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Approximation of maps into spheres by regulous maps

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Abstract. Let X be a compact real algebraic set of dimension n. We prove that every Euclidean continuous map from X into the unit n-sphere can be approximated by a regulous map. This strengthens and generalizes previously known results.

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1. Introduction. A recent direction of research in real algebraic geometry is to study intermediate classes of maps between continuous and regular maps. Such classes as continuous rational maps, stratified-regular maps, and regulous maps (which often coincide) have been studied in a series of papers [3,7,8,11–27]. The aim of this note is to strengthen a certain related result of [21] and [15].

We begin by fixing some terminology. A *real algebraic variety* is a locally ringed space isomorphic to some algebraic subset of $\mathbb{R}^n$, for some positive integer n, endowed with the Zariski topology and the sheaf of regular functions. It is worth recalling that this class is identical with the class of quasi-projective real varieties, for more detail and information see [4]. A morphism of real algebraic varieties is called a *regular map*. We will be also interested in the Euclidean topology of such varieties and this is the topology we will mean, unless explicitly stated otherwise, when using topological notions. By a smooth map we understand a map of class $\mathcal{C}^\infty$.

Let X be a real algebraic variety. A *stratification* $\mathcal{S}$ of X is a finite collection of pairwise disjoint Zariski locally closed subvarieties whose union is equal to X. A map $f\colon X \to Y$ of real algebraic varieties is said to be *regulous* if it is continuous and if there exists some stratification $\mathcal{S}$ of X such that $f|_S$ is a regular map for every $S \in \mathcal{S}$. We denote the set of all regulous maps between X and Y by $\mathcal{R}^0(X,Y)$. We shall treat $\mathcal{R}^0(X,Y)$ as a subspace of the space

$\mathcal{C}(X,Y)$ of all continuous maps endowed with the compact-open topology. Note that regulous maps in the sense of our definition were called stratified-regular in [21] and the follow-up papers [20,22]. This definition is different but equivalent to that of [7] where the terminology was introduced, see [21, Remark 2.3] or [24].

Each regulous map is also *continuous rational*—i.e. f is continuous and $f|_{X^0}$ is regular for some Zariski open dense subset $X^0 \subset X$. While the converse is false in general, it is true if X is nonsingular, see [11].

Let us first recall the following result contained in [15].

Theorem 1.1. *Let X be a compact nonsingular real algebraic variety of dimension $p \geq 1$. Then the set $\mathcal{R}^0(X, \mathbb{S}^p)$ is dense in $\mathcal{C}(X, \mathbb{S}^p)$.*

A related weaker result allowing for a singular X is contained in [21]:

Theorem 1.2. *Let X be a compact real algebraic variety of dimension $p \geq 1$. Then any continuous map $X \to \mathbb{S}^p$ is homotopic to a regulous map.*

Our aim is to strenghten Theorem 1.2 by showing that the nonsingularity assumption of Theorem 1.1 is unnecessary.

Theorem 1.3. *Let X be a compact real algebraic variety of dimension $p \geq 1$. Then the set $\mathcal{R}^0(X, \mathbb{S}^p)$ is dense in $\mathcal{C}(X, \mathbb{S}^p)$.*

It is well known that analogous results do not hold if regulous maps in Theorems 1.1, 1.2, and 1.3 are replaced with regular maps. For example, a continuous map $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^2$ is homotopic to a regular map if and only if it is null-homotopic, cf. [5].

2. Proof of the main theorem. We shall use the concept of the algebraic cohomology classes of a real algebraic variety which we recall now. Let X be a compact nonsingular real algebraic variety. A class in $H^*(X; \mathbb{Z}/2)$ is said to be algebraic if it is Poincaré dual to a homology class in $H_*(X, \mathbb{Z}/2)$ represented by an algebraic subset. The set $H^*_{\mathrm{alg}}(X; \mathbb{Z}/2)$ of all algebraic cohomology classes is a subring of $H^*(X; \mathbb{Z}/2)$ and if $f\colon X \to Y$ is a regular map, then the induced map f^* in cohomology maps $H^*_{\mathrm{alg}}(Y; \mathbb{Z}/2)$ into $H^*_{\mathrm{alg}}(X, \mathbb{Z}/2)$, cf. [2,4,6].

An important tool we need is [15, Lemma 2.2], which allows controlled approximation of continuous maps into projective space by regular maps. We restate it here for convenience.

Lemma 2.1. *Let X be a compact nonsingular real algebraic variety, and let A be a Zariski closed subvariety of X. Let $f\colon X \to \mathbb{P}^n(\mathbb{R})$ be a continuous map whose restriction $f|_A\colon A \to \mathbb{P}^n(\mathbb{R})$ is a regular map. Assume that*

$$f^*(H^1(\mathbb{P}^n(\mathbb{R}); \mathbb{Z}/2)) \subset H^1_{\mathrm{alg}}(X; \mathbb{Z}/2).$$

Then one can find a regular $g\colon X \to \mathbb{P}^n(\mathbb{R})$ arbitrarily close to f and satisfying $g|_A = f|_A$ (i.e. every neigborhood of f in $\mathcal{C}(X, \mathbb{P}^n(\mathbb{R}))$ contains such a map).

We are now ready to prove Theorem 1.3

Proof. Let f be any map in $\mathcal{C}(X,\mathbb{S}^p)$. Treating X as a closed subset of $\mathbb{R}^m$ for some $m \in \mathbb{N}$, one can find a smooth map $f_0\colon U \to \mathbb{S}^p$ defined on some neighborhood U of X in $\mathbb{R}^m$ such that $f_0|_X$ is arbitrarily close to f. Let Σ denote the singular locus of X. Then $f_0(\Sigma) \subsetneq \mathbb{S}^p$, since $\dim \Sigma < p$. This allows us to approximate $f_0|_\Sigma$ by regular maps using the stereographic projection and Weierstrass approximation theorem. We can therefore reduce the proof (by replacing f with suitably modified f_0) to the case where f is a restriction of a smooth map defined on a neighborhood of X in $\mathbb{R}^m$ and $f|_\Sigma$ is regular with $f(\Sigma) \subsetneq \mathbb{S}^p$. Then, by Sard's theorem, there exists an $s_0 \in \mathbb{S}^p \setminus f(\Sigma)$ which is a regular value of the smooth map $f|_{X\setminus\Sigma}$.

By Hironaka's resolution of singularities theorem [9,10], there exists a finite composition of blowups $\pi\colon Y \to X$ over Σ with Y nonsingular. The restriction $\bar{\pi}\colon Y \setminus \pi^{-1}(\Sigma) \to X \setminus \Sigma$ of π is then a biregular isomorphism and s_0 is a regular value of the smooth map $f \circ \bar{\pi}$. Letting $F = (f \circ \pi)^{-1}(s_0) = (f \circ \bar{\pi})^{-1}(s_0)$, consider the blowup of Y with center F, which we will denote $\sigma\colon B(Y,F) \to Y$, and the blowup $\tau\colon B(\mathbb{S}^p, s_0) \to \mathbb{S}^p$ of $\mathbb{S}^p$ over s_0. Since $\dim X = p$, the set F is finite as a fiber of the smooth map $f \circ \bar{\pi}$ over its regular value s_0, hence $B(Y,F)$ is a real algebraic variety. Finiteness of F and the fact that $f \circ \pi$ is also a smooth map allow us to apply [1, Lemma 2.5.9] in order to construct a smooth lifting g of $f \circ \pi$ to a map between $B(Y,F)$ and $B(\mathbb{S}^p, s_0)$ making the following diagram commute:

$$\begin{array}{ccc}
B(Y,F) & \xrightarrow{\ g\ } & B(\mathbb{S}^p, s_0) \\
{\scriptstyle\sigma}\downarrow & & {\scriptstyle\tau}\downarrow \\
Y & \xrightarrow{\ f\circ\pi\ } & \mathbb{S}^p \\
{\scriptstyle\pi}\downarrow & & \Vert \\
X & \xrightarrow{\ f\ } & \mathbb{S}^p
\end{array}$$

Our aim for now is to find a regular map $H\colon B(Y,F) \to B(\mathbb{S}^p, s_0)$ arbitrarily close to g in $\mathcal{C}(B(Y,F), B(\mathbb{S}^p, s_0))$ in such a way that the map $\tilde{f}\colon X \to \mathbb{S}^p$ making the following diagram commute will be regulous and close to f:

$$\begin{array}{ccc}
B(Y,F) & \xrightarrow{\ H\ } & B(\mathbb{S}^p, s_0) \\
{\scriptstyle\pi\circ\sigma}\downarrow & & {\scriptstyle\tau}\downarrow \\
X & \xrightarrow{\ \tilde{f}\ } & \mathbb{S}^p
\end{array}$$

Let $D = \sigma^{-1}(F)$ and $E = \tau^{-1}(s_0)$. Then D and E are nonsingular algebraic hypersurfaces in the respective blowups. Let $u \in H^1(B(Y,F);\mathbb{Z}/2)$ be the cohomology class Poincaré dual to the homology class in $H_*(B(Y,F);\mathbb{Z}/2)$ represented by D and let v be the class in $H^1(B(\mathbb{S}^p, s_0);\mathbb{Z}/2)$ dual to the class in $H_*(B(\mathbb{S}^p, s_0);\mathbb{Z}/2)$ represented by E. Recall that there exists a biregular isomorphism $\varphi\colon B(\mathbb{S}^p, s_0) \to \mathbb{P}^p(\mathbb{R})$ such that

$$\varphi(E) = \mathbb{P}^{p-1}(\mathbb{R}) \subset \mathbb{P}^p(\mathbb{R}). \tag{1}$$

Hence $H^1(B(\mathbb{S}^p, s_0); \mathbb{Z}/2) = \varphi^*(H^1(\mathbb{P}^p(\mathbb{R}); \mathbb{Z}/2)) \cong \mathbb{Z}/2$ with v the generator. Since g is transverse to E and $D = g^{-1}(E)$ we have $u = g^*(v)$ (cf. [6, Proposition 2.15]) and it follows that

$$g^*(H^1(B(\mathbb{S}^p, s_0); \mathbb{Z}/2)) \subset H^1_{\mathrm{alg}}(B(Y; F); \mathbb{Z}/2). \tag{2}$$

Let $i\colon D \to B(Y, F)$ and $j\colon E \to B(\mathbb{S}^p, s_0)$ be the inclusion maps and let $\bar{g}\colon D \to E$ be the map induced by the restriction of g to D. Then, since $g \circ i = j \circ \bar{g}$ we have $\bar{g}^*(j^*(v)) = i^*(g^*(v)) = i^*(u)$. This allows us to apply Lemma 2.1 to approximate $\bar{g}$ by regular maps. Indeed, since by (1) $H^1(E; \mathbb{Z}/2)$ is generated by $j^*(v)$ and by definition $i^*(u)$ is in $H^1_{\mathrm{alg}}(D; \mathbb{Z}/2)$, we have

$$\bar{g}^*(H^1(E; \mathbb{Z}/2)) \subset H^1_{\mathrm{alg}}(D; \mathbb{Z}/2).$$

Therefore, there exists a regular map $r\colon D \to E$ arbitrarily close to $\bar{g}$ in $\mathcal{C}(D, E)$.

Let $L = (\pi \circ \sigma)^{-1}(\Sigma)$ and let $h\colon \Sigma \to B(\mathbb{S}^p, s_0)$ be the regular map given by $(\tau|_{\tau^{-1}(\mathbb{S}^p \setminus \{s_0\})})^{-1} \circ (f|_\Sigma)$. We now extend $j \circ r$ to a smooth map $G\colon B(Y, F) \to B(\mathbb{S}^p, s_0)$ close to g such that $G|_L = (h \circ \pi \circ \sigma)|_L$ (this is possible since the sets E and L are disjoint). If G is close enough to g, we have $g^* = G^*$ and hence by (2) we can apply Lemma 2.1 to G with $A = D \cup L$. This gives a regular map $H\colon B(Y, F) \to B(\mathbb{S}^p, s_0)$ which is close to g and satisfies $H|_L = (h \circ \pi \circ \sigma)|_L$ and $H(D) \subset E$. The latter shows that the map $\bar{H}\colon Y \to \mathbb{S}^p$ given by

$$\bar{H}(y) = \begin{cases} s_0, & \text{for } y \in F,\\ \tau(H(\sigma^{-1}(y))), & \text{for } y \notin F, \end{cases}$$

is continuous and well-defined. Moreover $\bar{H}(y) = f \circ \pi(y)$ for $y \in \pi^{-1}(\Sigma)$. Also note that $\bar{H}$ is close to $f \circ \pi$, since $(\tau \circ H)|_{Y \setminus F}$ is close to $\tau \circ f \circ \pi$. Therefore, the map $\tilde{f}\colon X \to \mathbb{S}^p$ given by

$$\tilde{f}(x) = \begin{cases} f(x), & \text{for } x \in \Sigma,\\ \bar{H}(\pi^{-1}(x)), & \text{for } x \notin \Sigma, \end{cases}$$

is well-defined, continuous, and can be chosen arbitrarily close to f.

It remains to check that $\tilde{f}$ is regulous. Since $\tilde{f}|_\Sigma$ is regular as is $\tilde{f}$ in restriction to the finite set $f^{-1}(s_0)$, it is enough to show $\tilde{f}|_{X \setminus A}$ is regular where $A = \Sigma \cup f^{-1}(s_0)$. Since we have $\tilde{f}|_{X \setminus A} = \tau \circ H \circ (\sigma|_{Y \setminus F})^{-1} \circ (\pi|_{X \setminus \Sigma})^{-1}$ with all the maps on the right-hand side regular, the proof is finished. □

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