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Constraints on the applicability range of pressure-sensitive yield/failure criteria: strong orthotropy or transverse isotropy

Abstract Yield/failure initiation criteria discussed in this paper account for the three following effects: the hydrostatic pressure dependence, the tension/compression asymmetry, and isotropic or anisotropic material response. For isotropic materials, criteria accounting for pressure/compression asymmetry (strength differential effect) have to include all three stress invariants (Iyer and Lissenden in Int J Plast 19:2055–2081, 2003; Gao et al. in Int J Plast 27:217–231, 2011; Yoon et al. in Int J Plast 56:184–202, 2014; Coulomb–Mohr's, cf. Chen and Han in Plasticity for structural engineers. Springer, Berlin, 1995 criteria). In narrower case when only pressure sensitivity is accounted for, rotationally symmetric surfaces independent of the third invariant are considered and broadly discussed (Burzyński in study on strength hypotheses (in Polish). Akad Nauk Tech Lwów, 1928; Drucker and Prager in Q Appl Math 10:157–165, 1952 criteria). For anisotropic materials, the explicit formulation based on either all three common invariants (Goldenblat and Kopnov in Stroit Mekh 307–319, 1966; Kowalsky et al. in Comput Mater Sci 16:81–88, 1999) or first and second common invariants (extended von Mises–type Tsai–Wu's criterion in Int J NumerMethods Eng 38:2083–2088, 1971) is addressed especially for the case of transverse isotropy, when difference between tetragonal versus hexagonal symmetry is highlighted. The classical Tsai and Wu criterion involves Hill's type fourth-rank tensor inheriting a possibility of convexity loss in case of strong orthotropy, as discussed by Ganczarski and Skrzypek (Acta Mech 225:2563–2582, 2014). In order to overcome this defect, in the present paper the new Mises-based Tsai–Wu's criterion is proposed and exemplary implemented for the columnar ice. A mixed way to formulate pressure-sensitive tension/compression asymmetric failure criteria-capable of describing fully distorted limit surfaces, which are based on both all stress invariants and the second common invariant (Khan and Liu in Int J Plast 38:14–26, 2012; Yoon et al. in Int J Plast 56:184–202, 2014), is revised and addressed to orthotropic materials for which the fourth-order linear transformation tensors are used to achieve extension of the isotropic criterion.

1 Introduction

Yield conditions discussed in [16] are applicable for ductile materials in which it is justified to ignore both the tension/compression asymmetry and the hydrostatic pressure sensitivity. In a majority of metallic polycrystalline materials, a termination of the elastic range corresponds to initiation of plastic microslips. In the case of brittle materials, such as the majority of ceramic materials, rocks, concrete, Ceramic Matrix Composites CMC, columnar ice, material failure is initiated not by the plastic slips but by microcracks (damage) which in the way of evolution and aggregation processes may lead to initiation and formation of macrocracks (failure).

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Since the present paper is an extension of previous work [16] for a case of limit surfaces of pressure-sensitive materials, it is essential to invoke the general formulation of limit surface based on the Goldenblat and Kopnov concept of common invariants.

In a most general case of both elastic and plastic material anisotropy, extension of the isotropic plastic yield initiation criterion to the anisotropic flow by the use of common invariants of the stress tensor and of the structural tensors of plastic anisotropy (cf. Hill [24], Sayir [50], Betten [3], Życzkowski [62]), can be shown in a general fashion

$$f = f\left(\mathbf{\Pi}^{(0)}, \mathbf{\Pi}^{(2)} : \boldsymbol{\sigma}, \boldsymbol{\sigma} : \mathbf{\Pi}^{(4)} : \boldsymbol{\sigma}, \boldsymbol{\sigma} : \mathbf{\Pi}^{(6)} : \boldsymbol{\sigma} : \boldsymbol{\sigma}, \dots\right) = 0 \tag{1}$$

where Einstein's summation holds. Note that the general form Eq. (1) uses unspecified representation of an unlimited number of common invariants. In such a case, initiation of plastic flow is governed by the structural tensors of plastic anisotropy of even-ranks: $\mathbf{\Pi}^{(0)} = \Pi$, $\mathbf{\Pi}^{(2)}$, $\mathbf{\Pi}^{(4)}$, $\mathbf{\Pi}^{(6)}$, ..., etc. Equation (1) owns a general representation, but its practical identification is limited by a large number of required material tests and, additionally, because the components of the structural tensors are temperature dependent, which makes identification much more complicated (cf. e.g. Herakovich and Aboudi [23], Tamma and Avila [56]). Hence, the general form (1) is usually specialized for engineering needs.

In a particular case, Goldenblat and Kopnov [21], and later Sayir [50], proposed a polynomial representation for Eq. (1) which controls initiation of anisotropic plastic flow or failure in a material by the tensorial-polynomial anisotropic criterion

$$(\boldsymbol{\pi} : \boldsymbol{\sigma})^{\alpha} + (\boldsymbol{\sigma} : \mathbf{\Pi} : \boldsymbol{\sigma})^{\beta} + (\boldsymbol{\sigma} : \mathbb{III} : \boldsymbol{\sigma} : \boldsymbol{\sigma})^{\gamma} + \cdots - 1 = 0. \tag{2}$$

The even-rank structural anisotropy tensors $\boldsymbol{\pi} = \mathbf{\Pi}^{(2)}/\Pi$, $\mathbf{\Pi} = \mathbf{\Pi}^{(4)}/\Pi$, $\mathbb{III} = \mathbf{\Pi}^{(6)}/\Pi$, ..., in Eq. (2) are normalized by the common constant Π, whereas α, β, γ ..., etc., are arbitrary exponents of a polynomial representation. Assuming, further, $\alpha = 1$, $\beta = 1/2$, $\gamma = 1/3$, and limiting an infinite form (2) to the equation that contains only three common invariants of the stress and structural anisotropy tensors of appropriate ranks, we arrive at the simpler form, which satisfies the homogeneity of three polynomial components, known as the Goldenblat and Kopnov criterion (cf. Goldenblat and Kopnov [21])

$$\boldsymbol{\pi} : \boldsymbol{\sigma} + (\boldsymbol{\sigma} : \mathbf{\Pi} : \boldsymbol{\sigma})^{1/2} + (\boldsymbol{\sigma} : \mathbb{III} : \boldsymbol{\sigma} : \boldsymbol{\sigma})^{1/3} - 1 = 0. \tag{3}$$

Equation (3), when limited only to three common invariants of the stress tensor $\boldsymbol{\sigma}$ and structural anisotropy tensors of even orders: 2nd $\boldsymbol{\pi}$, 4th $\mathbf{\Pi}$ and 6th $\mathbb{III}$, is not the most general one, in the meaning of the representation theorems, which determine the most general irreducible representation of the scalar and tensor functions that satisfy the invariance with respect to a change of coordinates and material symmetry properties (cf. e.g. Spencer [52], Rymarz [49], Rogers [48]). However, 2nd, 4th and 6th order structural anisotropy tensors, which are used in (3), are found satisfactory for describing fundamental transformation modes of limit surfaces caused by plastic or damage processes, namely: isotropic change of size, kinematic translation and rotation, as well as surface distortion (cf. Kowalsky et al. [32], Betten [3]).

Goldenblat and Kopnov's equation (3) is quite general, too. Hence, for some engineering applications, its further reduction is performed. Namely, assuming $\alpha = \beta = \gamma = 1$, a following format of anisotropic failure initiation criterion is obtained:

$$\boldsymbol{\pi} : \boldsymbol{\sigma} + \boldsymbol{\sigma} : \mathbf{\Pi} : \boldsymbol{\sigma} + \boldsymbol{\sigma} : \mathbb{III} : \boldsymbol{\sigma} : \boldsymbol{\sigma} - 1 = 0. \tag{4}$$

When compared to the analogous polynomial format used as yield criterion [16] in Eq. (4), all three terms are saved, and the first invariant $\boldsymbol{\pi} : \boldsymbol{\sigma}$ plays an essential role when pressure-sensitive materials are considered.

Although the general format of the failure initiation criterion (4) is analogous to the previously introduced criterion of the yield initiation [16], further reduction of this equation to be applicable for brittle materials has to be performed applying different assumptions from those used for ductile materials. Plastic yield initiation in metallic polycrystalline materials is traditionally characterized by the following features (compare [16]):

– yield initiation is usually hydrostatic pressure insensitive (independent of the first common invariant),
– the criterion of yield initiation exhibits symmetry with respect to tension and compression (no strength differential effect),
– yield surface has to be convex (Drucker's postulate).

On the other hand, failure initiation in some metallic or non-metallic materials is commonly described including the more complex behaviour:

- failure initiation is hydrostatic pressure sensitive,
- the criterion of failure initiation exhibits asymmetry with respect to tension and compression (also called strength differential effect), since for the majority of brittle materials strength resistance for compression is frequently much higher than for tension (essential strength differential effect),
- the failure surface has to be convex (positive definiteness of tangent stiffness matrix in Sylvester's sense–see "Appendix D").

The above features are briefly discussed by Ganczarski and Skrzypek in [18]. In the light of experimental observation of metal alloys, a distinction between the plastic mechanism and the brittle mechanism of failure initiation may not be essential and justified.

Indeed, the majority of the pressure-insensitive metallic materials shows rather plastic yield initiation mechanism of either isotropic nature (NiTi shape memory alloys, Raniecki and Mróz [46]; Mg, Mg–Th or Mg–Li alloys, Cazacu and Barlat [6]; $4\text{Al}-\frac{1}{4}\text{O}_2$ titanium alloy, Cazacu et al. [7]) or anisotropic nature (Al 6061–T6511 alloy, Cazacu and Barlat [6]; Ti–6Al-4V titanium alloy, Khan et al. [28]; Al 6260–T4 alloy, Korkolis and Kyriakides [33]).

However, some experimental evidences for pressure-sensitive metallic alloys show rather combined plastic/failure mechanism with pronounced tension/compression asymmetry effect of either isotropic nature (Nickel-base Inconel 718, Iyer [26], Pęcherski et al. [42]; 5083 aluminium alloy, Gao et al. [20]) or anisotropic response (Ti–6Al–4V titanium alloy, Khan, Yu, Liu [30], Khan, Liu [29]; AA2008–T4 aluminium alloy; AZ31 magnesium alloy, Yoon et al. [60]).

Nevertheless, it should be pointed out that in all described cases of physically different coupled mechanisms the limit surface of yield and/or failure initiation must definitely satisfy the convexity requirement in the sense of either Drucker's or Sylvester's material stability postulates.

2 Remarks on isotropic yield/failure criteria accounting for the pressure sensitivity and strength differential effect

In general case of the yield/failure initiation in isotropic materials, the equation of the limit surface in terms of the three basic stress invariants $J_{1\sigma}$, $J_{2\sigma}$, $J_{3\sigma}$ takes a form

$$f(J_{1\sigma}, J_{2\sigma}, J_{3\sigma}; k_i) = 0 \tag{5}$$

where symbol k_i stands for a set of material constants, for instance k_t, k_c, k_s. This equation describes the general class of isotropic hydrostatic pressure-sensitive materials; see items C1–C5 in Table 7.

In the narrower case of failure in isotropic materials, the equation of the limit surface is commonly written in terms of the following three invariants: the first stress tensor invariant $J_{1\sigma}$, and the second J_{2s}, and the third J_{3s} stress deviator invariants

$$f(J_{1\sigma}, J_{2s}, J_{3s}; k_i) = 0. \tag{6}$$

The advantage of Eq. (6) is that it separates the hydrostatic stress $J_{1\sigma}$ from the influence of deviatoric stresses expressed by J_{2s} and J_{3s}. Such surface is no longer cylindrical, and hence it naturally exhibits tension/compression asymmetry. All limit surfaces belonging to the class considered exhibit certain sectorial symmetry with respect to the hydrostatic axis (see discussion in [16]); however, only in a particular case of independence of the third invariant J_{3s} (or θ), it is fully rotationally symmetric.

The Ottosen and Ristinmaa [40] mixed format separates the influence of the hydrostatic pressure $J_{1\sigma}$ from the deviatoric stress represented by J_{2s} and $\cos 3\theta$,

$$f(J_{1\sigma}, J_{2s}, \cos 3\theta; k_i) = 0, \tag{7}$$

where the following definitions hold:

$$\cos(3\theta) = \frac{3\sqrt{3}}{2} \frac{J_{3s}}{(J_{2s})^{3/2}}, \quad J_{2s} = \frac{1}{2}\boldsymbol{s} \cdot \boldsymbol{s}, \quad J_{3s} = \frac{1}{3}\boldsymbol{s} \cdot \boldsymbol{s} \cdot \boldsymbol{s}. \tag{8}$$

The invariants $J_{1\sigma}$, J_{2s} and $\cos 3\theta$ have clear interpretation: $J_{1\sigma}$ tells about the hydrostatic pressure sensitivity (non-cylindrical limit surface), J_{2s} represents the distance of the point at the deviatoric limit curve (magnitude of the deviatoric stress), and $\cos 3\theta$ informs about the influence of the direction of deviatoric stress (asymmetry in so-called strength differential materials), cf. Ottosen and Ristinmaa [40]. Applying consistently a system of

the first stress invariant and the second and the third stress deviator invariants, Eq. (7) can also be written as (6).

The other stress invariant systems are broadly discussed by Haigh [22], Westergaard [59], Novoshilov [39], Życzkowski [61], and summarized by Altenbach et al. [2] in a recent important monograph. In the last one monograph, the authors provide a complete review and comparison of a variety of phenomenological yield/failure pressure-insensitive versus pressure-sensitive isotropic criteria. In particular for novel materials, the following phenomena are taken into account: tension/compression asymmetry, also called strength differential effect, hydrostatic pressure sensitivity, the Poynting–Swift effect as well as the Kelvin effect. In contrast to conventional material behaviour, described by tensorial linear equations, the above phenomena require tensorial nonlinear equations; therefore, they may be classified as the second-order effects according to the nomenclature by Reiner and Abir [47].

The general format of the power threshold function applicable to advanced metals, having not only three threshold parameters a, b and c but also including two independent powers p and r in the yield/failure criterion for isotropic materials, can be furnished as follows, see Ganczarski and Skrzypek [15] and Skrzypek and Ganczarski [51]:

$$\left(aJ_{1\sigma}^{2p} + bJ_{2s}^{p} + cJ_{3s}^{2p/3}\right)^{1/r} - 1 = 0. \tag{9}$$

It will be shown below that the majority of criteria met in the literature to predict onset of yield, failure or even phase transformation (Iyer [26], Pęcherski et al. [42], Gao et al. [20], Iyer and Lissenden [27], Brünig et al. [4], Raniecki and Mróz [46]) can be captured as the specific cases of this general format (9).

If $a = 0$ and $r = 1$ whereas p is arbitrary, the Raniecki and Mróz cylindrical surface is recovered from (9),

$$bJ_{2s}^{p} + cJ_{3s}^{2p/3} - 1 = 0, \tag{10}$$

or equivalently, if the original notation is saved: $p = 3n/2$, $c/b = -C$, $1/b = k^{3n}$, we arrive at the Raniecki–Mróz equation

$$J_{2s}^{3n/2} - CJ_{3s}^{n} - k^{3n} = 0 \tag{11}$$

that represents a rounded yield surface with trigonal symmetry, see Fig. 1b. If, additionally, $n = 2$, the classical Drucker's [10] format is recovered,

$$J_{2s}^{3} - CJ_{3s}^{2} - k^{6} = 0. \tag{12}$$

Assuming namely $r = p$ in Eq. (9), the Iyer [26] yield/failure onset criterion is recovered,

$$\left(aJ_{1\sigma}^{2p} + bJ_{2s}^{p} + cJ_{3s}^{2p/3}\right)^{1/p} - 1 = 0. \tag{13}$$

Equation (13) is successfully used to describe the Nickel-based alloy Inconel 718 at elevated temperature 650°C by Iyer and Lissenden [27].

If $r = p = 1$, the narrower format used by Iyer and Lissenden [27], Pęcherski et al. [42] is recovered,

$$aJ_{1\sigma}^{2} + bJ_{2s} + cJ_{3s}^{2/3} - 1 = 0. \tag{14}$$

An equation of this type, which represents an asymmetric either paraboloidal or ellipsoidal surface, was calibrated by Pęcherski et al. [42] for Inconel 718. Note that the Huber–von Mises $f(J_{2s})$, the Drucker–Prager $f(J_{1\sigma}, J_{2s})$ and the Drucker $f(J_{2s}, J_{3s})$ yield functions can be obtained as special cases.

Assuming another combination of powers $p/r = 1/2$ and $r = 6$, we arrive at the Gao et al. [20] yield function

$$c_1\left(a_1 J_{1\sigma}^{6} + 27J_{2s}^{3} + b_1 J_{3s}^{2}\right)^{1/6} - k = 0 \tag{15}$$

calibrated and verified for the 5083 aluminium alloy. The constant c_1 can be found by introducing the uniaxial condition onto equation (15),

$$c_1 = \left(a_1 + \frac{4}{729}b_1 + 1\right)^{-1/6}. \tag{16}$$

Another special case of Eq. (9), when $p/r = 1/2$ and $r = 1$, was considered by Brünig et al. [4],

$$aJ_{1\sigma} + \sqrt{J_{2s}} + b\sqrt[3]{J_{3s}} = c. \tag{17}$$

who confirmed its applicability for failure initiation in aluminium alloys and high-strength steels. An equation of this type represents an asymmetric cone, and it is capable of capturing yield onset in the high-strength 4310 and 4330 steels. If $a = \alpha$, $b = 0$ and $c = k$, Eq. (17) reduces to the well-known Drucker–Prager equation

$$a J_{1\sigma} + \sqrt{J_{2s}} = k. \tag{18}$$

An asymmetric yield function of pressure-sensitive materials, which cannot be obtained from Eq. (9), is due to Yoon et al. [60],

$$a \left[b J_{1\sigma} + \left(J_{2s}^{3/2} - c J_{3s} \right)^{1/3} \right] - 1 = 0. \tag{19}$$

Equation (19) is in fact an extension of Drucker's equation (12), by use of an additional term linear with respect to the first invariant $J_{1\sigma}$, according to experimental results by Spitzig et al. [53] as well as Spitzig and Richmond [54]. This format inherits after Drucker's trigonal symmetry that enables to describe the SD effect, but the presence of first invariant $J_{1\sigma}$ leads to a conical surface, instead of a cylindrical in Drucker's case. Let us mention that from among three material constants a, b and c in (19), only two are independent since the third a has to capture the uniaxial tensile test, namely

$$a = \frac{1}{b + \sqrt[3]{\frac{1}{3\sqrt{3}} - \frac{c}{27}}}. \tag{20}$$

Additionally, the convexity of the proposed yield function is satisfied if $c \in [\frac{-3\sqrt{3}}{4}, \frac{3\sqrt{3}}{4}]$. This isotropic equation is also extended to material anisotropy in order to properly describe various metals of AA 2008-T4, high-purity α-titanium and the AZ31 magnesium alloy. In case if pressure sensitivity is ignored ($b = 0$ and $a = k$), the Cazacu and Barlat [6] format is recovered,

$$J_{2s}^{3/2} - c J_{3s} - k^3 = 0. \tag{21}$$

3 Influence of stress invariants on a description of pressure sensitivity, asymmetry and anisotropy: explicit or implicit approaches

The "sharp" distinction between the ductile and brittle materials mentioned in Sect. 1 is often not exactly justified, especially when more advanced structural material behaviours are considered. Numerous experimental findings referred to novel materials suggest another classification with respect to distinct mechanical responses, among which the three are of particular importance:

– hydrostatic pressure sensitivity,
– tension versus compression asymmetry,
– material anisotropy.

For the purpose of the present paper, the hydrostatic pressure dependence is considered as the most important among the three. In this sense, materials can be classified into two groups: hydrostatic pressure-dependent and hydrostatic pressure-independent materials, also called pressure-sensitive and pressure-insensitive ones. Traditionally, ductile materials (majority of metals) can be considered as hydrostatic pressure independent. On the other hand, brittle materials (rocks, ceramics, etc.) should be treated as hydrostatic pressure-dependent ones. Hydrostatic pressure dependence of isotropic or anisotropic limit criteria can be captured in the two different manners:

1. direct dependence by both first and second invariants:
   1a. the first stress invariant plus the second deviatoric invariant

$$f \left[\mathrm{tr}(\boldsymbol{\sigma}), \frac{1}{2}\mathrm{tr}(s) \right] \qquad \text{isotropy,} \tag{22}$$

   1b. the first common invariant plus the second common deviatoric invariant

$$f \left[\boldsymbol{\pi} : \boldsymbol{\sigma}, s : \boldsymbol{\Pi} : s \right] \qquad \text{anisotropy.} \tag{23}$$

2. indirect dependence by the second invariants with the first invariants ignored:

Table 1 Hydrostatic pressure dependence of yield/failure criteria

| | Dependence on σ_h | | |
	Direct	Indirect	Lack of dependence
Isotropy	Drucker–Prager $\alpha J_{1\sigma} + \sqrt{J_{2s}} = k$ Tsai–Wu	Beltrami $\sqrt{3J_{2\sigma}} = k$ von Mises	Huber–von Mises $\sqrt{3J_{2s}} = k$ Hill
Anisotropy	$\boldsymbol{\pi} : \boldsymbol{\sigma} + \boldsymbol{s} : \boldsymbol{\Pi} : \boldsymbol{s} = 1$	$\boldsymbol{\sigma} : \boldsymbol{\Pi} : \boldsymbol{\sigma} = 1$	$\boldsymbol{s} : \boldsymbol{\Pi}^\mathrm{H} : \boldsymbol{s} = 1$ where $\boldsymbol{\Pi}^\mathrm{H} = \boldsymbol{\Pi}^\mathrm{TW}$

2a. the second stress invariant

$$f\left[\frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma} \cdot \boldsymbol{\sigma})\right] \qquad (\boldsymbol{\sigma} = \boldsymbol{s} + \sigma_\mathrm{h}\mathbf{1}) \qquad \text{isotropy,} \tag{24}$$

2b. the second common stress invariant

$$f(\boldsymbol{\sigma} : \boldsymbol{\Pi} : \boldsymbol{\sigma}) \qquad (\boldsymbol{\sigma} = \boldsymbol{s} + \sigma_\mathrm{h}\mathbf{1}) \qquad \text{anisotropy.} \tag{25}$$

However, there exists a broad class of engineering materials which do not exhibit any dependence on the hydrostatic pressure, either direct or indirect. This means that in case of the isotropic hydrostatic pressure-independent materials the corresponding limit surfaces have to include the second deviatoric invariant exclusively. In case of anisotropy limit, surfaces can include the second common deviatoric invariant and additionally the first common deviatoric invariant. In all cases considered, any equation of limit surface has to include the second stress or the second common invariants, which results from the quadratic form of energy representation. Exemplary equations of limit surfaces that found application in engineering materials are presented in Table 1 according to the aforementioned classification.

The above Table shows a comparison between the pairs of appropriate isotropic and anisotropic criteria that correspond to: the direct dependence on hydrostatic pressure, e.g. Drucker–Prager's criterion versus Tsai–Wu criterion, the indirect dependence on hydrostatic pressure, e.g. Beltrami criterion versus von Mises criterion, and independence of the hydrostatic pressure, e.g. Huber–von Mises criterion versus Hill criterion. The oldest criterion based on total elastic energy, formulated by Beltrami in 1885, is invoked in this Table although it has no experimental evidence, cf. Życzkowski [61].

Tension/compression asymmetry, also called strength differential effect (see for instance Altenbach et al. [2]), is included in a natural way in limit criteria for anisotropic materials. In the case of limit criteria for isotropic materials, this effect manifests itself by the presence of first stress invariant $J_{1\sigma}$ and/or the third stress invariant J_{3s}, as shown in Fig. 1. Note that in the case of the first stress invariant-dependent surface, compressive and tensile meridians are in identical distance from the centre of the limit curve, but the axis of the limit surface is shifted in Fig. 1a. In other case, when the limit surface is the third stress invariant-dependent function, compressive and tensile meridians are not in identical distance from the centre of the limit curve, but the axis of the limit surface remains at the position of the hydrostatic axis, see Fig. 1b. In our opinion, there is no sharp distinction between two terms appearing in the literature: the tension/compression asymmetry and the strength differential effect, such that both terms can be used equivalently. This might be confirmed by many authors: Raniecki and Mróz (SD effect due to J_{3s}) [46], Cazacu et al. [7] (tension/compression asymmetry due to J_{3s}), Khan et al. [30] (tension/compression asymmetry due to J_{3s}), Altenbach et al. [2] (tension/compression asymmetry = SD effect), to mention only some.

As it was aforementioned, the limit criteria have to include appropriate second invariants. However, limit surfaces based on the second invariants exclusively (stress invariants or common invariants) are capable to capture neither tension/compression asymmetry nor any shape change due to distortion. By contrast, the limit criteria based on the second and the third invariants (stress invariants or common invariants) are capable to capture both tension/compression asymmetry and distortion. Table 2 shows a comparison between the pairs of selected isotropic and anisotropic criteria that correspond to: the lack of tension/compression asymmetry and distortion, e.g. Huber–von Mises' criterion vs. Hill's criterion, tension/compression asymmetry with distortion ruled out, c.f. Drucker–Prager's criterion versus Tsai–Wu's criterion, and tension/compression asymmetry with distortion accounted for, c.f. Drucker's criterion versus Kowalsky's et al. criterion. To illustrate the classification described in Table 2, a comparison between asymmetry without distortion and asymmetry with distortion accounted for is presented in Fig. 2. In the case of isotropy Fig. 2a, the Drucker criterion is compared with the
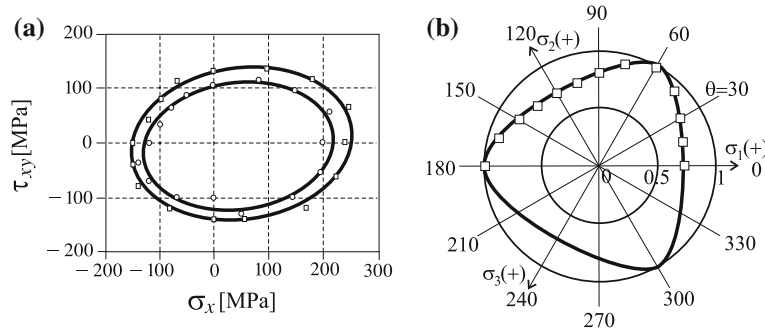
Fig. 1 Tension/compression asymmetry caused by: **a** first stress invariant—low carbon steel 18G2A subjected to monotonic prestrain (plastic offsets: $circle \varepsilon_{\text{off}} = 1 \times 10^{-5}$, $square\ \varepsilon_{\text{off}} = 5 \times 10^{-5}$) after Kowalewski and Śliwowski [31], **b** third stress invariant—TiNi alloy after Raniecki and Mróz [46]
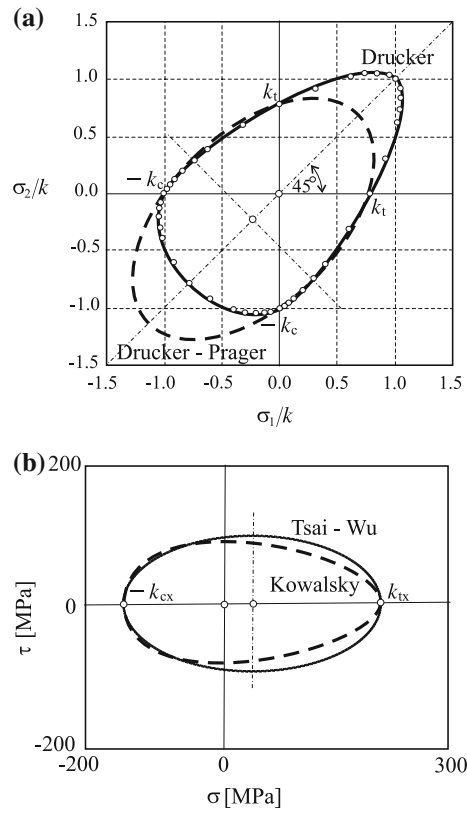
Table 2 Effect of first and third invariants on tension/compression asymmetry and distortion of limit surfaces

	Lack of asymmetry and distortion	Asymmetry without distortion	Asymmetry and distortion
Isotropy	Huber–von Mises $\sqrt{3J_{2s}} = k$ Hill	Drucker–Prager $\alpha J_{1\sigma} + \sqrt{J_{2s}} = k$ Tsai–Wu	Drucker $J_{2s}^3 - c J_{3s}^2 = k^6$ Kowalsky et al.
Anisotropy	$s : \Pi^{\text{H}} : s = 1$	$\pi : \sigma +$ $s : \Pi^{\text{H}} : s = 1$	$s : \Pi : s +$ $s : \mathbb{III} : s : s = 1$

Drucker–Prager criterion. In the case of Drucker–Prager's criterion, based on the first and the second stress invariants, tension/compression asymmetry appears independently from shape distortion. By contrast, when Drucker's criterion is used, both effects are coupled by the third invariant, and hence they appear simultaneously. In the case of anisotropy Fig. 2b, the Tsai–Wu criterion is compared with the Kowalsky et al. criterion. Tsai–Wu's criterion accounts for tension/compression asymmetry without distortion (only translation by the first common invariant is accounted for). By contrast, when the Kowalsky et al. sixth-order criterion is used, the tension/compression asymmetry and shape distortion are coupled in an anisotropic fashion by the third common invariant.

In general, material anisotropy can be captured by use of the two approaches. In the first mathematically consistent approach, here called the explicit anisotropy approach, the system of stress invariants $J_{1\sigma}$, J_{2s} and J_{3s} is substituted by the corresponding system of common invariants $\pi : \sigma, s : \Pi : s$ and $s : \mathbb{III} : s : s$. In the other currently dynamically developed approach, here called the implicit anisotropy approach, developed in Barlat and Khan schools, either the second J_{2s} and the third J_{3s} stress invariants are substituted by the corresponding transformed deviatoric invariants J_{2s}^0, J_{3s}^0, or the stress deviator is transformed by use of the two independent fourth-rank transformation tensors $\Sigma = C : s$ and $\Sigma' = C' : s$, and next they are inserted to one of the well-known isotropic criteria, either Drucker's one or Hosford's one, respectively. These linear transformations correspond to mapping of the deviatoric Cauchy stress tensor s to the other two deviatoric stresses Σ and Σ' referring to the material anisotropy (orthotropy) frame. The implicit approach is able to capture the full material orthotropy (with distortion effect included) by use of two fourth-rank orthotropic transformation tensors C, C' (containing $2 \times 9 = 18$ independent material constants), in contrast to the explicit common invariant-based approach which requires in the case of material orthotropy, a fourth-rank tensor Π and sixth-rank tensor $\mathbb{III}$ (containing $9 + 56 = 65$ material constants). Although the explicit approach is mathematically more rigorous than the implicit one, simultaneously it is much more cumbersome and compliant to misunderstandings. Both approaches, the explicit and the implicit, are alternatively used, but obviously they lead to different approximations. Comparison of the explicit and the implicit approaches to capture anisotropy is schematically presented for selected criteria in Table 3.

A major difficulty when describing the limit yield/failure is caused by the coupling between anisotropy and strong tension/compression asymmetry, as discussed by Khan et al. [30]. Such significant coupling can lead to a total distortion of the limit surface (possible lack of any axis of symmetry), as it is presented in Fig. 3, based on Luo et al. [35] experimental findings for AZ31B Mg alloy, and fitted by Plunkett et al. [44].

Fig. 2 Comparison of tension/compression asymmetry and distortion of limit curves in case of: **a** isotropy (Drucker–Prager's criterion versus Drucker's criterion), **b** anisotropy (Tsai–Wu criterion versus Kowalsky et al. criterion)

Another interesting "hybrid" concept is applied by Yoon et al. [60]. It is a combination of both the explicit approach in the case of first common invariant and simultaneously the implicit approach with regard to second and third transformed invariants. In Yoon et al. [60], the orthotropic yield criterion and the isotropic criterion being its original pattern are compared in Table 4.

This highly extended yield criterion is capable of capturing all three features: anisotropy, tension/compression asymmetry and hydrostatic pressure sensitivity of various metals like: AA 2008-T4, high-purity α-titanium, and AZ31 magnesium alloy. Excellent fitting of the proposed yield criterion and experimental data for AZ31 alloy is shown in Fig. 3. This sufficiently general form can unconditionally be recommended as very effective and especially addressed to model the totally distorted response of cold rolled metals.

However, some limitations for the implicit approach can be pointed out. In the case of the explicit formulations, the conditions that guarantee convexity of the anisotropic limit surface are known, see Ottosen and Ristinmaa [40], Ganczarski and Skrzypek [19]. In the other case of an implicit formulation, general conditions for convexity are very difficult and cumbersome to reach, Cazacu et al. [6,7]. This is due to a substitution of isotropic stress invariants by the linearly transformed stress, Plunkett et al. [44], Yoon et al. [60].

Additionally, the use of doubled linear stress transformation has no clear physical sense, although this guarantees perfect fitting to experimental data. Finally, although both curves in Fig. 3 describe the same material in the σ_{xx}, σ_{yy} plane, corresponding 3D limit surfaces belong to different classes: cylindrical surface for pressure sensitivity vs. conical surface for pressure sensitivity.

Reassuming the above broad discussion concerning three essential aspects: hydrostatic pressure sensitivity, strength differential effect and material anisotropy, emphasizes that there exists strong coupling between them. In general, at least two ways to capture these coupled features: the conventional common invariants formulation or the linear transformed stresses to account for anisotropy, can be distinguished. Coupling between all three mechanical responses manifests strongly when implicit formulation is used. This is due to the non-separateness of the above effects involved by the transformation tensors applied to all invariants. By contrast, when the explicit formulation is applied, it is possible to point out that the first common invariant is directly responsible for hydrostatic pressure sensitivity and strength differential effects. However, to account for the distortion

Table 3 Explicit or implicit anisotropy of limit surfaces

	Explicit		Implicit
	Huber–von Mises		Hosford [25]
Isotropy	$s : \mathbf{I} : s = k^2$	Iyer and Lissenden [27] $J_{2s}^{3/2} - c J_{3s} = k^3$	$|s_1 - s_2|^a + |s_2 - s_3|^a +$ $|s_3 - s_1|^a = 2k^a$
		Cazacu–Barlat [6]	Plunkett et al. [44]
Anisotropy	$s : \mathbf{\Pi}^{\mathrm{H}} : s = 1$	$(J_2^0)^{3/2} - c J_3^0 = k^3$	$\sum_{i=1}^{3} [(|\Sigma_i| - \widehat{k}\,\Sigma_i)^a +$ $(|\Sigma_i'| - \widehat{k}'\,\Sigma_i')^a] = 2k^a$ where $\mathbf{\Sigma} = \mathbf{C} : s$, $\mathbf{\Sigma}' = \mathbf{C}' : s$

where

$$J_2^0 = \frac{1}{6}\left[a_1(\sigma_x - \sigma_y)^2 + a_2(\sigma_y - \sigma_z)^2 + a_3(\sigma_z - \sigma_x)^2\right]$$
$$+ a_4\tau_{xy}^2 + a_5\tau_{xz}^2 + a_6\tau_{zy}^2$$

$$J_3^0 = \frac{1}{27}\Big\{(b_1 + b_2)\sigma_x^3 + (b_3 + b_4)\sigma_y^3 + \left[2(b_1 + b_4) - b_2 - b_3\right]\sigma_z^3\Big\} + 2b_{11}\tau_{xy}\tau_{yz}\tau_{zx} + \frac{1}{9}\Big\{2(b_1 + b_2)\sigma_x\sigma_y\sigma_z$$
$$- (b_1\sigma_y + b_2\sigma_z)\sigma_x^2 - (b_3\sigma_z + b_2\sigma_x)\sigma_y^2$$
$$- \left[(b_1 - b_2 + b_4)\sigma_x + (b_1 + b_3 + b_4)\sigma_y\right]\sigma_z^2\Big\}$$
$$- \frac{1}{3}\Big\{\tau_{yz}^2\left[(b_6 + b_7)\sigma_x - b_6\sigma_y - b_7\sigma_z\right]$$
$$- \tau_{zx}^2\left[2b_9\sigma_y - b_8\sigma_z - (2b_9 - b_8)\sigma_x\right]$$
$$- \tau_{xy}^2\left[2b_{10}\sigma_z - b_5\sigma_y - (2b_{10} - b_5)\sigma_x\right]\Big\}$$

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & & & \\ C_{12} & C_{22} & C_{23} & & & \\ C_{13} & C_{23} & C_{33} & & & \\ & & & C_{44} & & \\ & & & & C_{55} & \\ & & & & & C_{66} \end{bmatrix}$$

$$\mathbf{C}' = \begin{bmatrix} C_{11}' & C_{12}' & C_{13}' & & & \\ C_{12}' & C_{22}' & C_{23}' & & & \\ C_{13}' & C_{23}' & C_{33}' & & & \\ & & & C_{44}' & & \\ & & & & C_{55}' & \\ & & & & & C_{66}' \end{bmatrix}$$

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Table 4 Hybrid (explicit/implicit) Yoon et al. anisotropic limit surfaces

Isotropy	Yoon et al. [60] $a[bJ_{1\sigma} + (J_{2s}^{3/2} - cJ_{3s})^{1/3}] - 1 = 0$ $J_{1\sigma} = \mathrm{tr}(\boldsymbol{\sigma}),\ J_{2s} = \frac{1}{2}\boldsymbol{s}:\boldsymbol{s},\ J_{3s} = \det(\boldsymbol{s})$
Anisotropy	Yoon et al. [60] $\widetilde{I}_1 + (J_2'^{3/2} - J_3'')^{1/3} - 1 = 0$ $\widetilde{I}_1 = h_x\sigma_{xx} + h_y\sigma_{yy} + h_z\sigma_{zz},\ J_{2s}' = \frac{1}{2}\boldsymbol{s}':\boldsymbol{s}',\quad J_{3s}'' = \det(\boldsymbol{s}'')$ $\boldsymbol{s}' = \mathbf{L}':\boldsymbol{\sigma},\ \boldsymbol{s}'' = \mathbf{L}'':\boldsymbol{\sigma}$ $\mathbf{L}^{(i)} = \begin{bmatrix} \frac{C_2^{(i)}+C_3^{(i)}}{3} & -\frac{C_3^{(i)}}{3} & -\frac{C_2^{(i)}}{3} & & & \\ -\frac{C_3^{(i)}}{3} & \frac{C_3^{(i)}+C_1^{(i)}}{3} & -\frac{C_1^{(i)}}{3} & & & \\ -\frac{C_1^{(i)}}{3} & -\frac{C_1^{(i)}}{3} & \frac{C_1^{(i)}+C_2^{(i)}}{3} & & & \\ & & & C_4^{(i)} & & \\ & & & & C_5^{(i)} & \\ & & & & & C_6^{(i)} \end{bmatrix}$
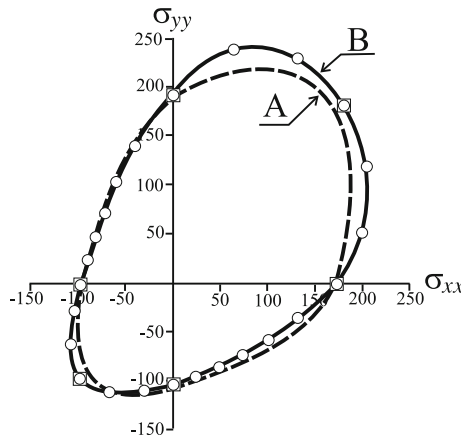
Fig. 3 Comparison of the different approaches to modelling limit curves for AZ31B Mg alloy: **a** hydrostatic pressure-insensitive criterion by Plunkett et al. [44], fitting of Luo et al. [35] experimental data, and **b** hydrostatic pressure-sensitive criterion by Yoon et al. [60], (unit MPa)

effect it is necessary to apply at least the third common invariant. In what follows, the explicit approach will consistently be applied.

4 Von Mises–Tsai–Wu-type yield/failure criteria

In what follows, the narrower format of Eq. (4) with the third common invariant $\boldsymbol{\sigma}:\mathbb{III}:\boldsymbol{\sigma}:\boldsymbol{\sigma}$ neglected is used,

$$\boldsymbol{\pi}:\boldsymbol{\sigma} + \boldsymbol{\sigma}:\boldsymbol{\Pi}:\boldsymbol{\sigma} - 1 = 0. \tag{26}$$

Removal of the third common invariant excludes distortion of the limit surface. In other words, limit curves are ellipses instead of ear-like shapes, as shown in Fig. 3. The structural tensors of the second $\boldsymbol{\pi}$ and fourth $\boldsymbol{\Pi}$ orders appearing in Eq. (26) stand for two independent yield/failure anisotropy tensors, the identification of which has to be performed on the basis of respective yield/failure tests in an analogous way as that discussed in [16]. However, in the present case two anisotropy tensors have to be calibrated, and hence the appropriate number of tests increases, such that the difference between the tension and the compression uniaxial tests can be captured. Substituting for convenience Voigt's vector-matrix notation, both above tensors can be represented in the vector-matrix forms

$$[\boldsymbol{\pi}] = \begin{bmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ & \pi_{22} & \pi_{23} \\ & & \pi_{33} \end{bmatrix} \tag{27}$$

or

$$\{\boldsymbol{\pi}\}^{\mathrm{T}} = \left\{\pi_{11}, \quad \pi_{22}, \quad \pi_{33}, \quad \pi_{23}, \quad \pi_{13}, \quad \pi_{12}\right\}^{\mathrm{T}} \tag{28}$$

and

$$[\boldsymbol{\Pi}] = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} & \Pi_{15} & \Pi_{16} \\ & \Pi_{22} & \Pi_{23} & \Pi_{24} & \Pi_{25} & \Pi_{26} \\ & & \Pi_{33} & \Pi_{34} & \Pi_{35} & \Pi_{36} \\ \hline & & & \Pi_{44} & \Pi_{45} & \Pi_{46} \\ & & & & \Pi_{55} & \Pi_{56} \\ & & & & & \Pi_{66} \end{bmatrix} \tag{29}$$

where yield/failure loci are determined by two yield/failure characteristic matrices $[\boldsymbol{\pi}]$ of the dimension (3×3) and $[\boldsymbol{\Pi}]$ of the dimension (6×6). Hence, in the considered case of general anisotropy, the number of moduli defining yield/failure initiation is equal to $27 = 6 + 21$.

The condition of yield/failure initiation in anisotropic materials (26) takes in Voigt's notation the equivalent format

$$\{\boldsymbol{\pi}\}\{\boldsymbol{\sigma}\} + \{\boldsymbol{\sigma}\}^{\mathrm{T}}[\boldsymbol{\Pi}]\{\boldsymbol{\sigma}\} - 1 = 0 \tag{30}$$

where

$$\{\boldsymbol{\sigma}\} = \left\{\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{zx}, \tau_{xy}\right\}^{\mathrm{T}} \tag{31}$$

being an extension of the anisotropic von Mises' criterion of plastic materials, however, enriched by the linear term.

In what follows, the extension of the anisotropic von Mises' criterion, enhanced by linear terms (30), is considered. Assume the deviatoric form of the extended von Mises criterion [16]

$$\{\boldsymbol{\pi}\}\{\boldsymbol{\sigma}\} + \{\boldsymbol{s}\}^{\mathrm{T}}[_{\mathrm{dev}}\boldsymbol{\Pi}]\{\boldsymbol{s}\} - 1 = 0 \tag{32}$$

where

$$[_{\mathrm{dev}}\boldsymbol{\Pi}] = \begin{bmatrix} -\Pi_{12} - \Pi_{13} & \Pi_{12} & \Pi_{13} & -\Pi_{24} - \Pi_{34} \\ & -\Pi_{12} - \Pi_{23} & \Pi_{23} & \Pi_{24} \\ & & -\Pi_{13} - \Pi_{23} & \Pi_{34} \\ \hline & & & \Pi_{44} \\ \\ & & & \\ & & \Pi_{15} & \Pi_{16} \\ & -\Pi_{15} - \Pi_{35} & \Pi_{26} \\ & \Pi_{35} & -\Pi_{16} - \Pi_{26} \\ \hline \Pi_{45} & \Pi_{46} \\ \Pi_{55} & \Pi_{56} \\ \Pi_{66} \end{bmatrix}. \tag{33}$$

When the engineering format is used, we arrive at

$$\begin{aligned}
&-\Pi_{12}\left(\sigma_x - \sigma_y\right)^2 - \Pi_{13}\left(\sigma_x - \sigma_z\right)^2 - \Pi_{23}\left(\sigma_y - \sigma_z\right)^2 \\
&+ 2\left\{\tau_{yz}\left[\Pi_{24}\left(\sigma_y - \sigma_x\right) + \Pi_{34}\left(\sigma_z - \sigma_x\right)\right]\right. \\
&+ \tau_{zx}\left[\Pi_{15}\left(\sigma_x - \sigma_y\right) + \Pi_{35}\left(\sigma_z - \sigma_y\right)\right] \\
&+ \tau_{xy}\left[\Pi_{16}\left(\sigma_x - \sigma_z\right) + \Pi_{26}\left(\sigma_y - \sigma_z\right)\right] \\
&+ \left. \Pi_{45}\tau_{yz}\tau_{zx} + \Pi_{46}\tau_{xy}\tau_{yz} + \Pi_{56}\tau_{zx}\tau_{xy}\right\} \\
&+ \Pi_{44}\tau_{yz}^2 + \Pi_{55}\tau_{zx}^2 + \Pi_{66}\tau_{xy}^2 \\
&+ \pi_{11}\sigma_x + \pi_{22}\sigma_y + \pi_{33}\sigma_z + \pi_{12}\tau_{xy} + \pi_{13}\tau_{zx} + \pi_{23}\tau_{yz} = 1.
\end{aligned} \tag{34}$$

Note that this form is strictly pressure insensitive only in the quadratic terms, but it is pressure sensitive as far as the linear terms are concerned. The fully deviatoric format of Eq. (32) by an additional constraint applied for the linear term

$$\pi_{11} + \pi_{22} + \pi_{33} = 0 \tag{35}$$

was discussed by Szczepiński [55], whereas an experimental verification for low carbon steel 18G2A is due to Kowalewski and Śliwowski [31].

Limiting further considerations to orthotropic materials, both characteristic matrices $[_{\text{ort}}\boldsymbol{\pi}]$ (27) and $[_{\text{ort}}\boldsymbol{\Pi}]$ (29) take the following forms valid for principal directions of orthotropy:

$$[_{\text{ort}}\boldsymbol{\pi}] = \begin{bmatrix} \pi_{11} & 0 & 0 \\ & \pi_{22} & 0 \\ & & \pi_{33} \end{bmatrix}, \quad [_{\text{ort}}\boldsymbol{\Pi}] = \left[\begin{array}{ccc|ccc} \Pi_{11} & \Pi_{12} & \Pi_{13} & 0 & 0 & 0 \\ & \Pi_{22} & \Pi_{23} & 0 & 0 & 0 \\ & & \Pi_{33} & 0 & 0 & 0 \\ \hline & & & \Pi_{44} & 0 & 0 \\ & & & & \Pi_{55} & 0 \\ & & & & & \Pi_{66} \end{array}\right]. \tag{36}$$

The second-rank matrix $[_{\text{ort}}\boldsymbol{\pi}]$ is of the dimension 3×3, whereas the fourth-rank matrix $[_{\text{ort}}\boldsymbol{\Pi}]$ is of dimension 6×6. The matrix $[_{\text{ort}}\boldsymbol{\pi}]$ has diagonal form, and the matrix $[_{\text{ort}}\boldsymbol{\Pi}]$ is of the identical symmetry as the von Mises plastic orthotropy matrix. Both matrices (36) are defined by $12 = 3 + 9$ modules.

The condition of yield/failure initiation for orthotropic materials takes a form typical for the rotationally symmetric group

$$\{_{\text{ort}}\boldsymbol{\pi}\}\{\boldsymbol{\sigma}\} + \{\boldsymbol{\sigma}\}^{\text{T}} [_{\text{ort}}\boldsymbol{\Pi}] \{\boldsymbol{\sigma}\} - 1 = 0, \tag{37}$$

being an extension of the von Mises orthotropic yield condition which is pressure insensitive, for pressure-sensitive materials. The von Mises orthotropic yield/failure initiation criterion (37) can be written down in the following extended format:

$$\Pi_{11}\sigma_x^2 + \Pi_{22}\sigma_y^2 + \Pi_{33}\sigma_z^2 + 2\left(\Pi_{12}\sigma_x\sigma_y + \Pi_{23}\sigma_y\sigma_z + \Pi_{31}\sigma_z\sigma_x\right)$$
$$+ \Pi_{44}\tau_{yz}^2 + \Pi_{55}\tau_{zx}^2 + \Pi_{66}\tau_{xy}^2 + \pi_{11}\sigma_x + \pi_{22}\sigma_y + \pi_{33}\sigma_z - 1 = 0. \tag{38}$$

Note that the above equation represents the fully tensorial form of the orthotropic yield/failure criterion, contrary to the deviatoric form which is characteristic for the Hill yield criterion. This means that $12 = 3 + 9$ material moduli defining the yield/failure material characteristic tensors $_{\text{ort}}\boldsymbol{\pi}$ and $_{\text{ort}}\boldsymbol{\Pi}$ are required for its identification.

Consider now a reduction of the orthotropic criterion (37) to a narrower format known in the literature as the Tsai–Wu orthotropic criterion of failure. The Tsai–Wu criterion is characterized simultaneously by the strength differential effect and pressure insensitivity of $[_{\text{ort}}\boldsymbol{\Pi}]$ in Eq. (36) such that $[_{\text{ort}}\boldsymbol{\Pi}] \longrightarrow [\boldsymbol{\Pi}^{\text{TW}}]$, whereas $\{_{\text{ort}}\pi\} = \{\pi^{\text{TW}}\}$

$$\{\boldsymbol{\pi}^{\text{TW}}\}\{\boldsymbol{\sigma}\} + \{s\}^{\text{T}}\left[\boldsymbol{\Pi}^{\text{TW}}\right]\{s\} - 1 = 0. \tag{39}$$

This leads to the following narrower representation of both characteristic matrices:

$$\left[\boldsymbol{\pi}^{\text{TW}}\right] = \begin{bmatrix} \pi_{11} & 0 & 0 \\ & \pi_{22} & 0 \\ & & \pi_{33} \end{bmatrix},$$

$$\left[\boldsymbol{\Pi}^{\text{TW}}\right] = \left[\begin{array}{ccc|ccc} -\Pi_{12} - \Pi_{13} & \Pi_{12} & \Pi_{13} & 0 & 0 & 0 \\ & -\Pi_{12} - \Pi_{23} & \Pi_{23} & 0 & 0 & 0 \\ & & -\Pi_{13} - \Pi_{23} & 0 & 0 & 0 \\ \hline & & & \Pi_{44} & 0 & 0 \\ & & & & \Pi_{55} & 0 \\ & & & & & \Pi_{66} \end{array}\right]. \tag{40}$$

The form of Eq. (39) and its representation (40) reflects a "hybrid notation" in the following sense: the first term represents the linear common invariant of the stress tensor $\boldsymbol{\sigma}$ and the structural tensor $\boldsymbol{\pi}^{\text{TW}}$ (analogy to the pressure sensitivity in case of isotropic material), whereas the second term represents the quadratic common invariant of the stress deviator s and the structural tensor $\boldsymbol{\Pi}^{\text{TW}}$ (defining shape and orientation of the surface in the stress space). The criterion (39) takes therefore the explicit format of the nine-parameter Tsai–Wu's criterion [58],

$$-\left[\Pi_{23}\left(\sigma_y - \sigma_z\right)^2 + \Pi_{13}\left(\sigma_z - \sigma_x\right)^2 + \Pi_{12}\left(\sigma_x - \sigma_y\right)^2\right] + \Pi_{44}\tau_{yz}^2$$
$$+ \Pi_{55}\tau_{zx}^2 + \Pi_{66}\tau_{xy}^2 + \pi_{11}\sigma_x + \pi_{22}\sigma_y + \pi_{33}\sigma_z - 1 = 0. \tag{41}$$

As a matter of fact, any addition of a hydrostatic pressure to all normal stresses $\sigma_x \rightarrow \sigma_x \pm \sigma_{\text{h}}$ does not change the magnitude of quadratic terms in condition (41), but simultaneously causes the linear terms still to depend

on σ_h. Hence, finally the Tsai–Wu criterion in the format given by (41) remains the pressure-sensitive one by the linear terms.

This condition can be considered as a special case of the n-type yield functions presented by Pariseau [41], see Chen and Han [8]. The Pariseau condition predicts a generalized paraboloidal increase of strength with hydrostatic pressure. However, the conventional quadratic format of the Pariseau criterion, commonly named the Tsai–Wu criterion [58], was identified and implemented to ice crushing failure by Ralston [45].

5 Transversely isotropic Tsai–Wu-type format: tetragonal versus hexagonal symmetry criteria: convexity analysis

Similarly to [16], a further reduction of the nine-parameter yield/failure orthotropic Tsai–Wu's criterion (41) to the narrower format of transverse isotropy requires a precise distinction between the tetragonal and hexagonal symmetry classes. Assuming, after Chen and Han [8], the plane of transverse isotropy xy, the fourth-rank orthotropy matrix $[\mathbf{\Pi}^{\mathrm{TW}}]$ (40) reduces to the transversely isotropic format $[_{\mathrm{tris}}\mathbf{\Pi}^{\mathrm{TW}}]$, analogously to the transversely isotropic Hill criterion possessing only four independent material constants, whereas the second-rank transversely isotropic matrix $[_{\mathrm{tris}}\boldsymbol{\pi}^{\mathrm{TW}}]$ reduces to a form possessing only two independent material constants. Namely, assuming $\Pi_{23} = \Pi_{13}$, $\Pi_{44} = \Pi_{55}$ and $\pi_{11} = \pi_{22}$ in (40), we arrive at the six-parameter form of the transversely isotropic yield/failure matrices of tetragonal symmetry

$$
\left[_{\mathrm{tris}}\boldsymbol{\pi}^{\mathrm{TW}}\right] = \begin{bmatrix} \pi_{11} & 0 & 0 \\ & \pi_{11} & 0 \\ & & \pi_{33} \end{bmatrix},
$$

$$
\left[_{\mathrm{tris}}\mathbf{\Pi}^{\mathrm{TW}}\right] = \left[\begin{array}{ccc|ccc} -\Pi_{12} - \Pi_{13} & \Pi_{12} & \Pi_{13} & 0 & 0 & 0 \\ & -\Pi_{12} - \Pi_{13} & \Pi_{13} & 0 & 0 & 0 \\ & & -2\Pi_{13} & 0 & 0 & 0 \\ \hline & & & \Pi_{44} & 0 & 0 \\ & & & & \Pi_{44} & 0 \\ & & & & & \Pi_{66} \end{array}\right]. \tag{42}
$$

In case when the non-abbreviated notation is used, the six-parameter transversely isotropic Tsai–Wu yield/failure criterion of tetragonal symmetry takes the following format:

$$
-\Pi_{13}\left[(\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2\right] - \Pi_{12}(\sigma_x - \sigma_y)^2 + \Pi_{44}\left(\tau_{yz}^2 + \tau_{zx}^2\right)
$$
$$
+ \Pi_{66}\tau_{xy}^2 + \pi_{11}(\sigma_x + \sigma_y) + \pi_{33}\sigma_z - 1 = 0. \tag{43}
$$

Equation (43) is expressed in terms of six material anisotropy modules: Π_{12}, Π_{13}, Π_{44}, Π_{66}, π_{11} and π_{33}, which will be determined in terms of six independent material coefficients, referring to appropriate tensile and compressive strengths $k_{\mathrm{tx}}, k_{\mathrm{cx}}, k_{\mathrm{tz}}, k_{\mathrm{cz}}$ and shear strengths k_{zx}, k_{xy}. In order to calibrate them, the following tests have to be performed:

$$
\begin{aligned}
\sigma_x &= k_{\mathrm{tx}}, & \sigma_y &= \cdots = \tau_{zx} = 0, \\
\sigma_x &= -k_{\mathrm{cx}}, & \sigma_y &= \cdots = \tau_{zx} = 0, \\
\sigma_z &= k_{\mathrm{tz}}, & \sigma_x &= \cdots = \tau_{zx} = 0, \\
\sigma_z &= -k_{\mathrm{cz}}, & \sigma_x &= \cdots = \tau_{zx} = 0, \\
\tau_{zx} &= k_{zx}, & \sigma_x &= \cdots = \tau_{yz} = 0, \\
\tau_{xy} &= k_{xy}, & \sigma_x &= \cdots = \tau_{yz} = 0
\end{aligned} \tag{44}
$$

which lead to the system of equations:

$$
\begin{aligned}
(-\Pi_{13} - \Pi_{12})\, k_{\mathrm{tx}}^2 + \pi_{11} k_{\mathrm{tx}} &= 1, \\
(-\Pi_{13} - \Pi_{12})\, k_{\mathrm{cx}}^2 - \pi_{11} k_{\mathrm{cx}} &= 1, \\
-2\Pi_{13} k_{\mathrm{tz}}^2 + \pi_{33} k_{\mathrm{tz}} &= 1, \\
-2\Pi_{13} k_{\mathrm{cz}}^2 - \pi_{33} k_{\mathrm{cz}} &= 1, \\
\Pi_{44} k_{zx}^2 &= 1, \\
\Pi_{66} k_{xy}^2 &= 1.
\end{aligned} \tag{45}
$$

The solution of Eqs. (45), with respect to $\Pi_{13}, \Pi_{12}, \Pi_{44}, \Pi_{66}, \pi_{11}$ and π_{33}, yields

$$-\Pi_{13} = \frac{1}{2k_{tz}k_{cz}}, \quad -\Pi_{12} = \frac{1}{k_{tx}k_x} - \frac{1}{2k_{tz}k_{cz}}, \quad \Pi_{44} = \frac{1}{k_{zx}^2},$$

$$\Pi_{66} = \frac{1}{k_{xy}^2}, \quad \pi_{11} = \frac{1}{k_{tx}} - \frac{1}{k_{cx}}, \quad \pi_{33} = \frac{1}{k_{tz}} - \frac{1}{k_{cz}}. \tag{46}$$

The final form of the tetragonal transversely isotropic Tsai–Wu criterion in terms of $k_{tx}, k_{cx}, k_{tz}, k_{cz}, k_{zx}$ and k_{xy} is

$$\frac{\sigma_x^2 + \sigma_y^2}{k_{tx}k_{cx}} + \frac{\sigma_z^2}{k_{tz}k_{cz}} - \left(\frac{2}{k_{tx}k_{cx}} - \frac{1}{k_{tz}k_{cz}}\right)\sigma_x\sigma_y - \frac{\sigma_y\sigma_z + \sigma_x\sigma_z}{k_{tz}k_{cz}}$$

$$+ \frac{\tau_{yz}^2 + \tau_{zx}^2}{k_{zx}^2} + \frac{\tau_{xy}^2}{k_{xy}^2} + \left(\frac{1}{k_{tx}} - \frac{1}{k_{cx}}\right)(\sigma_x + \sigma_y) + \left(\frac{1}{k_{tz}} - \frac{1}{k_{cz}}\right)\sigma_z = 1. \tag{47}$$

Except the above tetragonal format of the transversely isotropic Tsai–Wu criterion Eq. (47), there exists also its hexagonal format, in which the number of essentially independent moduli is equal to five. In case if the narrower hexagonal symmetry sub-class is assumed, the sixth diagonal modulus Π_{66} has to satisfy the relationship (cf. Chen and Han [8], Ganczarski and Skrzypek [14])

$$\Pi_{66} = -2(\Pi_{13} + 2\Pi_{12}). \tag{48}$$

The constraint (48) satisfies the reducibility of the criterion (47) to the format invariant with respect to equivalent simple shear stress states: $\tau_{xy} = \sigma$ and $\sigma_x = \sigma, \sigma_y = -\sigma$, which must hold in the transverse isotropy xy plane.

The sixth material modulus Π_{66} is equal to

$$\Pi_{66} = \frac{4}{k_{tx}k_{cx}} - \frac{1}{k_{tz}k_{cz}}. \tag{49}$$

The final format of the hexagonal transversely isotropic Tsai–Wu criterion in terms of $k_{tx}, k_{cx}, k_{tz}, k_{cz}$ and k_{zx} is furnished as

$$\frac{\sigma_x^2 + \sigma_y^2}{k_{tx}k_{cx}} + \frac{\sigma_z^2}{k_{tz}k_{cz}} - \left(\frac{2}{k_{tx}k_{cx}} - \frac{1}{k_{tz}k_{cz}}\right)\sigma_x\sigma_y - \frac{\sigma_y\sigma_z + \sigma_x\sigma_z}{k_{tz}k_{cz}}$$

$$+ \frac{\tau_{yz}^2 + \tau_{zx}^2}{k_{zx}^2} + \left(\frac{4}{k_{tx}k_{cx}} - \frac{1}{k_{tz}k_{cz}}\right)\tau_{xy}^2$$

$$+ \left(\frac{1}{k_{tx}} - \frac{1}{k_{cx}}\right)(\sigma_x + \sigma_y) + \left(\frac{1}{k_{tz}} - \frac{1}{k_{cz}}\right)\sigma_z = 1. \tag{50}$$

Both Tsai–Wu limit equations: the tetragonal format (47) and the hexagonal format (50) may suffer from a possible convexity loss, analogously as in the case of corresponding transversely isotropic Hill's criteria, as discussed in detail in [16].

However, in the previous case the strength differential effect was not taken into account, such that the known inequality bounds by Ottosen and Ristinmaa [40] could be applicable,

$$\frac{1}{k_z^2}\left(\frac{4}{k_x^2} - \frac{1}{k_z^2}\right) > 0. \tag{51}$$

For the present convexity analysis, the extended inequality bounds the range of applicability of the transversely isotropic Tsai–Wu criteria (47) and (50),

$$\frac{1}{k_{tz}k_{cz}}\left(\frac{4}{k_{tx}k_{cx}} - \frac{1}{k_{tz}k_{cz}}\right) > 0, \tag{52}$$

where the strength differential effect is included. It should be noticed that by satisfying the condition (52) convexity of both criteria (47) and (50) is guaranteed. In other words, criterion (52) means that the material coefficients in the xy transverse isotropy plane, preceding both terms $\sigma_x\sigma_y$ and the τ_{xy}^2 in (50), cannot change signs.
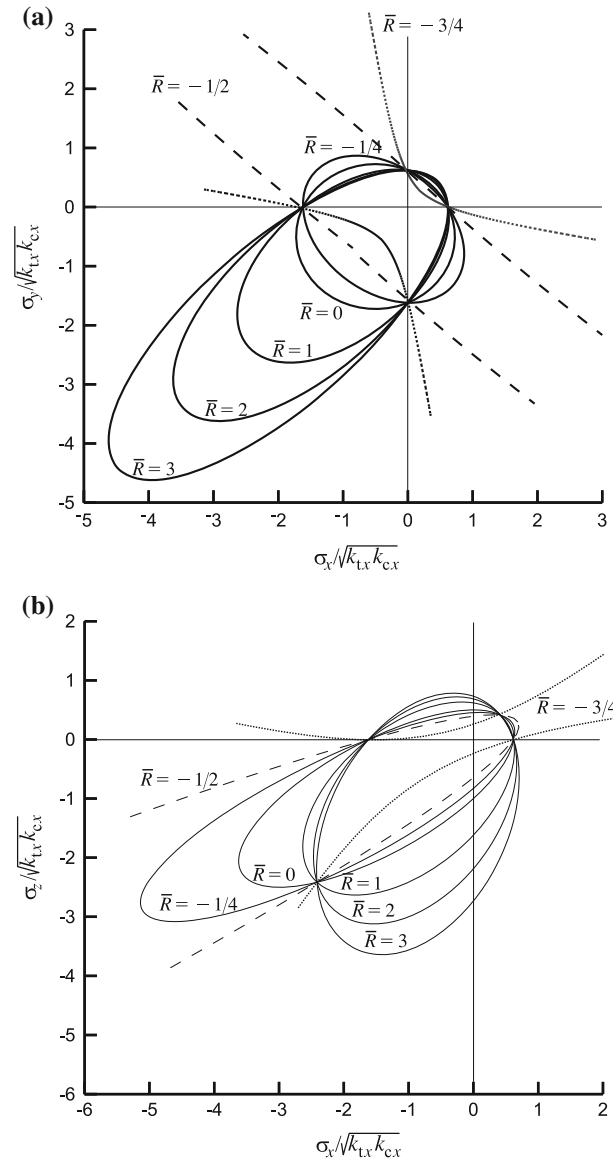
Fig. 4 Degeneration of the Tsai–Wu limit surface with the magnitude of the generalized Hosford and Backhofen parameter $\overline{R}$: **a** transverse isotropy plane, **b** orthotropy plane, after Ganczarski and Adamski [17], Skrzypek and Ganczarski [51]

A general proof of convexity of the Tsai–Wu criterion (41), as well as its particular case of transverse isotropy (43), is presented in Appendix A.

Substitution of the dimensionless parameter

$$\overline{R} = 2\frac{k_{tz}k_{cz}}{k_{tx}k_{cx}} - 1 \tag{53}$$

being an extension of the Hosford and Backhofen parameter leads to the simplified format of (52)

$$\overline{R} > -0.5. \tag{54}$$

If the above inequality (54) does not hold, elliptic cross sections of the limit surfaces degenerate to two hyperbolic branches such that loss of convexity occurs.

To illustrate what happens if criterion (54) is violated, the yield/failure curves in two planes: the transverse isotropy plane (σ_x, σ_y)

$$\sigma_x^2 - \frac{2\overline{R}}{1+\overline{R}}\sigma_x\sigma_y + \sigma_y^2 + (k_{cx} - k_{tx})(\sigma_x + \sigma_y) = k_{tx}k_{cx} \tag{55}$$

and the orthotropy plane (σ_x, σ_z)

$$\sigma_x^2 - \frac{2}{1+\overline{R}}\sigma_x\sigma_z + \frac{2}{1+\overline{R}}\sigma_z^2 + (k_{cx} - k_{tx})\sigma_x + k_{tz}k_{cz}\sigma_z = k_{tx}k_{cx} \tag{56}$$

for various $\overline{R}$-values, are sketched in Figs. 4a and 4b, respectively. Note that in the case considered, when shear terms which distinguish both formulas, tetragonal (47) and hexagonal (50), do not appear, Eqs. (55) and (56) are identical for both formulations. It is observed that, when $\overline{R}$ is ranging from $\overline{R} = 3$ to $\overline{R} = -0.5$, the limit curves degenerate from closed ellipses to two parallel lines, whereas for $\overline{R} < -0.5$, the inadmissible concave hyperbolas appear.

6 New formulation of unconditionally convex von Mises–Tsai–Wu-type criterion

Both previously discussed criteria: the tetragonal (47) and the hexagonal (50) suffer from possible convexity loss, moreover criterion (50) exhibits an intrinsic contradiction since it promises transverse isotropy of hexagonal type, whereas the ratio between the ellipse semi-axes in this plane is not equal to the corresponding ratio of the isotropic Huber–von Mises ellipse, analogously as it was proposed in [16].

In order to avoid both inconveniences, in what follows the other hexagonal transversely isotropic von Mises–Tsai–Wu yield/failure criterion, see Ganczarski and Adamski [17], Skrzypek and Ganczarski [51], is proposed. This is unconditionally convex and simultaneously preserves the Huber–von Mises ratio of ellipses' semi-axes. In order to do this, consider the more general transversely isotropic von Mises–Tsai–Wu criterion of the format

$$\Pi_{11}\left(\sigma_x^2 + \sigma_y^2\right) + \Pi_{33}\sigma_z^2 + 2\Pi_{12}\sigma_x\sigma_y + 2\Pi_{13}\left(\sigma_x + \sigma_y\right)\sigma_z$$
$$+ \Pi_{44}\left(\tau_{yz}^2 + \tau_{zx}^2\right) + \Pi_{66}\tau_{xy}^2 + \pi_{11}\left(\sigma_x + \sigma_y\right) + \pi_{33}\sigma_z = 1. \tag{57}$$

Equation (57) contains $8 = 6 + 2$ modules, and it is a straightforward simplification of the orthotropic von Mises–Tsai–Wu criterion (38) by assuming $\Pi_{11} = \Pi_{22}$, $\Pi_{23} = \Pi_{31}$, $\Pi_{44} = \Pi_{55}$ and $\pi_{11} = \pi_{22}$. For calibration of (57), the following tests can be performed: four uniaxial tension/compression tests and one out-of-plane shear test,

$$\sigma_x = k_{tx}, \quad \sigma_y = \cdots = \tau_{zx} = 0 \quad \longrightarrow \quad \Pi_{11}k_{tx}^2 + \pi_{11}k_{tx} = 1,$$
$$\sigma_x = -k_{cx}, \quad \sigma_y = \cdots = \tau_{zx} = 0 \quad \longrightarrow \quad \Pi_{11}k_{cx}^2 - \pi_{11}k_{cx} = 1,$$
$$\sigma_z = k_{tz}, \quad \sigma_x = \cdots = \tau_{zx} = 0 \quad \longrightarrow \quad \Pi_{33}k_{tz}^2 + \pi_{33}k_{tz} = 1,$$
$$\sigma_z = -k_{cz}, \quad \sigma_x = \cdots = \tau_{zx} = 0 \quad \longrightarrow \quad \Pi_{33}k_{cz}^2 - \pi_{33}k_{cz} = 1, \tag{58}$$
$$\tau_{zx} = k_{zx}, \quad \sigma_x = \cdots = \tau_{yz} = 0 \quad \longrightarrow \quad \Pi_{44}k_{zx}^2 = 1. \tag{59}$$

Additionally, three lacking conditions: one in-plane shear condition and two biaxial conditions that allow to capture magnitudes of Π_{66}, Π_{12} and Π_{13},

$$\tau_{xy} = \sqrt{\frac{k_{tx}k_{cx}}{3}}, \sigma_x = \cdots = \tau_{yz} = 0 \longrightarrow \frac{\Pi_{66}k_{tx}k_{cx}}{3} = 1,$$
$$\sigma_x = \sigma_y = c_{(xy)} = -(k_{cx} - k_{tx}) \mp \sqrt{\Delta_1}, \quad \sigma_z = \cdots = \tau_{yz} = 0$$
$$\longrightarrow \frac{2c_{(xy)}^2}{k_{tx}k_{cx}} - 2\Pi_{12}c_{(xy)}^2 + \frac{k_{cx} - k_{tx}}{k_{tx}k_{cx}}c_{(xy)} = 1,$$
$$\sigma_x = \sigma_z = c_{(xz)} = -\frac{1}{2}\left[(k_{cx} - k_{tx}) + (k_{cz} - k_{tz})\frac{k_{tx}k_{cx}}{k_{tz}k_{cz}} \pm \sqrt{\Delta_2}\right],$$
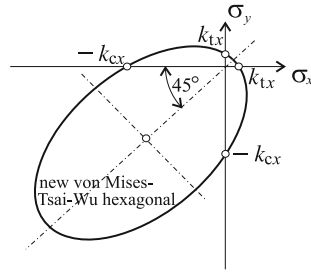
Fig. 5 Illustration of the limit curve of Eq. (63) in the plane of transverse isotropy

$$\sigma_y = \cdots = \tau_{yz} = 0 \longrightarrow \frac{c_{(xz)}^2}{k_{tx}k_{cx}} + \frac{c_{(xz)}^2}{k_{tz}k_{cz}} - 2\Pi_{13}c_{(xz)}^2, \tag{60}$$

$$+ \left(\frac{1}{k_{tx}} - \frac{1}{k_{cx}}\right)c_{(xz)} + \left(\frac{1}{k_{tz}} - \frac{1}{k_{cz}}\right)c_{(xz)} = 1, \tag{61}$$

are assumed. Symbols Δ_1 and Δ_2, used for brevity, denote: $\Delta_1 = (k_{cx} - k_{tx})^2 + k_{tx}k_{cx}$ and $\Delta_2 = [(k_{cx} - k_{tx}) + (k_{cz} - k_{tz})\frac{k_{tx}k_{cx}}{k_{tz}k_{cz}}]^2 + 4k_{tx}k_{cx}$. The solution of Eqs. (58–60) with respect to eight material moduli $\Pi_{11}, \Pi_{12}, \Pi_{13}, \Pi_{33}, \Pi_{44}, \Pi_{66}, \pi_{11}$ and π_{33} yields

$$\begin{aligned}
\Pi_{11} &= \frac{1}{k_{tx}k_{cx}}, & \Pi_{12} &= -\frac{1}{2k_{tx}k_{cx}}, & \Pi_{13} &= -\frac{1}{2k_{tz}k_{cz}}, \\
\Pi_{33} &= \frac{1}{k_{tz}k_{cz}}, & \Pi_{44} &= \frac{1}{k_{zx}^2}, & \Pi_{66} &= \frac{3}{k_{tx}k_{cx}}, \\
\pi_{11} &= \frac{1}{k_{tx}} - \frac{1}{k_{cx}}, & \pi_{33} &= \frac{1}{k_{tz}} - \frac{1}{k_{cz}}
\end{aligned} \tag{62}$$

which finally leads to the new hexagonal transversely isotropic von Mises–Tsai–Wu failure criterion also in terms of 5 independent material constants $k_{tx}, k_{cx}, k_{tz}, k_{cz}$ and k_{zx}, but different from hexagonal format of the Tsai–Wu criterion (50),

$$\frac{\sigma_x^2 + \sigma_y^2}{k_{tx}k_{cx}} + \frac{\sigma_z^2}{k_{tz}k_{cz}} - \frac{\sigma_x\sigma_y}{k_{tx}k_{cx}} - \frac{\sigma_y\sigma_z + \sigma_x\sigma_z}{k_{tz}k_{cz}} + \frac{\tau_{yz}^2 + \tau_{zx}^2}{k_{zx}^2} + 3\frac{\tau_{xy}^2}{k_{tx}k_{cx}}$$
$$+ \left(\frac{1}{k_{tx}} - \frac{1}{k_{cx}}\right)(\sigma_x + \sigma_y) + \left(\frac{1}{k_{tz}} - \frac{1}{k_{cz}}\right)\sigma_z = 1, \tag{63}$$

Criterion (63) ensures unconditional convexity, regardless of the magnitude of orthotropy ratio, and required coincidence of the semi-axes ratio, in the isotropy plane with that expected from the isotropic Huber–von Mises criterion, see Fig. 5.

Both Tsai–Wu transversely isotropic failure criteria: the hexagonal Eq. (50) and new hexagonal type Eq. (63), with conditions (58) and (60) applied, are compared for columnar ice, the experimental data of which were established by Ralston [45] in Table 5, in the plane of transverse isotropy (σ_x, σ_y) and the shear plane (σ_x, τ_{xy}), see Fig. 6, and in the plane of orthotropy (σ_x, σ_z), see Fig. 7. Corresponding cross sections of the limit surface are ellipses that exhibit strong oblateness, the centres of which are shifted outside the origin of the coordinate system towards the third quarter. In the case of cross section by the plane of transverse isotropy, in Fig. 6, the symmetry axis is inclined at 45°. In other words, it overlaps the projection of the hydrostatic axis at the transverse isotropy plane (σ_x, σ_y), contrary to the cross section by the plane of orthotropy, in Fig. 7, where the main semi-axis of the ellipse is inclined by 71.1°. It has to be emphasized that, in the case of columnar ice, the compressive strength along the orthotropy axis k_{cz} is more than 10 times larger than tensile strength k_{tz}, whereas the analogous ratio k_{cx}/k_{tx} is approximately equal to 7 in the case of the transverse isotropy plane. Both the hexagonal transversely isotropic Tsai–Wu failure criterion Eq. (50) and the new hexagonal transversely isotropic von Mises–Tsai–Wu failure criterion Eq. (63) contain the same number of 5 independent strengths $k_{tx}, k_{cx}, k_{tz}, k_{cz}$ and k_{zx}; however, only the criterion (63) is unconditionally convex.

Table 5 Experimental data for columnar ice after Ralston [45]

Tensile strength				Compressive strength			
k_{tx}	1.01 MPa	k_{tz}	1.21 MPa	k_{cx}	7.11 MPa	k_{cz}	13.5 MPa
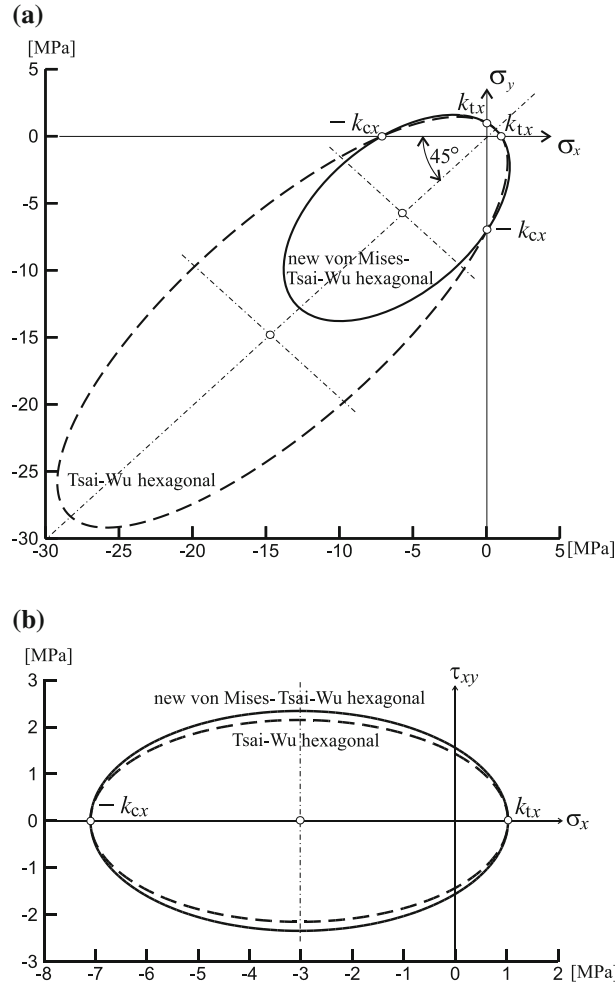
Fig. 6 Comparison of the hexagonal isotropic Tsai–Wu criterion (50) and the new hexagonal transversely isotropic von Mises–Tsai–Wu criterion (63) for columnar ice: **a** plane of transverse isotropy (σ_x, σ_y), **b** shear plane (σ_x, τ_{xy}), after Ganczarski and Adamski [17], Skrzypek and Ganczarski [51]

7 Remarks on the implicit formulation of pressure-sensitive anisotropic failure criteria: Khan's concept

In what follows, selected examples of implementation of the implicit approach to the broader class accounting for anisotropy, tension/compression asymmetry, and pressure sensitivity are thoroughly considered.

In some cases of anisotropic alloys exhibiting tension/compression asymmetry, it is convenient to consider a scalar function of selected (mixed) stress invariants and common invariants (cf. Khan and Liu [29], etc.),

$$f\left[\mathrm{tr}(\boldsymbol{\sigma}),\ \frac{1}{2}\mathrm{tr}(\boldsymbol{s}\cdot\boldsymbol{s}),\ \frac{1}{3}\mathrm{tr}(\boldsymbol{s}\cdot\boldsymbol{s}\cdot\boldsymbol{s});\ \ldots,\ \boldsymbol{\sigma}:\boldsymbol{\Pi}:\boldsymbol{\sigma},\ \ldots\right]. \tag{64}$$

The particular format of (64) can be recognized as a combination of (6) and (25).

Khan and Liu [29] applied the following extension of the nine-parameter orthotropic von Mises criterion to describe the ductile failure of the Ti–6Al–4V alloy accounting for: hydrostatic pressure sensitivity, anisotropy and significant tension/compression asymmetry effect:
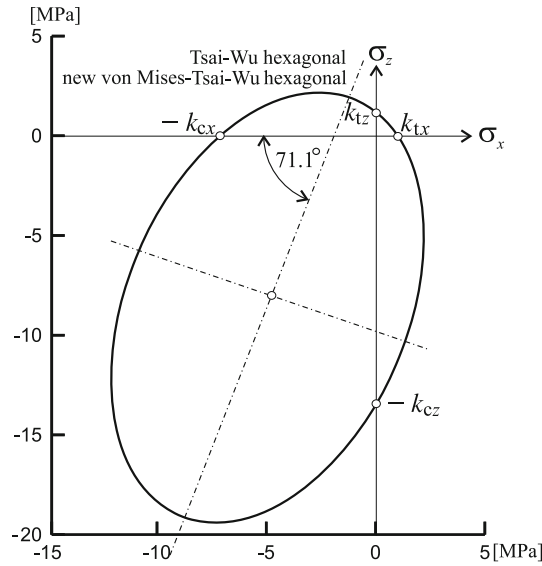
Fig. 7 Identity of the hexagonal isotropic Tsai–Wu criterion (50) and the new hexagonal transversely isotropic von Mises–Tsai–Wu criterion (63) for columnar ice in the plane of orthotropy (σ_x, σ_z), after Ganczarski and Adamski [17], Skrzypek and Ganczarski [51]

$$\sqrt{\exp[C(\zeta + 1)] \left(F\sigma_1^2 + G\sigma_2^2 + H\sigma_3^2 + L\sigma_1\sigma_2 + M\sigma_2\sigma_3 + N\sigma_1\sigma_3 + P\sigma_{12}^2 + Q\sigma_{13}^2 + R\sigma_{23}^2 \right)} = \exp\left(c_1 \frac{I_1}{\sqrt{3}} \right). \tag{65}$$

Both the hydrostatic pressure dependence I_1 and the tension/compression asymmetry J_3 are included in an implicit fashion as arguments of two exponential functions, appearing as multipliers at the right- and the left-hand sides of the orthotropic von Mises' equation. According to the authors' interpretation, the main advantage of such formulation is that the anisotropy and tension/compression asymmetry are uncoupled into separate multiplicative terms, which allow the anisotropic parameters and tension/compression asymmetry coefficient to be determined independently. The following definitions hold: F, G, H, L, M, N, P, Q and R are anisotropic parameters, C is the tension/compression asymmetry coefficient, ζ denotes the Lode parameter $\zeta = \cos 3\theta = \frac{27}{2} \frac{J_3}{(\sqrt{3J_2})^3}$, where θ is the Lode angle, I_1 is the first stress invariant, whereas J_2 and J_3 are the second and the third invariants of the deviatoric stress tensor. Although the general form of limit criterion (65), accounts for all three features: anisotropy, tension/compression asymmetry and hydrostatic pressure dependence, in fact its calibration performed by the authors leads to the form capturing only the tension/compression asymmetry and hydrostatic pressure dependence. As a consequence, the limit curve of Al2024-T351 alloy exhibits only one axis of symmetry, which means that this corresponds to the case of a partly distorted limit surface. By the use of the above formula, the authors succeeded with fitting experimental data in rolling direction (RD), transverse to rolling direction (TD), and the thickness direction (ND).

However, hydrostatic pressure dependence introduced by the use of the right-hand-side exponential function leads to loss of convexity of the failure surface along the meridional direction as it was shown by Khan and Liu [29] in Fig. 10b in Appendix B. This property was broadly discussed, where an analogy between anisotropic Khan–Liu's and isotropic Burzyński's surfaces was presented. An essential difference between these two cases is pointed out, because the Burzyński surface exhibits only conditional meridional convexity loss, whereas the Khan–Liu surface exhibits unconditional concavity built-in. The convexity loss discussed in this case is significant only from the theoretical point of view because in such a case Drucker's postulate is violated. However, for the data cited by the authors the concave meridian effect is very small such that it can probably be ignored from the engineering point of view for the considered material data. Nevertheless, in spite of possible convexity loss along the meridian, none convexity loss along circumference is observed although there exists a second exponential function dependent on J_2 and J_3, being a multiplier of the Hill form on the left-hand side of Eq. (65).

In another paper by Khan et al. [30], the direct hydrostatic pressure dependence (by I_1) is dropped; however, both significant anisotropy (fully anisotropic calibration of all material constants F, G, ..., R) and tension/
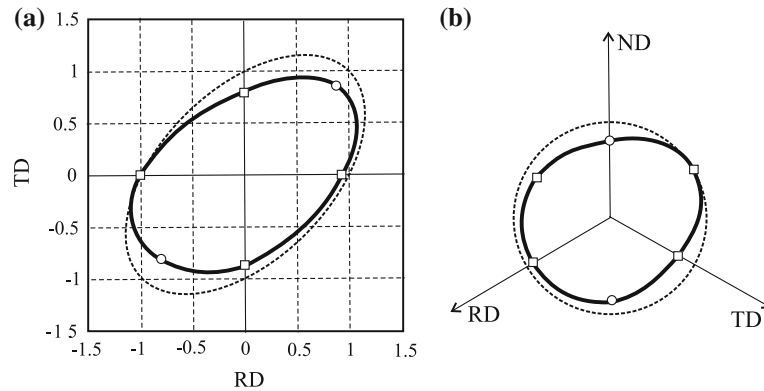
Fig. 8 Correlation of the 0.2 % yield loci of Ti–6Al–4V alloy (◦—experimental data points, □—points calculated from ND experimental data) with the yield function proposed by Khan et al. [30] (*solid line*) and Huber–von Mises criterion (*dashed line*): **a** comparison in RD-TD plane, **b** projection on deviatoric plane

compression asymmetry are saved,

$$\exp[-C(\zeta+1)]\left(F\sigma_1^2 + G\sigma_2^2 + H\sigma_3^2 + L\sigma_1\sigma_2 + M\sigma_2\sigma_3 + N\sigma_1\sigma_3 + P\sigma_{12}^2 + Q\sigma_{13}^2 + R\sigma_{23}^2\right) = 1. \quad (66)$$

Although the general form of the limit criterion (66) accounts for nine independent anisotropy parameters, in the example considered by the authors in [30], due to calibration the material constant G is determined from the equi-biaxial compression test; so, it depends on three compression limits like in the case of Hill's criterion. Under the assumption of plane stress state, it reduces to the four-parameter orthotropic Hill's condition. Fitting of experimental data for Ti–6Al–4V alloy at different strain rates and temperatures shows excellent coincidence between the experimental findings and simulation. By contrast to the previous formulation (65), the symmetry of the limit curve is completely lost (no axis of symmetry exists), as shown in Fig. 8.

8 Conclusions

The present contribution is a straightforward continuation of the earlier work by Ganczarski and Skrzypek [16]. This is an essential extension for pressure-sensitive materials, instead of the previously discussed pressure-insensitive materials. Detailed discussion on a possible convexity loss of the conventional Tsai–Wu criterion is done.

The Tsai–Wu- and Pariseau-type criteria describe materials in which failure limits in tension and compression may be essentially different. Hence, these criteria are strongly exposed to a possible convexity loss, see Eq. (50). To avoid the aforementioned deficiency of convexity lack, the new von Mises–Tsai–Wu failure criterion, which is free from the possibility of inadmissible degeneration of the single convex, simply connected elliptical surface into two concave hyperbolic surfaces as shown in Eq. (63), is proposed. This was successfully done by relaxing the constraint (48) for Π_{66}, which is no longer dependent on Π_{13} and Π_{12}.

The paper is focused on the three aspects: pressure sensitivity, strength differential effect, and anisotropy. The authors are convinced of the conventional approach, which is based on rigorous application of common invariants formalism, see Goldenblat and Kopnov [21], Sayir [50], and others.

Due to dynamic growth in the field of material fabrication and design, including metallic ones that exhibit all three properties, the parallel approach based on the transformed stress applied to isotropic limit criteria is developed, see Barlat et al. [6,7,44], Khan et al. [30], and others. Also, other mixed-type criteria, which combine some elements from the theory of common invariants and the approach based on transformed stress, are developed in parallel, see Yoon et al. [60]. Having this in mind, in this paper attention is paid to compare the traditional rigorous approach versus the transformed stress approach, see Appendix C.

General conclusions can be drawn as follows:

– newly formulated constitutive laws must follow fundamental laws of symmetry and convexity,
– clear physical bases are necessary to include,
– even impressive fitting of a proposed single curve to experimental data may not lead to a correct description of the limit surface as a whole,

– the particular case of transverse isotropy should be considered carefully, because of its important place in the field of composites, and requires the precise distinction between symmetry groups and classes, the reducibility to well-established isotropic formulations,
– continuous symmetry (infinite number of symmetry axes) should obey in the transverse isotropy plane for the criterion proposed, preserving the ratio of semi-axes identical to the isotropic Huber–von Mises ellipse.

As a matter of fact, this paper addresses the initial limit surfaces only, but not a loading process on which changes in the subsequent limit surfaces occur. It means that the convexity requirement (Sylvester or Drucker) is guaranteed strictly at the beginning of the hardening process that produces microstructural changes. Hence, the convexity requirement for the initial limit surface may but not has to guarantee convexity of subsequent surfaces. On the other hand, a concavity of the initial surface must result in a lack of convexity of subsequent surfaces.

Appendix A: Conditions of convexity of the Tsai–Wu yield/failure function

The Tsai–Wu yield/failure condition takes the format

$$f\left(\boldsymbol{\sigma}, \boldsymbol{\pi}^{\mathrm{TW}}, \boldsymbol{\Pi}^{\mathrm{TW}}\right) = \left\{\boldsymbol{\pi}^{\mathrm{TW}}\right\}\{\boldsymbol{\sigma}\} + \{s\}^{\mathrm{T}}\left[\boldsymbol{\Pi}^{\mathrm{TW}}\right]\{s\} - 1 = 0 \tag{A.1}$$

where $\boldsymbol{\sigma}$ stands for stress tensor, whereas $\boldsymbol{\pi}^{\mathrm{TW}}$ and $\boldsymbol{\Pi}^{\mathrm{TW}}$ denote structural tensors of material anisotropy of second and fourth rank, respectively.

The condition of convexity for the yield/failure function (A.1) is that the Hessian matrix ought to be positive semi-definite,

$$H_{ij} = \frac{\partial^2 f}{\partial \sigma_i \partial \sigma_j} \geq 0. \tag{A.2}$$

To save the generality of convexity proof, in what follows, the arbitrary stress components are considered, instead of the conventionally used principal stress components, see Cazacu et al. [7].

The format of the Tsai–Wu yield/failure function (A.1), including linear and quadratic terms with respect to the stress tensor, inserted to (A.2), shows that only the fourth-rank structural tensor of material anisotropy decides convexity,

$$[\mathbf{H}] = \left[\boldsymbol{\Pi}^{\mathrm{TW}}\right] =$$
$$\begin{bmatrix} -\Pi_{12} - \Pi_{13} & \Pi_{12} & \Pi_{13} & 0 & 0 & 0 \\ & -\Pi_{12} - \Pi_{23} & \Pi_{23} & 0 & 0 & 0 \\ & & -\Pi_{13} - \Pi_{23} & 0 & 0 & 0 \\ \hline & & & \Pi_{44} & 0 & 0 \\ & & & & \Pi_{55} & 0 \\ & & & & & \Pi_{66} \end{bmatrix}, \tag{A.3}$$

and simultaneously requirement of semi-definiteness (A.2) means that all eigenvalues of the Hessian (A.3) must be non-negative. Since the format of (A.3) exhibits symmetry typical for material orthotropy, the eigenvalues of the lower right block ($i, j = 4, 5, 6$) are obviously given by inequalities

$$\begin{aligned} H_{44} &= \Pi_{44} > 0, \\ H_{55} &= \Pi_{55} > 0, \\ H_{66} &= \Pi_{66} > 0 \end{aligned} \tag{A.4}$$

whereas eigenvalues of the upper left block ($i, j = 1, 2, 3$) have to be found as roots of the appropriate secular equation, the coefficients (invariants) of which are as follows:

$$\begin{aligned} \mathrm{tr}\left(H_{ij}\right) &= -2\left(\Pi_{12} + \Pi_{13} + \Pi_{23}\right), \\ \mathrm{tr}\left(H_{ij} H_{ji}\right) &= 3\left(\Pi_{12}\Pi_{23} + \Pi_{12}\Pi_{13} + \Pi_{13}\Pi_{23}\right), \\ \mathrm{tr}\left(H_{ij} H_{ik} H_{jk}\right) &= 0. \end{aligned} \tag{A.5}$$

Note that the third (cubic) invariant in (A.5) is equal to zero, which confirms hydrostatic pressure independence of the Hessian and leads to the following spectral decomposition (eigenvalues):

$$\lambda_1 = 0,$$

$$\lambda_{2,3} = -(\Pi_{12} + \Pi_{13} + \Pi_{23}) \mp \sqrt{\frac{1}{2}\left[(\Pi_{23} - \Pi_{12})^2 + (\Pi_{13} - \Pi_{23})^2 + (\Pi_{12} - \Pi_{13})^2\right]}. \qquad (A.6)$$

It is visible that only one of three eigenvalues (A.6) may be negative, and hence the fourth inequality completing (A.4) is as follows:

$$2(\Pi_{12} + \Pi_{13} + \Pi_{23})^2 > (\Pi_{23} - \Pi_{12})^2 + (\Pi_{13} - \Pi_{23})^2 + (\Pi_{12} - \Pi_{13})^2. \qquad (A.7)$$

The conditions of convexity (A.4) and (A.7) may be expressed in terms of tension, compression and shear strength, namely performing 9 known tests magnitudes of moduli of material orthotropy Π_{ij} can be established as follows:

$$-\Pi_{12} = \frac{1}{2}\left(\frac{1}{k_{tx}k_{cx}} + \frac{1}{k_{ty}k_{cy}} - \frac{1}{k_{tz}k_{cz}}\right), \qquad (A.8.1)$$

$$-\Pi_{13} = \frac{1}{2}\left(\frac{1}{k_{tx}k_{cx}} - \frac{1}{k_{ty}k_{cy}} + \frac{1}{k_{tz}k_{cz}}\right), \qquad (A.8.2)$$

$$-\Pi_{23} = \frac{1}{2}\left(-\frac{1}{k_{tx}k_{cx}} + \frac{1}{k_{ty}k_{cy}} + \frac{1}{k_{tz}k_{cz}}\right), \qquad (A.8.3)$$

$$\Pi_{44} = \frac{1}{k_{yz}^2}, \quad \Pi_{55} = \frac{1}{k_{zx}^2}, \quad \Pi_{66} = \frac{1}{k_{xy}^2}, \qquad (A.8.4\text{--}6)$$

$$\Pi_{11} = \frac{1}{k_{tx}} - \frac{1}{k_{cx}}, \quad \Pi_{22} = \frac{1}{k_{ty}} - \frac{1}{k_{cy}}, \quad \Pi_{33} = \frac{1}{k_{tz}} - \frac{1}{k_{cz}} \qquad (A.8.7\text{--}9)$$

which inserted to (A.4) and (A.7) leads to the inequalities

$$\frac{2}{(k_{tx}k_{cx})(k_{ty}k_{cy})} + \frac{2}{(k_{tx}k_{cx})(k_{tz}k_{cz})} + \frac{2}{(k_{tz}k_{cz})(k_{tx}k_{cx})} > \frac{1}{(k_{tx}k_{cx})^2} + \frac{1}{(k_{ty}k_{cy})^2} + \frac{1}{(k_{tz}k_{cz})^2} \qquad (A.9.1)$$

$$\frac{1}{k_{yz}^2} > 0, \quad \frac{1}{k_{zx}^2} > 0, \quad \frac{1}{k_{xy}^2} > 0 \qquad (A.9.2\text{--}4)$$

which turn out to be generalizations of Ottosen and Ristinmaa's [40] bounds.

In the case of transverse isotropy, the symmetry conditions $k_{tx} = k_{ty}$ and $k_{cx} = k_{cy}$ reduce Eq. (A.9.1) to

$$\frac{1}{k_{tz}k_{cz}}\left(\frac{4}{k_{tx}k_{cx}} - \frac{1}{k_{tz}k_{cz}}\right) > 0, \qquad (A.10)$$

and simultaneously $\Pi_{66} = -2(\Pi_{13} + 2\Pi_{12})$ inserted to (A.8.6) leads to

$$H_{66} = \frac{4}{k_{tx}k_{cx}} - \frac{1}{k_{tz}k_{cz}} > 0 \qquad (A.11)$$

which is obviously the sub-case of (A.9.1). Finally, substitution of the generalized Hosford and Backhofen parameter

$$\overline{R} = 2\frac{k_{tz}k_{cz}}{k_{tx}k_{cx}} - 1 \qquad (A.12)$$

simplifies condition (A.9.1) to the format

$$\overline{R} > -0.5. \qquad (A.13)$$

Appendix B: Comparison of possible meridional convexity loss of limit surfaces by Khan–Liu and Burzyński

In Sect. 6, the unconditionally convex von Mises–Tsai–Wu criterion has been formulated and discussed, in contrast to Sect. 5, where the Tsai–Wu criterion exhibiting possible convexity loss in hoop direction has been discussed in detail. The majority of researchers notices the case of inadmissible hoop convexity loss, in the light of Drucker's postulate sense, for instance Cazacu and Barlat [6], Ottosen and Ristinmaa [40], Cazacu et al. [7], Ganczarski and Lenczowski [13], Ganczarski and Skrzypek [19], Skrzypek and Ganczarski [51], to mention only some of them. However, very few concepts deal with the other case of meridional convexity loss. In what follows, the above feature incorporated to both the anisotropic Khan–Liu concept and chronologically earlier isotropic Burzyński criterion is discussed. Either Khan–Liu or Burzyński criteria are obviously pressure-sensitive ones. It will be shown that Burzyński's criterion, which is usually convex, may loose convexity in particular cases, whereas Khan–Liu's criterion possesses a concavity built-in.

The three-parameter isotropic criterion, rotationally symmetric yield/failure surface, originally introduced by Burzyński [5], is

$$A3J_{2s} + B\left(\frac{J_{1\sigma}}{3}\right)^2 + C\left(\frac{J_{1\sigma}}{3}\right) - 1 = 0 \tag{B.1}$$

where A, B, and C are the material constants. These constants are determined based on three tests: the uniaxial tension (k_t), the uniaxial compression (k_c), and the simple shear (k_s). These calibrations lead to the general format of the three-parameter Burzyński's criterion,

$$\frac{k_t k_c}{3k_s^2} J_{2s} + \left(9 - \frac{3k_t k_c}{k_s^2}\right)\left(\frac{J_{1\sigma}}{3}\right)^2 + (k_c - k_t) J_{1\sigma} - k_t k_c = 0. \tag{B.2}$$

Such formulation presumes not only tension/compression asymmetry $k_t \neq k_c$, but also the third shear limit point k_s is considered as independent. Equation (B.2) represents different types of Burzyński's rotationally symmetric surface depending on mutual relationships between k_t, k_c and k_s. In case if the shear yield/failure strength k_s is larger than $\sqrt{k_t k_c/3}$, Eq. (B.2) represents a rotationally symmetric ellipsoid, if the shear yield strength is equal to $\sqrt{k_t k_c/3}$ it represents a rotationally symmetric paraboloid, whereas if the shear yield strength is less than $\sqrt{k_t k_c/3}$ it represents a twofold rotationally symmetric hyperboloid. If k_s reaches its lower admissible bound $\frac{2k_t k_c}{\sqrt{3}(k_t + k_c)}$, the Burzyński hyperboloid transforms to the Drucker–Prager's cone. Shear yield strengths less than $\frac{2k_t k_c}{\sqrt{3}(k_t + k_c)}$ are inadmissible in the sense of Drucker's postulate, since in such case Eq. (B.2) represents a onefold concave hyperboloid (Fig. 9). In the case of a twofold hyperboloid, only one fold that includes stress origin has physical sense.

The two-parameter paraboloidal approximation of Burzyński's surface ($k_s = \sqrt{k_t k_c/3}$) can be written as follows:

$$3J_{2s} + (k_c - k_t) J_{1\sigma} - k_t k_c = 0. \tag{B.3}$$

It is experimentally verified for metallic alloys, see for instance material constants for Inconel 718 cited by Pęcherski et al. [42],

$$k_t = 779 \text{ MPa}, \qquad k_c = 878 \text{ MPa}, \qquad k_s = 473 \text{ MPa}. \tag{B.4}$$

Note that the yield/failure strength satisfies condition $k_s = \sqrt{k_t k_c/3}$.

In the other particular case, if

$$k_s = \frac{2k_t k_c}{\sqrt{3}(k_t + k_c)} \tag{B.5}$$

the two-parameter conical approximation of Burzyński's surface is furnished,

$$\sqrt{3J_{2s}} + \frac{k_c - k_t}{k_t + k_c} J_{1\sigma} - 2\frac{k_t k_c}{k_t + k_c} = 0. \tag{B.6}$$

The above condition can be reduced to the Drucker–Prager's condition commonly met in the literature. Note, however, that in the light of the above discussion the Drucker–Prager's condition can be considered as the limit case for the applicability of Burzyński's criterion. Below this limit, when $k_s < \frac{2k_t k_c}{\sqrt{3}(k_t + k_c)}$, the conical surface deforms into the concave onefold hyperboloid, which is inadmissible following Drucker's stability postulate.
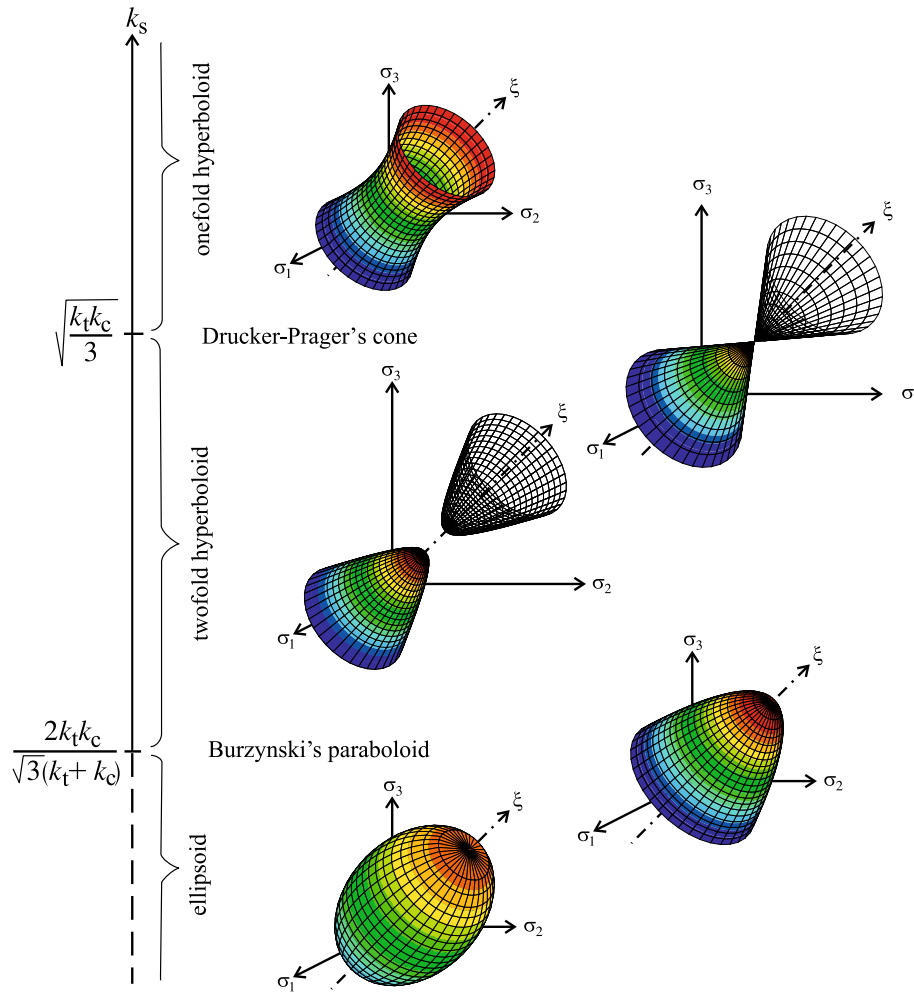
Fig. 9 Different types of Burzyński's rotationally symmetric yield/failure surface versus mutual relationships between k_t, k_c and k_s ($\xi = J_{1\sigma}/\sqrt{3}$ Haigh–Westergaard's coordinate along the hydrostatic axis)
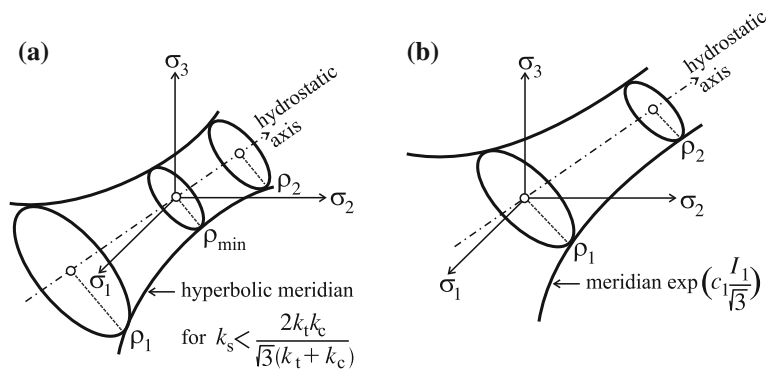
Fig. 10 Schematics of: **a** Burzyński's onefold hyperboid, and **b** Khan–Liu's failure surfaces in the principal stress frame

Table 6 Discussed limit criteria exhibiting convexity loss of the meridional type

Isotropy	Burzyński (B.1)	Conditional
	$\frac{k_t k_c}{3 k_s^2} J_{2s} + \left(9 - \frac{3 k_t k_c}{k_s^2}\right) \left(\frac{J_{1\sigma}}{3}\right)^2$	Meridional convexity
	$+ (k_c - k_t) J_{1\sigma} - k_t k_c = 0$	loss if $k_s < \frac{2 k_t k_c}{\sqrt{3}(k_t + k_c)}$
Anisotropy	Khan–Liu (65)	Unconditional
	$\{\exp[C(\zeta + 1)]\,(F\sigma_1^2 + G\sigma_2^2 + H\sigma_3^2$	meridional
	$+ L\sigma_1\sigma_2 + M\sigma_2\sigma_3 + N\sigma_1\sigma_3 + P\sigma_{12}^2$	concavity
	$+ Q\sigma_{13}^2 + R\sigma_{23}^2)\}^{1/2} = \underline{\exp\left(c_1 \frac{I_1}{\sqrt{3}}\right)}$	
	$\zeta = \frac{27}{2} \frac{J_3}{(\sqrt{3 J_2})^3}$	

A comparison of the isotropic Burzyński conditionally convex yield surface and the anisotropic Khan–Liu unconditionally concave failure surface is shown in Fig. 10.

Corresponding underlined terms which are directly responsible for convexity loss in both criteria: Burzyński's or Khan–Liu's, are demonstrated in Table 6.

Appendix C: Concise review of isotropic and anisotropic explicit or implicit failure criteria

In this Section, a brief review of the selected pressure-sensitive yield/failure criteria is presented. Contrary to the survey given in [16], in the case considered here, the review of pressure-sensitive criteria has to account for three characteristic properties:

– the first stress $J_{1\sigma}$ or the first common $\boldsymbol{\pi} : \boldsymbol{\sigma}$ invariants have to be present in the yield/failure criterion,
– isotropic versus anisotropic formulation,
– direct versus indirect dependence on the stress invariants or the common invariants.

In the case of isotropic pressure-sensitive criteria, the attention is paid to an invariant representation of invoked criteria. Selected isotropic yield/failure criteria are collected in Table 7. All cited criteria depend on both the first stress invariant and the second deviatoric invariant, but, additionally, they may also depend on the third deviatoric invariant. Criteria C1 Iyer [26], Gao et al. [20] and C3 Iyer and Lissenden [27], Pęcherski et al. [42] are special cases of the general criterion C4, Eq. (9). Criterion C2 by Yoon et al. [60] has slightly different format and cannot be derived from the general criterion C4 as a particular case, but it can be considered as the extension of the Cazacu and Barlat [6] pressure-insensitive yield criterion A2 to the case of hydrostatic pressure sensitivity. Chronologically first, yield/failure Coulomb–Mohr's criterion C5 has been presented in the three equivalent formats: the original Coulomb format, the Mohr format explicitly expressed in terms of principal stresses, and the mixed invariant format, in which definition $\cos(3\theta) = \frac{3\sqrt{3}}{2} \frac{J_{3s}}{(J_{2s})^{3/2}}$ holds such that explicit dependence on the third stress deviator invariant is visible, see Chen and Han [8]. All above criteria C1–C5 represent in the Haigh–Westergaard space asymmetric yield/failure surfaces, and hence tensile and compressive meridians are positioned at different distance from the hydrostatic axis. Next, criterion C6 originated by Burzyński [5] represents in the Haigh–Westergaard space a rotationally symmetric surface of various shapes: ellipsoidal, paraboloidal, hyperboloidal or conical. A hypothetically possible onefold hyperboloidal surface has to be excluded on the basis of Drucker's convexity postulate. The last one admissible bound of the Burzyński criterion, which satisfies Drucker's convexity postulate, is Drucker–Prager's [11] criterion C7 that represents the conical surface. Note that both Burzyński's and Drucker–Prager's criteria degenerate to the Huber–von Mises' cylindrical surface in the case when dependence on hydrostatic pressure is neglected.

Selected pressure-sensitive anisotropic yield/failure criteria are written down in Table 8. Most of the criteria presented in this Table, namely items D1–D13 deal with the explicit formulation of the anisotropic yield/failure criteria, being consistently formulated in the frame of common stress and structural tensors $\boldsymbol{\pi} : \boldsymbol{\sigma}, \boldsymbol{\sigma} : \boldsymbol{\Pi} : \boldsymbol{\sigma}$, and $\boldsymbol{\sigma} : \mathbb{III} : \boldsymbol{\sigma} : \boldsymbol{\sigma}$. On the other hand, the last two items D14 and D15 comprise exemplary anisotropic yield/failure criteria based on the implicit formulation, where anisotropy is introduced by linear transformation imposed on the stress tensor. Next, the generalization of the known pressure-sensitive isotropic criteria is done, by replacing stresses or stress invariants by transformed ones. All aforementioned criteria include first and second common or transformed invariants by definition. The presence of the first invariant is necessary in

Table 7 Review of isotropic pressure-sensitive yield/failure criteria

C.	Author(s)	Stress invariants
C1	Iyer [26], Gao et al. [20]	$\left(a J_{1\sigma}^{2p} + b J_{2s}^{p} + c J_{3s}^{2p/3}\right)^{1/p} = 1$
C2	Yoon et al. [60]	$a[b J_{1\sigma} + (J_{2s}^{3/2} - c J_{3s})^{1/3}] = 1$
C3	Iyer and Lissenden [27],	
	Pęcherski et al. [42]	$a J_{1\sigma}^{2} + b J_{2s} + c J_{3s}^{2/3} = 1$
	Extension of C1 and C3	
C4	Formats, Skrzypek and	$\left(a J_{1\sigma}^{2p} + b J_{2s}^{p} + c J_{3s}^{2p/3}\right)^{1/r} - 1 = 0$
	Ganczarski [51]	
	Coulomb [9], Mohr [38]	$\|\tau\| = c - \sigma \tan\phi$
		$\frac{1}{2}(\sigma_1 - \sigma_3)\cos\phi$
C5	Principal stress format	$= c - \left[\frac{1}{2}(\sigma_1 + \sigma_3) + \frac{\sigma_1 - \sigma_3}{2}\sin\phi\right]\tan\phi$
	Mixed invariant	$\frac{1}{3}J_{1\sigma}\sin\phi + \sqrt{J_{2s}}\sin\left(\theta + \frac{\pi}{3}\right)$
	Format, Chen and Han [8]	$+\sqrt{\frac{1}{3}J_{2s}}\cos\left(\theta + \frac{\pi}{3}\right)\sin\phi - c\cos\phi = 0$
C6	Burzyński [5]	$A 3 J_{2s} + B\left(\frac{J_{1\sigma}}{3}\right)^2 + C\left(\frac{J_{1\sigma}}{3}\right) = 1$
C7	Drucker and Prager [11]	$\alpha J_{1\sigma} + \sqrt{J_{2s}} = k$

Table 8 Review of anisotropic pressure-sensitive yield/failure criteria

D.	Author(s)	Common invariants
D1	Goldenblat–Kopnov [21],	$(\boldsymbol{\pi} : \boldsymbol{\sigma})^{\alpha} + (\boldsymbol{\sigma} : \boldsymbol{\Pi} : \boldsymbol{\sigma})^{\beta}$
	Sayir [50]	$+ \left(\boldsymbol{\sigma} : \mathbb{III} : \boldsymbol{\sigma} : \boldsymbol{\sigma}\right)^{\gamma} + \ldots = 1$
D2	Życzkowski [61]	$\boldsymbol{\pi} : \boldsymbol{\sigma} + (\boldsymbol{\sigma} : \boldsymbol{\Pi} : \boldsymbol{\sigma})^{1/2}$
		$+ \left(\boldsymbol{\sigma} : \mathbb{III} : \boldsymbol{\sigma} : \boldsymbol{\sigma}\right)^{1/3} = 1$
D3	Kowalsky et al. [32]	$\mathbf{h}^{(0)} + \mathbf{h}_{ij}^{(1)} : \boldsymbol{s} + \boldsymbol{s} : \mathbf{h}^{(2)} : \boldsymbol{s}$
		$+ \boldsymbol{s} : \mathbf{h}^{(3)} : \boldsymbol{s} : \boldsymbol{s} = 0$
D4	Życzkowski [62]	$\boldsymbol{\pi} : \boldsymbol{\sigma} + \sqrt{\boldsymbol{\sigma} : \boldsymbol{\Pi} : \boldsymbol{\sigma}} = 1$
D5	Ganczarski and Lenczowski [13],	
	Ganczarski and Skrzypek [15]	$\boldsymbol{\pi} : \boldsymbol{\sigma} + \boldsymbol{\sigma} : \boldsymbol{\Pi} : \boldsymbol{\sigma} = 1$
D6	Ganczarski and Skrzypek [15]	$\boldsymbol{\pi} : \boldsymbol{\sigma} + \boldsymbol{s} : \boldsymbol{\Pi} : \boldsymbol{s} = 1$
D7	Orthotropic von Mises–Tsai–Wu	$_{\text{ort}}\boldsymbol{\pi} : \boldsymbol{\sigma} + \boldsymbol{\sigma} : {}_{\text{ort}}\boldsymbol{\Pi} : \boldsymbol{\sigma} = 1$
D8	von Mises [36], [37]	$\boldsymbol{\sigma} : \boldsymbol{\Pi} : \boldsymbol{\sigma} = 1$
D9	Khan et al. [30]	$\boldsymbol{\sigma} : {}_{\text{ort}}\boldsymbol{\Pi} : \boldsymbol{\sigma} = 1$
D10	Theocaris [57], Liu et al. [34]	$_{\text{ort}}\boldsymbol{\pi} : \boldsymbol{\sigma} + \sqrt{\boldsymbol{s} : \boldsymbol{\Pi}^{\text{H}} : \boldsymbol{s}} = 1$
D11	Tsai–Wu [58]	$_{\text{ort}}\boldsymbol{\pi} : \boldsymbol{\sigma} + \boldsymbol{s} : \boldsymbol{\Pi}^{\text{H}} : \boldsymbol{s} = 1$
D12	Tetragonal transversely	$_{\text{tris}}\boldsymbol{\pi}^{\text{TW}} : \boldsymbol{\sigma} + \boldsymbol{s} : {}_{\text{tris}}^{\text{tet}}\boldsymbol{\Pi}^{\text{TW}} : \boldsymbol{s} = 1$
	Isotropic Tsai–Wu	(6 material constants)
	Hexagonal transversely	
D13	Isotropic Tsai–Wu,	$_{\text{tris}}\boldsymbol{\pi}^{\text{TW}} : \boldsymbol{\sigma} + \boldsymbol{s} : {}_{\text{tris}}^{\text{hex}}\boldsymbol{\Pi}^{\text{TW}} : \boldsymbol{s} = 1$
	Ganczarski and Adamski [17]	(5 material constants)
D14	Khan and Liu [29],	$\sqrt{e^{C(\zeta+1)}\boldsymbol{\sigma} : {}_{\text{ort}}\boldsymbol{\Pi} : \boldsymbol{\sigma}} = e^{c_1 I_1/3}$
	Khan et al. [30]	$e^{-C(\zeta+1)}\boldsymbol{\sigma} : {}_{\text{ort}}\boldsymbol{\Pi} : \boldsymbol{\sigma} = 1$
D15	Yoon et al. [60]	$\widetilde{I}_1 + (J_2'^{3/2} - J_3'')^{1/3} = 1$
D16	Pietruszczak and Mróz [43]	$\sqrt{J_{2s}} - \eta g(\theta) I_1/3 = 0$

order to account for hydrostatic pressure sensitivity. The second invariant ensures energy-based interpretation of the limit criterion, whereas the third common or transformed invariant is optional.

The most general form D1, originated by Goldenblat and Kopnov [21], Sayir [50], etc., is written in a polynomial format, where the exponents α, β, γ, ... are arbitrary constants and the number of terms is arbitrarily chosen, but usually limited to the first three terms. Two particular cases of the criterion D1 are of special interest. Assuming $\alpha = 1$, $\beta = 1/2$, $\gamma = 1/3$, the homogeneity of the polynomial function on the left-hand side is assured, e.g. Życzkowski [61] D2. On the other hand, the criterion D3 used by Kowalsky et al. [32] does not satisfy the homogeneity requirement where $\alpha = \beta = \gamma = 1$ holds. In the criteria D1–D2, all three common invariants are saved; hence, the total number of independent material constants corresponding to the first $\boldsymbol{\pi} : \boldsymbol{\sigma}$, the second $\boldsymbol{\sigma} : \boldsymbol{\Pi} : \boldsymbol{\sigma}$, and the third $\boldsymbol{\sigma} : \mathbb{III} : \boldsymbol{\sigma} : \boldsymbol{\sigma}$ common invariant is equal to $6 + 21 + 56 = 83$. Both criteria D1–

D2 are formulated in the space of stress tensor components. However, when the majority of metallic materials is considered, the stress deviator space is more adequate to formulate limit criteria. Criterion D3 described in this space, having a reduced total number of independent material constants, was proposed by Kowalsky et al. [32]. Engineering application of the full format including all three common invariants is very complicated because it requires identification of a large number of moduli of the third common invariant (up to 56 in general case). The third common invariant is responsible for distortion of the limit surface; hence, in all cases where distortion is not very significant, it is reasonable to neglect the third common invariant. Items D4–D13 take advantage of the aforementioned simplification what means that only the first two common invariants are saved, which drastically reduces the number of independent material constants down to $6 + 21 = 27$. Both items D4 and D5 are consequently written down in the stress space. However, item D4 represents the conical type limit surface, being an anisotropic generalization of the isotropic Drucker–Prager's cone, whereas item D5 represents a paraboloidal type limit surface, being an anisotropic generalization of isotropic rotationally symmetric Burzyński's paraboloid. Of course, due to anisotropy, both discussed criteria do not satisfy the rotational symmetry property. In some cases, it is justified to use the deviatoric format of the second common invariant only, which leads to some reduction in the number of independent material constants $6 + 15 = 21$. The representative of such limit criterion is item D6. If fully deviatoric format of both the first and the second common invariant is used, we arrive at the hydrostatic pressure-independent criterion considered by Szczepiński [55] which has $5 + 15 = 20$ independent material constants. This criterion does not appear in Table 8 since it is pressure independent; however, its simplified form that has not the first deviatoric common invariant (35) is briefly discussed in Sect. 5.

Criterion D7 may be considered as an extension of the orthotropic von Mises criterion by use of the first common invariant described by $(3 + 9 = 12)$ independent material constants. The criteria presented in items D8 and D9 do not contain the first common invariant, and they are written down in the stress space. This means that both discussed criteria are pressure-sensitive ones. The general von Mises criterion D8 is described by 21 independent material constants, whereas the criterion D9, suggested by Khan et al. [30], contains only 9 independent material constants since it describes material orthotropy. The next two items, namely D10 and D11, can be considered as narrower formats of items D4 and D5. Both criteria are determined by $3 + 6 = 9$ independent material constants. This reduction is furnished by simultaneous use of two substitutions: first, substitution of Hill's structural tensor (6 independent material constants), instead of Mises' tensor (21 independent material constants), and second, substitution of the stress deviator by the stress tensor in the second common invariant.

Criteria D12 and D13 describe yield/failure surfaces in the case of transverse isotropy of tetragonal Eq. (47) and hexagonal Eq. (50) symmetry; however, they differ each from the other in this sense that format Eq. (47) is described by $2 + 4 = 6$ independent material constants in contrast to the format Eq. (50), in which $2 + 3 = 5$ independent material constants are present. In such a way, the narrower hexagonal form assures its reducibility to the shifted Huber–von Mises type ellipse in the plane of transverse isotropy.

Criteria D14 and D15 belong to a separate type of limit criteria in that sense that they are neither the common invariant-based explicit equations nor linear transformation-based implicit generalization of chosen isotropic criteria. These original mixed concepts are difficult to be classified in sense of either implicit or explicit approaches because there are involved simultaneously all three invariants. In D14 criterion, suggested by Khan and Liu [29], Khan et al. [30], the second common invariant $\boldsymbol{\sigma} : \boldsymbol{\Pi} : \boldsymbol{\sigma}$ and all stress invariants I_1, J_2, J_3 are involved. In D15 criterion, proposed by Yoon et al. [60], the first common invariant $\boldsymbol{\pi} : \boldsymbol{\sigma}$ together with the second and the third transformed invariants J_2', J_3' is used. The format with J_2', J_3' turns out to be the anisotropic extension of Drucker's criterion, which appears in power 1/3 to assure dimension homogeneity with the first common invariant.

Inherent anisotropy in geomaterials may be described by a concept by Pietruszczak and Mróz D16 [43] where the failure function involves basic stress invariants and the second mixed invariant of stress and structure orientation tensors, which reflect the orientation-dependent nature of the geomaterial strength, the Mohr–Coulomb extension to the case of anisotropy. Symbols η and $g(\theta)$ stand for an orientation-dependent parameter and a function of Lode's angle θ, respectively.

Appendix D: Convexity

Consider the narrower format of the anisotropic yield/failure criterion (4) with the third common invariant neglected, known as von Mises–Tsai–Wu criterion,

$$f(\boldsymbol{\sigma}) = \boldsymbol{\pi} : \boldsymbol{\sigma} + \boldsymbol{\sigma} : \boldsymbol{\Pi} : \boldsymbol{\sigma} - 1 = 0. \tag{D.1}$$

In what follows, Sylvester's stability formulation will be used in order to guarantee convexity of the yield/failure limit surface. According to Sylvester's formulation, positive definiteness of the quadratic form has to be satisfied,

$$\mathrm{d}\boldsymbol{\sigma} : \frac{\partial^2 f}{\partial \boldsymbol{\sigma} \partial \boldsymbol{\sigma}} : \mathrm{d}\boldsymbol{\sigma} > 0. \tag{D.2}$$

The matrix of components

$$[H_{ij}] = \left[\frac{\partial^2 f}{\partial \sigma_i \partial \sigma_j} \right] \tag{D.3}$$

is known as the Hessian matrix of the loading function $f(\boldsymbol{\sigma})$ (cf. Chen and Han [8]). The first term in (D.1) has no influence onto the form (D.3) after double partial differentiation with respect to $\boldsymbol{\sigma}$. Hence, (D.1) may be rewritten into the equivalent form

$$[H_{ij}] = [\Pi_{ij}] \tag{D.4}$$

or

$$\mathrm{d}\sigma_i \Pi_{ij} \mathrm{d}\sigma_j > 0. \tag{D.5}$$

According to Sylvester's criterion, a necessary and sufficient condition for positive definiteness of the quadratic form (D.5) takes the format

$$\det[\boldsymbol{\Pi}]_k > 0 \qquad k = 1, 2, \ldots, 6 \tag{D.6}$$

for arbitrary arguments $\dot{\sigma}_i$ and $\dot{\sigma}_j$, where $[\boldsymbol{\Pi}]_k$ stands for subsequent minors of the fourth-order structural tensor $\boldsymbol{\Pi}$,

$$
\begin{aligned}
&\Pi_{11} > 0 \\
&\det \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ & \Pi_{22} \end{bmatrix} > 0 \\
&\det \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} \\ & \Pi_{22} & \Pi_{23} \\ & & \Pi_{33} \end{bmatrix} > 0 \\
&\vdots \\
&\det \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} & \Pi_{15} & \Pi_{16} \\ & \Pi_{22} & \Pi_{23} & \Pi_{24} & \Pi_{25} & \Pi_{26} \\ & & \Pi_{33} & \Pi_{34} & \Pi_{35} & \Pi_{36} \\ & & & \Pi_{44} & \Pi_{45} & \Pi_{46} \\ & & & & \Pi_{55} & \Pi_{56} \\ & & & & & \Pi_{66} \end{bmatrix} > 0,
\end{aligned} \tag{D.7}
$$

The conventional Drucker's postulate for elastic-plastic material

$$\mathrm{d}\boldsymbol{\sigma} : \mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{p}} \geq 0 \tag{D.8}$$

is weaker than Sylvester's stability postulate (D.6). Drucker's postulate is formulated in the mixed stress–strain space (see Chen and Han [8]), whereas Sylvester's postulate is consistently defined in the stress space. Moreover, Sylvester's postulate requires positive definiteness, whereas Drucker's postulate requires semi-positive definiteness only.

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