FUZZY DIFFERENTIAL EQUATIONS BY USING MODIFIED ROMBERG'S METHOD

Noor'ani Bt Ahmad[1] , J. Kavikumar[2], Mustafa Mamat[3] & Nor Shamsidah Bt Amir Hamzah[4]

[1, 2, 4] *Centre for Science Studies, Universiti Tun Hussein Onn Malaysia, 86400 Parit Raja, Johor, Malaysia*
nooreini@uthm.edu.my, kavi@uthm.edu.my, shamsidah@uthm.edu.my

[3] *Department of Mathematic Science, Faculty of Science and Technology, Universiti Malaysia Terengganu, 21030 Kuala Terengganu, Terengganu, Malaysia*
mus@umt.edu.my

In this paper, the numerical method for solving the fuzzy differential equations is studied by an application of the Romberg's method. The formula for the modified Romberg's method was derived from the modified Two-step Simpson's 1/3 method. The results obtained from this method are much better compared to Euler's method.

Keywords: Fuzzy Differential Equations (FDE); Romberg's method; Modified Two-step Simpson's 1/3 Method; Euler's Method.

1. INTRODUCTION

The concept of the fuzzy derivative was first introduced by Chang and Zadeh (1972). The topic of FDE has been rapidly growing in recent years. It was followed up by Dubois and Prade (1982), who used the extension principle in their approach. Other methods have been discussed by Puri & Ralescu (1983) and Goetschel & Voxman (1986). Kandel & Byatt (1978; 1980), applied the concept of FDE to the analysis of fuzzy dynamical problems. The FDE with the initial value problem were rigorously treated by Kaleva (1987; 1990), Seikkala (1987), Ouyang & Wu (1989), Kloeden (1991) and Menda (1988). The numerical methods for solving FDE are introduced in Abbasbandy & Allahviranloo (2002), Abbasbandy *et al.* (2004), Allahviranloo *et al.* (2007) and Buckley & Feuring (2001) for solving nth-order linear differential equations with fuzzy initial conditions. Kaleva (1987, 1990), Lakshmikantham *et al.* (2001), Lakshmikantham & Mohapatra (2003) have studied initial and boundary value problems associated with first and second order FDE on the metric space (E^n, D) of normal fuzzy convex sets with the distance D given by the supreme of the Hausdorff distance between the corresponding r-level sets.

FDE referring to differential equations where some coefficient or initial conditions are uncertain and defined as fuzzy numbers (Wu & Song, 1996; Abbasbandy & Allahviranloo, 2002; Abbasbandy *et al.*, 2005; Stefanini, 2007; Allahviranloo *et al.*, 2007; Bede, 2008; Chalco-Cano & Roman-Flores, 2008; Chalco-Cano *et al.*, 2008; Babolian *et al.*, 2009; Ahmad & Hasan, 2010; 2011). Its solution is then of a fuzzy region of uncertainty. As an example, consider a differential equation with fuzzy initial values as follows :

$$x'(t) = f(t,x), \quad x(t_0) = x_0,$$

In this case, the initial value may not be known exactly and the function f may contain an uncertain parameter.

The study and solution of FDE is extremely important in applications, especially when it involves uncertain parameters or uncertain initial conditions. The uncertainties can be written in the fuzzy number form. Fields of FDE applications come from computational physics, engineering and system control, economics and finance (Oberguggenberger & Pittschmann, 1999; Chalco-Cano & Roman-Flores, 2008; Allahviranloo, 2009). The numerical method can be used to solve the model with uncertainty or imprecise information in the parameters or variables. In this paper, we have developed Modified Romberg's Method for FDE. By using numerical method to solve FDE, the stable algorithms is developed. Then, the numerical solutions of the approximate solution can be found as well as providing useful information to problem solver.

2. PRELIMINARIES

In this section, we briefly elaborate some important concepts of fuzzy set and fuzzy initial value problem.

2.1 Fuzzy Numbers

Let R be the universal set. y is called a fuzzy set in R if y is a set of order pairs.

$$\bar{y} = \{(x, \mu_y(x)) | x \in R\},$$

where $\mu_y(x)$ is the membership function of x in y. The closer $\mu_y(x)$ is to 1 the more x belongs to y and the closer it is to 0 means the less it belongs to y and satisfying the following properties :

(i) y is normal, that is there exist $r_0 \in R$ with $y\ (r_0) = 1$,

(ii) y is convex fuzzy set, that is
$y(tp + (1-t)q) \geq \min\{y(p), y(q)\}$ for all $t \in [0,1]$ and $p, q \in R$,

(iii) y is upper semi continuous on R,

(iv) $[y]^0 = \{x \in R; y(x) > 0\}$ is compact subset of R.

2.2 r-Level Sets

Suppose R_F is the space of fuzzy numbers see (Kaleva, 1990) with membership function μ and $r \in [0,1]$. Here $R \subset R_F$ is understood $R_F = \{\chi_{\{x\}} : x \text{ is real number}\}$.

Meanwhile the r-level sets can be defined are as follows :

$$[y]_r = \{x \in R; y(x) \geq r\}, \quad 0 < r \leq 1;$$

$$[y]_0 = \{x \in R; y(x) > r\} \text{ is compact.}$$

Then it is well known that for each $r \in [0,1]$, $[y]_r$ is bounded closed interval. Where denoted as $[y]_r = [\underline{y}(r), \overline{y}(r)]$.

It is clear that the following statements are true :

(i) $\underline{y}(r)$ is a bounded left continuous non decreasing function over $[0,1]$,

(ii) $\overline{y}(r)$ is a bounded right continuous non increasing function over $[0,1]$,

(iii) $\underline{y}(r) \leq \overline{y}(r)$ for all $r \in (0,1]$, for more details see Buckley & Eslami, (2001) and Buckley *et al.* (2002a).

2.3 Triangular Fuzzy Number

Mosleh *et al.* (2008) stated that the popular fuzzy number is in the form of triangular fuzzy numbers, $\overline{A} = (a,b,c)$. The membership function of the triangular fuzzy numbers are defined as follows :

$$\mu_{\overline{A}}(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \leq x \leq b, \\[2ex] \dfrac{x-c}{b-c}, & b \leq x \leq c, \\[2ex] 0 & \text{otherwise.} \end{cases}$$

and its $r-cuts$ are simply $[\mathrm{A}]^r = [(r) = a + (b-a)r, \quad (r) = c - (c-b)r]$.

2.4 Fuzzy Initial Value Problem

Consider a first-order differential equation with fuzzy initial value problem (IVP) given by

$$\begin{cases} y'(t) = g(t, y(t)), & t \in [0,1] \\ y(t_0) = y_0 \end{cases} \tag{1}$$

where y is a fuzzy function of t, $g(t, y)$ is a fuzzy function of crisp variable t and the fuzzy variable y, y' is the fuzzy derivative of y and $y(t_0) = y_0$ is a triangular or a triangular shaped fuzzy number. Thus a fuzzy initial value problem (Kaleva, 1990), denoted the fuzzy function by $y = [\underline{y}, \overline{y}]$. It means that the r-level sets of $y(t)$ for $t \in [0,1]$ is $[y(t)]_r = [\underline{y}(t;r), \overline{y}(t;r)]$. Also,

$[y'(t)]_r = [\underline{y}'(t;r), \overline{y}'(t;r)]$, $[\mathrm{g}[t, y(t)]_r = [\underline{g}(t, y(t); r), \overline{g}(t, y(t); r)]$.

It can also be written as,

$\mathrm{g}[t, y] = [\underline{g}(t, y), \overline{g}(t, y)]$ and $\underline{\mathrm{g}}[t, y] = F[t, \underline{y}, \overline{y}]$, $\overline{\mathrm{g}}[t, y] = G[t, \underline{y}, \overline{y}]$.

Because of $y' = g(t, y)$ then there is,

$$\underline{y}'[t;r] = \underline{g}(t, y(t); r) = F[t, \underline{y}(t;r), \overline{y}(t;r)]. \tag{2}$$

$$\overline{y}'[t;r] = \overline{g}(t, y(t); r) = G[t, \underline{y}(t;r), \overline{y}(t;r)]. \tag{3}$$

Also,

$$[y(t_0)]_r = [\underline{y}(t_0;r), \overline{y}(t_0;r)], \quad [y_0]_r = [\underline{y_0}(r), \overline{y_0}(r)]$$

$$\underline{y}(t_0;r) = \underline{y}_0(r), \quad \overline{y}(t_0;r) = \overline{y}_0(r)$$

By using the extension principal of Zadeh (1965), the membership function is

$$g(t; y(t))(s) = \sup\{(y(t)(\tau) | s = g(t,\tau), s \in \Im\}.$$

So, $g(t, y(t))$ is a fuzzy number. From this, it follows that,

$$[g(t, y(t))]_r = [\underline{g}(t, y(t); r), \overline{g}(t, y(t); r)], \; r \in [0,1]$$

where

$$\underline{g}(t, y(t); r) = \min\{g(t,u) | u \in [\underline{y}(t;r), \overline{y}(t;r)]\},$$

$$\overline{g}(t, y(t); r) = \max\{g(t,u) | u \in [\underline{y}(t;r), \overline{y}(t;r)]\}.$$

Throughout this work, fuzzy functions which are continuous in metric D are also considered. Then the continuity of $g(t, y(t); r)$ guarantees the existence of the solution of $g(t, y(t); r)$ for $t \in [t_0, T]$ and $r \in [0,1]$.

3. Modified Two-step Simpson's Method for Numerical Solution of Fuzzy Differential Equations (Moghadam & Dahaghin, 2004)

Let $Y = [\underline{Y}, \overline{Y}]$ be the exact solution and $y = [\underline{y}, \overline{y}]$ be the approximate solution of the initial value problem (1) by using modified Two-step Simpson's method.

Let

$$[Y(t)]_r = [\underline{Y}(t;r), \overline{Y}(t;r)], \quad [y(t)]_r = [\underline{y}(t;r), \overline{y}(t;r)].$$

It can be noted that throughout each integration step, the value of r is unchanged. The exact and approximate solution at t_n are denoted by; $[Y_n]_r = [\underline{Y}_n(r), \overline{Y}_n(r)]$, $[y_n]_r = [\underline{y}_n(r), \overline{y}_n(r)]$, $(0 \le n \le N)$, respectively. The grid points at which the solution is calculated are $h = \dfrac{T - t_0}{N}$, $t_i = t_0 + ih, \; 0 \le i \le N$.

The following is obtained by using the modified Two-step Simpson's method:

$$\underline{Y}_{i+1}(r) = \underline{Y}_{i-1}(r) + \frac{h}{3} F(t_{i-1}, \underline{Y}_{i-1}(r), \overline{Y}_{i-1}(r)) + \frac{4h}{3} F(t_i, \underline{Y}_i(r), \overline{Y}_i(r)) +$$
$$\frac{h}{3} F[t_{i+1}, \underline{Y}_i(r), + hF(t_i, \underline{Y}_i(r), \overline{Y}_i(r)), \overline{Y}_i(r) + hG(t_i, \underline{Y}_i(r), \overline{Y}_i(r))] + h^3 \underline{A}(r) \tag{4}$$
and

$$\overline{Y}_{i+1}(r) = \overline{Y}_{i-1}(r) + \frac{h}{3} G(t_{i-1}, \underline{Y}_{i-1}(r), \overline{Y}_{i-1}(r)) + \frac{4h}{3} G(t_i, \underline{Y}_i(r), \overline{Y}_i(r)) +$$
$$\frac{h}{3} G[t_{i+1}, \underline{Y}_i(r), + hF(t_i, \underline{Y}_i(r), \overline{Y}_i(r)), \overline{Y}_i(r) + hG(t_i, \underline{Y}_i(r), \overline{Y}_i(r))] + h^3 \overline{A}(r) \tag{5}$$

Also

$A = [\underline{A}, \overline{A}], [A]_r = [\underline{A}(r), \overline{A}(r)]$ and $[\mathrm{A}]_r = [\frac{1}{6} g'(\xi_2, y(\xi_2)) g_y(t_{i+1}, \xi_3)$

$-\frac{h^2}{90} g^{(4)}(\xi_1, y(\xi_1))]$ where $\xi_1 = (\underline{\xi}_1, \overline{\xi}_1)$, $t_{i-1} \le \xi_1 \le t_{i+1}$, $\xi_2 = [\underline{\xi}_2, \overline{\xi_2}]$,

$\xi_2 \in [t_i, t_{i+1})$ and $\xi_3 = (\underline{\xi}_3, \overline{\xi}_3)$ is in between $\underline{Y}(t_i; r) + hF(t_i, \underline{Y}(t_i; r), \overline{Y}(t_i, r))$

and $\underline{Y}(\mathrm{t_i}; r) + hF(t_i, \underline{Y}(t_i; r), \overline{Y}(t_i, r)) + \frac{h^2}{2} F'(\xi_2, \underline{Y}(\xi_2), \overline{Y}(\xi_2))$ and it is in between

$\overline{\mathrm{Y}}(\mathrm{t_i}; r) + hG(t_i, \underline{Y}(t_i; r), \overline{Y}(t_i, r))$

and $\overline{\mathrm{Y}}(\mathrm{t_i}; r) + hG(t_i, \underline{Y}(t_i; r), \overline{Y}(t_i, r)) + \frac{h^2}{2} G'(\xi_2, \underline{Y}(\xi_2), \overline{Y}(\xi_2))$.

Also obtain,

$$\underline{y}_{i+1}(r) = \underline{\mathrm{y}}_{i-1}(r) + \frac{h}{3} F(t_{i-1}, \underline{y}_{i-1}(r), \overline{y}_{i-1}(r)) + \frac{4h}{3} F(t_i, \underline{y_i}(r), \overline{y_i}(r)) +$$
$$\frac{h}{3} F[t_{i+1}, \underline{y}_i(r), + hF(t_i, \underline{y_i}(r), \overline{y_i}(r)), \overline{y_i}(r) + hG(t_i, \underline{y_i}(r), \overline{y_i}(r))], \tag{6}$$

and

$$\overline{y}_{i+1}(r) = \overline{y}_{i-1}(r) + \frac{h}{3} G(t_{i-1}, \underline{y}_{i-1}(r), \overline{y}_{i-1}(r)) + \frac{4h}{3} G(t_i, \underline{y_i}(r), \overline{y_i}(r)) +$$
$$\frac{h}{3} G[t_{i+1}, \underline{y}_i(r), + hF(t_i, \underline{y_i}(r), \overline{y_i}(r)), \overline{y_i}(r) + hG(t_i, \underline{y_i}(r), \overline{y_i}(r))]. \tag{7}$$

4. MODIFIED ROMBERG'S METHOD

In this section, we present analytical numerical solution of fuzzy differential equations.

4.1 Modified Romberg's Method by Using Modified Two-step Simpson's 1/3 Method

The total error in the modified Two-step Simpson's method, for an interval $[t_0, T]$ of size h is,

$$E = \frac{(T - t_0)h^2}{6} g'(\xi_2, y(\xi_2)) g_y(t_{p+i}, \xi_3) - \frac{(T\text{-}t_0)h^4}{90} g^{(4)}(\xi_1, y(\xi_1)),$$

where $t_{p-1} \le \xi_1 \le t_{p+1}$, $t_p \le \xi_2 \le t_{p+1}$ and ξ_3 is between $y(t_p) + hg(t_p, y(t_p))$ and

$y(t_p) + hg(t_p, y(t_p)) + \frac{h^2}{2} g'(\xi_2, y(\xi_2))$.

Let $E = Jh^2 + Kh^4$, where $J = \frac{(T-t_0)}{6} g'(\xi_2, y(\xi_2)) g_y(t_{p+1}, \xi_3)$ and $K = \frac{(T\text{-}t_0)h^4}{90} g^{(4)}(\xi_1, y(\xi_1))$ may be chosen as a constant if $g'(\xi_2, y(\xi_2)), g_y(t_{p+1}, \xi_3)$ and $g^{(4)}(\xi_1, y(\xi_1))$ are reasonable constant. Evaluate $\int_{t_{p-1}}^{t_{p+1}} y'(s)ds$ by using the modified two-step Simpson's method with three different subintervals h_1, h_2 and h_3. Let I_1, I_2 and I_3 be the approximations with errors E_1, E_2 and E_3. respectively. Then,

$$I = I_1 + E_1 = I_1 + Jh_1^2 + Kh_1^4, \tag{8}$$

$$I = I_2 + E_2 = I_2 + Jh_2^2 + Kh_2^4, \tag{9}$$

$$I = I_3 + E_3 = I_3 + Jh_3^2 + Kh_3^4, \tag{10}$$

From (8) and (9), K can be written as:

$$K = \frac{I_1 - I_2 + J(h_1^2 - h_2^2)}{h_2^4 - h_1^4} \tag{11}$$

From (8) and (10), K can be written as:

$$K = \frac{I_1 - I_3 + J(h_1^2 - h_3^2)}{h_3^4 - h_1^4} \tag{12}$$

From (11) and (12) after simplification, J can be written as:

$$J = \frac{I_1(h_2^4 - h_3^4) - I_2(h_1^4 - h_3^4) + I_3(h_1^4 - h_2^4)}{h_1^4 h_2^2 - h_1^4 h_3^2 - h_2^2 h_3^4 + h_2^4 h_3^2 + h_1^2 h_3^4 - h_1^2 h_2^4}$$

$$K = \frac{I_1 - I_3 + \frac{I_1(h_2^4 - h_3^4) - I_2(h_1^4 - h_3^4) + I_3(h_1^4 - h_2^4)}{h_1^4 h_2^2 - h_1^4 h_3^2 - h_2^2 h_3^4 + h_2^4 h_3^2 + h_1^2 h_3^4 - h_1^2 h_2^4}(h_1^2 - h_3^2)}{h_3^4 - h_1^4}$$

By putting the above values of J and K in (8) then it will be a better approximation to I then I_1, I_2 and I_3.

In this case, to evaluate systematically, put the values of J, K, $h_1 = 4h, h_2 = 2h$ and $h_3 = h$ in (8) then after simplification, obtain,

$$I = \frac{I_1 - 20I_2 + 64I_3}{45} \tag{13}$$

This result was obtained by applying modified two-step Simpson's method thrice. By applying the method several times, every time halving h, a sequence of L_1, L_2, L_3, L_4 and $L_5 \ldots$ is obtained and by applying the formula (13) again to the three successive values i.e. $L_1, L_2, L_3; L_2, L_3, L_4;\ L_3, L_4,\ L_5, \ldots,$ to get improved results

M_1, M_2, M_3, M_4 and $M_5,...,$. Again applying the formula (13) to the three successive values i.e. $M_1, M_2, M_3; M_2, M_3, M_4; M_3, M_4, M_5,...,$ and better results $N_1, N_2, N_3,...,$ were obtained. This process was continued until two successive values are close to each other.

4.2 Modified Romberg's Method for Numerical Solution of Fuzzy Differential Equations

In this section, discussion will be done for an interval $[t_0, T]$ of size h, the total error of the initial value problem (1) by using the modified two-step Simpson's method is $[E]_r = [\underline{E}(r), \overline{E}(r)]$, where

$$\underline{E}(r) = \frac{(T-t_0)h^2}{6}\underline{g}'(\xi_2, Y(\xi_2))\underline{g}_y(t_{i+1}, \xi_3) - \frac{(T\text{-}t_0)h^4}{90}\underline{g}^{(4)}(\xi_1, Y(\xi_1)),$$

$$\overline{E}(r) = \frac{(T-t_0)h^2}{6}\overline{g}'(\xi_2, Y(\xi_2))\overline{g}_y(t_{i+1}, \xi_3) - \frac{(T\text{-}t_0)h^4}{90}\overline{g}^{(4)}(\xi_1, Y(\xi_1)),\quad \xi_1 = (\underline{\xi}_1, \overline{\xi}_1),$$

$t_{i-1} \le \xi_1 \le t_{i+1}$, $\xi_2 = [\underline{\xi}_2, \overline{\xi}_2]$, $\xi_2 \in [t_i, t_{i+1})$ and $\xi_3 = (\underline{\xi}_3, \overline{\xi}_3)$ is in between $\underline{Y}(t_i; r)$ + $hF(t_i, \underline{Y}(t_i; r), \overline{Y}(t_i, r))$ and $\underline{Y}(t_i; r) + hF(t_i, \underline{Y}(t_i; r), \overline{Y}(t_i, r))$ $+\frac{h^2}{2}F'(\xi_2, \underline{Y}(\xi_2), \overline{Y}(\xi_2))$ and it is in between $\overline{Y}(t_i; r) + hG(t_i, \underline{Y}(t_i; r), \overline{Y}(t_i, r))$ and $\overline{Y}(t_i; r) + hG(t_i, \underline{Y}(t_i; r), \overline{Y}(t_i, r))$ $+\frac{h^2}{2}G'(\xi_2, \underline{Y}(\xi_2), \overline{Y}(\xi_2))$.

Let,

$$\underline{E}(r) = \underline{R}(r)h^2 + \underline{S}(r)h^4,$$

where $\underline{R}(r) = \frac{(T-t_0)}{6}\underline{g}'(\xi_2, Y(\xi_2))\underline{g}_y(t_{i+1}, \xi_3)$ and $\underline{S}(r) = -\frac{(T\text{-}t_0)h^4}{90}\underline{g}^{(4)}(\xi_1, Y(\xi_1))$ may be chosen as a constant if $\underline{g}'(\xi_2, Y(\xi_2)), \underline{g}_y(t_{i+1}, \xi_3)$ and $\underline{g}^{(4)}(\xi_1, Y(\xi_1))$, are reasonably constant. Suppose, evaluate $\underline{I}(r) = \int_{t_{i-1}}^{t_{i+1}} Y'(s; r)ds$ by using the modified two-step Simpson's method with three different subintervals h_1, h_2 and h_3. Let $\underline{I}_1(r), \underline{I}_2(r)$ and $\underline{I}_3(r)$ be the approximations with errors $\underline{E}_1(r), \underline{E}_2(r)$ and $\underline{E}_3(r)$ respectively. Then,

$$\underline{I}(r) = \underline{I}_1(r) + \underline{E}_1(r) = \underline{I}_1(r) + \underline{R}(r)h_1^2 + \underline{S}(r)h_1^4 \qquad (14)$$

$$\underline{I}(r) = \underline{I}_2(r) + \underline{E}_2(r) = \underline{I}_2(r) + \underline{R}(r)h_2^2 + \underline{S}(r)h_2^4 \qquad (15)$$

$$\underline{I}(r) = \underline{I}_3(r) + \underline{E}_3(r) = \underline{I}_3(r) + \underline{R}(r)h_3^2 + \underline{S}(r)h_3^4 \qquad (16)$$

From (14) and (15) the following is obtained,

$$\underline{S}(r)=\frac{\underline{I}_1(r)-\underline{I}_2(r)+\underline{R}(r)(h_1^{\ 2}-h_2^2)}{h_2^{\ 4}-h_1^4} \tag{17}$$

From (14) and (16) the following is obtained,

$$\underline{S}(r)=\frac{\underline{I}_1(r)-\underline{I}_3(r)+\underline{R}(r)(h_1^{\ 2}-h_3^2)}{h_3^{\ 4}-h_1^4} \tag{18}$$

From (17) and (18) after simplification, the following is obtained,

$$\underline{R}(r)=\frac{\underline{I}_1(r)(h_2^{\ 4}-h_3^4)-\underline{I}_2(r)(h_1^{\ 4}-h_3^4)+\underline{I}_3(r)(h_1^{\ 4}-h_2^4)}{h_1^{\ 4}h_2^{\ 2}-h_1^{\ 2}h_2^{\ 4}-h_2^{\ 2}h_3^{\ 4}+h_2^{\ 4}h_3^{\ 2}+h_1^{\ 2}h_3^{\ 4}-h_1^{\ 4}h_3^{\ 2}},$$

$$\underline{S}(r)=\frac{\underline{I}_1(r)-\underline{I}_3(r)+\left[\dfrac{\underline{I}_1(r)(h_2^{\ 4}-h_3^4)-\underline{I}_2(r)(h_1^{\ 4}-h_3^4)+\underline{I}_3(r)(h_1^{\ 4}-h_2^4)}{h_1^{\ 4}h_2^{\ 2}-h_1^{\ 2}h_2^{\ 4}-h_2^{\ 2}h_3^{\ 4}+h_2^{\ 4}h_3^{\ 2}+h_1^{\ 2}h_3^{\ 4}-h_1^{\ 4}h_3^{\ 2}}(h_1^2-h_3^2)\right]}{h_3^4-h_1^4}$$

By putting the above values of $\underline{R}(r)$ and $\underline{S}(r)$ in (14), then it will be a better approximation to $\underline{I}(r)$ than $\underline{I}_1(r), \underline{I}_2(r)$ and $\underline{I}_3(r)$. In order to evaluate systematically, put $h_1=4h, h_2=2h$ and $h_3=h,$ and values of $\underline{R}(r)$ and $\underline{S}(r)$ in (14) then after simplification, the following is obtained,

$$\underline{I}(r)=\frac{\underline{I}_1(r)-20\underline{I}_2(r)+64\underline{I}_3(r)}{45} \tag{19}$$

This result was obtained by applying modified two-step Simpson's method thrice. By applying the method several times, every time halving h, a sequence of results $F_1, F_2, F_3, F_4, F_5,\ldots$ was acquired, then the formula (19) was applied again to the three successive values i.e. $F_1, F_2, F_3; F_2, F_3, F_4; F_3, F_4, F_5,\ldots,$ to get improved results $S_1, S_2, S_3, S_4, S_5,\ldots$. Again, by applying the formula (19) to the three successive values i.e. $S_1, S_2, S_3; S_2, S_3, S_4; S_3, S_4, S_5,\ldots,$ still better results $T_1, T_2, T_3,\ldots$ were achieved. This process was continued until two successive values are close to each other. Similarly,

$$\overline{I}(r)=\frac{\overline{I}_1(r)-20\overline{I}_2(r)+64\overline{I}_3(r)}{45} \tag{20}$$

4.3 Algorithms

The Modified Romberg's method algorithms to find the numerical solution of FDE are as follows :

The initial values, $y(t_0)=(\underline{y}_0, \overline{y}_0)$

$$(\underline{y}_1, \overline{y}_1)=(\underline{y}_0+h\underline{y}_0+\frac{h^2}{2}\underline{y}_0,\quad \overline{y}_0+h\overline{y}_0+\frac{h^2}{2}\overline{y}_0)$$

$$(\underline{y}_2,\ \overline{y}_2)=(\underline{y}_0+\frac{h}{3}\underline{y}_0+\frac{4h}{3}\underline{y}_1+\frac{h}{3}\underline{y}_1,\ \overline{y}_0+\frac{h}{3}\overline{y}_0+\frac{4h}{3}\overline{y}_1+\frac{h}{3}\overline{y}_1)$$

$\vdots$

$$(\underline{y}_{p+1},\overline{y}_{p+1}) = (\ \underline{y}_{p-1} + \frac{h}{3}g(t_{p-1},\underline{y}_{p-1}) + \frac{4h}{3}g(t_p,\underline{y}_p) + \frac{h}{3}g(t_{p+1},\underline{y}_p + hg(t_p,\underline{y}_p))\ ,$$
$$\overline{y}_{p-1} + \frac{h}{3}g(t_{p-1},\overline{y}_{p-1}) + \frac{4h}{3}g(t_p,\overline{y}_p) + \frac{h}{3}g(t_{p+1},\overline{y}_p + hg(t_p,\overline{y}_p)))$$

and

$$(\underline{I},\overline{I}) = (\frac{\underline{I}_1(r) - 20\underline{I}_2(r) + 64\underline{I}_3(r)}{45},\ \frac{\overline{I}_1(r) - 20\overline{I}_2(r) + 64\overline{I}_3(r)}{45}).$$

5. NUMERICAL EXAMPLE

In this section, a numerical example by using the Modified Romberg's method will be presented. The finding of the theoretical exact solution and the numerical solution via Romberg's method in this study are shown in Table 5.1 and Table 5.2 for Example 5.1 respectively. In order to compare the accuracy, Tables 5.3 is devoted for the corresponding errors of Example 5.1. Table 5.4 is shown the results of Example 5.1 by using the Standard Euler's method. Meanwhile Table 5.5 is shown corresponding error between the exact solution and the Standard Euler's method. It should be mentioned that the difference of two r-level sets $[c_1,d_1]$ and $[c_2,d_2]$ is denoted by $D_1 = |c_1 - c_2| + |d_1 - d_2|$.

Example 5.1

This Example refer to the growth and decay problem. Let $y(t)$ denote the amount of population that is growing. If $y'(t)$ is the time rate of change of this amount of population is proportional to the amount of population present, then we can write the problem in the form of FDE with fuzzy initial value problem as follow :

$$\begin{cases} y'(t) = y(t) & t \in [0,1] \\ y(0) = (0.75 + 0.25r, 1.125 - 0.125r) \end{cases} \tag{21}$$

The exact solution at $t=1$ is given by

$$y(1;r) = [(0.75 + 0.25r)e, (1.125 - 0.125r)e], \quad 0 \le r \le 1 \tag{22}$$

Table 5.1 shows the exact solution of the problem from Equation (22), Table 5.2 shows the results of equation (21) using modified Romberg's method and Table 5.3 shows the corresponding errors.

Table 5.1: Exact Solution of Equation (22) (when $t=1$)

r	Exact solution
0	2.038711371344280, 3.058067057016430
0.2	2.174625462767240, 2.990110011304950
0.4	2.310539554190190, 2.922152965593470
0.6	2.446453645613140, 2.854195919882000
0.8	2.582367737036090, 2.786238874170520
1.0	2.718281828459050, 2.718281828459050

Meanwhile, for $h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9$ each time halving the value of h. If $h_1 = 0.25$ then by using modified Romberg's method for the different values of r, the following is achieved:

Table 5.2: Equation (21) Results by Using Modified Romberg's Method ($h_1 = 0.25$)

r	Method values
0	2.03871137132046, 3.05806705698069
0.2	2.17462546274181, 2.99011001127002
0.4	2.31053955416321, 2.92215296555930
0.6	2.44645364558455, 2.85419591984867
0.8	2.58236773700594, 2.78623887413795
1.0	2.71828182842730, 2.71828182842730

Table 5.3 : Corresponding Errors between Modified Romberg's Method with Exact Solution of Equation (21)

r	Corresponding errors
0	$2.382000000000 \times 10^{-11}$, $3.574000000000 \times 10^{-11}$
0.2	$2.543000000000 \times 10^{-11}$, $3.493000000000 \times 10^{-11}$
0.4	$2.698000000000 \times 10^{-11}$, $3.417000000000 \times 10^{-11}$
0.6	$2.859000000000 \times 10^{-11}$, $3.333000000000 \times 10^{-11}$
0.8	$3.015000000000 \times 10^{-11}$, $3.257000000000 \times 10^{-11}$
1.0	$3.175000000000 \times 10^{-11}$, $3.175000000000 \times 10^{-11}$

Table 5.4 shows the results of Equation (21) by using the Standard Euler's method (Ma *et al.*, 1999) with the step size, h = 0.25 and Table 5.5 shows the corresponding errors between Standard Euler's method and exact solution.

Table 5.4 : Equation (21) Results by Using Standard Euler's Method

r	Euler's method ($h = 0.25$)
0	1.831054687500000, 2.746582031250000
0.2	1.953125000000000, 2.685546875000000
0.4	2.075195312500000, 2.624511718750000
0.6	2.197265625000000, 2.563476562500000
0.8	2.319335937500000, 2.502441406250000
1.0	2.441406250000000, 2.441406250000000

Table 4.5 : Corresponding Errors between Standard Euler's Method with Exact Solution of Equation (21)

r	Corresponding Error Euler's method ($h = 0.25$)
0	$2.076566835000 \times 10^{-1}$, $3.114850252500 \times 10^{-1}$
0.2	$2.215004624000 \times 10^{-1}$, $3.045631358000 \times 10^{-1}$
0.4	$2.353442413000 \times 10^{-1}$, $2.976412463500 \times 10^{-1}$
0.6	$2.491880202000 \times 10^{-1}$, $2.907193569000 \times 10^{-1}$
0.8	$2.630317991000 \times 10^{-1}$, $2.837974674500 \times 10^{-1}$
1.0	$2.768755780000 \times 10^{-1}$, $2.768755780000 \times 10^{-1}$

6. CONCLUSION

In this study Modified Romberg's method (which is originally customized by using Modified Two-step Simpson's 1/3 method) for numerical solution of FDE with fuzzy initial value problem has been successfully derived. It has been shown that the Modified Romberg's method can be applied to estimate the integral of a function more effectively than Euler's method (Ma *et al.*, 1999).

The calculation and observation have shown that by using the Standard Euler's method the results of the errors are greater than 10^{-1} in Example 5.1. Meanwhile by using the Romberg's method the results of errors is smaller which is 10^{-11}. Therefore, Modified Romberg's method gave the best solution and more accurate answer compared to Standard Euler's method.

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