in } D.$$

□

Now we are ready to prove Theorem 14:

Proof Let u be a nonnegative continuous solution of Eq. 1.8.

Let (D_n) be a sequence of bounded regular domains exhausting Ω i.e.

$$\bar{D}_n \subset D_{n+1} \text{ and } \cup D_n = \Omega.$$

Since by Theorem 8, $u \in \mathcal{C}^+(\partial D_n)$, we have:

$$u = H_{D_n}(u) + G_{D_n}(\varphi(\cdot, u)) \;\; in \; D_n.$$

On one hand, $(H_{D_n}(u))$ is nonincreasing. Indeed,

$$\begin{cases} L(H_{D_n}(u) - H_{D_{n+1}}(u)) = 0, & \text{in } D_n; \\ H_{D_n}(u) - H_{D_{n+1}}(u) = u - H_{D_{n+1}} u \ge 0, & \text{on } \partial D_n, \end{cases}$$

which implies by the maximal principle $H_{D_n}(u) \geq H_{D_{n+1}}(u) \geq 0$, $\lim_{n\to\infty} H_{D_n}(u)(x) = h(x)$ exists for every $x \in \Omega$ and $Lh = 0$.

On the other hand, by the monotone convergence theorem we have, $(G_{D_n}(\varphi(\cdot, u))) \nearrow G_\Omega(\varphi(\cdot, u))$.
Hence we may conclude that

$$u = h(x) + G_\Omega(\varphi(\cdot, u)).$$

Now we turn our attention to prove that h is the maximal L-harmonic solution dominated by u: suppose that there is another L-harmonic nonnegative function h_1 such that:

$$0 < h_1 \leq u \ \ in \ \Omega.$$

Or $h_1 \leq H_{D_n}(u) \ \ in \ D_n$ because:

$$\begin{cases} L(h_1 - H_{D_n}(u)) = 0, & \text{in } D_n; \\ h_1 - H_{D_n}(u) = h_1 - u \leq 0, & \text{on } \partial D_n. \end{cases}$$

When n tends to $+\infty$, we get:

$$h_1 \leq h.$$

Let h be a nonnegative L-harmonic function in Ω.

By Theorem 8, we know that there is a unique continuous solution u_n of:

$$\begin{cases} Lu + \varphi(\cdot, u) = 0, & \text{in } D_n; \\ u = h, & \text{on } \partial D_n, \end{cases}$$

and

$$u_n = h + G_{D_n}(\varphi(\cdot, u_n)) \text{ in } D_n.$$

In addition the sequence (u_n) is not decreasing. In fact, by Lemma 9:

$$\begin{cases} Lu_n + \varphi(\cdot, u_n) = Lu_{n+1} + \varphi(\cdot, u_{n+1}) = 0, & \text{in } D_n; \\ u_n - u_{n+1} = h - u_{n+1} \leq 0, & \text{on } \partial D_n, \end{cases}$$

and so $u_n \leq u_{n+1}$ in D_n.

So by Lemma 16 $\sup_n u_n = u$ can be $+\infty$ almost everywhere or a solution of the equation. Therefore we have to prove that u is finite in Ω.

First case: Suppose $\inf_\Omega h = \epsilon > 0$:

Then we have:

$$0 < \epsilon \leq h \leq u_n.$$

So: $\varphi(\cdot, u_n) \leq \varphi(\cdot, \epsilon)$ which implies:

$$0 < u_n \leq h + G_{D_n}(\varphi(\cdot, \epsilon)).$$

Using (H_3) and tending n to $+\infty$ we get:

$$0 \leq u \leq h + G_\Omega(\varphi(\cdot, \epsilon)).$$

We can conclude that u is finite which implies that u is a continuous solution satisfying:

$$u = h + G_\Omega(\varphi(\cdot, u)).$$

Second case: Let h be just a nonnegative L-harmonic function:

Then we take

$$h_k = h + \frac{1}{k} \qquad , k \in \mathbb{N}^*.$$

Using $L1 = 0$, h_k is L-harmonic too so by the first step we get a continuous solution u_k such that :

$$u_k = h_k + G_\Omega(\varphi(\cdot, u_k)) \text{ in } \Omega.$$

Or $h_k \geq h_{k+1}$ so by Lemma 15 u_k is a nonincreasing sequence in Ω. We denote $u = \lim u_k$. As before, by upper and lower semi-continuity, we deduce that u is continuous in Ω. Now, we turn to prove that:

$$u > 0, \text{ in } \Omega. \tag{4.1}$$

Notice here if h is a nonnegative nontrivial L-harmonic function in Ω then $h > 0$ and so $u > 0$ in Ω. Otherwise, $h \equiv 0$, then we suppose that there exists $x_0 \in \Omega$ such that $u(x_0) = 0$. Though

$$0 \leq u_n - h_n \leq u - h = u, \text{ in } \Omega.$$

Hence

$$\begin{cases} L(u_n - h_n) = -\varphi(\cdot, u_n) \leq 0 & \text{in } \Omega; \\ (u_n - h_n)(x_0) = 0, \end{cases}$$

However, any L-superharmonic function nonconstant cannot attain its negative minimum inside Ω, so $u_n = h_n = \frac{1}{n}$ which implies that $G_{D_n}\varphi(\cdot, \frac{1}{n})(x_0) = 0$. This is a contradiction with H_4 and hence (4.1) holds. Following this, $\lim_{n\to+\infty} \varphi(\cdot, u_n) = \varphi(\cdot, u) < \infty$ so by the monotone convergence theorem, we get:

$$u = h + G_\Omega(\varphi(\cdot, u)) \text{ in } \Omega.$$

Further, for every compact set K in Ω, there exists $\alpha > 0$ such that $u(x) > \alpha$ for $x \in K$. Therefore, $\varphi(\cdot, u) \leq \varphi(\cdot, \alpha) \in L^1(K)$. Also, $G_\Omega(\varphi(\cdot, u)) = u - h$ is continuous, by Corollary 39 we may conclude

$$L(G_\Omega(\varphi(\cdot, u))) = -\varphi(\cdot, u), \quad in\ \Omega, \text{ in the sense of distributions.}$$

Finally, u is the minimal solution satisfying $u > h$: Let v be a continuous function on Ω satisfying:

$$\begin{cases} Lv + \varphi(\cdot, v) = 0, & \text{in } \Omega \\ h < v, & \text{in } \Omega. \end{cases}$$

By the first part of the proof we have

$$v = \lim_{n\to\infty} H_{D_n}(v) + G_\Omega(\varphi(\cdot, v)).$$

And $h_v = \lim_{n\to\infty} H_{D_n}(v)$ satisfies $h_v \geq h$. So by Lemma 15 we get $v \geq u$. This shows minimality of u. □

Remark 17 Theorem 14 generalizes Theorem 2.1 in [7].

The above proof suggests also the following corollary about bounded solutions.

Corollary 18 *Let h be a nonnegative bounded L-harmonic function in Ω. Suppose that the assumptions of Theorem 14, and in addition that there is $c > 0$ such that $G_\Omega(\varphi(\cdot, c)) \in L^\infty(\Omega)$. Then the continuous solution of Eq. 1.8 in Ω given by $u = h + G_\Omega(\varphi(\cdot, u))$ is bounded in Ω. Following this, there is one-to-one correspondence between L-harmonic nonnegative bounded functions and nonnegative bounded continuous solutions of Eq. 1.8.*

Proof By the proof of Theorem 14, u is the limit of a decreasing sequence of solution of Eq. 1.8 given by $u_n = h + \frac{1}{n} + G_\Omega(\varphi(\cdot, u_n))$. Let u_c be the solution of Eq. 1.8 given by $u_c = h + c + G_\Omega(\varphi(\cdot, u_c))$. Then by Lemma 15 $u_c \geq u_n$ for n big enough. So, $0 < u \leq u_n \leq u_c \leq h + c + G_\Omega(\varphi(\cdot, c))$. Moreover, by assumption h and $G_\Omega(\varphi(\cdot, c))$ are bounded in Ω. Hence we may conclude that u is bounded too. □

We obtain also a statement about regularity of solutions in a Greenian domain analogous to that for a bounded regular domain.

Theorem 19 *Suppose the same hypotheses as in Theorem 14, and assume in addition that* $\varphi(\cdot, c) \in L^{\infty}_{loc}(\Omega)$*, for every* $c > 0$*. Then for every continuous solution of Eq. 1.8 in* Ω *we have* $u \in \mathcal{C}^1(\Omega)$*. Furthermore, if we suppose that* $\varphi \in \mathcal{C}^{\alpha}_{loc}(\Omega \times]0, \infty[)$ *then* $u \in \mathcal{C}^{2,\alpha}_{loc}(\Omega)$.

Proof Let u be a continuous solution of Eq. 1.8 in Ω. Let D a bounded regular domain such that $\bar{D} \subset \Omega$. We denote $f = u_{\partial D}$. By Theorem 8

$$u = H_D f + G_D(\varphi(\cdot, u)).$$

By Theorem 12, $u \in \mathcal{C}^1(\Omega)$. Now, if $\varphi \in \mathcal{C}^{\alpha}_{loc}(\Omega \times]0, \infty[)$ then $\varphi(\cdot, u) \in \mathcal{C}^{\alpha}(D)$ and by Theorem 12, $u \in \mathcal{C}^{2,\alpha}_{loc}(\Omega) \cap \mathcal{C}(\bar{\Omega})$. □

5 Boundary Condition

In this section, for a given a nonnegative L-harmonic function h in a Greenian domain Ω, we give a sufficient and necessary conditions in order that the corresponding solution of the Eq. 1.8 takes the same values of h at the boundary, see Theorem 24 where $\varphi(x, t) = p(x)\psi(t)$. However, the same conditions become just sufficient for non-product φ:

Theorem 20 *Suppose that for every* $c > 0$*,* $G_{\Omega}(\varphi(\cdot, c))$ *vanishes at* $\partial\Omega$ *and that the assumptions of Theorem 14 are satisfied. Then for for every nonnegative* L*-harmonic function* h *there exists a unique nonnegative continuous solution of the problem* (1.1).

Proof Let h be a L-harmonic positive function, by Theorem 14, there exists a positive continuous solution u such that

$$u = h + G_{\Omega}(\varphi(\cdot, u)).$$

Thanks to (H_4), $u > h$ because $u - h = G_{\Omega}(\varphi(\cdot, u)) > 0$.

Furthermore, we denote $h_k = h + \frac{1}{k}$ which L-harmonic then there exists u_k a positive continuous solution of Eq. 1.8 such that $u_k = h_k + G_{\Omega}(\varphi(\cdot, u_k))$. Hence $0 < u_k - h_k = G_{\Omega}(\varphi(\cdot, u_k)) \leq G_{\Omega}(\varphi(\cdot, \frac{1}{k}))$ which vanishes at $\partial\Omega$. In addition, by Lemma 15

$$0 < u - h < u_k - h = u_k - h_k + \frac{1}{k}.$$

Then $0 \leq \limsup_{x \to \partial\Omega} (u - h)(x) \leq \frac{1}{k}$. By tending k to ∞ we obtain $u - h = 0$ on $\partial\Omega$. Moreover, by using Lemma 15, we conclude the uniqueness of solution. □

Remark 21 The theorem remains true if we replace $\partial\Omega$ by $\partial\Omega \cup \{\infty\}$ provided that for every $c > 0$, $G_{\Omega}(\varphi(\cdot, c)) \in C_0(\Omega)$. Unfortunately we cannot prove the converse statement. We have only the following theorem.

Theorem 22 *Let* Ω*,* φ *as in Theorem 14. We suppose that there exists a positive continuous solution of problem* (1.2)*. Then for all* $c \geq \sup_{x \in \Omega} u(x)$*,* $G_{\Omega}(\varphi(\cdot, c))$ *vanishes at the boundary.*

Proof u is continuous vanishes at $\partial\Omega$ so it is clear that it is bounded in Ω. We denote $M = \sup\limits_{x\in\Omega} u(x)$. Then

$$0 \le G_\Omega(\varphi(\cdot, c)) \le G_\Omega(\varphi(\cdot, M)) \le G_\Omega(\varphi(\cdot, u)) \in C_0(\Omega).$$

□

Remark 23 As we have seen until now $G_\Omega(\varphi(\cdot, c))$ vanishing at the boundary for every $c > 0$ is just a sufficient condition for the existence of solution. However, in the special case when $\varphi(x, y) = p(x)\psi(y)$, $p \in K_d^{loc}(\Omega)$ positive and ψ is a positive continuous decreasing function (as in [7]) one can easily formulate a necessary and sufficient condition for the solution of Eq. 1.1. The following theorem generalizes Theorem 6.1 in [7].

Theorem 24 *Under the same hypotheses as in Theorem 14, with $\varphi(x, t) = p(x)\psi(t)$ the problem (1.2) has a solution if and only if $G_\Omega(p)$ vanishes at the boundary.*

Proof By Theorem 20, we know that if $G_\Omega(p)$ vanishes at the boundary the problem (1.2) has a solution. For the converse implication let u be the solution of Eq. 1.2 then u is bounded. We denote $M = \sup\limits_{x\in\Omega} u(x)$. Then

$$0 \le G_\Omega(p)(x) \le \int_\Omega G_\Omega(x, y)p(y)\frac{\psi(u)(y)}{\psi(M)}\,dy = \frac{u(x)}{\psi(M)} \in \mathcal{C}_0(\Omega).$$

□

Appendix

Let Ω be a domain in $\mathbb{R}^d$, $d \ge 3$ and let L be a second order elliptic operator with smooth coefficients defined in Ω i.e.

$$L = \sum_{1\le i,j\le d} a_{i,j}(x)\partial_i\partial_j + \sum_{1\le i\le d} b_i(x)\partial_i + c(x),$$

with $a_{i,j}(x) = a_{j,i}(x)$, $1 \le i, j \le d$ and for every $x \in \Omega$ the quadratic form

$$\sum_{i,j} a_{i,j}(x)\xi_i\xi_j$$

is strictly positive definite i.e $\sum\limits_{i,j} a_{i,j}(x)\xi_i\xi_j > 0$ for every $x \in \Omega$ and $\xi \in \mathbb{R}^d\backslash\{0\}$.

Remark 25 Notice that L is locally uniformly elliptic in Ω.

Throughout all this section, we suppose that $L1 \le 0$ in Ω and there exists $s \in \mathcal{C}^\infty(\Omega)$ such that $s > 0$ and $Ls < 0$ in Ω. Such function can always exist if Ω is bounded and $L1 \le 0$.

We consider at first D a bounded domain contained with its closure in Ω with smooth boundary ($C^{1,\alpha}$ boundary is enough) where we represent basic notions of potential theory and most important properties of the corresponding Green function G_D. Afterwards, we justify the existence of Green function in Ω denoted G_Ω and we discuss her properties.

In a Regular Bounded Domain

The operator L satisfies the following properties in D provided that $L1 \leq 0$ in D and D is a bounded domain with smooth boundary contained with its closure in Ω:

Basic Properties

Theorem 26 (Strong maximum and minimum principle). (see [10] p.34 or [1] p.37) *Let $u \in \mathcal{C}^2(D) \cap \mathcal{C}(\overline{D})$ such that $Lu \geq 0 (\leq 0)$ in D. then u cannot achieve a nonnegative maximum (nonpositive minimum) in the interior of D unless it is constant.*

Proposition 27 (Weak maximum principle) *[see [10] p.31] Let $u \in \mathcal{C}^2(D) \cap \mathcal{C}(\bar{D})$ such that $Lu \leq 0$ in D and $u_{/\partial D} \geq 0$ then*

$$u \geq 0.$$

Theorem 28 (Harnack inequality. (see [2] p.299 or [10] p.40)) *For every compact set K contained in D, there exists a constant $\alpha_K > 0$ such that such for every positive L-harmonic function h in D and every multiindex I*

$$\sup_{y \in K} |\partial^I h(y)| \leq \alpha \inf_{y \in K} h(y)$$

Theorem 29 (Ameliorate version of Dirichlet problem). (see [19] p.75 or [10] p.101) *Let $f \in \mathcal{C}^\alpha(\overline{D})$ and $\psi \in \mathcal{C}(\partial D)$ then there exist a unique solution $u \in \mathcal{C}^{2,\alpha}(D) \cap \mathcal{C}(\overline{D})$ of*

$$\begin{cases} Lu = -f, & \textit{in the sense of distributions in } D; \\ u = \psi, & \textit{in } \partial D. \end{cases}$$

Remark 30 For $f \in \mathcal{C}(\bar{D})$, we can use a standard approximation of f which is explained e.g. in [2].

Definition 31 For $\psi = 0$ and $f \in \mathcal{C}(\overline{D})$ there exists a unique solution of

$$\begin{cases} Lu = -f, & \text{in the sense of distributions in } D; \\ u = 0, & \text{in } \partial D. \end{cases}$$

denoted $G_D f$. Then we can define the Green operator by:

$$\begin{aligned} G_D : \quad \mathcal{C}(\overline{D}) &\to \mathcal{C}(\overline{D}) \\ f &\mapsto G_D f \end{aligned}$$

Theorem 32 (Existence of Green function). (See [2] p.295 or [19] p.20) *There exists a function $G_D(x, y)$ called a Green function $\mathcal{C}^\infty$ outside the diagonal such that for every $f \in \mathcal{C}(\overline{D})$*

$$G_D f(x) = \int_D G_D(x, y) f(y)\, dy.$$

Furthermore, for every $y \in D$:

- $L(G_D(\cdot, y)) = -\epsilon_y$, *in the sense of distributions in D.*
- $\lim\limits_{x \to \partial D} G_D(\cdot, y) = 0.$

Now using the weak maximum principle, we can easily prove that for $\psi \in \mathcal{C}_c^\infty(D)$

$$G_D(L\psi) = -\psi, \text{ in the sense of distributions in } D,$$

In other words, G_D commute with L.

The Adjoint Operator

We denote L^* the adjoint operator of L.

Proposition 33 *Let $x \in D$ then*

$$L^*(G_D(x, \cdot)) = -\epsilon_x, \textit{ in the sense if distributions in } D.$$

Proof Let $\psi \in \mathcal{C}_c^\infty(D)$ then

$$\begin{aligned}\int_D L^*(G_D(x, y))\psi(y)\, dy &= \int_D G_D(x, y) L\psi(y)\, dy \\ &= G_D(L\psi)(x) = -\psi(x)\end{aligned}$$

□

Now, we denote

$$s^*(y) = \int_D G_D(x, y)\, dx \text{ for } y \in D.$$

It is clear that

$$L^* s^* = -\mathbb{1}_D, \text{ in the sense of distributions.}$$

where $\mathbb{1}$ is the characteristic function of D. So s^* is a smooth function satisfying $s^* > 0$ and $L^* s^* < 0$.

Such s^* can be constructed in any bounded regular domain which allows us to conjugate L^* and to obtain the corresponding preceding properties i.e. we define a new operator L_1^* by

$$L_1^* u = \frac{1}{s^*} L^*(s^* u).$$

We have $L_1^* 1 < 0$, so all the preceding properties are true for L_1^*. Following this we can also obtain for L^* the weak maximum principle, solvability of the Dirichlet problem and the Green function G_D^* in D.

Preceding as in [2] we prove that:

Proposition 34

$$G_D^*(x, y) = G_D(y, x). \tag{1}$$

Properties of the Green Function in Bounded Regular Domain

Proposition 35 *G_D is uniformly continuous in every compact set contained in $D \times \overline{D} \backslash \Delta$, where $\Delta = \{(x, y) \in D \times D, \ x = y\}$.*

Proof It is enough to prove that if $(x_n, y_n) \to (x, y) \in D \times \partial D$ then $G_D(x_n, y_n) \to G_D(x, y)$. We have

$$|G_D(x_n, y_n) - G_D(x, y)| \\ \leq |G_D(x_n, y_n) - G_D(x, y_n)| + |G_D(x, y_n) - G_D(x, y)|$$

We choose γ enough small such that $\overline{B(x,\gamma)} \subset D$ and $x_n \in \overline{B(x, \frac{\gamma}{2})}$ for every $n \geq n_0$. By Harnack inequality applied to the family $G_D(\cdot, y_n)$ of L-harmonic functions, we get

$$|G_D(x_n, y_n) - G_D(x, y_n)| \leq \sup_{t \in \overline{B(x,\frac{\gamma}{2})}} |\nabla G_D(t, y_n)||x_n - x| \\ \leq C_x G_D(x, y_n)|x_n - x|$$

Also

$$\lim_{n\to+\infty} G_D(x, y_n) = G_D(x, y), \text{ for every } t \in D,$$

and so we can deduce the result. □

In what follows we recall some estimations of the Green function in D which facilitate the generalisation from the case of Laplace operator to the general elliptic operator.

Proposition 36 • *There is $C > 0$ such that*

$$0 \leq G_D(x, y) \leq \frac{C}{|x-y|^{d-2}}, \qquad \textit{for every } x, y \in D, \tag{2}$$

(see [16]*).*

• *Let $x \in D$ and D_1 a compact set in D then there is $C > 0$ such that*

$$0 \leq |\partial_{x_i} G_D(x, y)| \leq \frac{C}{|x-y|^{d-1}}, \qquad \textit{for every } y \in D_1. \tag{3}$$

(see [19]*).*

Using (2) we obtain

Proposition 37 *Let $f \in L^\infty(D)$ then*

$$G_D f \in \mathcal{C}_0(D).$$

Proposition 38 *Let $f \in L^1(D)$. Then $G_D f \in L^1(D)$ and :*

$$L(G_D f) = -f, \textit{ in } D \textit{ in the sense of distributions.}$$

Proof First, we'll prove that $G_D|f| \in L^1(D)$. Indeed, $G_D^*(\mathbb{1}_D)$ is a continuous function on $\bar{D}$ so

$$\int_D |f(y)| G_D^*(\mathbb{1}_D)(y)\, dy < \infty.$$

By the Fubini theorem we get

$$\int_D G_D|f(x)|\, dx < \infty.$$

Secondly, let $\psi \in C_c^\infty(\bar{D})$. As before,

$$\int_D G_D^*(|L^*\psi|)(y)|f(y)|\,dy < \infty$$

and in addition, $G_D^*(L^*(\psi)) = -\psi$. Again writing the Fubini theorem, we obtain:

$$\int_D L(G_D f)(x)\psi(x)\,dx = -\int_D f(y)\psi(y)\,dy.$$

□

Corollary 39 *By the same proof we obtain*

$$L(G_D(f)) = -f, \text{ in the sense of distributions, in } D \tag{4}$$

for $f \in L^1_{loc}(D)$ *and* $G_D(|f|) \in L^1_{loc}(D)$.

In a Greenian Domain Ω

We consider (D_n) an increasing sequence of regular bounded domains exhausting Ω i.e.

$$\overline{D_n} \subset D_{n+1} \text{ and } \cup D_n = \Omega.$$

Existence of the Green Function in Ω

Using weak maximum principle we may easily justify:

Proposition 40 • (G_{D_n}) *is an increasing sequence.*
• $G_{D_n}(-Ls) \le s$ *for every* $n \in \mathbb{N}$.

Proposition 41 *Let* $f \in \mathcal{C}_c^+(\Omega)$. *Then* $(G_{D_n} f)$ *is convergent.*

Proof It is clear that $(G_{D_n} f)$ is increasing, so it is enough to prove that $(G_{D_n} f)$ is bounded. We denote k_f the support of f. The function $x \mapsto -Ls(x)$ is continuous on k_f so it is bounded on k_f i.e. there exists a constant $c_k > 0$ such that for every $x \in k_f$ $c_k \le -Ls(x)$, and then

$$\begin{aligned} 0 \le G_{D_n} f &\le \sup_{x\in\Omega} f(x) \;\; G_{D_n}\Big(\frac{-Ls}{c_k}\Big) \\ &\le \sup_{x\in\Omega} f(x) \;\; \frac{s}{c_k} \end{aligned}$$

□

It follows by Riez theorem that there exists a function G_Ω such that $\lim_{n\to+\infty} G_{D_n}(x, y) = G_\Omega(x, y)$. It follows that $G_\Omega^*(x, y) = G_\Omega(y, x)$ where G_Ω^* is the Green function corresponding to L^* in Ω. We can check easily by monotone convergence theorem that

- $G_\Omega(L\psi) = -\psi$, in the sense of distributions in Ω for every $\psi \in \mathcal{C}_c^\infty(\Omega)$.
- $L(G_\Omega(\cdot, y)) = -\epsilon_y$ in the sense of distributions in Ω for every $y \in \Omega$.
- $L^*(G_\Omega(x, \cdot)) = -\epsilon_x$ in the sense of distributions in Ω for every $x \in \Omega$.

Properties in a Greenian Domain Ω

In a Greenian domain Ω, we can obtain a generalized version of the maximum principle as follows:

Proposition 42 *Let* $f \in L^1_{loc}(\Omega)$, $Lf \in L^1_{loc}(\Omega)$, $Lf \leq 0$ *as a distributions and* $\liminf_{z\to\partial\Omega} f(z) \geq 0$. *Then*

$$f \geq 0 \text{ in } \Omega.$$

Proof Let D be a bounded regular domain. By Corollary 39

$$L(G_D(-Lf) - f) = 0.$$

Therefore, there is a L-harmonic function h such that

$$f = G_D(-Lf) - h$$

which implies that f is lower semi-continuous and satisfies the super mean value property. The result follows by the minimum principle for so called L-superharmonic functions in the sense of the classical potential theory (see [14] p.427-8). □

Now we focus on properties of the Green function G_Ω. First we recall the definition of potential in Ω.

Definition 43 We say that a function p is a potential if p is L-superharmonic positive and if

$$\begin{cases} 0 \leq h \leq p, & \text{in } \Omega, \\ Lh = 0 \end{cases}$$

then $h = 0$. We denote $p \in \mathcal{P}(\Omega)$.

As examples, we can mention that G_{D_n} and G_Ω are potentials.

Proposition 44 *Let* K *be a closed set in* Ω, f *L-superharmonic positive in* $\Omega - K$, f *lower semi-continuous on* $\overline{\Omega - k}$, p *a potential in* Ω, $f \geq p$ *on* ∂K, $p \in C(\overline{\Omega - K})$, p *is L-harmonic in* $\Omega - K$ *then* $f \geq p$ *on* $\Omega - K$. *See* [14] *p.429.*

Proposition 45 *Let* $f \in L^1_{loc}(\Omega)$ *and* $G_\Omega(|f|) \in L^1_{loc}(\Omega)$. *Then:*

$$L(G_\Omega f) = -f, \text{ in the sense of distributions.}$$

Proof $L(G_{D_n} f) = -f$, in the sense of distributions on D_n and for $\psi \in C_c^\infty$ we have

$$|(G_{D_n}(f) - G_\Omega(f))L^*\psi| \leq 2G_\Omega(|f|)|L^*\psi|,$$

which is integrable. Moreover, $\lim_{n\to+\infty} G_{D_n}(|f|) = G_\Omega(|f|)$. Hence by the dominated convergence theorem we may conclude:

$$\lim_{n\to+\infty} L(G_{D_n}(f)) = L(G_\Omega(f)), \text{ in the sense of distributions in } \Omega.$$

So:

$$L(G_\Omega f) = -f, \text{ in the sense of distributions in } \Omega\ .$$

□

Proposition 46 *Let* $f \in L^{\infty}_{loc}(\Omega)$*, and* $G_{\Omega}(|f|) \in L^{1}_{loc}(\Omega)$ *then* $G_{\Omega} f$ *is continuous in* Ω*.*

Proof By the previous proposition, $L(G_{\Omega} f) = -f$, in the sense of distributions in Ω. Hence for any bounded regular domain D such that $\bar{D} \subset \Omega$, $L(G_{\Omega} f - G_D f) = 0$, in the sense of distributions $in\ D$.

So $G_{\Omega} f - G_D f$ is L-harmonic in D and $G_D f$ is continuous which allows us to conclude. □

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