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**Aequationes Mathematicae**

# Some generalization of the quadratic and Wilson's functional equation

Radosław Łukasik

**Abstract.** We find the solutions $f, g, h\colon G \to X$, $\varphi\colon G \to \mathbb{K}$ of each of the functional equation

$$\sum_{\lambda\in K} f(x+\lambda y) = |K|\varphi(y)g(x) + |K|h(y), \quad x, y \in G,$$

where $(G, +)$ is an abelian group, $K$ is a finite, abelian subgroup of the automorphism group of $G$, $X$ is a linear space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

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## 1. Introduction

The generalization of the quadratic functional equation

$$f(x+y) + f(x+\sigma y) = 2f(x) + 2f(y), \quad x, y \in G,$$

where $\sigma$ is automorphism of an abelian group $G$ such that $\sigma^2 = id_G$, $f, g\colon G \to \mathbb{C}$, was investigated by Stetkær [13].

In another work [14] he solved the functional equation

$$\frac{1}{N}\sum_{n=0}^{N-1} f(z+\omega^n\zeta) = g(z) + h(\zeta), \quad z, \zeta \in \mathbb{C},$$

where $N \in \{2, 3, \dots\}$, $\omega$ is the primitive $Nth$ root of unity, $f, g, h\colon \mathbb{C} \to \mathbb{C}$ are continuous.

Łukasik [8] showed the solution of the functional equation

$$\sum_{\lambda\in K} f(x+\lambda y) = |K|g(x) + |K|h(y), \quad x, y \in S,$$

where $(S, +)$ is an abelian semigroup, $K$ is a finite subgroup of the automorphism group of $S$, $(H, +)$ is an abelian group.

Birkhäuser

The functional equation

$$\sum_{\lambda\in K} f(x+\lambda y) = |K|g(x)h(y), \quad x,y\in G,$$

where $(G,+)$ is an abelian group, $K$ is a finite subgroup of the automorphism group on $G$, $f,g,h\colon G\to\mathbb{C}$, was studied by Förg-Rob and Schwaiger [5], Gajda [6], Stetkær [11,12], Badora [2].

## 2. Main result

Throughout the present paper, we assume that $X$ is a linear space over the field $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, $(G,+)$ is an abelian group, $K$ is a finite, abelian subgroup of the automorphism group of $G$, $L:=\text{card } K$.

We give the complete solution of the following functional equation

$$\sum_{\lambda\in K} f(x+\lambda y) = L\varphi(y)g(x) + Lh(y), \quad x,y\in S.$$

In this work we use the following two theorems and one lemma:

**Theorem 1** (Shinya [9, Corollary 3.12], Kannappan [7], Czerwik [4], Sinopoulos [10], Chojnacki [3]). *Let $(G,+)$ be an abelian, locally compact, Hausdorff topological group, $K$ be a compact Hausdorff topological transformation group of $G$ acting by automorphisms on $G$. Let further $d\lambda$ be a normalized Haar measure on $K$. If $\varphi\in C(G)$ is a nonzero solution of*

$$\int_{\lambda\in K} \varphi(x+\lambda y)d\lambda = \varphi(x)\varphi(y), \quad x,y\in G,$$

*then there exists a continuous homomorphism $\chi\colon G\to\mathbb{C}^*$ such that*

$$\varphi(x) = \int_{\lambda\in K} \chi(\lambda x)d\lambda, \quad x\in G.$$

*If $\varphi$ is bounded, then $\chi$ may be taken as a unitary character.*

**Lemma 1** [1, Lemma 14.1]. *Let $\Omega$ be a nonempty set, $n\in\mathbb{N}$. Functions $f_1,\ldots,f_n\colon\Omega\to\mathbb{C}$ are linearly dependent if and only if for all $x_1,\ldots,x_n \in\Omega$*

$$\begin{vmatrix} f_1(x_1) & f_1(x_2) & \ldots & f_1(x_n) \\ f_2(x_1) & f_2(x_2) & \ldots & f_2(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ f_n(x_1) & f_n(x_2) & \ldots & f_n(x_n) \end{vmatrix} = 0.$$

**Theorem 2** [8, Theorem 5]. *Let $(S,+)$ be an abelian semigroup, $K$ be a finite subgroup of the automorphism group of $S$, $L :=$ card $K$, $(H,+)$ be an abelian group uniquely divisible by $L!$. Then the function $f\colon S \to H$ satisfies the equation*

$$\sum_{\lambda\in K} f(x+\lambda y) = Lf(x), \quad x,y \in S$$

*if and only if there exist $k$-additive, symmetric mappings $A_k\colon S^k \to H$, $k \in \{1,\dots L-1\}$ and $A_0 \in H$ such that*

$$f(x) = A_0 + A_1(x) + \cdots + A_{L-1}(x,\cdots,x), \quad x \in S,$$
$$\sum_{\lambda\in K} A_k(x,\dots,x,\underbrace{\lambda y,\dots,\lambda y}_{i}) = 0, \quad x,y \in S,\ 1 \le i \le k \le L-1.$$

First we show some lemmas.

**Lemma 2.** *Let $m\colon G \to \mathbb{C}^*$ be a homomorphism, $K_1 \subset K$ be a set such that $\{m\circ\lambda :\ \lambda \in K_1\}$ is linearly independent. If for some $\mu \in K$*

$$m(\mu x) = \sum_{\lambda\in K_1} a_\lambda m(\lambda x), \quad x \in G, \tag{1}$$

*then $m\circ\mu = m\circ\lambda$ for some $\lambda \in K_1$.*

*Proof.* Let

$$m(\mu x) = \sum_{\lambda\in K_1} a_\lambda m(\lambda x), \quad x \in G.$$

Therefore, for $x,y \in G$, we have

$$\sum_{\lambda\in K_1} a_\lambda m(\lambda y)m(\lambda x) = m(\mu(y+x)) = m(\mu y)m(\mu x) = m(\mu y)\sum_{\lambda\in K_1} a_\lambda m(\lambda x),$$

which means that

$$0 = \sum_{\lambda\in K_1} a_\lambda[m(\lambda y) - m(\mu y)]m(\lambda x).$$

From the linear independence we obtain

$$0 = a_\lambda[m(\lambda y) - m(\mu y)], \quad y \in G, \quad \lambda \in K_1,$$

since there exists $\lambda \in K_1$ such that $a_\lambda \neq 0$,

$$m(\lambda y) = m(\mu y), \quad y \in G.$$

□

**Lemma 3.** *Let $m\colon G \to \mathbb{C}^*$ be a homomorphism. Then the set $K_0 := \{\lambda \in K :\ m\circ\lambda = m\}$ is a subgroup of $K$.*

**Lemma 4.** *Let* $K_0 := \{\lambda \in K : \ m \circ \lambda = m\}$, $K_1 \subset K$ *be a minimal set such that* $K = K_0 \circ K_1$. *Then* $K_1$ *is a maximal set such that the set* $\{m \circ \lambda : \ \lambda \in K_1\}$ *is linearly independent.*

*Proof.* In view of Lemma 2, if $\{m \circ \lambda : \ \lambda \in K_1\}$ is linearly dependent, then $m \circ \lambda = m \circ \mu$ for some $\lambda, \mu \in K_1$, so we get $\lambda \circ \mu^{-1} \in K_0$, which means that $\lambda \in K_0 \circ \mu$, which contradicts the definition of $K_1$.

On the other hand each $\lambda \in K$ has the form $\lambda = \lambda_0 \circ \lambda_1$, where $\lambda_0 \in K_0$, $\lambda_1 \in K_1$, hence $m \circ \lambda = m \circ \lambda_0 \lambda_1 = m \circ \lambda_1$, which gives us the maximality of $K_1$. □

**Theorem 3.** *Let* $\varphi\colon G \to \mathbb{C}$, $\varphi \neq 0$, *satisfy the equation*

$$\sum_{\lambda \in K} \varphi(x + \lambda y) = L\varphi(x)\varphi(y), \quad x, y \in G. \tag{2}$$

*Then there exist a homomorphism* $m\colon G \to \mathbb{C}^*$, $\beta_\lambda \in \mathbb{C}^*, b_\lambda \in G, \lambda \in K_1$, *such that*

$$\varphi(x) = \frac{1}{L} \sum_{\lambda \in K} m(\lambda x), \quad x \in G, \tag{3}$$

$$\sum_{\lambda \in K_1} \beta_\lambda m(\mu b_\lambda) = \begin{cases} |K_1|, & \mu \in K_0 \\ 0, & \mu \notin K_0 \end{cases} \tag{4}$$

$$m(x) = \sum_{\lambda \in K_1} \beta_\lambda \varphi(x + b_\lambda), \quad x \in G, \tag{5}$$

*where* $K_0 := \{\lambda \in K : \ m \circ \lambda = m\}$, $K_1 \subset K$ *is a minimal set such that* $K = K_0 \circ K_1$.

*Proof.* If we take a discrete topology on group $(G, +)$ and a counting measure on $K$ divided by $L$, then the assumptions of Theorem 1 are fulfilled. Hence there exists a homomorphism $m\colon G \to \mathbb{C}^*$ satisfying (3).

Let $K_0$ and $K_1$ be as in the statement of this theorem. From the linear independence of the set $\{m \circ \lambda : \ \lambda \in K_1\}$ and Lemma 1 there exist $b_\lambda \in G, \lambda \in K_1$ such that the matrix $[m(\lambda b_\mu)]_{\lambda, \mu \in K_1}$ has a nonzero determinant. Hence, there exist $\beta_\lambda \in \mathbb{C}^*, \lambda \in K_1$, which satisfy (4).
We notice that

$$\begin{aligned} m(x) &= \frac{1}{|K_0|} \sum_{\mu \in K_0} m(\mu x) = \frac{1}{|K_0| \cdot |K_1|} \sum_{\mu \in K} m(\mu x) \sum_{\lambda \in K_1} \beta_\lambda m(\mu b_\lambda) \\ &= \sum_{\lambda \in K_1} \beta_\lambda \frac{1}{L} \sum_{\mu \in K} m(\mu(x + b_\lambda)) = \sum_{\lambda \in K_1} \beta_\lambda \varphi(x + b_\lambda), \quad x \in G, \end{aligned}$$

which proves (5). □

Now we prove a generalization of Wilson's functional equation.

**Theorem 4.** *Assume that $X$ is complex. Functions $f\colon G \to X, f \neq 0$, $\varphi\colon G \to \mathbb{C}$ satisfy the equation*

$$\sum_{\lambda\in K} f(x+\lambda y) = L\varphi(y)f(x), \quad x,y\in G, \tag{6}$$

*if and only if there exist a homomorphism $m\colon G \to \mathbb{C}^*$, $A_0^\lambda \in X$, $k$-additive, symmetric mappings $A_k^\lambda\colon G^k \to X, k\in\{1,\dots|K_0|-1\}, \lambda\in K_1$ such that*

$$\varphi(x) = \frac{1}{L}\sum_{\lambda\in K} m(\lambda x), \quad x\in G, \tag{7}$$

$$f(x) = \sum_{\lambda\in K_1} m(\lambda x)\left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x,\dots,x)\right], \quad x\in G, \tag{8}$$

$$\sum_{\mu\in K_0} A_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i}) = 0, \quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k\le |K_0|-1, \tag{9}$$

*where $K_0 := \{\lambda\in K:\ m\circ\lambda = m\}$, $K_1 \subset K$ is a minimal set such that $K = K_0\circ K_1$.*

*Proof.* It is easy to check that if functions $f$ and $\varphi$ satisfy conditions (7), (8), (9), then they satisfy Eq. (6).

Assume that $f$ and $\varphi$ satisfy Eq. (6). Since $f\neq 0$, $\varphi\neq 0$. Note that for $x,y,z\in G$ we have

$$\begin{aligned} L\sum_{\mu\in K}\varphi(y+\mu z)f(x) &= \sum_{\mu\in K}\sum_{\lambda\in K} f(x+\lambda(y+\mu z)) = \sum_{\mu\in K}\sum_{\lambda\in K} f(x+\lambda y+\mu z)\\ &= \sum_{\lambda\in K} L\varphi(z)f(x+\lambda y) = L^2\varphi(z)\varphi(y)f(x). \end{aligned}$$

Taking $x\in G$ such that $f(x)\neq 0$ we obtain that $\varphi$ satisfies Eq. (2). In view of Theorem 3 we get (7). Now we show that

$$|K_1|\sum_{\sigma\in K_0} f(x+\sigma y) = \sum_{\rho\in K} m(\rho y)\sum_{\nu\in K_1}\beta_\nu f(x+\rho^{-1}b_\nu), \quad x,y\in G. \tag{10}$$

Indeed, we have the following sequence of identities

$$\begin{aligned} |K_1|\sum_{\sigma\in K_0} f(x+\sigma y) &= \sum_{\lambda\in K}\sum_{\mu\in K_1}\beta_\mu m(\lambda b_\mu)f(x+\lambda y)\\ &= \sum_{\lambda\in K}\sum_{\mu\in K_1}\sum_{\nu\in K_1}\beta_\mu\beta_\nu\varphi(\lambda b_\mu+b_\nu)f(x+\lambda y)\\ &= \sum_{\lambda\in K}\sum_{\mu\in K_1}\sum_{\nu\in K_1}\sum_{\rho\in K}\frac{1}{L}\beta_\mu\beta_\nu f(x+\lambda y+\rho\lambda b_\mu+\rho b_\nu) \end{aligned}$$

$$= \sum_{\rho\in K}\sum_{\mu\in K_1}\sum_{\nu\in K_1}\beta_\mu\beta_\nu\varphi(y+\rho b_\mu)f(x+\rho b_\nu)$$
$$= \sum_{\rho\in K}\sum_{\nu\in K_1}\beta_\nu m(\rho^{-1}y)f(x+\rho b_\nu)$$
$$= \sum_{\rho\in K}m(\rho y)\sum_{\nu\in K_1}\beta_\nu f(x+\rho^{-1}b_\nu),\quad x,y\in G.$$

For each $\tau\in K_1$ we define $g_\tau\colon G\to X$ by the formula

$$g_\tau(x) := \frac{1}{Lm(\tau x)}\sum_{\nu\in K_1}\beta_\nu\sum_{\sigma\in K_0}f(x+\sigma\tau^{-1}b_\nu),\quad x\in G.$$

From equality (10) we obtain

$$Lm(\tau x+\tau y)\sum_{\sigma\in K_0}g_\tau(x+\sigma y)$$
$$= \sum_{\sigma\in K_0}Lm(\tau(x+\sigma y))g_\tau(x+\sigma y)$$
$$= \sum_{\sigma\in K_0}\sum_{\nu\in K_1}\beta_\nu\sum_{\rho\in K_0}f(x+\sigma y+\rho\tau^{-1}b_\nu)$$
$$= \sum_{\nu\in K_1}\beta_\nu\sum_{\rho\in K_0}\sum_{\sigma\in K_0}f(x+\sigma y+\sigma\rho\tau^{-1}b_\nu)$$
$$= \sum_{\nu\in K_1}\beta_\nu\sum_{\rho\in K_0}\frac{1}{|K_1|}\sum_{\mu\in K}m(\mu(y+\rho\tau^{-1}b_\nu))\sum_{\sigma\in K_1}\beta_\sigma f(x+\mu^{-1}b_\sigma)$$
$$= \sum_{\mu\in K}m(\mu y)\sum_{\nu\in K_1}\beta_\nu\sum_{\rho\in K_0}\frac{1}{|K_1|}m(\mu\rho\tau^{-1}b_\nu)\sum_{\sigma\in K_1}\beta_\sigma f(x+\mu^{-1}b_\sigma)$$
$$= \sum_{\mu\in K}m(\mu y)\frac{|K_0|}{|K_1|}\sum_{\nu\in K_1}\beta_\nu m(\mu\tau^{-1}b_\nu)\sum_{\sigma\in K_1}\beta_\sigma f(x+\mu^{-1}b_\sigma)$$
$$= |K_0|\sum_{\mu\in K_0}m(\mu\tau y)\sum_{\sigma\in K_1}\beta_\sigma f(x+\tau^{-1}\mu^{-1}b_\sigma)$$
$$= |K_0|m(\tau y)\sum_{\sigma\in K_1}\beta_\sigma\sum_{\mu\in K_0}f(x+\mu\tau^{-1}b_\sigma)$$
$$= |K_0|\cdot Lm(\tau y)m(\tau x)g_\tau(x),\quad x\in G.$$

Hence

$$\sum_{\sigma\in K_0}g_\tau(x+\sigma y) = |K_0|g_\tau(x),\quad x\in G,\quad \tau\in K_1. \tag{11}$$

In view of Theorem 2, for each $\lambda \in K_1$ there exist $A_0^\lambda \in X$, k-additive, symmetric mappings $A_k^\lambda\colon S^k \to X, k \in \{1, \dots |K_0|-1\}$ such that

$$g_\lambda(x) = A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x,\dots,x), \quad x \in G,$$

$$\sum_{\mu \in K_0} A_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i}) = 0, \quad x,y \in G,\ 1 \le i \le k \le |K_0|-1.$$

We observe that

$$\begin{aligned} L\sum_{\lambda\in K_1} m(\lambda x) g_\lambda(x) &= \sum_{\lambda\in K_1}\sum_{\nu\in K_1}\beta_\nu \sum_{\sigma\in K_0} f(x+\sigma\lambda^{-1}b_\nu) \\ &= \sum_{\nu\in K_1}\beta_\nu\sum_{\lambda\in K} f(x+\lambda b_\nu) = \sum_{\nu\in K_1}\beta_\nu L\varphi(b_\nu)f(x) \\ &= Lm(0)f(x) = Lf(x), \quad x \in G, \end{aligned}$$

which ends the proof. □

**Theorem 5.** *Assume that $X$ is real. Functions $f\colon G \to X, f \neq 0, \varphi\colon G \to \mathbb{R}$ satisfy the equation*

$$\sum_{\lambda\in K} f(x+\lambda y) = L\varphi(y)f(x), \quad x,y \in G, \tag{12}$$

*if and only if there exist a homomorphism $m\colon G \to \mathbb{C}^*, A_0^\lambda \in X,\ B_0^\lambda \in X$, k-additive, symmetric mappings $A_k^\lambda, B_k^\lambda\colon G^k \to X, k \in \{1, \dots |K_0|-1\}, \lambda \in K_1$ such that*

$$\varphi(x) = \frac{1}{L}\sum_{\lambda\in K} m(\lambda x), \quad x \in G, \tag{13}$$

$$\begin{aligned} f(x) = \sum_{\lambda\in K_1}\Bigg( &Re\ m(\lambda x)\left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x,\dots,x)\right] \\ &-Im\ m(\lambda x)\left[B_0^\lambda + \sum_{i=1}^{|K_0|-1} B_i^\lambda(x,\dots,x)\right]\Bigg), \quad x \in G, \end{aligned} \tag{14}$$

$$\sum_{\mu\in K_0} A_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i}) = 0, \quad x,y \in G,\ \lambda \in K_1,\ 1 \le i \le k \le |K_0|-1, \tag{15}$$

$$\sum_{\mu\in K_0} B_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i}) = 0, \quad x,y \in G,\ \lambda \in K_1,\ 1 \le i \le k \le |K_0|-1, \tag{16}$$

*where $K_0 := \{\lambda \in K :\ m \circ \lambda = m\}$, $K_1 \subset K$ is a minimal set such that $K = K_0 \circ K_1$.*

*Proof.* It is easy to check that if functions $f$ and $\varphi$ satisfy conditions (13), (14), (15), then they satisfy Eq. (12).

Assume that $f$ and $\varphi$ satisfy Eq. (12). Since $f \neq 0$, $\varphi \neq 0$. We observe that for $x, y, z \in G$ we have

$$L \sum_{\mu\in K} \varphi(y+\mu z) f(x) = \sum_{\mu\in K}\sum_{\lambda\in K} f(x+\lambda(y+\mu z)) = \sum_{\mu\in K}\sum_{\lambda\in K} f(x+\lambda y+\mu z)$$
$$= \sum_{\lambda\in K} L\varphi(z) f(x+\lambda y) = L^2\varphi(z)\varphi(y)f(x).$$

Taking $x \in G$ such that $f(x) \neq 0$ we obtain that $\varphi$ satisfies Eq. (2). In view of Theorem 3 we get (13). From equalities (4) and (5) we have

$$|K_1| \sum_{\sigma\in K_0} f(x+\sigma y) = \sum_{\rho\in K_1}\sum_{\nu\in K_1} \mathrm{Re}\,(\beta_\nu m(\rho y)) \sum_{\sigma\in K_0} f(x+\sigma\rho^{-1}b_\nu), \quad x, y \in G, \tag{17}$$

$$0 = \sum_{\rho\in K_1}\sum_{\nu\in K_1} \mathrm{Im}\,(\beta_\nu m(\rho y)) \sum_{\sigma\in K_0} f(x+\sigma\rho^{-1}b_\nu), \quad x, y \in G. \tag{18}$$

Indeed, for $x, y \in G$ we have

$$\begin{aligned}|K_1| \sum_{\sigma\in K_0} f(x+\sigma y) &= \sum_{\lambda\in K} \mathrm{Re}\left(\sum_{\mu\in K_1} \beta_\mu m(\lambda b_\mu)) f(x+\lambda y\right)\\
&= \sum_{\lambda\in K}\sum_{\mu\in K_1}\sum_{\nu\in K_1} \mathrm{Re}\,(\beta_\mu\beta_\nu)\varphi(\lambda b_\mu + b_\nu) f(x+\lambda y)\\
&= \sum_{\lambda\in K}\sum_{\mu\in K_1}\sum_{\nu\in K_1}\sum_{\rho\in K} \frac{1}{L}\mathrm{Re}\,(\beta_\mu\beta_\nu) f(x+\lambda y+\rho\lambda b_\mu+\rho b_\nu)\\
&= \sum_{\rho\in K}\sum_{\mu\in K_1}\sum_{\nu\in K_1} \mathrm{Re}\,(\beta_\mu\beta_\nu)\varphi(y+\rho b_\mu) f(x+\rho b_\nu)\\
&= \sum_{\rho\in K}\sum_{\nu\in K_1} \mathrm{Re}\,(\beta_\nu m(\rho^{-1}y)) f(x+\rho b_\nu)\\
&= \sum_{\rho\in K_1}\sum_{\nu\in K_1} \mathrm{Re}\,(\beta_\nu m(\rho y)) \sum_{\sigma\in K_0} f(x+\sigma\rho^{-1}b_\nu),\end{aligned}$$

and

$$\begin{aligned}0 &= \sum_{\lambda\in K} \mathrm{Im}\left(\sum_{\mu\in K_1} \beta_\mu m(\lambda b_\mu)\right) f(x+\lambda y)\\
&= \sum_{\lambda\in K}\sum_{\mu\in K_1}\sum_{\nu\in K_1} \mathrm{Im}\,(\beta_\mu\beta_\nu)\varphi(\lambda b_\mu + b_\nu) f(x+\lambda y)\\
&= \sum_{\lambda\in K}\sum_{\mu\in K_1}\sum_{\nu\in K_1}\sum_{\rho\in K} \frac{1}{L}\mathrm{Im}\,(\beta_\mu\beta_\nu) f(x+\lambda y+\rho\lambda b_\mu+\rho b_\nu)\end{aligned}$$

$$= \sum_{\rho\in K}\sum_{\mu\in K_1}\sum_{\nu\in K_1} \mathrm{Im}\ (\beta_\mu\beta_\nu)\varphi(y+\rho b_\mu)f(x+\rho b_\nu)$$
$$= \sum_{\rho\in K}\sum_{\nu\in K_1} \mathrm{Im}\ (\beta_\nu m(\rho^{-1}y))f(x+\rho b_\nu)$$
$$= \sum_{\rho\in K_1}\sum_{\nu\in K_1} \mathrm{Im}\ (\beta_\nu m(\rho y))\sum_{\sigma\in K_0} f(x+\sigma\rho^{-1}b_\nu),$$

For each $\tau\in K_1$ we define $g_\tau, h_\tau\colon G\to X$ by the formula

$$g_\tau(x) := \sum_{\nu\in K_1} \mathrm{Re}\ \frac{\beta_\nu}{Lm(\tau x)}\sum_{\sigma\in K_0} f(x+\sigma\tau^{-1}b_\nu),\quad x\in G,$$
$$h_\tau(x) := \sum_{\nu\in K_1} \mathrm{Im}\ \frac{\beta_\nu}{Lm(\tau x)}\sum_{\sigma\in K_0} f(x+\sigma\tau^{-1}b_\nu),\quad x\in G.$$

From equalities (17) and (18) we obtain

$$\sum_{\sigma\in K_0} g_\tau(x+\sigma y) = |K_0|g_\tau(x),\quad x,y\in G,\quad \tau\in K_1, \tag{19}$$
$$\sum_{\sigma\in K_0} h_\tau(x+\sigma y) = |K_0|h_\tau(x),\quad x,y\in G,\quad \tau\in K_1. \tag{20}$$

Indeed, we have the following sequence of identities

$$|K_1|\sum_{\lambda\in K_0} g_\tau(x+\lambda y)$$
$$= |K_1|\sum_{\lambda\in K_0}\sum_{\nu\in K_1} \mathrm{Re}\ \frac{\beta_\nu}{Lm(\tau(x+\lambda y))}\sum_{\sigma\in K_0} f(x+\lambda y+\sigma\tau^{-1}b_\nu)$$
$$= \sum_{\nu\in K_1} \mathrm{Re}\ \frac{\beta_\nu}{Lm(\tau(x+y))}\sum_{\sigma\in K_0} |K_1|\sum_{\lambda\in K_0} f(x+\lambda(y+\sigma\tau^{-1}b_\nu))$$
$$= \sum_{\nu\in K_1} \mathrm{Re}\ \frac{\beta_\nu}{Lm(\tau(x+y))}\sum_{\sigma\in K_0}\sum_{\rho\in K_1}\sum_{\mu\in K_1} \mathrm{Re}\ (\beta_\mu m(\rho(y+\sigma\tau^{-1}b_\nu)))$$
$$\cdot\sum_{\lambda\in K_0} f(x+\lambda\rho^{-1}b_\mu)$$
$$= \sum_{\nu\in K_1} \mathrm{Re}\ \frac{|K_0|\beta_\nu}{Lm(\tau(x+y))}\sum_{\rho\in K_1}\sum_{\mu\in K_1} \mathrm{Re}\ (\beta_\mu m(\rho y)m(\rho\tau^{-1}b_\nu))$$
$$\cdot\sum_{\lambda\in K_0} f(x+\lambda\rho^{-1}b_\mu) - \sum_{\nu\in K_1} \mathrm{Im}\ \frac{|K_0|\beta_\nu}{Lm(\tau(x+y))}$$
$$\cdot\sum_{\rho\in K_1}\sum_{\mu\in K_1} \mathrm{Im}\ (\beta_\mu m(\rho y)m(\rho\tau^{-1}b_\nu))\sum_{\lambda\in K_0} f(x+\lambda\rho^{-1}b_\mu)$$

$$= |K_0| \sum_{\rho\in K_1} \sum_{\mu\in K_1} \mathrm{Re}\,(\frac{\beta_\mu m(\rho y)}{Lm(\tau(x+y))} \sum_{\nu\in K_1} \beta_\nu m(\rho\tau^{-1}b_\nu)) \sum_{\lambda\in K_0} f(x+\lambda\rho^{-1}b_\mu)$$
$$= |K_0|\cdot|K_1| \sum_{\mu\in K_1} \mathrm{Re}\,(\frac{\beta_\mu m(\tau y)}{Lm(\tau(x+y))}) \sum_{\lambda\in K_0} f(x+\lambda\tau^{-1}b_\mu)$$
$$= |K_0|\cdot|K_1| \sum_{\mu\in K_1} \mathrm{Re}\,\frac{\beta_\mu}{Lm(\tau x)} \sum_{\lambda\in K_0} f(x+\lambda\tau^{-1}b_\mu)$$
$$= |K_0|\cdot|K_1| g_\tau(x), \quad x\in G.$$

and

$$|K_1| \sum_{\lambda\in K_0} h_\tau(x+\lambda y)$$
$$= |K_1| \sum_{\lambda\in K_0} \sum_{\nu\in K_1} \mathrm{Im}\,\frac{\beta_\nu}{Lm(\tau(x+\lambda y))} \sum_{\sigma\in K_0} f(x+\lambda y+\sigma\tau^{-1}b_\nu)$$
$$= \sum_{\nu\in K_1} \mathrm{Im}\,\frac{\beta_\nu}{Lm(\tau(x+y))} \sum_{\sigma\in K_0} |K_1| \sum_{\lambda\in K_0} f(x+\lambda(y+\sigma\tau^{-1}b_\nu))$$
$$= \sum_{\nu\in K_1} \mathrm{Im}\,\frac{\beta_\nu}{Lm(\tau(x+y))} \sum_{\sigma\in K_0} \sum_{\rho\in K_1} \sum_{\mu\in K_1} \mathrm{Re}\,(\beta_\mu m(\rho(y+\sigma\tau^{-1}b_\nu)))$$
$$\cdot \sum_{\lambda\in K_0} f(x+\lambda\rho^{-1}b_\mu)$$
$$= \sum_{\nu\in K_1} \mathrm{Im}\,\frac{|K_0|\beta_\nu}{Lm(\tau(x+y))} \sum_{\rho\in K_1} \sum_{\mu\in K_1} \mathrm{Re}\,(\beta_\mu m(\rho y) m(\rho\tau^{-1}b_\nu))$$
$$\cdot \sum_{\lambda\in K_0} f(x+\lambda\rho^{-1}b_\mu) + \sum_{\nu\in K_1} \mathrm{Re}\,\frac{|K_0|\beta_\nu}{Lm(\tau(x+y))}$$
$$\cdot \sum_{\rho\in K_1} \sum_{\mu\in K_1} \mathrm{Im}\,(\beta_\mu m(\rho y) m(\rho\tau^{-1}b_\nu)) \sum_{\lambda\in K_0} f(x+\lambda\rho^{-1}b_\mu)$$
$$= |K_0| \sum_{\rho\in K_1} \sum_{\mu\in K_1} \mathrm{Im}\,(\frac{\beta_\mu m(\rho y)}{Lm(\tau(x+y))} \sum_{\nu\in K_1} \beta_\nu m(\rho\tau^{-1}b_\nu)) \sum_{\lambda\in K_0} f(x+\lambda\rho^{-1}b_\mu)$$
$$= |K_0|\cdot|K_1| \sum_{\mu\in K_1} \mathrm{Im}\,(\frac{\beta_\mu m(\tau y)}{Lm(\tau(x+y))}) \sum_{\lambda\in K_0} f(x+\lambda\tau^{-1}b_\mu)$$
$$= |K_0|\cdot|K_1| \sum_{\mu\in K_1} \mathrm{Im}\,\frac{\beta_\mu}{Lm(\tau x)} \sum_{\lambda\in K_0} f(x+\lambda\tau^{-1}b_\mu)$$
$$= |K_0|\cdot|K_1| h_\tau(x), \quad x\in G.$$

In view of Theorem 2, for each $\lambda \in K_1$ there exist $A_0^\lambda, B_0^\lambda \in X$, k-additive, symmetric mappings $A_k^\lambda, B_k^\lambda\colon S^k \to X, k \in \{1, \dots |K_0|-1\}$ such that

$$g_\lambda(x) = A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \dots, x), \quad x \in G,$$

$$\sum_{\mu \in K_0} A_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_{i}) = 0, \quad x, y \in G,\ 1 \le i \le k \le |K_0|-1,$$

$$h_\lambda(x) = B_0^\lambda + \sum_{i=1}^{|K_0|-1} B_i^\lambda(x, \dots, x), \quad x \in G,$$

$$\sum_{\mu \in K_0} B_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_{i}) = 0, \quad x, y \in G,\ 1 \le i \le k \le |K_0|-1.$$

We observe that

$$\begin{aligned}
&\sum_{\tau \in K_1} \text{Re } m(\tau x) g_\tau(x) - \sum_{\tau \in K_1} \text{Im } m(\tau x) h_\tau(x) \\
&\quad = \sum_{\tau \in K_1} \Bigg[ \text{Re } m(\tau x) \sum_{\nu \in K_1} \text{Re } \frac{\beta_\nu}{Lm(\tau x)} \sum_{\sigma \in K_0} f(x + \sigma\tau^{-1} b_\nu) \\
&\qquad - \text{Im } m(\tau x) \sum_{\nu \in K_1} \text{Im } \frac{\beta_\nu}{Lm(\tau x)} \sum_{\sigma \in K_0} f(x + \sigma\tau^{-1} b_\nu) \Bigg] \\
&\quad = \sum_{\tau \in K_1} \sum_{\nu \in K_1} \text{Re } \left( m(\tau x) \frac{\beta_\nu}{Lm(\tau x)} \right) \sum_{\sigma \in K_0} f(x + \sigma\tau^{-1} b_\nu) \\
&\quad = \sum_{\tau \in K_1} \sum_{\sigma \in K_0} \sum_{\nu \in K_1} \text{Re } \left( \frac{\beta_\nu}{L} \right) f(x + \sigma\tau^{-1} b_\nu) \\
&\quad = \sum_{\lambda \in K} \sum_{\nu \in K_1} \text{Re } \left( \frac{\beta_\nu}{L} \right) f(x + \lambda b_\nu) \\
&\quad = \sum_{\nu \in K_1} \text{Re } \beta_\nu \varphi(b_\nu) f(x) = m(0) f(x) = f(x), \quad x \in G,
\end{aligned}$$

which ends the proof. □

**Corollary 1.** *Functions $f, g\colon G \to X, \varphi\colon G \to \mathbb{K}, f \neq 0$, satisfy the equality*

$$\sum_{\lambda \in K} f(x + \lambda y) = L\varphi(y) g(x), \quad x, y \in G, \tag{21}$$

*if and only if there exists a homomorphism $m\colon G \to \mathbb{C}^*$ such that*

$$\varphi(x) = \varphi(0) \frac{1}{L} \sum_{\lambda \in K} m(\lambda x), \quad x \in G,$$

*and (i) if $X$ is real, then there exist $A_0^\lambda, B_0^\lambda \in X$, $k$-additive, symmetric mappings $A_k^\lambda, B_k^\lambda \colon G^k \to X, k \in \{1, \ldots |K_0|-1\}, \lambda \in K_1$ such that*

$$f(x) = \varphi(0) \sum_{\lambda \in K_1} \left( Re\ m(\lambda x) \left[ A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \ldots, x) \right] - Im\ m(\lambda x) \left[ B_0^\lambda + \sum_{i=1}^{|K_0|-1} B_i^\lambda(x, \ldots, x) \right] \right), \quad x \in G,$$

$$g(x) = \sum_{\lambda \in K_1} \left( Re\ m(\lambda x) \left[ A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \ldots, x) \right] - Im\ m(\lambda x) [B_0^\lambda + \sum_{i=1}^{|K_0|-1} B_i^\lambda(x, \ldots, x)] \right), \quad x \in G,$$

$$\sum_{\mu \in K_0} A_k^\lambda\ (x, \ldots, x, \underbrace{\mu y, \ldots, \mu y}_{i}) = 0, \quad x, y \in G,\ \lambda \in K_1,\ 1 \le i \le k \le |K_0|-1,$$

$$\sum_{\mu \in K_0} B_k^\lambda\ (x, \ldots, x, \underbrace{\mu y, \ldots, \mu y}_{i}) = 0, \quad x, y \in G,\ \lambda \in K_1,\ 1 \le i \le k \le |K_0|-1,$$

*(ii) if $X$ is complex, then there exist $A_0^\lambda \in X$, $k$-additive, symmetric mappings $A_k^\lambda \colon G^k \to X, k \in \{1, \ldots |K_0|-1\}, \lambda \in K_1$ such that*

$$f(x) = \varphi(0) \sum_{\lambda \in K_1} m(\lambda x) \left[ A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \ldots, x) \right], \quad x \in G,$$

$$g(x) = \sum_{\lambda \in K_1} m(\lambda x) \left[ A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \ldots, x) \right], \quad x \in G,$$

$$\sum_{\mu \in K_0} A_k^\lambda(x, \ldots, x, \underbrace{\mu y, \ldots, \mu y}_{i}) = 0, \quad x, y \in G,\ \lambda \in K_1,\ 1 \le i \le k \le |K_0|-1,$$

*where $K_0 := \{\lambda \in K :\ m \circ \lambda = m\}$, $K_1 \subset K$ is a minimal set such that $K = K_0 \circ K_1$.*

*Proof.* Putting $y = 0$ in (21) we have

$$Lf(x) = L\varphi(0)g(x), \quad x \in G.$$

Since $f \neq 0$, $g \neq 0$, $\varphi(0) \neq 0$ and

$$\varphi(0) \sum_{\lambda \in K} g(x + \lambda y) = L\varphi(y)g(x), \quad x, y \in G,$$

hence for $\varphi_0 := \frac{\varphi}{\varphi(0)}$ we have

$$\sum_{\lambda\in K} g(x+\lambda y) = L\varphi_0(y)g(x), \quad x,y\in G.$$

In view of Theorems 4 and 5 accordingly we obtain (ii) and (i) of the theorem. □

**Theorem 6.** *Functions $f\colon G\to X, \varphi\colon G\to \mathbb{K}, f\neq 0, \varphi\neq const$, satisfy the equality*

$$\sum_{\lambda\in K} f(x+\lambda y) = L\varphi(y)f(x) + \sum_{\lambda\in K} f(\lambda y), \quad x,y\in G, \tag{22}$$

*if and only if there exists a homomorphism $m\colon G\to\mathbb{C}^*$ such that*

$$\varphi(x) = \frac{1}{L}\sum_{\lambda\in K} m(\lambda x), \quad x\in G,$$

*and*
*(i) if $X$ is real, then there exist $A_0^\lambda, B_0^\lambda\in X$, $k$-additive, symmetric mappings $A_k^\lambda, B_k^\lambda\colon G^k\to X, k\in\{1,\dots|K_0|-1\}, \lambda\in K_1$ such that*

$$f(x) = \sum_{\lambda\in K_1}\left( Re\ m(\lambda x)\left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x,\dots,x)\right]\right.$$
$$\left. - Im\ m(\lambda x)\left[B_0^\lambda + \sum_{i=1}^{|K_0|-1} B_i^\lambda(x,\dots,x)\right]\right) - \sum_{\lambda\in K_1} A_0^\lambda, \quad x\in G,$$

$$\sum_{\mu\in K_0} A_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i}) = 0, \quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k\le |K_0|-1,$$

$$\sum_{\mu\in K_0} B_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i}) = 0, \quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k\le |K_0|-1,$$

*(ii) if $X$ is complex, then there exist $A_0^\lambda\in X$, $k$-additive, symmetric mappings $A_k^\lambda\colon G^k\to X, k\in\{1,\dots|K_0|-1\}, \lambda\in K_1$ such that*

$$f(x) = \sum_{\lambda\in K_1} m(\lambda x)\left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x,\dots,x)\right] - \sum_{\lambda\in K_1} A_0^\lambda, \quad x\in G,$$

$$\sum_{\mu\in K_0} A_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i}) = 0, \quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k\le |K_0|-1,$$

*where $K_0 := \{\lambda\in K:\ m\circ\lambda = m\}$, $K_1 \subset K$ is a minimal set such that $K = K_0\circ K_1$.*

*Moreover*

$$\sum_{\lambda\in K} f(\lambda x) = L(\varphi(x)-1)\sum_{\lambda\in K_1} A_0^\lambda, \quad x\in G. \tag{23}$$

*Proof.* Putting $x=0$ in (22) we have

$$\sum_{\lambda\in K} f(\lambda y) = L\varphi(y)f(0) + \sum_{\lambda\in K} f(\lambda y), \quad y\in G.$$

Since $\varphi \neq const$, $f(0)=0$. Putting $y=0$ in (22) we get

$$Lf(x) = L\varphi(0)f(x) + Lf(0), \quad x\in G,$$

hence we obtain $\varphi(0)=1$. We observe that

$$\begin{aligned} L\varphi(x)\sum_{\lambda\in K} f(\lambda y) + L\sum_{\lambda\in K} f(\lambda x) &= \sum_{\mu\in K}\sum_{\lambda\in K} f(\lambda y+\mu x) = \sum_{\mu\in K}\sum_{\lambda\in K} f(\mu x+\lambda y)\\ &= L\varphi(y)\sum_{\mu\in K} f(\mu x) + L\sum_{\lambda\in K} f(\lambda y), \quad x,y\in G. \end{aligned}$$

Hence we have

$$L(\varphi(y)-1)\sum_{\mu\in K} f(\mu x) = L(\varphi(x)-1)\sum_{\lambda\in K} f(\lambda y), \quad x,y\in G,$$

which means that

$$\sum_{\lambda\in K} f(\lambda y) = L(\varphi(y)-1)A_0, \quad y\in G, \tag{24}$$

for some $A_0\in X$. Therefore we obtain that $f\neq const$. Putting the above equality in (22) we get

$$\sum_{\lambda\in K} f(x+\lambda y) = L\varphi(y)f(x) + L(\varphi(y)-1)A_0, \quad x,y\in G,$$

hence, for the function $g\colon G\to X$ given by the formula $g := f + A_0$, we obtain

$$\sum_{\lambda\in K} g(x+\lambda y) = L\varphi(y)g(x), \quad x,y\in G.$$

In view of Theorems 4 and 5 there exists a homomorphism $m\colon G\to\mathbb{C}^*$ such that

$$\varphi(x) = \frac{1}{L}\sum_{\lambda\in K} m(\lambda x), \quad x\in G,$$

and
(i) if $X$ is real, then there exist $A_0^\lambda, B_0^\lambda\in X$, k-additive, symmetric mappings $A_k^\lambda, B_k^\lambda\colon G^k\to X, k\in\{1,\dots|K_0|-1\}, \lambda\in K_1$ such that

$$g(x) = \sum_{\lambda\in K_1}\left(\mathrm{Re}\; m(\lambda x)\left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x,\dots,x)\right]\right.$$
$$\left.-\mathrm{Im}\; m(\lambda x)\left[B_0^\lambda + \sum_{i=1}^{|K_0|-1} B_i^\lambda(x,\dots,x)\right]\right), \quad x\in G,$$
$$\sum_{\mu\in K_0} A_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i}) = 0, \quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k\le |K_0|-1,$$
$$\sum_{\mu\in K_0} B_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i}) = 0, \quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k\le |K_0|-1,$$

(ii) if $X$ is complex, then there exist $A_0^\lambda\in X$, k-additive, symmetric mappings $A_k^\lambda\colon G^k\to X, k\in\{1,\dots|K_0|-1\},\lambda\in K_1$ such that

$$g(x) = \sum_{\lambda\in K_1} m(\lambda x)\left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x,\dots,x)\right], \quad x\in G,$$
$$\sum_{\mu\in K_0} A_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i}) = 0, \quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k\le |K_0|-1.$$

Since $f(0)=0$ and $g=f+A_0$, we have

$$A_0 = g(0) = \sum_{\lambda\in K_1} A_0^\lambda,$$

which together with (24) gives equality (23) and the form of the function $f$. □

**Corollary 2.** *Functions* $f\colon G\to X,\varphi\colon G\to\mathbb{K}, f\neq 0,\varphi\neq const$, *satisfy the equation*

$$\sum_{\lambda\in K} f(x+\lambda y) = L\varphi(y)f(x)+Lf(y), \quad x,y\in G, \tag{25}$$

*if and only if there exist a homomorphism* $m\colon G\to\mathbb{C}^*$, *and* $A\in X$ *such that*

$$\varphi(x) = \frac{1}{L}\sum_{\lambda\in K} m(\lambda x), \quad x\in G,$$
$$f(x) = (\varphi(x)-1)A, \quad x\in G.$$

*Proof.* It is easy to check that if functions $f$ and $\varphi$ satisfy the above conditions, then they satisfy Eq. (25).

Assume that $f$ and $\varphi$ satisfy Eq. (25). Putting $x=0$ in (25) we have

$$\sum_{\lambda\in K} f(\lambda y) = L\varphi(y)f(0)+Lf(y), \quad y\in G.$$

Putting $y=0$ in (25) we get

$$Lf(x)=L\varphi(0)f(x)+Lf(0),\quad x\in G.$$

If $\varphi(0)=0$, then $f=f(0)$ and from the above equalities $f=0$ which gives a contradiction. Hence, since $\varphi\neq const$, $f(0)=0$, $\varphi(0)=1$ and $\sum_{\lambda\in K}f(\lambda y)=Lf(y)$. In particular $f$ and $\varphi$ satisfy Eq. (22). In view of Theorem 6 we have

$$Lf(x)=\sum_{\mu\in K}f(\mu x)=L(\varphi(x)-1)A,\quad x,y\in G,$$

for some $A\in X$. □

**Theorem 7.** *Functions* $f,g,h\colon G\to X,\varphi\colon G\to\mathbb{K}, f\neq 0,\varphi\neq const,\varphi(0)\neq 0,$ *satisfy the equation*

$$\sum_{\lambda\in K}f(x+\lambda y)=L\varphi(y)g(x)+Lh(y),\quad x,y\in G,\tag{26}$$

*if and only if there exists a homomorphism* $m\colon G\to\mathbb{C}^*, A,B\in X$ *such that*

$$\varphi(x)=\varphi(0)\frac{1}{L}\sum_{\lambda\in K}m(\lambda x),\quad x\in G,$$

*and*
*(i) if* $X$ *is real, then there exist* $A_0^\lambda,B_0^\lambda\in X$, *k-additive, symmetric mappings* $A_k^\lambda,B_k^\lambda\colon G^k\to X,k\in\{1,\dots|K_0|-1\},\lambda\in K_1$ *such that*

$$\begin{aligned}f(x)=\varphi(0)\sum_{\lambda\in K_1}&\left(\mathit{Re}\ m(\lambda x)\left[A_0^\lambda+\sum_{i=1}^{|K_0|-1}A_i^\lambda(x,\dots,x)\right]\right.\\&\left.-\mathit{Im}\ m(\lambda x)\left[B_0^\lambda+\sum_{i=1}^{|K_0|-1}B_i^\lambda(x,\dots,x)\right]\right)+A-\varphi(0)\sum_{\lambda\in K_1}A_0^\lambda,\quad x\in G,\\g(x)=\sum_{\lambda\in K_1}&\left(\mathit{Re}\ m(\lambda x)\left[A_0^\lambda+\sum_{i=1}^{|K_0|-1}A_i^\lambda(x,\dots,x)\right]\right.\\&\left.-\mathit{Im}\ m(\lambda x)\left[B_0^\lambda+\sum_{i=1}^{|K_0|-1}B_i^\lambda(x,\dots,x)\right]\right)+B-\sum_{\lambda\in K_1}A_0^\lambda,\quad x\in G,\\h(x)=\varphi(x)&\left(\sum_{\lambda\in K_1}A_0^\lambda-B\right)+\left(A-\varphi(0)\sum_{\lambda\in K_1}A_0^\lambda\right),\quad x\in G,\end{aligned}$$

$$\sum_{\mu\in K_0} A_k^\lambda(x,\ldots,x,\underbrace{\mu y,\ldots,\mu y}_{i}) = 0, \quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k\le |K_0|-1,$$

$$\sum_{\mu\in K_0} B_k^\lambda(x,\ldots,x,\underbrace{\mu y,\ldots,\mu y}_{i}) = 0, \quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k\le |K_0|-1,$$

*(ii) if $X$ is complex, then there exist $A_0^\lambda\in X$, $k$-additive, symmetric mappings $A_k^\lambda\colon G^k\to X, k\in\{1,\ldots|K_0|-1\}, \lambda\in K_1$ such that*

$$f(x)=\varphi(0)\left(\sum_{\lambda\in K_1} m(\lambda x)\left[A_0^\lambda+\sum_{i=1}^{|K_0|-1} A_i^\lambda(x,\ldots,x)\right]-\sum_{\lambda\in K_1} A_0^\lambda\right)+A, \quad x\in G,$$

$$g(x)=\sum_{\lambda\in K_1} m(\lambda x)\left[A_0^\lambda+\sum_{i=1}^{|K_0|-1} A_i^\lambda(x,\ldots,x)\right]+B-\sum_{\lambda\in K_1} A_0^\lambda, \quad x\in G,$$

$$h(x)=\varphi(x)\left(\sum_{\lambda\in K_1} A_0^\lambda - B\right)+\left(A-\varphi(0)\sum_{\lambda\in K_1} A_0^\lambda\right), \quad x\in G,$$

$$\sum_{\mu\in K_0} A_k^\lambda(x,\ldots,x,\underbrace{\mu y,\ldots,\mu y}_{i}) = 0, \quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k\le |K_0|-1,$$

*where $K_0 := \{\lambda\in K:\ m\circ\lambda=m\}$, $K_1 \subset K$ is a minimal set such that $K=K_0\circ K_1$.*

*Proof.* Putting $x=0$ in (26) we have

$$\sum_{\lambda\in K} f(\lambda y) = L\varphi(y)g(0)+Lh(y), \quad y\in G.$$

Putting $y=0$ in (26) we get

$$Lf(x)=L\varphi(0)g(x)+Lh(0), \quad x\in G.$$

Hence we have

$$\begin{aligned}\varphi(0)\sum_{\lambda\in K} g(x+\lambda y) &= \sum_{\lambda\in K} f(x+\lambda y)-Lh(0)\\ &= L\varphi(y)g(x)+Lh(y)-Lh(0)\\ &= L\varphi(y)g(x)+\sum_{\lambda\in K} f(\lambda y)-L\varphi(y)g(0)-Lh(0)\\ &= L\varphi(y)(g(x)-g(0))+\varphi(0)\sum_{\lambda\in K} g(\lambda y), \quad x,y\in G,\end{aligned}$$

which gives us that functions $g_0=g-g(0)$, $\varphi_0=\frac{\varphi}{\varphi(0)}$ satisfy the equation

$$\sum_{\lambda\in K} g_0(x+\lambda y) = L\varphi_0(y)g_0(x)+\sum_{\lambda\in K} g_0(\lambda y), \quad x,y\in G.$$

In view of Theorem 6 there exists a homomorphism $m\colon G \to \mathbb{C}^*$ such that

$$\varphi_0(x) = \frac{1}{L}\sum_{\lambda\in K} m(\lambda x), \quad x \in G,$$

and
(i) if $X$ is real, then there exist $A_0^\lambda, B_0^\lambda \in X$, k-additive, symmetric mappings $A_k^\lambda, B_k^\lambda\colon G^k \to X, k \in \{1,\dots|K_0|-1\}, \lambda \in K_1$ such that

$$g_0(x) = \sum_{\lambda\in K_1}\left(\mathrm{Re}\ m(\lambda x)\left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x,\dots,x)\right]\right.$$
$$\left. -\mathrm{Im}\ m(\lambda x)\left[B_0^\lambda + \sum_{i=1}^{|K_0|-1} B_i^\lambda(x,\dots,x)\right]\right) - \sum_{\lambda\in K_1} A_0^\lambda, \quad x \in G,$$

$$\sum_{\mu\in K_0} A_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i}) = 0, \quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k\le |K_0|-1,$$

$$\sum_{\mu\in K_0} B_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i}) = 0, \quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k\le |K_0|-1,$$

(ii) if $X$ is complex, then there exist $A_0^\lambda \in X$, k-additive, symmetric mappings $A_k^\lambda\colon G^k \to X, k \in \{1,\dots|K_0|-1\}, \lambda \in K_1$ such that

$$g_0(x) = \sum_{\lambda\in K_1} m(\lambda x)[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x,\dots,x)] - \sum_{\lambda\in K_1} A_0^\lambda, \quad x\in G,$$

$$\sum_{\mu\in K_0} A_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i}) = 0, \quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k\le |K_0|-1,$$

Moreover

$$\sum_{\lambda\in K} g_0(\lambda x) = L(\varphi_0(x)-1)\sum_{\lambda\in K_1} A_0^\lambda, \quad x\in G.$$

Hence, putting $B := g(0)$, we obtain the form of $\varphi$ and $g$. Since

$$Lf(x) = L\varphi(0)g(x) + Lh(0) = L\varphi(0)g_0(x) + Lf(0), \quad x\in G,$$

and

$$Lh(x) = \sum_{\lambda\in K} f(\lambda x) - L\varphi(x)g(0) = \varphi(0)\sum_{\lambda\in K} g(\lambda x) + Lh(0) - L\varphi(x)g(0)$$
$$= \varphi(0)\sum_{\lambda\in K} g_0(\lambda x) + Lf(0) - L\varphi(x)g(0), \quad x\in G,$$

from the form of $g_0$ we obtain the form of $f$ and $h$ in the real and the complex case of the space $X$. □

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Radosław Łukasik
Institute of Mathematics
University of Silesia
ul. Bankowa 14
40-007 Katowice
Poland
e-mail: rlukasik@math.us.edu.pl

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