A First Approximation for Quantization of Singular Spaces

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Abstract

Many mathematical models of physical phenomena that have been proposed in recent years require more general spaces than manifolds. When taking into account the symmetry group of the model, we get a reduced model on the (singular) orbit space of the symmetry group action. We investigate quantization of singular spaces obtained as leaf closure spaces of regular Riemannian foliations on compact manifolds. These contain the orbit spaces of compact group actions and orbifolds. Our method uses foliation theory as a desingularization technique for such singular spaces. A quantization procedure on the orbit space of the symmetry group - that commutes with reduction - can be obtained from constructions which combine different geometries associated with foliations and new techniques originated in Equivariant Quantization. The present paper contains the first of two steps needed to achieve these just detailed goals.

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1 Introduction

Quantization of singular spaces is an emerging issue that has been addressed in an increasing number of recent works, see e.g. [BHP06], [Hue02], [Hue06], [HRS07], [Hui07], [Pfl02] ...

One of the reasons for this growing popularity originates from current developments in Theoretical Physics related with reduction of the number of degrees of freedom of a dynamical system with symmetries. Explicitly, if a symmetry Lie group acts on the phase space or the configuration space of a general mechanical system, the quotient space is usually a singular space, an orbifold or a stratified space ... The challenge consists in the quest for a quantization procedure for these singular spaces that in addition commutes with reduction.

In this work, we investigate quantization of singular spaces obtained as leaf closure spaces of regular Riemannian foliations of compact manifolds. These contain the orbit spaces of compact group actions (see [Rich01]). We build a quantization that commutes by construction with projection onto the quotient.

Our method uses the foliation as desingularization of the orbit space $M/\bar{\mathcal{F}}$, where $\bar{\mathcal{F}}$ is the singular Riemannian foliation made up by the closures of the leaves of the regular Riemannian foliation $\mathcal{F}$ on manifold M. More precisely, we combine Foliation Theory with recent techniques from Natural and Equivariant Quantization. Close match can indeed be expected, as both topics are tightly connected

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with natural bundles and natural operators.

Equivariant quantization, in the sense of C. Duval, P. Lecomte, and V. Ovsienko, developed as from 1996, see [LMT96], [LO99], [DLO99], [Lec00], [BM01], [DO01], [BHMP02], [BM06]. This procedure requires equivariance of the quantization map with respect to the action of a finite-dimensional Lie subgroup of the symmetry group $\mathrm{Diff}(\mathbb{R}^n)$ of configuration space $\mathbb{R}^n$. Equivariant quantization has first been studied in Euclidean space, mainly for the projective and conformal subgroups, then extended in 2001 to arbitrary manifolds, see [Lec01]. An equivariant, or better, a natural quantization on a smooth manifold M is a vector space isomorphism

$$Q[\nabla] : \mathrm{Pol}(T^*M) \ni s \to Q[\nabla](s) \in \mathcal{D}(M)$$

that verifies some normalization condition and maps, in this paper, a smooth function $s \in \mathrm{Pol}(T^*M)$ of "phase space" T^*M, which is polynomial along the fibers, to a differential operator $Q[\nabla](s) \in \mathcal{D}(M)$ that acts on functions $f \in C^\infty(M)$ of "configuration space" M. The quantization map $Q[\nabla]$ depends on the projective class $[\nabla]$ of an arbitrary torsionless covariant derivative ∇ on M, and it is natural with respect to all its arguments and for the action of the group $\mathrm{Diff}(M)$ of all local diffeomorphisms of M, i.e.

$$Q[\phi^*\nabla](\phi^*s)(\phi^*f) = \phi^*\left(Q[\nabla](s)(f)\right),$$

$\forall s \in \mathrm{Pol}(T^*M), \forall f \in C^\infty(M), \forall \phi \in \mathrm{Diff}(M)$. Existence of such natural and projectively invariant quantizations has been investigated in several works, see e.g. [Bor02], [MR05], [Han06].

In Foliation Theory, one distinguishes different geometries associated with a foliated manifold $(M, \mathcal{F})$ (defined by a Hæfliger cocycle), namely adapted geometry, foliated geometry, and transverse geometry. We denote in this introduction objects of the adapted (resp. foliated, transverse) "world" by O_3 (resp. O_2, O_1), whereas objects of leaf closure space $M/\bar{\mathcal{F}}$ are denoted by O_0. Ideally, geometric structures of level i project onto geometric structures of level $i-1$, so that $p(O_i) = O_{i-1}$, if we agree to denote temporarily any of these projections by p. Let us also recall that, roughly, adapted objects are objects on M with some special properties, foliated objects are locally constant along the leaves and live in the normal bundle of the foliation, and that transverse objects are objects on the transverse manifold N, which are $\mathcal{H}$-invariant, where transverse manifold N and the holonomy pseudo-group $\mathcal{H}$ depend on the chosen defining cocycle of foliation $\mathcal{F}$. In order to build a quantization Q_0 on $M/\bar{\mathcal{F}}$, which commutes with the projection onto this singular space, we construct adapted, foliated, and transverse quantizations Q_3, Q_2, and Q_1, in such a way that

$$Q_{i-1}[p\,\nabla_i](p\,s_i)(p\,f_i) = p\left(Q_i[\nabla_i](s_i)(f_i)\right), \quad \forall i \in \{1, 2, 3\}. \tag{1}$$

Hence,

$$Q_0[\nabla_0](s_0)(f_0) = Q_0[p^3\nabla_3](p^3 s_3)(p^3 f_3) = p^3\left(Q_3[\nabla_3](s_3)(f_3)\right).$$

Observe that adapted quantization Q_3 quantizes objects on M, whereas singular quantization Q_0 only quantizes the objects of $M/\bar{\mathcal{F}}$. Eventually, quantization actually commutes with projection onto the quotient.

The proofs of the three stages mentioned in Equation (1) are not equally hard. Since foliated geometric objects on a foliated manifold $(M, \mathcal{F})$ are in 1-to-1 correspondence with $\mathcal{H}$-invariant geometric objects on the transverse manifold N associated with the chosen cocycle, it is clear that stage $Q_2 - Q_1$ is quite obvious. The passages $Q_3 - Q_2$ between the "big" adapted and "small" foliated quantizations, as well as transition $Q_1 - Q_0$ from transverse quantization to singular quantization are much more intricate.

In order to limit the length of the article, we publish the stages $Q_3 - Q_2$ and $Q_1 - Q_0$ in two different works. This publication deals with the *first approximation $Q_3 - Q_2$ for quantization of singular spaces.*

2 Natural and projectively invariant quantization

The constructions of Q_3 and Q_2 are nontrivial extensions to the adapted and foliated contexts of the proof of existence of natural and projectively invariant quantization maps on an arbitrary smooth manifold, see [MR05]. In the present section, we concisely describe the basic ideas of this technique. We refer the reader to [MR05] for more details.

Throughout this note, we denote by M a smooth, Hausdorff and second countable manifold of dimension n.

We denote by $\mathcal{C}_M$ the space of torsion-free linear connections on M. A *quantization* on a manifold M is a linear bijection Q_M from the space of symbols $\mathcal{S}(M)$ to the space of differential operators $\mathcal{D}(M)$ such that

$$\sigma(Q_M(s)) = s, \quad \forall s \in \mathcal{S}^k(M),\ \forall k \in \mathbb{N},$$

where σ denotes the principal symbol operator. A *natural and projectively equivariant quantization* is a collection of maps (defined for every manifold M)

$$Q_M : \mathcal{C}_M \times \mathcal{S}(M) \to \mathcal{D}(M)$$

such that

- For all ∇ in $\mathcal{C}_M$, $Q_M(\nabla)$ is a quantization,
- If ϕ is a local diffeomorphism from M to N, then one has

$$Q_M(\phi^*\nabla)(\phi^* s) = \phi^*(Q_N(\nabla)(s)), \quad \forall \nabla \in \mathcal{C}_N, \forall s \in \mathcal{S}(N).$$

- One has $Q_M(\nabla) = Q_M(\nabla')$ whenever ∇ and ∇' are projectively equivalent torsion-free linear connections on M.

Recall that ∇ and ∇' are projectively equivalent if they fulfill the relation

$$\nabla'_X Y = \nabla_X Y + \alpha(X)Y + \alpha(Y)X,$$

where α is a one-form on M.

The method used in [MR05] to solve the problem of the natural and projectively equivariant quantization can be divided into four steps.

In a first step, one associates in a natural and bijective way to the projective class of ∇ a reduction P of the second order frame bundle P^2M. This reduction is called *Cartan fiber bundle* and its structural group is $H(n+1, \mathbb{R})$, the isotropy subgroup at the origin of the projective space $\mathbb{R}P^n$ of the projective group $\mathrm{PGL}(n+1, \mathbb{R})$.

Next, one can associate to the symbol s an equivariant function on P in a natural and bijective way.

In a third step, one associates naturally to the projective class of ∇ a Cartan connection on P called the *normal Cartan connection* ω.

Finally, thanks to an operation called *invariant differentiation* builded from ω, one constructs a formula on P expressing the natural and projectively equivariant qunatization, this formula being exactly the same as the formula giving the projectively equivariant quantization on $\mathbb{R}^n$ if one replaces the invariant differentiation by the partial derivatives.

3 Adapted and foliated quantizations

In this section, we are going to define precisely the notions of adapted and foliated objects in order to define the problems of adapted and foliated quantizations.

3.1 Foliations

Let $(M, \mathcal{F})$ be a foliated manifold, more precisely, let M be an n-dimensional smooth manifold endowed with a *regular foliation* $\mathcal{F}$ of dimension p (and codimension $q = n - p$). It is well-known that such a foliation can be defined as an involutive subbundle $T\mathcal{F} \subset TM$ of constant rank p.

Foliation $\mathcal{F}$ can also be viewed as a partition into (maximal integral) p-dimensional smooth submanifolds or *leaves*, such that in appropriate or *adapted charts* (U_i, ϕ_i) the connected components of the traces on U_i of these leaves lie in M as $\mathbb{R}^p$ in $\mathbb{R}^n$ [pages of a book], with transition diffeomorphisms of type $\psi_{ji} = \phi_j \circ \phi_i^{-1} : \phi_i(U_{ij}) \ni (x, y) \to (\psi_{ji,1}(x, y), \psi_{ji,2}(y)) \in \phi_j(U_{ji})$, $U_{ij} = U_i \cap U_j$ [the ψ_{ji} map a page onto a page]. The pages provide by transport to manifold M the so-called *plaques* or *slices* and these glue together from chart to chart—in the way specified by the transition diffeomorphisms—to give maximal connected injectively immersed submanifolds, precisely the leaves of the foliation.

Eventually, foliation $\mathcal{F}$ can be described by means of a Hæfliger *cocycle* $\mathcal{U} = (U_i, f_i, g_{ij})$ modelled on a q-dimensional smooth manifold N_0. The U_i form an open cover of M and the $f_i : U_i \to f_i(U_i) =: N_i \subset N_0$ are submersions that have connected fibers [the connected components of the traces on the U_i of the leaves of $\mathcal{F}$] and are subject to the transition conditions $g_{ji} f_i = f_j$, where the $g_{ji} : f_i(U_{ij}) =: N_{ij} \to N_{ji} := f_j(U_{ji})$ are diffeomorphisms that verify the usual cocycle condition $g_{ij} g_{jk} = g_{ik}$. We refer to the disjoint union $N = \amalg_i N_i$ as the (smooth, q-dimensional) *transverse manifold* and to $\mathcal{H} :=\rangle g_{ij} \langle$ as the pseudogroup of (locally defined) diffeomorphisms or *holonomy pseudogroup* associated with the chosen cocycle $\mathcal{U}$.

A vector field $X \in \mathrm{Vect}(M)$, such that $[X, Y] \in \Gamma(T\mathcal{F})$, for all $Y \in \Gamma(T\mathcal{F})$, is said to be *adapted* (to the foliation). The space $\mathrm{Vect}_{\mathcal{F}}(M)$ of adapted vector fields is obviously a Lie subalgebra of the Lie algebra $\mathrm{Vect}(M)$, and the space $\Gamma(T\mathcal{F})$ of *tangent* (to the foliation) vector fields is an ideal of $\mathrm{Vect}_{\mathcal{F}}(M)$. The quotient algebra $\mathrm{Vect}(M, \mathcal{F}) = \mathrm{Vect}_{\mathcal{F}}(M)/\Gamma(T\mathcal{F})$ is the algebra of *foliated* vector fields.

Let (x, y) be local coordinates of M that are adapted to $\mathcal{F}$, i.e. $x = (x^1, \ldots, x^p)$ are leaf coordinates and $y = (y^1, \ldots, y^q)$ are transverse coordinates. The local form of an arbitrary (resp. tangent, adapted, foliated) vector field is then $X = \sum_{\iota=1}^p X^\iota(x, y)\partial_\iota + \sum_{\mathrm{i}=1}^q X^{\mathrm{i}}(x, y)\partial_{\mathrm{i}}$, $\partial_\iota = \partial_{x^\iota}$, $\partial_{\mathrm{i}} = \partial_{y^{\mathrm{i}}}$ (resp. $X = \sum_{\iota=1}^p X^\iota(x, y)\partial_\iota$,

$$X = \sum_{\iota=1}^{p} X^\iota(x, y)\partial_\iota + \sum_{\mathrm{i}=1}^{q} X^{\mathrm{i}}(y)\partial_{\mathrm{i}}, \tag{2}$$

$$[X] = [\sum_{\mathrm{i}=1}^{q} X^{\mathrm{i}}(y)\partial_{\mathrm{i}}], \tag{3}$$

where $[.]$ denotes the classes in the aforementioned quotient algebra).

A smooth function $f \in C^\infty(M)$ is foliated (or basic) if and only if $L_Y f = 0, \forall Y \in \Gamma(T\mathcal{F})$. We denote by $C^\infty(M, \mathcal{F})$ the space of all foliated functions of $(M, \mathcal{F})$. A differential k-form $\omega \in \Omega^k(M)$ is foliated (or basic) if and only if $\mathrm{i}_Y \omega = \mathrm{i}_Y \mathrm{d}\,\omega = 0, \forall Y \in \Gamma(T\mathcal{F})$, where notations are self-explaining. Again, we denote by $\Omega^k(M, \mathcal{F})$ the space of all foliated differential k-forms of $(M, \mathcal{F})$.

It is easily checked that $C^\infty(M, \mathcal{F}) \times \mathrm{Vect}(M, \mathcal{F}) \ni (f, [X]) \to f[X] := [fX] \in \mathrm{Vect}(M, \mathcal{F})$ defines a $C^\infty(M, \mathcal{F})$-module structure on $\mathrm{Vect}(M, \mathcal{F})$. Furthermore, $\mathrm{Vect}(M, \mathcal{F}) \times C^\infty(M, \mathcal{F}) \ni ([X], f) \to L_{[X]} f := L_X f \in C^\infty(M, \mathcal{F})$ is the natural action of foliated vector fields on foliated functions. Eventually, the contraction of a foliated 1-form $\alpha \in \Omega^1(M, \mathcal{F})$ and a foliated vector field $[X] \in \mathrm{Vect}(M, \mathcal{F})$ is a foliated function $\alpha([X]) := \alpha(X) \in C^\infty(M, \mathcal{F})$.

3.2 Adapted and foliated frame bundles

3.2.1 Adapted frame bundles

Since an adapted linear frame is a frame $(v_1, \ldots, v_{p+q})$ of a fiber $T_m M$, $m \in M$, the first vectors $(v_1, \ldots, v_p)$ of which form a frame of $T_m\mathcal{F}$, we denote by $P^r_{\mathcal{F}}M$ the principal bundle $P^r_{\mathcal{F}}M = \{j^r_0(f) | \, f : 0 \in U \subset \mathbb{R}^n \to M, T_0 f \in \mathrm{Isom}(\mathbb{R}^n, T_{f(0)}M), Tf(T\mathcal{F}_0) = T\mathcal{F}\}$, where $\mathcal{F}_0$ is the canonical regular p-dimensional foliation of $\mathbb{R}^n$. The structure group of $P^r_{\mathcal{F}}M$ is $G^r_{n,\mathcal{F}_0} = \{j^r_0(\varphi) | \, \varphi : 0 \in U \subset \mathbb{R}^n \to$

$\mathbb{R}^n, \varphi(0) = 0, T_0\varphi \in \mathrm{GL}(n,\mathbb{R}), T\varphi(T\mathcal{F}_0) = T\mathcal{F}_0\}$, its action on $P^r_{\mathcal{F}}M$ is canonical. We call $P^r_{\mathcal{F}}M$ the principal bundle of *adapted r-frames* on M. For instance, $P^1_{\mathcal{F}}M =: L_{\mathcal{F}}M$ is the bundle of adapted linear frames of M with structure group

$$G^1_{n,\mathcal{F}_0} \simeq \mathrm{GL}(n,q,\mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : A \in \mathrm{GL}(p,\mathbb{R}), B \in \mathrm{gl}(p \times q,\mathbb{R}), D \in \mathrm{GL}(q,\mathbb{R}) \right\}. \tag{4}$$

The foliation $\mathcal{F}$ induces a foliation $\mathcal{F}_{\underline{P}^r}$ on $P^r_{\mathcal{F}}M$ whose leaves are locally defined by the sets of frames $j^r_0 f$ such that $\underline{P}^r(y)(j^r_0 f)$ is constant, with

$$\underline{P}^r(y) : j^r_0 f \mapsto j^r_0(y \circ f \circ i_q),$$

where y denotes the passing to the transverse coordinates and where i_q denotes the canonical inclusion of $\mathbb{R}^q$ into $\mathbb{R}^n$.

3.2.2 Foliated frame bundles

Let U and V be neighborhoods of 0 in $\mathbb{R}^q$ and let $f : U \to M$, $g : V \to M$ be smooth maps transverse to $\mathcal{F}$ (it means that $im\, f_* \oplus T\mathcal{F} = TM$) with $f(0) = g(0) = x$. Let W be a neighborhood of x and let $F : W \to \mathbb{R}^q$ be a submersion constant along the leaves of $\mathcal{F}$. We say that f and g define the same transverse r-frame at x if $F \circ f$ and $F \circ g$ have the same partial derivatives up to order r at 0. This definition is independent of the choice of the submersion F (one can take F equal to the passing y to the transverse coordinates of an adapted system). Let $J^r_0(f)$ denote the transverse r-frame determined by f and let $P^r(M,\mathcal{F})$ be the set of transverse r-frames on M. Then $\pi^r : P^r(M,\mathcal{F}) \to M$, $\pi^r(J^r_0(f)) = x$, is a principal bundle over M with group G^r_q where G^r_q is the group of r-frames at $0 \in \mathbb{R}^q$. The right action of G^r_q on $P^r(M,\mathcal{F})$ is given by $J^r_0(f)j^r_0(g) = J^r_0(f \circ g)$, for $J^r_0(f) \in P^r(M,\mathcal{F})$, $j^2_0(g) \in G^r_q$.

One can view the frame $J^r_0(f)$ as the following set of q foliated vectors :

$$(\sum_{k=1}^{q} \partial_1(y \circ f)^k[\partial_{k+p}], \ldots, \sum_{k=1}^{q} \partial_q(y \circ f)^k[\partial_{k+p}]).$$

The foliation $\mathcal{F}$ induces a foliation $\mathcal{F}_{P^rN}$ on $P^r(M,\mathcal{F})$ whose leaves are locally defined by the sets of frames $J^r_0(f)$ such that $P^rN(y)(J^r_0(f))$ is constant, with

$$P^rN(y) : J^r_0 f \mapsto j^r_0(y \circ f).$$

3.3 Adapted and foliated connections

3.3.1 Adapted connections

Definition 1. *Let $(M,\mathcal{F})$ be a foliated manifold. An adapted connection $\nabla_{\mathcal{F}}$ is a linear torsion-free connection on M, such that $\nabla_{\mathcal{F}} : \mathrm{Vect}_{\mathcal{F}}(M) \times \Gamma(T\mathcal{F}) \to \Gamma(T\mathcal{F})$ and $\nabla_{\mathcal{F}} : \mathrm{Vect}_{\mathcal{F}}(M) \times \mathrm{Vect}_{\mathcal{F}}(M) \to \mathrm{Vect}_{\mathcal{F}}(M)$.*

Remark In the following, we use the Einstein summation convention, and, as already adumbrated above, Latin indices $i, k, l \ldots$ (resp. Greek indices $\iota, \kappa, \lambda \ldots$, German indices $\mathfrak{i}, \mathfrak{k}, \mathfrak{l} \ldots$) are systematically and implicitly assumed to vary in $\{1,\ldots,n\}$ (resp. $\{1,\ldots,p\}$, $\{1,\ldots,q\}$).

As torsionlessness means that $\nabla_{\mathcal{F},Y}X = \nabla_{\mathcal{F},X}Y + [Y,X]$, it follows that $\nabla_{\mathcal{F}} : \Gamma(T\mathcal{F}) \times \mathrm{Vect}_{\mathcal{F}}(M) \to \Gamma(T\mathcal{F})$.

Further, locally, in adapted coordinates, we have $\nabla_{\mathcal{F},X}Y = \left(X^i\partial_i Y^k + \Gamma^k_{il}X^iY^l\right)\partial_k$, so that condition $\nabla_{\mathcal{F}} : \mathrm{Vect}_{\mathcal{F}}(M) \times \Gamma(T\mathcal{F}) \to \Gamma(T\mathcal{F})$ means that

$$\Gamma^{\mathfrak{k}}_{i\lambda} = \Gamma^{\mathfrak{k}}_{\lambda i} = 0, \tag{5}$$

whereas condition $\nabla_{\mathcal{F}} : \mathrm{Vect}_{\mathcal{F}}(M) \times \mathrm{Vect}_{\mathcal{F}}(M) \to \mathrm{Vect}_{\mathcal{F}}(M)$ is then automatically verified provided that Christoffel's symbols $\Gamma^{\mathfrak{k}}_{\mathfrak{il}}$ are independent of x, $\Gamma^{\mathfrak{k}}_{\mathfrak{il}} = \Gamma^{\mathfrak{k}}_{\mathfrak{il}}(y)$.

Proposition 1. *If two adapted connections $\nabla_{\mathcal{F}}$ and $\nabla'_{\mathcal{F}}$ of a foliated manifold $(M,\mathcal{F})$ are projectively equivalent, the corresponding differential 1-form $\alpha \in \Omega^1(M)$ is foliated, i.e. $\alpha \in \Omega^1(M,\mathcal{F})$.*

Proof. In adapted local coordinates (x,y), projective equivalence of $\nabla_{\mathcal{F}}$ and $\nabla'_{\mathcal{F}}$ reads $(\Gamma'^k_{il} - \Gamma^k_{il})X^iY^l = \alpha_i X^i Y^k + \alpha_i Y^i X^k, \forall k$. When writing this equation for $X^i = \delta^i_\iota$, $Y^l = \delta^l_{\mathfrak{l}}$, and $k = \mathfrak{l}$, we get, in view of Equation (5), $\alpha_\iota = 0$. If we now choose $X^i = \delta^i_{\mathfrak{i}}$, $Y^l = \delta^l_{\mathfrak{l}}$, and $k = \mathfrak{i} \neq \mathfrak{l}$, we finally see that $\alpha_{\mathfrak{l}}$ is independent of x. □

3.3.2 Foliated connections

Definition 2. *Consider a foliated manifold $(M,\mathcal{F})$. A foliated torsion-free connection $\nabla(\mathcal{F})$ on $(M,\mathcal{F})$ is a bilinear map $\nabla(\mathcal{F}) : \mathrm{Vect}(M,\mathcal{F}) \times \mathrm{Vect}(M,\mathcal{F}) \to \mathrm{Vect}(M,\mathcal{F})$, such that, for all $f \in C^\infty(M,\mathcal{F})$ and all $[X],[Y] \in \mathrm{Vect}(M,\mathcal{F})$, the following conditions hold true:*

- $\nabla(\mathcal{F})_{f[X]}[Y] = f\nabla(\mathcal{F})_{[X]}[Y]$,
- $\nabla(\mathcal{F})_{[X]}(f[Y]) = \left(L_{[X]}f\right)[Y] + f\nabla(\mathcal{F})_{[X]}[Y]$,
- $\nabla(\mathcal{F})_{[X]}[Y] = \nabla(\mathcal{F})_{[Y]}[X] + [[X],[Y]]$.

In view of the above definitions, the local form (in adapted coordinates (x,y)) of a foliated vector field is $[X] = X^{\mathfrak{i}}[\partial_{\mathfrak{i}}]$, $X^{\mathfrak{i}} = X^{\mathfrak{i}}(y)$, and a foliated connection reads

$$\nabla(\mathcal{F})_{[X]}[Y] = X^{\mathfrak{i}}\left(L_{[\partial_{\mathfrak{i}}]}Y^{\mathfrak{l}}\right)[\partial_{\mathfrak{l}}] + X^{\mathfrak{i}}Y^{\mathfrak{l}}\,\Gamma(\mathcal{F})^{\mathfrak{k}}_{\mathfrak{i}\mathfrak{l}}\,[\partial_{\mathfrak{k}}], \quad \Gamma(\mathcal{F})^{\mathfrak{k}}_{\mathfrak{i}\mathfrak{l}} = \Gamma(\mathcal{F})^{\mathfrak{k}}_{\mathfrak{i}\mathfrak{l}}(y).$$

Definition 3. *Two foliated connections $\nabla(\mathcal{F})$ and $\nabla'(\mathcal{F})$ of a foliated manifold $(M,\mathcal{F})$ are projectively equivalent, if and only if there is a foliated 1-form $\alpha \in \Omega^1(M,\mathcal{F})$, such that, for all $[X],[Y] \in \mathrm{Vect}(M,\mathcal{F})$, one has $\nabla'(\mathcal{F})_{[X]}[Y] - \nabla(\mathcal{F})_{[X]}[Y] = \alpha([X])[Y] + \alpha([Y])[X]$.*

3.3.3 Link between adapted and foliated connections

Eventually, adapted connections induce foliated connections.

Proposition 2. *Let $(M,\mathcal{F})$ be a foliated manifold of codimension q. Any adapted connection $\nabla_{\mathcal{F}}$ of M induces a foliated connection $\nabla(\mathcal{F})$, defined by $\nabla(\mathcal{F})_{[X]}[Y] := [\nabla_{\mathcal{F},X}Y]$. In adapted coordinates, Christoffel's symbols $\Gamma(\mathcal{F})^{\mathfrak{i}}_{\mathfrak{k}\mathfrak{l}}$ of $\nabla(\mathcal{F})$ coincide with the corresponding Christoffel symbols $\Gamma^{\mathfrak{i}}_{\mathcal{F},\mathfrak{k}\mathfrak{l}}$ of $\nabla_{\mathcal{F}}$. Eventually, projective classes of adapted connections induce projective classes of foliated connections.*

Proof. It immediately follows from the definition of adapted connections that for any $[X],[Y] \in \mathrm{Vect}(M,\mathcal{F})$, the class $\nabla(\mathcal{F})_{[X]}[Y] := [\nabla_{\mathcal{F},X}Y] \in \mathrm{Vect}(M,\mathcal{F})$ is well-defined. All properties of foliated connections are obviously satisfied. If (x,y) are adapted coordinates, we have $\Gamma(\mathcal{F})^{\mathfrak{k}}_{\mathfrak{i}\mathfrak{l}}[\partial_{\mathfrak{k}}] = \nabla(\mathcal{F})_{[\partial_{\mathfrak{i}}]}[\partial_{\mathfrak{l}}] = [\nabla_{\mathcal{F},\partial_{\mathfrak{i}}}\partial_{\mathfrak{l}}] = [\Gamma^{\mathfrak{k}}_{\mathcal{F},\mathfrak{i}\mathfrak{l}}\partial_{\mathfrak{k}}] = \Gamma^{\mathfrak{k}}_{\mathcal{F},\mathfrak{i}\mathfrak{l}}[\partial_{\mathfrak{k}}]$, since $\Gamma^{\mathfrak{k}}_{\mathcal{F},\mathfrak{i}\mathfrak{l}} = \Gamma^{\mathfrak{k}}_{\mathcal{F},\mathfrak{i}\mathfrak{l}}(y)$. The remark on projective structures follows immediately from preceding observations. □

3.4 Adapted and foliated differential operators

3.4.1 Adapted differential operators

Definition 4. *An adapted differential operator of a foliated manifold $(M,\mathcal{F})$ (where $\mathcal{F}$ is of dimension p and codimension q) is an endomorphism $D \in \mathrm{End}_{\mathbb{R}}(C^\infty(M))$ that reads in any system of adapted coordinates $(x,y) = (x^1,\ldots,x^p,y^1,\ldots,y^q)$ over any open subset $U \subset M$,*

$$D|_U = \sum_{|\gamma|\le k} D_\gamma\, \partial^{\gamma^1}_{x^1} \ldots \partial^{\gamma^p}_{x^p} \partial^{\gamma^{p+1}}_{y^1} \ldots \partial^{\gamma^{p+q}}_{y^q},$$

where $k \in \mathbb{N}$ is independent of the considered adapted chart, where $D_\gamma \in C^\infty(U)$, and where the coefficients D_γ with $\gamma^1 = \ldots = \gamma^p = 0$ are locally defined foliated functions. The smallest possible integer k is called the order *of operator D.*

We denote by $\mathcal{D}_{\mathcal{F}}(M)$ the filtered space of all adapted differential operators on $(M,\mathcal{F})$.

3.4.2 Foliated differential operators

Definition 5. *A foliated differential operator of a foliated manifold* $(M, \mathcal{F})$ *(where* $\mathcal{F}$ *is of dimension* p *and codimension* q*) is an endomorphism* $D \in \mathrm{End}_{\mathbb{R}}(C^\infty(M, \mathcal{F}))$ *that reads in any system of adapted coordinates* $(x, y) = (x^1, \ldots, x^p, y^1, \ldots, y^q)$ *over any open subset* $U \subset M$*,*

$$D|_U = \sum_{|\gamma| \leq k} D_\gamma \, \partial_{y^1}^{\gamma^1} \ldots \partial_{y^q}^{\gamma^q},$$

where $k \in \mathbb{N}$ *is independent of the considered adapted chart and where the coefficients* $D_\gamma \in C^\infty(U, \mathcal{F})$ *are locally defined foliated functions. The smallest possible integer* k *is called the* order *of operator* D*.*

We denote by $\mathcal{D}(M, \mathcal{F})$ (resp. $\mathcal{D}^k(M, \mathcal{F})$) the space of all foliated differential operators (resp. all foliated differential operators of order $\leq k$). Of course, the usual filtration

$$\mathcal{D}(M, \mathcal{F}) = \cup_{k \in \mathbb{N}} \mathcal{D}^k(M, \mathcal{F}) \tag{6}$$

holds true.

3.4.3 Link between adapted and foliated differential operators

The space of adapted differential operators projects onto the space of foliated differential operators in the following way :

$$\sum_{|\gamma| \leq k} D_\gamma \, \partial_{x^1}^{\gamma^1} \ldots \partial_{x^p}^{\gamma^p} \partial_{y^1}^{\gamma^{p+1}} \ldots \partial_{y^q}^{\gamma^{p+q}} \mapsto \sum_{|\gamma| \leq k, \gamma^1 = \ldots = \gamma^p = 0} D_\gamma \, \partial_{y^1}^{\gamma^1} \ldots \partial_{y^q}^{\gamma^q}.$$

3.5 Adapted and foliated symbols

3.5.1 Adapted symbols

Definition 6. *The graded space* $\mathcal{S}_{\mathcal{F}}(M)$ *associated to* $\mathcal{D}_{\mathcal{F}}(M)$ *is the space of* adapted symbols *on* $(M, \mathcal{F})$*.*

One can view the symbol $[\sum_{|\gamma| \leq k} D_\gamma \, \partial_{x^1}^{\gamma^1} \ldots \partial_{x^p}^{\gamma^p} \partial_{y^1}^{\gamma^{p+1}} \ldots \partial_{y^q}^{\gamma^{p+q}}]$ as the symmetric contravariant tensor fields $\sum_{|\gamma| = k} D_\gamma \, \partial_{x^1}^{\gamma^1} \vee \ldots \vee \partial_{x^p}^{\gamma^p} \vee \partial_{y^1}^{\gamma^{p+1}} \ldots \vee \partial_{y^q}^{\gamma^{p+q}}$.

Theorem 1. *Let* $(M, \mathcal{F})$ *be a foliated manifold of codimension* q*. We then have a canonical vector space isomorphism:*

$$\sim : \mathcal{S}^k_{\mathcal{F}}(M) \ni s \mapsto \tilde{s} \in C^\infty_{\mathcal{F}}(L_{\mathcal{F}} M, S^k \mathbb{R}^n)_{\mathrm{GL}(n,q,\mathbb{R})}, \tag{7}$$

where the notation $C^\infty_{\mathcal{F}}$ *means that* $p_{n,q}\tilde{s}$ *is foliated for* $\mathcal{F}_{\underline{P}^1}$*, with* $p_{n,q}$ *denoting the canonical projection from* $S^k \mathbb{R}^n$ *to* $S^k \mathbb{R}^q$*.*

Proof. One associates to $s = \sum_{|\gamma| = k} D_\gamma \, \partial_{x^1}^{\gamma^1} \vee \ldots \vee \partial_{x^p}^{\gamma^p} \vee \partial_{y^1}^{\gamma^{p+1}} \vee \ldots \vee \partial_{y^q}^{\gamma^{p+q}}$ the function $\tilde{s}$ that associates $\sum_{|\gamma| = k} D_\gamma \, (A^{-1} e_1)^{\gamma^1} \vee \ldots \vee (A^{-1} e_p)^{\gamma^p} \vee (A^{-1} e_{p+1})^{\gamma^{p+1}} \vee \ldots \vee (A^{-1} e_{p+q})^{\gamma^{p+q}}$ to $j^1_0(f)$, where A denotes the representation of $j^1_0 f$ in the adapted coordinates system. Thanks to the fact that $A \in GL(n, q, \mathbb{R})$, one sees easily that D_γ is foliated for $\gamma^1 = \ldots = \gamma^p = 0$ if and only if $p_{n,q}\tilde{s}$ is foliated for $\mathcal{F}_{\underline{P}^1}$. □

3.5.2 Foliated symbols

Definition 7. *The graded space* $\mathcal{S}(M, \mathcal{F})$ *associated with the filtered space* $\mathcal{D}(M, \mathcal{F})$*,*

$$\mathcal{S}(M, \mathcal{F}) = \oplus_{k \in \mathbb{N}} \mathcal{S}^k(M, \mathcal{F}) = \oplus_{k \in \mathbb{N}} \mathcal{D}^k(M, \mathcal{F}) / \mathcal{D}^{k-1}(M, \mathcal{F}),$$

is the space of foliated symbols. The principal symbol of a foliated differential operator D *is then simply its class* $[D]$ *in the associated graded space.*

Theorem 2. *Let $(M, \mathcal{F})$ be a foliated manifold of codimension q. We then have a canonical vector space isomorphism:*

$$\sim : \mathcal{S}^k(M, \mathcal{F}) \ni s \mapsto \tilde{s} \in C^\infty(LN(M, \mathcal{F}), S^k\mathbb{R}^q; \mathcal{F}_{LN})_{\mathrm{GL}(q,\mathbb{R})}. \tag{8}$$

Proof. One associates to $s = [\sum_{|\gamma|\leq k} D_\gamma\, \partial_{y^1}^{\gamma^1} \dots \partial_{y^q}^{\gamma^q}]$ the function $\tilde{s}$ that associates $\sum_{|\gamma|=k} D_\gamma\, (A^{-1}e_1)^{\gamma^1} \vee \dots \vee (A^{-1}e_q)^{\gamma^q}$ to $J_0^1(f)$, where A denotes the representation of $J_0^1 f$ in the adapted coordinates system. It is then obvious that s is foliated if and only if $\tilde{s}$ is foliated for $\mathcal{F}_{LN}$. □

3.5.3 Link between adapted and foliated symbols

First define the following canonical projection :

$$^{\mathcal{F}}\!p^r : P^r_{\mathcal{F}}M \to P^r(M, \mathcal{F}) : j_0^r f \mapsto J_0^r(f \circ i_q).$$

Projection of adapted differential operators onto foliated differential operators induce a projection of adapted symbols onto foliated symbols that we will denote by $^{\mathcal{F}}\pi$. In fact, we have the

Proposition 3. *If $^{\mathcal{F}}\tilde{\pi}$ denotes projection $^{\mathcal{F}}\pi$ read through the isomorphism $\sim$ detailed in Theorems 2 and 1, we have, for any symbol $s \in \mathcal{S}^k_{\mathcal{F}}(M)$, $k \in \mathbb{N}$,*

$$\left(^{\mathcal{F}}\tilde{\pi}\tilde{s}\right) \circ {}^{\mathcal{F}}\!p^1 = p_{n,q} \circ \tilde{s}. \tag{9}$$

Proof. It is quite obvious thanks to the description of $^{\mathcal{F}}\pi$ and to the description of the equivariant functions given in theorems 1 and 2. □

3.6 Adapted and foliated quantizations

The definitions of adapted and foliated natural projectively equivariant quantizations that we will denote respectively by $Q_{\mathcal{F}}$ and $Q(\mathcal{F})$ are obtained simply by replacing the standard objects in the definition of the natural projectively equivariant quantization by adapted or foliated objects. Simply note that $Q_{\mathcal{F}}$ and $Q(\mathcal{F})$ have to commute with foliated morphisms as they are defined in [Wol89] page 340. Recall that ϕ is a foliated morphism between two foliated manifolds $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ if $\phi_* T\mathcal{F}_1 \subset T\mathcal{F}_2$ and $\phi^*\mathcal{F}_2 = \mathcal{F}_1$.

The aim is to show that this two quantizations "commute with the reduction". In other words :

$$Q_{\mathcal{F}}(\nabla_{\mathcal{F}})(s)(f) = Q(\mathcal{F})(\nabla(\mathcal{F}))(^{\mathcal{F}}\pi s)(f)$$

if $\nabla_{\mathcal{F}}$, s and f are respectively adapted connection, symbol and foliated function.

4 Construction of the Cartan fiber bundle

4.1 Construction in the adapted case

The isotropy subgroup of $[e_{n+1}]$ in the projective space $\mathbb{R}P^n$ for the natural action of

$$\mathrm{PGL}(n+1, q+1, \mathbb{R}) = \left\{ \begin{pmatrix} A & B & h' \\ 0 & D & h'' \\ 0 & \alpha'' & a \end{pmatrix} : A \in \mathrm{GL}(p, \mathbb{R}), B \in \mathrm{gl}(p \times q, \mathbb{R}), \right. \tag{10}$$
$$\left. h' \in \mathbb{R}^p, D \in \mathrm{GL}(q, \mathbb{R}), h'' \in \mathbb{R}^q, \alpha'' \in \mathbb{R}^{q*}, a \in \mathbb{R}_0 \right\} / \mathbb{R}_0 \,\mathrm{id}$$

on $\mathbb{R}P^n$ is

$$H(n+1, q+1, \mathbb{R}) = \left\{ \begin{pmatrix} A & B & 0 \\ 0 & D & 0 \\ 0 & \alpha'' & a \end{pmatrix} : A \in \mathrm{GL}(p, \mathbb{R}), B \in \mathrm{gl}(p \times q, \mathbb{R}), \right. \tag{11}$$
$$\left. D \in \mathrm{GL}(q, \mathbb{R}), \alpha'' \in \mathbb{R}^{q*}, a \in \mathbb{R}_0 \right\} / \mathbb{R}_0 \,\mathrm{id}.$$

The group $H(n+1,q+1,\mathbb{R})$ acts on $\mathbb{R}^n$ by linear fractional transformations that leave the origin fixed. This allows to view $H(n+1,q+1,\mathbb{R})$ as a subgroup of G^2_n (see [Koba72]).

Theorem 3. *For any foliated manifold $(M,\mathcal{F})$, there exists a natural application from the set of projective classes of adapted connections $[\nabla_{\mathcal{F}}]$ into the set of reductions $P_{\mathcal{F}}$ of the principal bundle $P^2_{\mathcal{F}}M$ of $\mathcal{F}$-adapted second order frames on M to structure group $H(n+1,q+1,\mathbb{R})$.*

Proof. The proof is exactly similar to that of Proposition 7.2 p.147 in [Koba72]. The reduction associated to the class of $[\nabla_{\mathcal{F}}]$ is equal over a point x to $j^2_0 f H(n+1,q+1,\mathbb{R})$, where $j^2_0 f$ is equal in an adapted coordinates system to

$$(x^i, \delta^i_k, -\Gamma^i_{kl}),$$

where the Γ^i_{kl} are Christoffel's symbols of $[\nabla_{\mathcal{F}}]$. □

4.2 Construction in the foliated case

Theorem 4. *For any foliated manifold $(M,\mathcal{F})$ of codimension q, there exists a natural application from the set of projective classes of foliated connections $[\nabla(\mathcal{F})]$ into the set of reductions $P(\mathcal{F})$ of $P^2(M,\mathcal{F})$ to structure group $H(q+1,\mathbb{R}) \subset G^2_q$.*

Proof. The proof is exactly similar to that of Theorem 3. The reduction associated to the class of $[\nabla(\mathcal{F})]$ is equal over a point x to $J^2_0 f H(q+1,\mathbb{R})$, where $J^2_0 f$ is equal in an adapted coordinates system to

$$(x^i, \delta^{\mathfrak{i}}_{\mathfrak{k}}, -\Gamma^{\mathfrak{i}}_{\mathfrak{k}\mathfrak{l}}),$$

where the $\Gamma^{\mathfrak{i}}_{\mathfrak{k}\mathfrak{l}}$ are Christoffel's symbols of $\nabla(\mathcal{F})$. □

4.3 Link between the adapted and the foliated Cartan fiber bundle

Proposition 4. *For any foliated manifold $(M,\mathcal{F})$ endowed with an adapted projective structure and the induced foliated projective structure, projection ${}^{\mathcal{F}}p^2$ restricts to a projection ${}^{\mathcal{F}}\mathfrak{p}^2 : P_{\mathcal{F}} \to P(\mathcal{F})$.*

Proof. It is easily seen thanks to proposition 2 and thanks to descriptions of $P_{\mathcal{F}}$ and $P(\mathcal{F})$ given in theorems 4 and 3. □

5 Lift of symbols

We denote by $p^r_{\mathcal{F}}$ (resp. $p^r(\mathcal{F})$), $r \geq 1$, the canonical projection

$$p^r_{\mathcal{F}} : P^r_{\mathcal{F}}M \to P^{r-1}_{\mathcal{F}}M : j^r_0(f) \mapsto j^{r-1}_0(f)\,(resp. p^r(\mathcal{F}) : P^r(M,\mathcal{F}) \to P^{r-1}(M,\mathcal{F}) : J^r_0(f) \mapsto J^{r-1}_0(f)).$$

5.1 Lift of adapted symbols

Theorem 5. *Let $(M,\mathcal{F})$ be a foliated manifold of codimension q endowed with an adapted projective structure $[\nabla_{\mathcal{F}}]$, and denote by $P_{\mathcal{F}}$ the corresponding reduction of $P^2_{\mathcal{F}}M$ to $H(n+1,q+1,\mathbb{R})$. We then have the following canonical vector space isomorphism:*

$$\wedge : \mathcal{S}^k_{\mathcal{F}}(M) \ni s \mapsto \hat{s} = \tilde{s} \circ p^2_{\mathcal{F}} \in C^\infty_{\mathcal{F}}(P_{\mathcal{F}}, S^k\mathbb{R}^n)_{H(n+1,q+1,\mathbb{R})}, \tag{12}$$

where the notation $C^\infty_{\mathcal{F}}$ means that $p_{n,q}\hat{s}$ is foliated for $\mathcal{F}_{\underline{P}^2}$.

Proof. Using Theorem 1, the proof is exactly similar to the proof of proposition 3 in [MR05]. □

5.2 Lift of foliated symbols

Theorem 6. *Let $(M,\mathcal{F})$ be a foliated manifold of codimension q endowed with a foliated projective structure $[\nabla(\mathcal{F})]$, and denote by $P(\mathcal{F})$ the corresponding reduction of $P^2(M,\mathcal{F})$ to $H(q+1,\mathbb{R}) \subset G^2_q$. The following canonical vector space isomorphism holds :*

$$\wedge : \mathcal{S}^k(M,\mathcal{F}) \ni s \mapsto \hat{s} = \tilde{s} \circ p^2(\mathcal{F}) \in C^\infty(P(\mathcal{F}), S^k\mathbb{R}^q; \mathcal{F}_{P^2N})_{H(q+1,\mathbb{R})}. \tag{13}$$

Proof. Using Theorem 2, the proof is exactly similar too to the proof of proposition 3 in [MR05]. □

5.3 Link between the lifts

Proposition 5. *If ${}^{\mathcal{F}}\hat{\pi}$ denotes projection ${}^{\mathcal{F}}\pi$ read through the isomorphism $\wedge$ detailed in Theorems 6 and 5, we have, for any symbol $s \in \mathcal{S}^k_{\mathcal{F}}(M)$, $k \in \mathbb{N}$,*

$$\left({}^{\mathcal{F}}\hat{\pi}\hat{s}\right) \circ {}^{\mathcal{F}}\mathfrak{p}^2 = p_{n,q} \circ \hat{s}. \tag{14}$$

Proof. It is obvious thanks to (9) and thanks to the fact that $p^2(\mathcal{F}) \circ {}^{\mathcal{F}}\mathfrak{p}^2 = {}^{\mathcal{F}}\mathfrak{p}^1 \circ p^2_{\mathcal{F}}$. □

6 Construction of the normal Cartan connection

The method exposed in [MR05] in order to solve the problem of the natural and projectively equivariant quantization uses the notion of normal Cartan connection. We are going to adapt this object firstly to the adapted situation and secondly to the foliated situation. Finally, in a third step, we are going to analyze the link between the adapted normal Cartan connection and the foliated one.

6.1 Construction in the adapted case

First, recall the notion of Cartan connection on a principal fiber bundle :

Definition 8. *Let G be a Lie group and H a closed subgroup. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras. Let $P \to M$ be a principal H-bundle over M, such that* $\dim M = \dim G/H$. *A Cartan connection on P is a $\mathfrak{g}$-valued one-form ω on P such that*

- *If R_a denotes the right action of $a \in H$ on P, then $R_a^*\omega = \mathrm{Ad}(a^{-1})\omega$,*
- *If k^* is the vertical vector field associated to $k \in \mathfrak{h}$, then $\omega(k^*) = k$,*
- *$\forall u \in P, \omega_u : T_uP \to \mathfrak{g}$ is a linear bijection.*

Recall too the definition of the curvature of a Cartan connection :

Definition 9. *If ω is a Cartan connection defined on a H-principal bundle P, then its curvature Ω is defined as usual by*

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]. \tag{15}$$

Next, one adapts Theorem 4.2. cited in [Koba72] p.135 in the following way :

Theorem 7. *Let $P_{\mathcal{F}}$ be an $H(n+1, q+1, \mathbb{R})$-principal fiber bundle on a manifold M. If one has a one-form ω_{-1} with values in $\mathbb{R}^n$ of components ω^i and a one-form ω_0 with values in $\mathrm{gl}(n,q,\mathbb{R})$ (the Lie algebra of $\mathrm{GL}(n,q,\mathbb{R})$) of components ω^i_j that satisfy the three following conditions :*

- *$\omega_{-1}(h^*) = 0, \quad \omega_0(h^*) = h_0, \quad \forall h \in \mathrm{gl}(n,q,\mathbb{R}) + \mathbb{R}^{q*}$, where h_0 is the projection with respect to $\mathrm{gl}(n,q,\mathbb{R})$ of h,*
- *$(R_a)^*(\omega_{-1} + \omega_0) = (\mathrm{Ad}\ a^{-1})(\omega_{-1} + \omega_0), \quad \forall a \in H(n+1, q+1, \mathbb{R})$, where $\mathrm{Ad}\ a^{-1}$ is the application from $\mathbb{R}^n + \mathrm{gl}(n,q,\mathbb{R}) + \mathbb{R}^{q*}/\mathbb{R}^{q*}$ in itself induced by the adjoint action $\mathrm{Ad}\ a^{-1}$ from $\mathbb{R}^n + \mathrm{gl}(n,q,\mathbb{R}) + \mathbb{R}^{q*}$ into $\mathbb{R}^n + \mathrm{gl}(n,q,\mathbb{R}) + \mathbb{R}^{q*}$,*
- *If $\omega_{-1}(X) = 0$, then X is vertical,*

and the following additional condition :

$$d\omega^i = -\sum \omega^i_k \wedge \omega^k, \tag{16}$$

then there is a unique Cartan connection $\omega = \omega_{-1} + \omega_0 + \omega_1$ whose curvature Ω of components $(0; \Omega^i_j; \Omega_j)$ satisfies the following property :

$$\sum_{i=p+1}^{n} K^i_{jil} = 0, \quad \forall j \in \{p+1, \ldots, n\}, \forall l,$$

where

$$\Omega^i_j = \sum \frac{1}{2} K^i_{jkl}\, \omega^k \wedge \omega^l.$$

Proof. The proof goes as in [Koba72]. Let $\omega = (\omega^i; \omega^i_j; \omega_j)$ and $\overline{\omega} = (\omega^i; \omega^i_j; \overline{\omega}_j)$ be two Cartan connections with the given $(\omega^i; \omega^i_j)$. If $\overline{\omega}_j - \omega_j = \sum A_{jk}\omega^k$, where the coefficients A_{jk} are functions on P, if $\overline{\Omega} = (0; \overline{\Omega}^i_j; \overline{\Omega}_j)$ denotes the curvature of $\overline{\omega}$ and if $\overline{\Omega}^i_j = \sum \frac{1}{2}\overline{K}^i_{jkl}\, \omega^k \wedge \omega^l$, one can prove in the same way as in [Koba72] that

$$\sum_{i=p+1}^{n} (\overline{K}^i_{ikl} - K^i_{ikl}) = (q+1)(A_{kl} - A_{lk}), \tag{17}$$

$$\sum_{i=p+1}^{n} (\overline{K}^i_{jil} - K^i_{jil}) = (q-1)A_{jl} + (A_{jl} - A_{lj}). \tag{18}$$

If ω and $\overline{\omega}$ are normal Cartan connections, i.e., $\sum_{i=p+1}^n \overline{K}^i_{jil} = \sum_{i=p+1}^n K^i_{jil} = 0$, then $A_{ij} = 0$ and hence $\omega = \overline{\omega}$. This prove the uniqueness of the normal Cartan connection.

To prove the existence, one assumes that there is a Cartan connection $\omega = (\omega^i; \omega^i_j; \omega_j)$ with the given $(\omega^i; \omega^i_j)$. The goal is then to find functions A_{jk} such that $\overline{\omega} = (\omega^i; \omega^i_j; \overline{\omega}_j)$ becomes a normal Cartan connection. If $1 \leq j \leq p$, A_{jk} is of course equal to zero. If $p+1 \leq j \leq n$ and if $p+1 \leq k \leq n$, one can view thanks to (17) and (18) that it suffices to set

$$A_{jk} = \frac{1}{(q+1)(q-1)} \sum_{i=p+1}^{n} K^i_{ijk} - \frac{1}{q-1} \sum_{i=p+1}^{n} K^i_{jik}. \tag{19}$$

If $1 \leq k \leq p$, one sees thanks to (18) that it suffices to set

$$A_{jk} = -\sum_{i=p+1}^{n} \frac{1}{q} K^i_{jik}. \tag{20}$$

The last step of the proof that shows the existence of one Cartan connection ω that "begins" by (ω^i, ω^i_j) is exactly similar to the corresponding step in [Koba72]. □

One can remark that the codimension of the foliation $\mathcal{F}$ has to be different from 1.

One can define on $P_{\mathcal{F}}$ an one-form in the following way :

Definition 10. *If $u = j^2_0 f$ is a point belonging to $P_{\mathcal{F}}$ and if X is a tangent vector to $P_{\mathcal{F}}$ at u, the canonical form $\theta_{\mathcal{F}}$ of $P_{\mathcal{F}}$ is the 1-form with values in $\mathbb{R}^n \oplus \mathrm{gl}(n, q, \mathbb{R})$ defined at the point u in the following way :*

$$\theta_{\mathcal{F};u}(X) = (P^1 f)^{-1}_{*e}(p^2_{\mathcal{F}*} X),$$

where e is the frame at the origin of $\mathbb{R}^n$ represented by the identity matrix.

Theorem 8. *One can associate to the projective class of an adapted connection $[\nabla_{\mathcal{F}}]$ a Cartan connection on $P_{\mathcal{F}}$ in a natural way. We will denote by $\omega_{\mathcal{F}}$ this Cartan connection.*

Proof. The canonical one-form defined above is the restriction to $P_{\mathcal{F}}$ of the canonical one-form of $P^2(M)$ defined in [Koba72] p.140. It is too the restriction to $P_{\mathcal{F}}$ of the restriction to P of the canonical one-form of $P^2(M)$, where P is the projective structure associated to $\nabla_{\mathcal{F}}$ defined in [Koba72]. Thanks to the fact that the canonical one-form on P satisfies the properties of Theorem 4.2. mentioned in [Koba72], $\theta_{\mathcal{F}}$ satisfies the properties mentioned in Theorem 7. One defines then the adapted normal Cartan connection $\omega_{\mathcal{F}}$ as the unique Cartan connection on $P_{\mathcal{F}}$ beginning by $\theta_{\mathcal{F}}$ and satisfying the property linked to the curvature cited in Theorem 7. Because of the naturality of this property, the naturality of $\theta_{\mathcal{F}}$ and the uniqueness of the Cartan connection mentioned in Theorem 7, $\omega_{\mathcal{F}}$ is a Cartan connection on $P_{\mathcal{F}}$ associated naturally to the class $[\nabla_{\mathcal{F}}]$. □

6.2 Construction in the foliated case

The reduction $P(\mathcal{F})$ is actually an example of a foliated bundle defined in [Blum84]. The Cartan connection that we are going to define on it is an example of a Cartan connection in a foliated bundle defined too in [Blum84]. It is the reason for which we are going first to recall the definitions of these notions.

Definition 11. *Let M be a manifold of dimension n and $\mathcal{F}$ a codimension q foliation of M. Let H be a Lie group and $\pi : P \to M$ be a principal H-bundle. We say that $\pi : P \to M$ is a foliated bundle if there is a foliation $\tilde{\mathcal{F}}$ of P satisfying*

- *$\tilde{\mathcal{F}}$ is H-invariant,*
- *$\tilde{E}_u \cap V_u = \{0\}$ for all $u \in P$,*
- *$\pi_{*u}(\tilde{E}_u) = T\mathcal{F}_{\pi(u)}$ for all $u \in P$,*

where $\tilde{E}$ is the tangent bundle of $\tilde{\mathcal{F}}$ and V is the bundle of vertical vectors.

Definition 12. *Let $\mathcal{F}$ be a codimension q foliation of M. Let G be a Lie group and let H be a closed subgroup of G with dimension$(G/H) = q$. Let $\pi : P \to M$ be a foliated principal H-bundle. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{h}$ be the Lie algebra of H. For each $A \in \mathfrak{h}$, let A^* be the corresponding fundamental vector field on P.*

A Cartan connection in the foliated bundle $\pi : P \to M$ is a $\mathfrak{g}$-valued one-form ω on P satisfying

- *$\omega(A^*) = A$ for all $A \in \mathfrak{h}$,*
- *$(R_a)^*\omega = \mathrm{Ad}(a^{-1})\omega$ for all $a \in H$ where R_a denotes the right translation by a acting on P and $\mathrm{Ad}(a^{-1})$ is the adjoint action of a^{-1} on $\mathfrak{g}$,*
- *For each $u \in P$, $\omega_u : T_uP \to \mathfrak{g}$ is onto and $\omega_u(\tilde{E}_u) = 0$,*
- *$L_X\omega = 0$ for all $X \in \Gamma(\tilde{E})$ where $\Gamma(\tilde{E})$ denotes the smooth sections of $\tilde{E}$.*

Theorem 9. *The reduction $P(\mathcal{F})$ is a foliated bundle.*

Proof. One can easily view that $\mathcal{F}_{P^2N}$ satisfies the properties of the definition of a foliated bundle : first, $\mathcal{F}_{P^2N}$ is $H(q+1,\mathbb{R})$-invariant because, if (U_i, f_i, g_{ij}) is a cocycle corresponding to the foliation $\mathcal{F}$, if $J_0^2 f \in P(\mathcal{F})$ and if $j_0^2(f_i \circ f)$ is constant, then $j_0^2(f_i \circ f \circ h) = j_0^2(f_i \circ f) \circ j_0^2(h)$ is constant, where $j_0^2(h) \in H(q+1,\mathbb{R})$.

If $X \in V_u$, then $X = \frac{d}{dt}u\exp(th)|_{t=0}$, where $h \in \mathfrak{h}(q+1,\mathbb{R})$. If $u = J_0^2(f)$ and if $\exp(th) = j_0^2(g_t)$, then $j_0^2(f_i \circ f \circ g_t)$ is constant if X is tangent to the foliation $\mathcal{F}_{P^2N}$. One has then that $J_0^2(f \circ g_t)$ is constant and then $X = 0$.

If X is tangent to the foliation $\mathcal{F}_{P^2N}$, then $X = \frac{d}{dt}\gamma(t)|_{t=0}$, where $\gamma(t) \in \mathcal{F}_{P^2N}$. Then $\pi^2_{*u}(X) = \frac{d}{dt}\pi^2(\gamma(t))|_{t=0}$, that belongs to $T\mathcal{F}_{\pi(u)}$ because $\pi^2(\gamma(t))$ belongs to $\mathcal{F}$. Indeed, if $\gamma(t) = J_0^2(f_t)$, $f_i \circ \pi^2(\gamma(t))$ is constant because $j_0^2(f_i \circ f_t)$ is constant. □

Theorem 10. *One can associate to the class of a foliated connection $[\nabla(\mathcal{F})]$ a Cartan connection on $P(\mathcal{F})$ in a natural way. We will denote this connection by $\omega(\mathcal{F})$.*

Proof. If (U_i, f_i, g_{ij}) is a Haefliger cocycle of $\mathcal{F}$, one can easily seen that the image by $P^2N(f_i)$ of $P(\mathcal{F})|_{U_i}$ is a reduction of $P^2N(f_i)(P^2M(U_i,\mathcal{F}))$ to $H(q+1,\mathbb{R})$. We will denote by $\overline{P}$ this reduction.

One builds locally the normal Cartan connection $\omega(\mathcal{F})$ on $P(\mathcal{F})$ in the following way : if $\overline{\omega}$ denotes the normal Cartan connection on $\overline{P}$, then $\omega(\mathcal{F})|_{P^2M(U_i,\mathcal{F})} := (P^2N(f_i))^*\overline{\omega}$.

One can show (see [Blum84]) that the connection $\omega(\mathcal{F})$ is a well-defined foliated Cartan connection.

Thanks to the naturality of the normal Cartan connection, $\omega(\mathcal{F})$ is associated naturally to the class of the foliated connection $[\nabla(\mathcal{F})]$. □

One can remark that, as the foliation $\mathcal{F}_{P^2N}$ is of dimension p, the third condition of the definition of a foliated Cartan connection implies that, in our case, the kernel of $\omega(\mathcal{F})_u$ will be exactly equal to the tangent space to $\mathcal{F}_{P^2N}$.

6.3 Link between adapted and foliated Cartan connections

Remark. The image by P^2y of $P(\mathcal{F})$ is a reduction of $P^2(U)$ to $H(q+1,\mathbb{R})$, where U is an open set of $\mathbb{R}^q$. We will denote by P_U this reduction of $P^2(U)$ to $H(q+1,\mathbb{R})$. If ω_U denotes the normal Cartan connection on P_U, then $\omega(\mathcal{F})_{(P^2y)^{-1}P_U} = (P^2y)^*\omega_U$. Indeed, if ϕ denotes the diffeomorphism such that $\phi \circ y = f_i$, then $P^2\phi(P_U) = \overline{P}$. By naturality of the normal Cartan connection, $\omega_U = (P^2\phi)^*\overline{\omega}$ and then $(P^2y)^*\omega_U = (P^2f_i)^*\overline{\omega}$.

Proposition 6. *If θ_U denotes the canonical one-form on P_U, then*

$$(P^2y \circ ({}^{\mathcal{F}}\mathfrak{p}^2))^*\theta_U = p_{n,q}\theta_{\mathcal{F}}.$$

Proof. It is a simple verification using the definitions of the canonical forms θ_U and $\theta_{\mathcal{F}}$. □

Theorem 11. *The connections $\omega_{\mathcal{F}}$ and $\omega(\mathcal{F})$ are linked by the following relation :*

$${}^{\mathcal{F}}\mathfrak{p}^{2*}\omega(\mathcal{F}) = p_{n,q}\omega_{\mathcal{F}}.$$

Proof. To prove that, it suffices to prove that

$$(P^2y \circ ({}^{\mathcal{F}}\mathfrak{p}^2))^*\omega_U = p_{n,q}\omega_{\mathcal{F}}.$$

If one denotes by ∇_U the connection on U whose Christoffel symbols are the Christoffel symbols of $\nabla(\mathcal{F})$ read through the passing to transverse coordinates y, if $(\epsilon^1,\ldots,\epsilon^n)$ denotes the canonical basis of $\mathbb{R}^{n*}$ (resp. $(\epsilon^1,\ldots,\epsilon^q)$ denotes the canonical basis of $\mathbb{R}^{q*}$), one has

$$\omega_{\mathcal{F}} = \tilde{\Upsilon}_{\mathcal{F}} - \sum_{j=p+1}^{n}\sum_{k=1}^{n}(\Gamma_{\mathcal{F}\ jk})(\theta^k_{\mathcal{F}-1})\epsilon^j$$

$$(resp.\ \omega_U = \tilde{\Upsilon}_U - \sum_{j=1}^{q}\sum_{k=1}^{q}(\Gamma_{U\ jk})(\theta^k_{U\ -1})\epsilon^j),$$

where $\tilde{\Upsilon}_{\mathcal{F}}$ (resp. $\tilde{\Upsilon}_U$) is the Cartan connection induced by $\nabla_{\mathcal{F}}$ (resp. ∇_U), $\Gamma_{\mathcal{F}}$ (resp. Γ_U) is the deformation tensor corresponding to $\nabla_{\mathcal{F}}$ (resp. ∇_U) (see [CSS97]).

One recalls that $\tilde{\Upsilon}_{\mathcal{F}}$ (resp. $\tilde{\Upsilon}_U$) is the unique Cartan connection such that its component with respect to $\mathbb{R}^{q*}$ vanishes on the section $(x^i,\delta^i_k,-\Gamma^i_{jk})$ (resp. $(x^{\mathfrak{i}},\delta^{\mathfrak{i}}_{\mathfrak{k}},-\Gamma^{\mathfrak{i}}_{\mathfrak{j}\mathfrak{k}})$). If $\sigma_{\mathcal{F}}$ (resp. σ_U) denotes the section $(x^i,\delta^i_k,-\Gamma^i_{jk})$ (resp. $(x^{\mathfrak{i}},\delta^{\mathfrak{i}}_{\mathfrak{k}},-\Gamma^{\mathfrak{i}}_{\mathfrak{j}\mathfrak{k}})$), the connection $\tilde{\Upsilon}_{\mathcal{F}}$ (resp. $\tilde{\Upsilon}_U$) is defined in this way :

$$\tilde{\Upsilon}_{\mathcal{F}\ u}(X) = \mathrm{Ad}(b^{-1})\theta_{\mathcal{F}}((\sigma_{\mathcal{F}}\circ\pi^2)_*X) + B,$$

$$(resp.\ \tilde{\Upsilon}_{U\ u}(X) = \mathrm{Ad}(b^{-1})\theta_U((\sigma_U\circ\pi^2)_*X) + B),$$

where π^2 is the projection on M (resp. U), $R_b(\sigma_{\mathcal{F}}(\pi^2(u))) = u$ (resp. $R_b(\sigma_U(\pi^2(u))) = u$) and $B^* = X - R_{b*}\sigma_{\mathcal{F}*}\pi^2_*X$ (resp. $B^* = X - R_{b*}\sigma_{U*}\pi^2_*X$).

The deformation tensor $\Gamma_{\mathcal{F}}$ (resp. Γ_U) is defined in this way :

$$\Gamma_{\mathcal{F}}(X) = (\tilde{\Upsilon}_{\mathcal{F}} - \omega_{\mathcal{F}})(\omega_{\mathcal{F}}^{-1}(X))$$

$$(resp.\ \Gamma_U(X) = (\tilde{\Upsilon}_U - \omega_U)(\omega_U^{-1}(X))).$$

In fact, the sections $(x^i,\delta^i_k,-\Gamma^i_{jk})$ and $(x^{\mathfrak{i}},\delta^{\mathfrak{i}}_{\mathfrak{k}},-\Gamma^{\mathfrak{i}}_{\mathfrak{j}\mathfrak{k}})$ correspond to the section σ of the end of the theorem 7, the connections $\tilde{\Upsilon}_{\mathcal{F}}$ and $\tilde{\Upsilon}_U$ correspond to the connection ω of the proof of this theorem, the connections $\omega_{\mathcal{F}}$ and ω_U correspond to the connection $\bar{\omega}$ whereas the Γ_{jk} correspond to the functions $-A_{jk}$.

We can easily prove that $(P^2y \circ ({}^{\mathcal{F}}\mathfrak{p}^2))^* \tilde{\Upsilon}_U = p_{n,q} \tilde{\Upsilon}_{\mathcal{F}}$ using proposition 6.

Next, prove that $(P^2y \circ ({}^{\mathcal{F}}\mathfrak{p}^2))^* \sum_{j=1}^q \sum_{k=1}^q (\Gamma_{U\ jk})(\theta^k_{U\ -1})\epsilon^j = \sum_{j=p+1}^n \sum_{k=1}^n (\Gamma_{\mathcal{F}\ jk})(\theta^k_{\mathcal{F}\ -1})\epsilon^j$.

One has $(P^2y \circ ({}^{\mathcal{F}}\mathfrak{p}^2))^* \sum_{j=1}^q \sum_{k=1}^q (\Gamma_{U\ jk})(\theta^k_{U\ -1})\epsilon^j = \sum_{j=1}^q \sum_{k=1}^q (P^2y \circ ({}^{\mathcal{F}}\mathfrak{p}^2))^* (\Gamma_{U\ jk})(\theta^{k+p}_{\mathcal{F}\ -1})\epsilon^{j+p}$.

It remains then to prove that $(P^2y \circ ({}^{\mathcal{F}}\mathfrak{p}^2))^*(\Gamma_{U\ jk}) = \Gamma_{\mathcal{F}\ j+p,k+p}$ and that $\Gamma_{\mathcal{F}\ jk} = 0$ if $1 \leq k \leq p$.

Indeed, if $1 \leq k \leq p$, $\Gamma_{\mathcal{F}\ jk} = \frac{1}{q} \sum_{l=p+1}^n R^l_{\mathcal{F}\ jlk}$, where $R_{\mathcal{F}}$ denotes the equivariant function on $P_{\mathcal{F}}$ representing the curvature tensor of $\nabla_{\mathcal{F}}$ thanks to the equation (20) of the Theorem 7 and thanks to the fact that the K^l_{ijk} represent the components of $R_{\mathcal{F}}$ (see [CSS97]). Thanks to the fact that $\nabla_{\mathcal{F}}$ is adapted, one can see that if $1 \leq k \leq p$, $\Gamma_{\mathcal{F}\ jk} = 0$. Moreover, $\Gamma_{U\ jk} = \frac{-1}{(q+1)(q-1)} \sum_{i=1}^q R^i_{U\ ijk} + \frac{1}{q-1} \sum_{i=1}^q R^i_{U\ jik}$, where R_U denotes the equivariant function on P_U representing the curvature tensor of ∇_U whereas if $p+1 \leq k \leq n$, $\Gamma_{\mathcal{F}\ jk} = \frac{-1}{(q+1)(q-1)} \sum_{i=p+1}^n R^i_{\mathcal{F}\ ijk} + \frac{1}{q-1} \sum_{i=p+1}^n R^i_{\mathcal{F}\ jik}$ thanks to the equation (19) of the Theorem 7. This allows to prove that $(P^2y \circ ({}^{\mathcal{F}}\mathfrak{p}^2))^*(\Gamma_{U\ jk}) = \Gamma_{\mathcal{F}\ j+p,k+p}$.

□

7 Construction of the quantization

In a first step, we are going to explain how to build the quantization in the adapted and foliated situations. In a second step, we are going to prove that the quantization commutes with the reduction. In other words, quantize adapted objects is equivalent to quantize the induced foliated objects.

7.1 Construction in the adapted situation

In the adapted situation, we can define the operator of invariant differentiation exactly in the same way as in the standard situation :

Definition 13. *Let V be a vector space. If $f \in C^\infty(P_{\mathcal{F}}, V)$, then the invariant differential of f with respect to $\omega_{\mathcal{F}}$ is the function $\nabla^{\omega_{\mathcal{F}}} f \in C^\infty(P_{\mathcal{F}}, \mathbb{R}^{n*} \otimes V)$ defined by*

$$\nabla^{\omega_{\mathcal{F}}} f(u)(X) = L_{\omega_{\mathcal{F}}^{-1}(X)} f(u) \quad \forall u \in P_{\mathcal{F}}, \quad \forall X \in \mathbb{R}^n.$$

We will also use an iterated and symmetrized version of the invariant differentiation

Definition 14. *If $f \in C^\infty(P_{\mathcal{F}}, V)$ then $(\nabla^{\omega_{\mathcal{F}}})^k f \in C^\infty(P_{\mathcal{F}}, S^k\mathbb{R}^{n*} \otimes V)$ is defined by*

$$(\nabla^{\omega_{\mathcal{F}}})^k f(u)(X_1, \ldots, X_k) = \frac{1}{k!} \sum_\nu L_{\omega_{\mathcal{F}}^{-1}(X_{\nu_1})} \circ \ldots \circ L_{\omega_{\mathcal{F}}^{-1}(X_{\nu_k})} f(u)$$

for $X_1, \ldots, X_k \in \mathbb{R}^n$.

Proposition 7. *If $v \in \mathbb{R}^n$ and if $p_{n,q}(v) = 0$, then $\omega_{\mathcal{F}}^{-1}(v)$ is tangent to $\mathcal{F}_{\underline{P}^2}$.*

Proof. Indeed, as $p_{n,q}\omega_{\mathcal{F}} = {}^{\mathcal{F}}\mathfrak{p}^{2*}\omega(\mathcal{F})$, one has $\omega(\mathcal{F})({}^{\mathcal{F}}\mathfrak{p}^2{}_*\omega_{\mathcal{F}}^{-1}(v)) = 0$. As ${}^{\mathcal{F}}\mathfrak{p}^2{}_*\omega_{\mathcal{F}}^{-1}(v)$ is then tangent to $\mathcal{F}_{P^2N}$, one can easily show that $\omega_{\mathcal{F}}^{-1}(v)$ is then tangent to $\mathcal{F}_{\underline{P}^2}$. □

In the adapted situation, the invariant differentiation has a particular property :

Proposition 8. *If f is a foliated function on $P_{\mathcal{F}}$, then*

$$(\nabla^{\omega^k_{\mathcal{F}}} f)(v_1, \ldots, v_k) = (\nabla^{\omega^k_{\mathcal{F}}} f)((0, p_{n,q}v_1), \ldots, (0, p_{n,q}v_k)).$$

Proof. Indeed, one can show that if f is constant along the leaves of $\mathcal{F}_{\underline{P}^2}$, then $L_{\omega_{\mathcal{F}}^{-1}(0,p_{n,q}v)}f$ is a foliated function too if $v \in \mathbb{R}^n$. Indeed, if X is tangent to $\mathcal{F}_{\underline{P}^2}$, then $L_X L_{\omega_{\mathcal{F}}^{-1}(0,p_{n,q}v)}f = 0$. To show that, it suffices to prove that $L_{[X,\omega_{\mathcal{F}}^{-1}(0,p_{n,q}v)]}f = 0$. The fact that $i_X\omega(\mathcal{F}) = i_X d\omega(\mathcal{F}) = 0$ if X is tangent to $\mathcal{F}_{P^2N}$, that $p_{n,q}\omega_{\mathcal{F}} = {}^{\mathcal{F}}\mathfrak{p}^{2*}\omega(\mathcal{F})$ and that ${}^{\mathcal{F}}\mathfrak{p}^2{}_*X$ is tangent to $\mathcal{F}_{P^2N}$ if X is tangent to $\mathcal{F}_{\underline{P}^2}$ implies that $i_X p_{n,q}\omega_{\mathcal{F}} = i_X p_{n,q} d\omega_{\mathcal{F}} = 0$ if X is tangent to $\mathcal{F}_{\underline{P}^2}$.

Remark that as the kernel of $p_{n,q}\omega_{\mathcal{F}}$ has a dimension equal to the dimension of $\mathcal{F}_{\underline{P}^2}$ (i.e. $p+np$), the kernel of $p_{n,q}\omega_{\mathcal{F}}$ is equal to the tangent space to $\mathcal{F}_{\underline{P}^2}$. One has then $0 = p_{n,q}d\omega_{\mathcal{F}}(X, \omega_{\mathcal{F}}^{-1}(0,p_{n,q}v)) = X.(p_{n,q}v) - \omega_{\mathcal{F}}^{-1}(0,p_{n,q}v).(p_{n,q}\omega_{\mathcal{F}}(X)) - p_{n,q}\omega_{\mathcal{F}}([X,\omega_{\mathcal{F}}^{-1}(0,p_{n,q}v)])$. As the first two terms are equal to 0, the third term vanishes too.

One has then that $[X,\omega_{\mathcal{F}}^{-1}(0,p_{n,q}v)]$ is tangent to $\mathcal{F}_{\underline{P}^2}$ and then $L_{[X,\omega_{\mathcal{F}}^{-1}(0,p_{n,q}v)]}f = 0$.

One concludes using the fact that $\omega_{\mathcal{F}}^{-1}(v)$ is tangent to $\mathcal{F}_{\underline{P}^2}$ if $p_{n,q}(v) = 0$. □

In the adapted situation, we define a divergence operator analogous to the divergence operator defined in [MR05].

We fix a basis $(e_1,\ldots,e_n)$ of $\mathbb{R}^n$ and we denote by $(\epsilon^1,\ldots,\epsilon^n)$ the dual basis in $\mathbb{R}^{n*}$.

Definition 15. *The* Divergence operator *with respect to the Cartan connection $\omega_{\mathcal{F}}$ is defined by*

$$\mathrm{Div}^{\omega_{\mathcal{F}}} : C^\infty(P_{\mathcal{F}}, S^k(\mathbb{R}^n)) \to C^\infty(P_{\mathcal{F}}, S^{k-1}(\mathbb{R}^n)) : S \mapsto \sum_{j=p+1}^{n} i(\epsilon^j)\nabla^{\omega_{\mathcal{F}}}_{e_j} S,$$

where i denotes the inner product.

Remark. If $S \in C^\infty(P_{\mathcal{F}}, S^k(\mathbb{R}^n))$ and if $f \in C^\infty(P_{\mathcal{F}}, \mathbb{R}; \mathcal{F}_{\underline{P}^2})$, thanks to Proposition 8, we have $\langle \mathrm{Div}^{\omega^l_{\mathcal{F}}} S, \nabla^{\omega^{k-l}_{\mathcal{F}}} f\rangle = \langle p_{n,q}\,\mathrm{Div}^{\omega^l_{\mathcal{F}}} S, p_{n,q}\nabla^{\omega^{k-l}_{\mathcal{F}}} f\rangle$.

One can then easily adapt Proposition 4, Lemma 7, Lemma 8, Propositions 9 and 10 from [MR05]:

Proposition 9. *Let (V,ρ) be a representation of* $\mathrm{GL}(n,q,\mathbb{R})$. *If f belongs to $C^\infty(P_{\mathcal{F}},V)_{\mathrm{GL}(n,q,\mathbb{R})}$, then $\nabla^{\omega_{\mathcal{F}}} f \in C^\infty(P_{\mathcal{F}}, \mathbb{R}^{n*}\otimes V)_{\mathrm{GL}(n,q,\mathbb{R})}$.*

Proof. The proof is exactly similar to the one of Proposition 4 in [MR05]. □

In the same way, we have the following result :

Proposition 10. *Let ρ be the action of* $\mathrm{GL}(q,\mathbb{R})$ *on $S^k(\mathbb{R}^q)$ and ρ' the induced action on $\mathbb{R}^{q*}\otimes S^k(\mathbb{R}^q)$. If $S \in C^\infty(P_{\mathcal{F}}, S^k(\mathbb{R}^n))$ is such that $p_{n,q}S$ is foliated and that $(p_{n,q}S)(ug) = \rho(p_{n,q}g^{-1})(p_{n,q}S(u))\ \forall g \in$ $\mathrm{GL}(n,q,\mathbb{R})$, then*

$$(p_{n,q}\nabla^{\omega_{\mathcal{F}}}S)(ug) = \rho'(p_{n,q}g^{-1})(p_{n,q}\nabla^{\omega_{\mathcal{F}}}S(u)).$$

Proof. The proof is analogous to the proof of the previous result.

$$(p_{n,q}\nabla^{\omega_{\mathcal{F}}}S)(ug) = \rho'(p_{n,q}g)^{-1}(p_{n,q}\nabla^{\omega_{\mathcal{F}}}S)(u)\ \forall u \in P_{\mathcal{F}}, \forall g \in \mathrm{GL}(n,q,\mathbb{R})$$

$$\Longleftrightarrow$$

$$(p_{n,q}\nabla^{\omega_{\mathcal{F}}}S)(ug)(X) = [\rho'(p_{n,q}g)^{-1}(p_{n,q}\nabla^{\omega_{\mathcal{F}}}S)(u)](X)\ \forall u \in P_{\mathcal{F}}, \forall g \in \mathrm{GL}(n,q,\mathbb{R}), \forall X \in \mathbb{R}^q$$

$$\Longleftrightarrow$$

$$(L_{\omega_{\mathcal{F}}^{-1}(0,X)}p_{n,q}S)(ug) = \rho(p_{n,q}g^{-1})(L_{\omega_{\mathcal{F}}^{-1}(0,(p_{n,q}g)X)}p_{n,q}S)(u)\ \forall u \in P_{\mathcal{F}}, \forall g \in \mathrm{GL}(n,q,\mathbb{R}), \forall X \in \mathbb{R}^q.$$

$$\Longleftrightarrow$$

$$(L_{\omega_{\mathcal{F}}^{-1}(0,X)}p_{n,q}S)(ug) = \rho(p_{n,q}g^{-1})(L_{\omega_{\mathcal{F}}^{-1}(g(0,X))}p_{n,q}S)(u)\ \forall u \in P_{\mathcal{F}}, \forall g \in \mathrm{GL}(n,q,\mathbb{R}), \forall X \in \mathbb{R}^q,$$

using the fact that $p_{n,q}S$ is foliated. If one denotes by φ_t the flow of $\omega_{\mathcal{F}}^{-1}(0,X)$ and by φ'_t the flow of $\omega_{\mathcal{F}}^{-1}(g(0,X))$, it suffices then to verify that

$$\frac{d}{dt}p_{n,q}S(\varphi_t(ug))|_{t=0} = \rho(p_{n,q}g^{-1})\frac{d}{dt}p_{n,q}S(\varphi'_t(u))|_{t=0}\ \forall u \in P_{\mathcal{F}}, \forall g \in \mathrm{GL}(n,q,\mathbb{R}).$$

One concludes using the fact that

$$\varphi_t(ug) = \varphi'_t(u)g$$

because the fields $\omega_{\mathcal{F}}^{-1}(g(0,X))$ and $\omega_{\mathcal{F}}^{-1}(0,X)$ are R_g-linked. □

Proposition 11. *Let ρ be the action of $\mathrm{GL}(q,\mathbb{R})$ on $S^k(\mathbb{R}^q)$ and ρ' the action on $S^{k-1}(\mathbb{R}^q)$. If $S \in C^\infty(P_{\mathcal{F}}, S^k(\mathbb{R}^n))$ is such that $p_{n,q}S$ is foliated and that $(p_{n,q}S)(ug) = \rho(p_{n,q}g^{-1})(p_{n,q}S(u))\ \forall g \in \mathrm{GL}(n,q,\mathbb{R})$, then*

$$(p_{n,q}\operatorname{Div}^{\omega_{\mathcal{F}}} S)(ug) = \rho'(p_{n,q}g^{-1})(p_{n,q}\operatorname{Div}^{\omega_{\mathcal{F}}} S(u)).$$

Proof. This can be checked directly from the definition of the divergence and from the proposition 10. □

Proposition 12. *For every $S \in C^\infty(P_{\mathcal{F}}, S^k(\mathbb{R}^n))$ such that $(p_{n,q}S)(ug) = \rho(p_{n,q}g^{-1})(p_{n,q}S(u))\ \forall g \in \mathrm{GL}(n,q,\mathbb{R})$, we have*

$$p_{n,q}[L_{h^*}\operatorname{Div}^{\omega_{\mathcal{F}}} S - \operatorname{Div}^{\omega_{\mathcal{F}}} L_{h^*}S] = (q+2k-1)p_{n,q}i(0,h)S,$$

for every $h \in \mathbb{R}^{q*}$.

Proof. First we remark that the Lie derivative with respect to a vector field commutes with the evaluation : if $\eta^1,\ldots,\eta^{k-1} \in \mathbb{R}^{q*}$, we have

$$\begin{array}{rcl}(L_{h^*}p_{n,q}\operatorname{Div}^{\omega_{\mathcal{F}}} S)(\eta^1,\ldots,\eta^{k-1}) & = & L_{h^*}(p_{n,q}\operatorname{Div}^{\omega_{\mathcal{F}}} S(\eta^1,\ldots,\eta^{k-1})) \\ & = & \sum_{j=p+1}^n (L_{h^*}L_{\omega_{\mathcal{F}}^{-1}(e_j)}p_{n,q}S(p_{n,q}\epsilon^j,\eta^1,\ldots,\eta^{k-1})).\end{array}$$

Now, the definition of a Cartan connection implies the relation

$$[h^*, \omega_{\mathcal{F}}^{-1}(X)] = \omega_{\mathcal{F}}^{-1}([h,X]), \quad \forall h \in \mathrm{gl}(n,q,\mathbb{R}) \oplus \mathbb{R}^{q*}, X \in \mathbb{R}^n,$$

where the bracket on the right is the one of $\mathrm{sl}(n+1,\mathbb{R})$. It follows that the expression we have to compute is equal to

$$\sum_{j=p+1}^n (L_{\omega_{\mathcal{F}}^{-1}(e_j)}L_{h^*}p_{n,q}S(p_{n,q}\epsilon^j,\eta^1,\ldots,\eta^{k-1}) + (L_{[h,e_j]^*}p_{n,q}S)(p_{n,q}\epsilon^j,\eta^1,\ldots,\eta^{k-1})).$$

Finally, we obtain

$$\begin{array}{rl} & p_{n,q}\operatorname{Div}^{\omega_{\mathcal{F}}}(L_{h^*}S)(\eta^1,\ldots,\eta^{k-1}) - \sum_{j=p+1}^n(\rho_*(p_{n,q}[h,e_j])p_{n,q}S)(p_{n,q}\epsilon^j,\eta^1,\ldots,\eta^{k-1}) \\ = & p_{n,q}\operatorname{Div}^{\omega_{\mathcal{F}}}(L_{h^*}S)(\eta^1,\ldots,\eta^{k-1}) + \sum_{j=p+1}^n(\rho_*(p_{n,q}(h\otimes e_j + \langle h,e_j\rangle Id))p_{n,q}S)(p_{n,q}\epsilon^j,\eta^1,\ldots,\eta^{k-1}).\end{array}$$

The result then easily follows from the definition of ρ on $S^k(\mathbb{R}^q)$. □

Proposition 13. *If S is an equivariant function on $P_{\mathcal{F}}$ representing an adapted symbol, we have*

$$p_{n,q}[L_{h^*}(\operatorname{Div}^{\omega_{\mathcal{F}}})^l S - (\operatorname{Div}^{\omega_{\mathcal{F}}})^l L_{h^*}S] = l(q+2k-l)p_{n,q}[i(h)(\operatorname{Div}^{\omega_{\mathcal{F}}})^{l-1}S],$$

for every $h \in \mathbb{R}^{q*}$.

Proof. For $l = 1$, this is simply the proposition 12. Then the result follows by induction, using propositions 11 and 12. □

Proposition 14. *If $f \in \mathcal{C}^\infty(P_\mathcal{F}, \mathbb{R})_{\mathrm{GL}(n,q,\mathbb{R})}$, then*

$$L_{h^*}(\nabla^{\omega_\mathcal{F}})^k f - (\nabla^{\omega_\mathcal{F}})^k L_{h^*} f = -k(k-1)(\nabla^{\omega_\mathcal{F}})^{k-1} f \vee h,$$

for every $h \in \mathbb{R}^{q*}$.

Proof. The proof goes exactly as in [MR05]. □

Theorem 12. *In the adapted situation, the formula giving the quantization $Q_\mathcal{F}$ is then the following :*

$$Q_\mathcal{F}(\nabla_\mathcal{F}, s)(f) = p_\mathcal{F}^{2*^{-1}}(\sum_{l=0}^{k} C_{k,l}\langle \mathrm{Div}^{\omega_\mathcal{F}^l}\hat{s}, \nabla_s^{\omega_\mathcal{F}^{k-l}}\hat{f}\rangle) \tag{21}$$

if

$$C_{k,l} = \frac{(k-1)\cdots(k-l)}{(q+2k-1)\cdots(q+2k-l)}\begin{pmatrix} k \\ l \end{pmatrix}, \forall l \geq 1, \quad C_{k,0} = 1.$$

Proof. The proof goes as in [MR05]. □

7.2 Construction in the foliated situation

In the foliated situation, one can define the invariant differentiation in this way :

Proposition 15. *The following definition makes sense : if f is a foliated function on $P(\mathcal{F})$, then*

$$(\nabla^{\omega(\mathcal{F})^k} f)(v_1, \ldots, v_k) = \frac{1}{k!}\sum_\nu L_{\omega(\mathcal{F})^{-1}(v_{\nu_1})} \circ \ldots \circ L_{\omega(\mathcal{F})^{-1}(v_{\nu_k})} f(u),$$

where $\omega(\mathcal{F})^{-1}(v)$ is a vector field such that its image by $\omega(\mathcal{F})$ is equal to v.

Proof. One has to show that the definition is independent of the choice of the vector field. Indeed, two such vector fields differ by a vector field tangent to $\mathcal{F}_{P^2N}$ and one can show that if f is constant along the leaves of $\mathcal{F}_{P^2N}$, then $L_{\omega(\mathcal{F})^{-1}(v)} f$ is a foliated function too if $v \in \mathbb{R}^q$. Indeed, if X is tangent to $\mathcal{F}_{P^2N}$, then $L_X L_{\omega(\mathcal{F})^{-1}(v)} f = 0$. To show that, it suffices to prove that $L_{[X,\omega(\mathcal{F})^{-1}(v)]} f = 0$.

One has $0 = d\omega(\mathcal{F})(X, \omega(\mathcal{F})^{-1}(v)) = X.v - \omega(\mathcal{F})^{-1}(v).\omega(\mathcal{F})(X) - \omega(\mathcal{F})([X, \omega(\mathcal{F})^{-1}(v)])$. As the first two terms are equal to 0, the third term vanishes too. One has then that $[X, \omega(\mathcal{F})^{-1}(v)]$ is tangent to $\mathcal{F}_{P^2N}$ and then $L_{[X,\omega(\mathcal{F})^{-1}(v)]} f = 0$.

□

In the foliated situation, we define the divergence operator in this way :

Definition 16. *The* Divergence operator *with respect to the Cartan connection $\omega(\mathcal{F})$ is defined by*

$$\mathrm{Div}^{\omega(\mathcal{F})} : C^\infty(P(\mathcal{F}), S^k(\mathbb{R}^q); \mathcal{F}_{P^2N}) \to C^\infty(P(\mathcal{F}), S^{k-1}(\mathbb{R}^q); \mathcal{F}_{P^2N}) : S \mapsto \sum_{j=1}^{q} i(\epsilon^j)\nabla_{e_j}^{\omega(\mathcal{F})} S.$$

One can then easily adapt the propositions 9, 11, 12, 13, 14. The proofs of these propositions are completely similar to the proofs of the corresponding results in [MR05].

Proposition 16. *If f is a $\mathrm{GL}(q,\mathbb{R})$-equivariant foliated function on $P(\mathcal{F})$ then $\nabla^{\omega(\mathcal{F})} f$ is $\mathrm{GL}(q,\mathbb{R})$-equivariant too.*

Proposition 17. *If $S \in C^\infty(P(\mathcal{F}), S^k(\mathbb{R}^q); \mathcal{F}_{P^2N})_{\mathrm{GL}(q,\mathbb{R})}$, then $\mathrm{Div}^{\omega(\mathcal{F})} S \in C^\infty(P(\mathcal{F}), S^{k-1}(\mathbb{R}^q); \mathcal{F}_{P^2N})_{\mathrm{GL}(q,\mathbb{R})}$.*

Proposition 18. *For every $S \in C^\infty(P(\mathcal{F}), S^k(\mathbb{R}^q); \mathcal{F}_{P^2N})_{\mathrm{GL}(q,\mathbb{R})}$ we have*

$$L_{h^*}\mathrm{Div}^{\omega(\mathcal{F})} S - \mathrm{Div}^{\omega(\mathcal{F})} L_{h^*} S = (q+2k-1)i(h)S,$$

for every $h \in \mathbb{R}^{q*}$.

Theorem 13. *For every $S \in C^\infty(P(\mathcal{F}), S^k(\mathbb{R}^q); \mathcal{F}_{P^2N})_{\mathrm{GL}(q,\mathbb{R})}$, we have*

$$L_{h^*}(\mathrm{Div}^{\omega(\mathcal{F})})^l S - (\mathrm{Div}^{\omega(\mathcal{F})})^l L_{h^*} S = l(q+2k-l) i(h)(\mathrm{Div}^{\omega(\mathcal{F})})^{l-1} S,$$

for every $h \in \mathbb{R}^{q*}$.

Theorem 14. *If $f \in \mathcal{C}^\infty(P(\mathcal{F}), \mathbb{R}; \mathcal{F}_{P^2N})_{\mathrm{GL}(q,\mathbb{R})}$, then*

$$L_{h^*}(\nabla^{\omega(\mathcal{F})})^k f - (\nabla^{\omega(\mathcal{F})})^k L_{h^*} f = -k(k-1)(\nabla^{\omega(\mathcal{F})})^{k-1} f \vee h,$$

for every $h \in \mathbb{R}^{q*}$.

Theorem 15. *In the foliated situation, the formula giving the quantization $Q(\mathcal{F})$ is the following :*

$$Q(\mathcal{F})(\nabla(\mathcal{F}), s)(f) = (p^2(\mathcal{F}))^{*^{-1}}(\sum_{l=0}^{k} C_{k,l} \langle \mathrm{Div}^{\omega(\mathcal{F})^l}\, \hat{s}, \nabla_s^{\omega(\mathcal{F})^{k-l}} \hat{f} \rangle)$$

if

$$C_{k,l} = \frac{(k-1)\cdots(k-l)}{(q+2k-1)\cdots(q+2k-l)} \begin{pmatrix} k \\ l \end{pmatrix}, \forall l \geq 1, \quad C_{k,0} = 1.$$

7.3 Quantization commutes with reduction

Proposition 19. *If f is an equivariant function on $P_{\mathcal{F}}$ representing a basic function, then*

$$(\nabla^{\omega_{\mathcal{F}}^k} f)(v_1, \ldots, v_k) = {}^{\mathcal{F}}\mathfrak{p}^{2^*}(\nabla^{\omega(\mathcal{F})^k}({}^{\mathcal{F}}\hat{\pi} f))(p_{n,q}v_1, \ldots, p_{n,q}v_k).$$

Proof. Indeed, one has first that ${}^{\mathcal{F}}\mathfrak{p}^2{}_* \omega_{\mathcal{F}}^{-1}(v)$ is equal to $\omega(\mathcal{F})^{-1}(p_{n,q}v)$ modulo a vector field tangent to $\mathcal{F}_{P^2N}$. By induction, if the proposition is true to $k-1$, it is true for k :

$$\begin{aligned}
(\nabla^{\omega_{\mathcal{F}}^k} f)(v_1, \ldots, v_k) &= L_{\omega_{\mathcal{F}}^{-1}(v_k)}(\nabla^{\omega_{\mathcal{F}}^{k-1}} f)(v_1, \ldots, v_{k-1}) \\
&= L_{\omega_{\mathcal{F}}^{-1}(v_k)} {}^{\mathcal{F}}\mathfrak{p}^{2^*}(\nabla^{\omega(\mathcal{F})^{k-1}}({}^{\mathcal{F}}\hat{\pi} f))(p_{n,q}v_1, \ldots, p_{n,q}v_{k-1}) \\
&= {}^{\mathcal{F}}\mathfrak{p}^{2^*}(L_{\omega(\mathcal{F})^{-1}(p_{n,q}v_k)} \nabla^{\omega(\mathcal{F})^{k-1}}({}^{\mathcal{F}}\hat{\pi} f))(p_{n,q}v_1, \ldots, p_{n,q}v_{k-1}) \\
&= {}^{\mathcal{F}}\mathfrak{p}^{2^*}(\nabla^{\omega(\mathcal{F})^k}({}^{\mathcal{F}}\hat{\pi} f))(p_{n,q}v_1, \ldots, p_{n,q}v_k).
\end{aligned}$$

□

In an other part,

Proposition 20. *If S is an equivariant function on $P_{\mathcal{F}}$ representing an adapted symbol, then*

$$p_{n,q}(\mathrm{Div}^{\omega_{\mathcal{F}}^l} S) = {}^{\mathcal{F}}\mathfrak{p}^{2^*}(\mathrm{Div}^{\omega(\mathcal{F})^l}({}^{\mathcal{F}}\hat{\pi} S)).$$

Proof. Indeed, by induction, if it is true to $l-1$, it is true for l :

$$\begin{aligned}
p_{n,q}(\mathrm{Div}^{\omega_{\mathcal{F}}^l} S) &= p_{n,q} \sum_{j=p+1}^{n} i(\epsilon^j) L_{\omega_{\mathcal{F}}^{-1}(e_j)}(\mathrm{Div}^{\omega_{\mathcal{F}}^{l-1}} S) \\
&= \sum_{j=p+1}^{n} L_{\omega_{\mathcal{F}}^{-1}(e_j)} p_{n,q} i(\epsilon^j)(\mathrm{Div}^{\omega_{\mathcal{F}}^{l-1}} S) \\
&= \sum_{j=p+1}^{n} L_{\omega_{\mathcal{F}}^{-1}(e_j)} i(p_{n,q}\epsilon^j) {}^{\mathcal{F}}\mathfrak{p}^{2^*}(\mathrm{Div}^{\omega(\mathcal{F})^{l-1}}({}^{\mathcal{F}}\hat{\pi} S)) \\
&= \sum_{j=p+1}^{n} {}^{\mathcal{F}}\mathfrak{p}^{2^*} L_{\omega(\mathcal{F})^{-1}(p_{n,q}e_j)} i(p_{n,q}\epsilon^j)(\mathrm{Div}^{\omega(\mathcal{F})^{l-1}}({}^{\mathcal{F}}\hat{\pi} S)).
\end{aligned}$$

□

Theorem 16. *If S is an equivariant function on $P_{\mathcal{F}}$ representing an adapted symbol and if f is an equivariant function on $P_{\mathcal{F}}$ representing a basic function, then*

$$\langle \mathrm{Div}^{\omega^l_{\mathcal{F}}} S, \nabla^{\omega^{k-l}_{\mathcal{F}}} f\rangle = {}^{\mathcal{F}}\!\mathfrak{p}^{2*}\langle \mathrm{Div}^{\omega(\mathcal{F})^l}({}^{\mathcal{F}}\hat{\pi}S), \nabla^{\omega(\mathcal{F})^{k-l}}({}^{\mathcal{F}}\hat{\pi}f)\rangle$$

if S is of degree k. The quantization commutes then with the reduction :

$$Q_{\mathcal{F}}(\nabla_{\mathcal{F}})(s)(f) = Q(\mathcal{F})(\nabla(\mathcal{F}))({}^{\mathcal{F}}\pi s)(f).$$

Proof. Indeed, if $\mathrm{Div}^{\omega^l_{\mathcal{F}}} S = v_1 \vee \ldots \vee v_{k-l}$,

$$\begin{aligned}\langle \mathrm{Div}^{\omega^l_{\mathcal{F}}} S, \nabla^{\omega^{k-l}_{\mathcal{F}}} f\rangle &= (\nabla^{\omega^{k-l}_{\mathcal{F}}} f)(v_1, \ldots, v_{k-l})\\ &= {}^{\mathcal{F}}\!\mathfrak{p}^{2*}(\nabla^{\omega(\mathcal{F})^{k-l}}({}^{\mathcal{F}}\hat{\pi}f))(p_{n,q}v_1, \ldots, p_{n,q}v_{k-l})\\ &= \langle p_{n,q}\,\mathrm{Div}^{\omega^l_{\mathcal{F}}} S, {}^{\mathcal{F}}\!\mathfrak{p}^{2*}\nabla^{\omega(\mathcal{F})^{k-l}}({}^{\mathcal{F}}\hat{\pi}f)\rangle.\end{aligned}$$

The conclusion follows then from Theorems 12 and 15. □

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