COLLISIONS OF PARTICLES IN LOCALLY ADS SPACETIMES II
MODULI OF GLOBALLY HYPERBOLIC SPACES

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Abstract. We investigate globally hyperbolic 3-dimensional AdS manifolds containing "particles", i.e., cone singularities of angles less than 2π along a time-like graph Γ. To each such space we associate a graph and a finite family of pairs of hyperbolic surfaces with cone singularities. We show that this data is sufficient to recover the space locally (i.e., in the neighborhood of a fixed metric). This is a partial extension of a result of Mess for non-singular globally hyperbolic AdS manifolds.

Contents

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1. Introduction

1.1. **The 3-dimensional AdS space.** The anti-de Sitter space, AdS_n, is a complete Lorentzian manifold of constant curvature -1. It can be defined as a quadric in the space $\mathbb{R}^{n-1,2}$, that is, $\mathbb{R}^{n+1}$ endowed with a symmetric bilinear form of signature $(n-1,2)$:

$$AdS_n = \{x \in \mathbb{R}^{n-1,2} \mid \langle x, x\rangle = -1\}\ .$$

In some ways, AdS_3 can be considered as a Lorentz analog of hyperbolic 3-space. We are interested here in manifolds endowed with a geometric structure which, outside some singular locus, is locally isometric to AdS_3.

1.2. **Globally hyperbolic AdS spacetimes.** A Lorentz 3-manifold M is AdS if it is locally modeled on AdS_3. Such a manifold is *globally hyperbolic maximal compact* (GHMC) if it contains a closed, space-like surface S, if any inextendible time-like curve in M intersects S exactly once, and if M is maximal under this condition (any isometric embedding of M in an AdS 3-manifold satisfying the same conditions is actually an isometry). Those manifolds can in some respects be considered as Lorentz analogs of quasifuchsian hyperbolic 3-manifolds.

Let S be a closed surface of genus at least 2. A well-known theorem of Bers [Ber60] asserts that the space of quasifuchsian hyperbolic metrics on $S \times \mathbb{R}$ (considered up to isotopy) is in one-to-one correspondence with $\mathcal{T}_S \times \mathcal{T}_S$, where $\mathcal{T}_S$ is the Teichmüller space of S.

In his 1990 IHES preprint, published only in 2007 [Mes07, ABB$^+$07], G. Mess discovered a remarkable analog of this theorem for globally hyperbolic maximal anti-de Sitter spacetimes: the space of GHM AdS metrics on $S\times\mathbb{R}$ is also parameterized by $\mathcal{T}_S\times\mathcal{T}_S$. Both the Bers and the Mess results can be described as "stereographic pictures": the full structure of a 3-dimensional constant curvature spacetime is encoded in a pair of hyperbolic metrics on a surface.

To understand this result more precisely, recall that the identity component of the isometry group of AdS_3, $O_0(2,2)$, is isomorphic, up to finite index, to $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$. Given a 3-dimensional globally hyperbolic maximal compact AdS spacetime M, it is homeomorphic to $S \times \mathbb{R}$, where S is a closed surface which can be chosen to be any Cauchy surface in M. Then M is isometric to $\Omega/h(\pi_1(S))$, where Ω is a convex subset of AdS_3 and $h : \pi_1(S) \to O_0(2,2)$ is a homomorphism, which can be written as (h_l, h_r) in the identification of $O_0(2,2)$ with $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$.

Theorem 1.1 (Mess [Mes07])**.** *The homomorphisms h_l and h_r are holonomy representations of hyperbolic metrics on S and can be identified with points in the Teichmüller space $\mathcal{T}_S$ of S. The map sending M to $(h_l, h_r) \in \mathcal{T}_S \times \mathcal{T}_S$ is a homeomorphism.*

The key point in the proof given by Mess is to show that h_l and h_r have maximal Euler number. By a celebrated result of Goldman [Gol88], this maximality implies that they are Fuchsian representations.

1.3. **Cone singularities.** Cone singularities along curves have been studied often in hyperbolic geometry, see e.g. [CHK00, BLP05]. A basic model space is the metric space $\mathbb{H}^3_\theta, \theta \in (0, 2\pi)$, defined as follows. Let Δ be a geodesic in $\mathbb{H}^3$, and let P, P' be two half-planes bounded by Δ, such that the oriented angle between P and P' is θ. Then $\mathbb{H}^3_\theta$ is obtained by "cutting out" the part of $\mathbb{H}^3$ bounded by P and P' (with angle θ along Δ) and isometrically gluing the boundary half-planes by the isometry which is the identity on Δ.

A hyperbolic cone-spacetime with singular locus a link is a metric space where each point has a neighborhood isometric to a neighborhood of $\mathbb{H}^3_\theta$, for some $\theta \in (0, 2\pi)$. A key rigidity result for closed 3-dimensional cone-spacetimes with singular locus a link and cone angles in $(0, 2\pi)$ was proved by Hodgson and Kerckhoff [HK98], and had a profound influence on hyperbolic geometry in recent years, see e.g. [BBES03]. More recently this rigidity result was extended to closed hyperbolic cone-spacetimes with singular set a graph, see [MM11, Wei]

1.4. **AdS spacetimes and particles.** 3-dimensional AdS spacetimes were first studied as a lower-dimensional toy model of gravity: they are solutions of Einstein's equation, with negative cosmological constant but without matter. A standard way to add physical relevance to this model is to consider in those AdS spacetimes some point particles, modeled by cone singularities along time-like lines (see e.g. [tH96, tH93]). They can be described as we did in Section 1.3 in the case of hyperbolic geometry: the geodesic Δ has to be a time-like geodesic in AdS_3 and P, P' are time-like totally geodesic half-planes bounding Δ, with cone angle θ. If we remove the region bounded by P and P' and glue the boundary half-plane by the unique AdS-isometry sending P and P' which is the identity on Δ, we obtain a spacetime containing a singular line.

Here we will call "massive particle" such a cone singularity, of angle less than 2π, along a time-like line. The condition that the angle is less than 2π is usually made by physicists, who consider it as corresponding to the positivity of mass. Here, as in [BBS11], it is also mathematically relevant.

Globally hyperbolic AdS spaces with such particles were considered in [BS09], when the cone angles are less than π. It was shown that a satisfactory extension of Theorem 1.1 exists in this setting, with elements of the Teichmüller space of S replaced by hyperbolic metrics with cone singularities, with cone angles equal to the angles at the "massive particles". There are corresponding results in the hyperbolic case [MS09, LS09] where the Bers double uniformization theorem extends to quasifuchsian manifolds with "particles": cone singularities along infinite lines, with angle in $(0, \pi)$. Here, by contrast, we allow cone angles to go all the way to 2π. This is a crucial difference since it allows massive particles to "interact" in interesting ways.

More general types of particles, including cone singularities along time-like or light-like lines and "black hole" singularities, are considered in [BBS11], where the reader can find a local description at the interaction points as well as some global examples and related constructions. Here we focus on massive particles only, our main goal is to show how the moduli space of globally hyperbolic AdS metrics with interacting massive particles, on a given 3-spacetime, can be locally parameterized by finite sequences of pairs of hyperbolic surfaces with cone singularities. This can be considered as a first step towards an extension of the Bers-type result of Mess [Mes07] quoted above. However the collisions between particles mean that the situation is much richer and more complex than for angles less than π as considered in [BS09], where no collision occurs.

1.5. **Left and right metrics of spacial slices.** A precise definition of the spacetimes with collision considered here is introduced in Definition 2.4. It is basically a pile of "spacial slices", each the product of a closed surface by an interval, containing particles but no collision, so that the collisions occur on the common boundary of two adjacent slices. We require that each such boundary surface contains a unique collision. In Section 2.4 we introduce a notion of m-spacetime, where m stands for "maximal", and prove that every admissible spacetime embeds in a unique m-spacetime satisfying a natural condition (Lemmas 2.6 and 2.7).

In Section 3 we study the holonomy representation of admissible spacetimes. We define a notion of admissible holonomy, and prove that small (admissible) deformations of an admissible spacetime are parameterized by small (admissible) deformations of its holonomy representation (Theorem 3.3).

This leads in Section 4 to the analysis of the left and right holonomies of a spacetime with collisions. In the non-singular setting, those left and right holonomies can be defined, as in [Mes07], using the decomposition of the identity component of $SO(2,2)$ as the product of two copies of $PSL(2,\mathbb{R})$. When cone singularities are present, however, this viewpoint is useful but sometimes not very convenient. In Section 4.1 we introduce two flat connections on a 3-dimensional AdS manifold and use them in Section 4.2 to construct a metric, locally isometric to $\mathbb{H}^2 \times \mathbb{H}^2$, on the space of time-like geodesics in a 3-dimensional AdS manifold.

In Section 4.3 we define a notion of "transverse vector field" along a space-like surface in an AdS manifold: it is basically a time-like vector field which behaves well at the particles and does not "rotate" too quickly. Using such a vector field along a space-like surface S, and the metric defined above on the space of time-like geodesics, it is possible to define on S two hyperbolic metrics, with cone singularities of equal angle at the intersection with the particles (see Proposition 4.20).

Those two metrics are called the left and right hyperbolic metrics of the slice, they do not depend on the choice of S or of the transverse vector field (see Lemma 4.23) and their holonomy representations are the left and right components of the holonomy representation of the AdS structure.

This construction based on a transverse vector field is more general and more flexible than that used in [KS07, BS09], which used space-like surfaces with more stringent constraints. The added flexibility is necessary here to understand how the two metrics change when the surface S moves across a particle interaction.

1.6. **Stereographic picture of spacetimes with colliding particles.** To each AdS spacetime with interacting particles is associated a sequence (or more precisely a graph) of "spacial slices", each corresponding to a domain where no interaction occurs. To each slice we associate as explained above a "stereographic picture": a "left" and a "right" hyperbolic metric, both with cone singularities of the same angles, which together are sufficient to reconstruct the spacial slice. This construction is described in Section 5.4.

A new but apparently natural notion occurs, that of a "good" spacial slice: one containing space-like surfaces with a transverse vector field. There are examples of spacetimes containing a "good" spacial slice which stop being "good" after a particle interaction. A GHMC AdS spacetimes with particles is "good" if it is made of good space-like slices, see Definition 5.11.

Adjacent spacial slices are "related" by a particle interaction. In Section 5.2 we show that the left and right hyperbolic metrics before and after the interaction in a good space-time are related by a surgery involving, for both the left and right metrics, the link of the interaction point: for both the left and right metrics, a topological disk is found, isometric to a large enough disk in the past component of the link of the interaction point, and each of those disks is replaced (in a compatible way) by a large enough disk in the future component of the link of the interaction point — see Definition 5.7 and Proposition 5.9. Since the same surgery is done on both the left and the right hyperbolic metrics, we use the term "double surgery".

As a consequence, to a good AdS space-time with particles, we can associate two distinct pieces of information:

- a "topological data", namely the position of the singular graph (the particles along with the interaction points) in the spacetime,
- a "geometric data", where to each spatial slice is associated a pair of hyperbolic surfaces with cone singularities (a "stereographic picture"), and to each interaction point is associated a double surgery (see Definition 5.16).

This is developed in Section 5.5.

1.7. **The stereographic picture is a complete description (locally).** In Section 6 we show that this locally provides a complete description of possible AdS spaces with interacting massive particles, i.e., given an AdS metric g with interacting particles, a small neighborhood g in the space of AdS metrics with interacting particles with the same singular graph is parameterized by the admissible deformations of the topological geometric data associated to the spacial slices. This is Theorem 6.1, which can be informally formulated here as follows.

Theorem 1.2. *Let M be an admissible AdS spacetime with interacting particles, and let (T, G) be the associated topological and geometric data. Then M is uniquely determined by (T, G). Moreover, any small enough deformation of G corresponds to an admissible AdS spacetime with interacting particles close to M, with the same topological data T.*

This statement is obviously informal since we did not introduce yet a number of notions necessary to make it precise, in particular concerning the space of topological and geometric data, etc. The precise form of the statement can be found below as Theorem 6.1.

In the non-singular case (Theorem 1.1) the parameterization of the space of GHMC AdS spacetimes by 2-dimensional data — a pair of hyperbolic metrics — is not only local, but global. It is of course natural to wonder whether it could be possible to extend Theorem 6.1 to a global existence theorem of an admissible AdS spacetime with particles having a given topological and geometric data. This question leads to new issues that we do not consider here.

1.8. **Contents.** The exposition below goes from the more general arguments to those tailored more specifically for AdS spacetimes with interacting particles.

In Section 2 we define the notion of admissible AdS spacetime with particles occuring in Theorem 1.2, and prove basic statements on the extension of isometries on AdS spacetimes with interacting particles.

In Section 3 we consider more specifically the holonomy representation of admissible AdS spacetimes with particles. The main result is Theorem 3.3, which states that small deformations of the AdS structure are in one-to-one correspondence to "admissible" deformations of the holonomy representation.

In Section 4, we define a notion of good spacelike slice and define the left and right hyperbolic metrics associated to such a slice. We then consider more specifically the properties of those left and right metrics in relation with a collision point in the boundary of the spacelike slice.

Section 5 deals with the change in the left and right metrics when a collision happens. The central notion of double surgery is introduced there. It is then possible to define precisely the topological and geometric data associated to an AdS spacetime with interacting particles.

Section 6 contains the main result of the paper, Theorem 6.1 (which is the same as Theorem 1.2 but stated more precisely).

The appendix contains a more technical development which is not necessary for the proof of the main result but which should clarify, for the more interested readers, the definition of a double surgery; it shows why this definition, which could at a first sight appear more complicated than necessary, is actually relevant.

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2. The space of maximal spacetimes with collisions

2.1. **Singular AdS spacetimes.** This paper is to some extend a continuation of [BBS11], where we studied the geometry of 3-dimensional AdS spacetimes with interacting particles. The particles considered in [BBS11] are more general than those under consideration here, since they are cone singularities on a space-like, a light-like, or a time-like curve, as well as more exotic objects (black holes or white holes). Here by contrast we only consider massive particles, that is, cone singularities along time-like segments.

However we will rely at some point on the analysis made in [BBS11] of the geometry near an "interaction of particles", that is, a vertex of the singular graph. Let us briefly recall here a simplified version of the notion of HS-surfaces introduced in [BBS11], suited to the purpose of the present paper, which allows to describe collisions of massive particles. Let p be a point in AdS_3. The tangent of space $T_p\,\mathrm{AdS}_3$ is a copy of Minkowski space $\mathbb{R}^{1,2}$. The link $L(p)$ at p is the space of non-oriented geodesic rays based at p; it is naturally identified with the space HS^2 of half-lines in the vector space $\mathbb{R}^{1,2}$. It admits a natural decomposition in five subsets:

- the domains $\mathbb{H}^2_+$ and $\mathbb{H}^2_-$ comprising respectively future oriented and past oriented time-like rays,
- the domain dS^2 comprising space-like rays,
- the two circles $\partial\mathbb{H}^2_+$ and $\partial\mathbb{H}^2_-$, boundaries of $\mathbb{H}^2_\pm$ in HS^2.

The domains $\mathbb{H}^2_\pm$ are notoriously Klein models of the hyperbolic plane, and dS^2 is the Klein model of de Sitter space of dimension 2. The group $\mathrm{SO}_0(1,2)$, i.e. the group of of time-orientation preserving and orientation preserving isometries of $\mathbb{R}^{1,2}$, acts naturally (and projectively) on HS^2, preserving this decomposition.

Definition 2.1. *A HS-surface is a topological surface endowed with a* $(\mathrm{SO}_0(1,2),\mathrm{HS}^2)$*-structure.*

A HS-surface admits a decomposition in hyperbolic and de Sitter regions, delimited by lines or circles of *photons*, corresponding to the circles $\partial\mathbb{H}^2_\pm$ in HS^2.

We now define the notion of *singular* HS-*surface.* Since here we only consider massive particles, we can restrict the definition given in [BBS11] and adopt the following definition:

Definition 2.2. *A singular HS-surface is a topological surface Σ containing a finite subset $P=\{p_1,...,p_k\}$ (the singular points) such that the regular part $\Sigma_{reg}=\Sigma\setminus P$ is a HS-surface. Moreover, we require that every singular point p_i admits an open neighborhood U_i such that $U_i\setminus\{p_i\}$ lies in the hyperbolic region of Σ, and is isometric to the neighborhood of the singular point in $\mathbb{H}^2_{\theta_i}$ for some θ_i in $[0,2\pi]$.*

Given a singular HS-surface Σ homeomorphic to the sphere $\mathbb{S}^2$, one can construct a 3-manifold $e(\Sigma)$ containing a closed subset $\mathcal{L}$ (the singular locus) such that $e(\Sigma)\setminus\mathcal{L}$ is a regular AdS-spacetime, and such that $\mathcal{L}$ is the union of a single point p_0 (the *collision point*) and singular rays which are massive particles based at p_0. More precisely, Σ can be interpreted as the space of geodesic rays starting from p_0, every singular point p_i corresponding to a massive particle beginning or finishing at p_0.

Mathematically speaking, the singularity along each singular ray in $\mathcal{L}$ is a cone singularity along a timelike line. If the cone angle is less than 2π there is a simple way to describe this singularity. Consider the region of the Anti de Sitter space U bounded by two timelike half-planes that meet along a time-like geodesic l and that form an angle θ. Then the space obtained by gluing the faces of U by a rotation around l is the model of a particle of cone angle θ.

Here we require all the masses to be positive, meaning that every θ_i is less than 2π. According to [BBS11, Theorem 5.6], and due to our restrictions here, there are only two possibilities:

- Σ has no de Sitter region, i.e. is reduced to its hyperbolic region. If this hyperbolic region is future (i.e. develops into $\mathbb{H}^+$), then $e(\Sigma)$ is contained in the future of p_0, which can be interpreted as a "Big Bang" singularity. If not, then p_0 is a "Big Crunch" singularity.
- Σ has one future hyperbolic region and a past hyperbolic region, both homeomorphic to the disc, and are connected by a unique de Sitter region homeomorphic to the annulus. The rays contained in the past of p_0 correspond to the elements of P lying in the future hyperbolic component; they are massive particles colliding at p_0. This collision produces a new set of massive particles which are the singular rays in the future of p_0.

Figure 1 illustrates the second situation: it represents the collision of two massive particles producing only one massive particle.

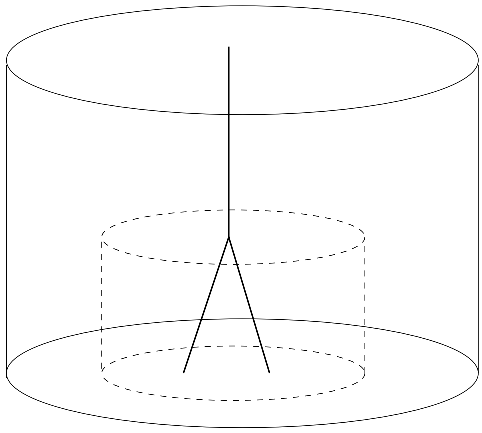

FIGURE 1. A collision of two particles.

Remark 2.3. Let θ_1, θ_2 denote the cone angle of the two massive particles in the past, and let θ be the cone angle of the massive particle in the future. Observe that the holonomy around the future singular point is a rotation of angle θ, that must be equal to the composition of one rotation of angle θ_1 and a rotation of angle θ_2 *with distinct centers of rotation.* Hence we have the inequality $\theta < \theta_1 + \theta_2$, which could at first glance appears as a violation of the conservation of mass. Actually there is no paradox here and this phenomena is well-known by physicists, the point is that the preserved quantity is the energy-momentum. Here we don't develop further this kind of consideration, and refer (for example) to [HM99, section 3].

2.2. **Globally hyperbolic AdS spacetimes with particles.** In [BBS11] we did a detailed study of the notion of global hyperbolicity in the case of AdS-spacetimes with particle. We proved in particular (Proposition 6.24) that a globally hyperbolic AdS-spacetime with particles and *no interaction* (i.e. without collision) admits a maximal globally hyperbolic extension, which is unique up to isometry. As pointed out in [BBS11, Remark 6.25], this result is completely false if one allows collisions. Indeed, once a collision occurs, there is no way to predict how many particles will be produced by the collision, and in which direction they will propagate (except in the special case where only one particle is produced). Hence, Proposition 6.24 in [BBS11] must be understood as a result on the uniqueness of maximal extensions of globally hyperbolic spacetimes, *as long as collisions do not occur.* Such a spacetime, which can be maximal among spacetimes without collision, may still be extended in a bigger globally hyperbolic spacetime, but where some collision must happen, and what happens after the collision is not uniquely determined.

The point here is that given a AdS spacetime M with interacting particles and a Cauchy surface S in M containing no collision, there is a unique maximal globally hyperbolic spacetime $M(S)$ with particles but containing no collision, which must coincide in the neighborhood of S to a neighborhood of S inside M. This observation will be crucial in the section 2.4 where we introduce the notion of m-spacetime.

2.3. **Admissible AdS spacetimes with interacting particles.** We consider in this paper AdS spacetimes with collisions of particles, defined as pairs (M, T) where

- M is a differentiable manifold and T is a closed graph embedded in M,
- a smooth AdS metric is defined on $M_{reg} = M \setminus T$,
- each edge of T is a massive particle, that is a cone singularity along a time-like curve;
- each vertex of T is a collision. This means that a neighborhood of each vertex isometrically embeds in a collision model $e(\Sigma)$ for some HS-surface Σ.

Notice that we do not consider here the cases where $e(\Sigma)$ is a Big Bang or a Big Crunch.

Given an AdS spacetime with particles (M, T), we define an isotopy in (M, T) as a homeomorphism $\phi : M \to M$ such that there exists a one-parameter family $(\phi_t)_{t\in[0,1]}$ of homeomorphisms from M to M, with $\phi_0 = Id, \phi_1 = \phi$, such that each ϕ_t sends the singular set of M to itself. Two domains in M are isotopic if there is an isotopy relative to T sending one to the other.

We will assume that the cone angle at each edge is in $(0, 2\pi)$ (positivity of the mass) and that the the link of any vertex of T is an HS-surface with positive mass (see Definition 4.2 of [BBS11]).

We will also require some natural causal properties. First we will consider the case where M is topologically the product $S_g \times \mathbb{R}$, where S_g is a closed surface of genus g. We require that there is a sequence of embedded closed space-like surfaces $S_1, S_2, \ldots, S_n$ in M which do not meet the vertices of T such that:

- S_{k+1} is contained in the future of S_k,
- the region bounded by S_k and S_{k+1} contains exactly one collision,
- the past of S_1 and the future of S_n do not contain any collision point,
- inextendible causal curves meet every S_k at one point.

Since the future and the past of each point of T is defined, curves ending at T are extendible. Moreover, since it is possible to construct an inextendible causal curve contained in T, T meets each surface S_k.

Definition 2.4. *An AdS spacetime with collisions is* **admissible** *if it satisfies the conditions above.*

Remark 2.5. In general, the sequence S_k is not unique up to isotopies of the pair (M, T). However S_1 and S_n are uniquely determined up to isotopy by the property that the past of S_1 and the future of S_n do not contain collision points.

2.4. **Maximal spacetimes.** Let M be an admissible spacetime with collisions. Let us consider a space-like surface S_i (resp. S_f) in M such that the past of S_i and the future of S_f do not contain any collision point.

The past of S_i in M can be thickened to a maximal globally hyperbolic spacetime without collision containing S_i, say M_i . Analogously the past of S_f can be thickened to a maximal globally hyperbolic spacetime without collision, say M_f. Note that the notion of maximal globally hyperbolic spacetime without collision used here is taken from [BBS11], that is, it corresponds to a spacetime which extends up to the point where a collision between the particles is bound to happen. The existence of the maximal globally hyperbolic spacetimes M_i and M_f follow from [BBS11, Proposition 6.24].

The maximality of M_i shows that $I^-_M(S_i)$ isometrically embeds in M_i. In general its image in M_i is contained in the past of S_i in M_i, but does not coincide with it.

We say that M is a m-spacetime, if the following conditions hold.

- $I^-_M(S_i)$ isometrically embeds in M_i and coincides with $I^-_{M_i}(S_i)$,
- $I^+_M(S_f)$ isometrically embeds in M_f and coincides with $I^+_{M_f}(S_f)$.

Lemma 2.6. *Every admissible spacetime with collisions embeds in a m-spacetime.*

Proof. Let S_i and S_f be space-like surfaces as above and denote by U_i (resp. U_f) the past (resp. the future) of S_i (resp. S_f) in M. Clearly U_i embeds in M_i. Let V_i be the past of U_i in M_i (and analogously let V_f be the future of U_f in M_f).

Then V_i and V_f can be glued to M by identifying U_i to its image in V_i and U_i to its image in U_f. The spacetime obtained in this way, say M', is clearly a m-spacetime. □

Let M be an admissible spacetime with collisions and T be its singular locus. We say that M is maximal if every isometric embedding

$$(M, T) \to (M', T')$$

that restricted to T is a bijection with the singular locus T' of M', is actually an isometry.

Lemma 2.7. *Every m-spacetime is maximal. Every admissible spacetime (M, T) isometrically embeds in a unique m-spacetime (M', T') such that each vertex of T' is the image of a vertex of T.*

Proof. We sketch the proof, leaving details to the reader.

Let (M, T) be an admissible spacetime and let $\pi : (M, T) \to (M_m, T_m)$ be the m-extension constructed in Lemma 2.6.

We need to prove that given any isometric embedding

$$\iota : (M, T) \to (M', T')$$

which restricts to a bijection between T and T', there is an embedding $\pi' : (M', T') \to (M_m, T_m)$ such that $\pi = \pi' \circ \iota$.

Let S_i, S_f be space-like surfaces in M as in Lemma 2.6. The embedding ι identifies S_i and S_f with disjoint space-like surfaces in M' (that with some abuse we will still denote by S_i, S_f). Moreover S_i is in the past of S_f in M'. Since ι bijectively sends T to T', the past of S_i in M' and the future of S_f in M' do not contain collision points and are globally hyperbolic domains.

Now the closure of the domains $\Omega = I^+_M(S_i) \cap I^-_M(S_f)$ and $\Omega' = I^+_{M'}(S_i) \cap I^-_{M'}(S_f)$ are both homeomorphic to $S \times [0, 1]$ and ι sends $\overline{\Omega}$ to $\overline{\Omega}'$ and $\partial\Omega$ onto $\partial\Omega'$. A standard topological argument shows that $\iota(\Omega) = \Omega'$.

Finally if M_i (resp. M_f) denotes the GHMC spacetime without collisions containing S_i (resp. S_f), we have that $I^-_{M'}(S_i)$ (resp. $I^+_{M'}(S_f)$) embeds in $I^-_{M_i}(S_i)$ (resp. $I^+_{M_f}(S_f)$).

Thus we can construct the map $\pi' : M' \to M_m$ in such a way that

- on $I^+_{M'}(S_i) \cap I^-_{M'}(S_f)$ we have $\pi' = \iota^{-1}$;
- on $I^-_{M'}(S_i)$ it coincides with the embedding in $I^-_{M_i}(S_i)$;
- on $I^+_{M'}(S_f)$ it coincides with the embedding in $I^+_{M_f}(S_f)$.

□

2.5. **The deformation space of maximal spacetimes.** In this section we introduce the space of deformations of a spacetime with collisions. Let us fix g, an oriented graph T in $S_g \times \mathbb{R}$ such that non-compact edges are properly embedded, and a family of numbers $\theta = (\theta_e)_{e \in T_1}$, where T_1 is the set of edges of T.

We consider the space $\tilde{\Omega}$ of maximal admissible AdS-structures on $S_g \times \mathbb{R}$ with singular locus T such that

- every edge e is a particle of angle θ_e;
- the orientation of e agrees with the time-orientation induced by the AdS structure;
- every vertex of T admits a neighborhood in M which embeds in $e(\Sigma)$ for some HS-surface Σ

We denote by $\mathcal{U}(g, T, \theta)$ the quotient space: an element of $\mathcal{U}(g, T, \theta)$ is a singular metric with the properties described above, up to isotopies relative to T.

There is a natural forgetful map from $\mathcal{U}(g, T, \theta)$ to the set of AdS structures on $(S_g \times \mathbb{R}) \setminus T$ up to isotopy. Proposition 2.8 ensures that this map is injective, so $\mathcal{U}(g, T, \theta)$ can be identified to a subset of the space of anti-de Sitter structures on $S \times \mathbb{R} \setminus T$. Thus $\mathcal{U}(g, T, \theta)$ inherits from this structure space a natural topology (see [CEG86, Section 1.5] for a discussion on the topology of the space of (G, X)-structures on a fixed manifold).

Proposition 2.8. *Let μ, ν be two singular metrics on $S \times \mathbb{R}$ with singular locus equal to T.*

Then any isometry

$$\psi : (S \times \mathbb{R} \setminus T, \mu) \to (S \times \mathbb{R} \setminus T, \nu)$$

extends to an isometry $\bar{\psi} : (S \times \mathbb{R}, \mu) \to (S \times \mathbb{R}, \nu)$.

Proof. Let us take $p \in T$. Consider some small space-like μ-geodesic $c : [0, 1] \to S \times \mathbb{R}$ such that $c(1) = p$ and $c([0, 1)) \cap T = \emptyset$. If c is small enough, we can find two points r_- and r_+ in $S \times \mathbb{R} \setminus T$ such that $c[0, 1) \subset I^+_\mu(r_-) \cap I^-_\mu(r_+)$.

Now let us consider the space-like ν-geodesic path $c'(t) = \psi(c(t))$ defined in $[0, 1)$. Notice that c' is an inextendible geodesic path in $S \times \mathbb{R} \setminus T$.

We know that c' is contained in $I^+_\nu(\psi(r_-)) \cap I^-_\nu(\psi(r_+))$. Thus if $S_\pm$ is a space-like surface through $r_\pm$, we have that $c' \subset I^+_\nu(S_-) \cap I^-_\nu(S_+)$ that is a compact region in $S \times \mathbb{R}$. Thus $c'(t)$ has accumulation points as $t \to 1$. All these accumulation points lie in T; if there are two different accumulation points, then there is a segment in T accumulated by c'. This is a contradiction since c' is space-like whereas T is time-like.

Hence, $c'(t)$ converges to some point in T as $t \to 1$. We define $\hat{c} = \lim_{t \to 1} c'(1)$.

To prove that ψ can be extended on T we have to check that $\hat{c}$ only depends on the endpoint p of c. In other words, if d is another space-like geodesic arc ending at p, we have to prove that $\hat{c}$ is equal to $\hat{d} = \lim_{t \to 1} \psi \circ d(t)$. By a standard connectedness argument, there is no loss of generality if we assume that d is close to c. In particular we may assume that there exists the space-like geodesic triangle Δ with vertices $c(0), d(0), p$.

Consider now the μ-geodesic segment in Δ, say I_t, with endpoints $c(t)$ and $d(t)$. The image $\psi(I_t)$ is a ν-geodesic segment contained in $S \times \mathbb{R} \setminus T$. Arguing as above, we can prove that all these segments $(\psi(I_t))_{t \in [0,1)}$ are contained in some compact region of $S \times \mathbb{R}$. Thus either they converge to a point (that is the case $\hat{c} = \hat{d}$), or they converge to some geodesic path in T with endpoints $\hat{d}$ and $\hat{c}$.

On the other hand, the ν-length of $\psi(I_t)$ goes to zero as $t \to 1$. Thus either $\psi(I_t)$ converges to a point or it converges to a lightlike path. Since T does not contain any lightlike geodesic it follows that $\psi(I_t)$ must converge to a point. Thus $\hat{d} = \hat{c}$.

Finally we can define

$$\psi(p) = \hat{c}$$

where c is any space-like μ-geodesic segment with endpoint equal to p.

Let us prove now that this extension is continuous. For a sequence of points p_n converging to $p \in T$ we have to check that $\psi(p_n) \to \psi(p)$. We can reduce to consider two cases:

- (p_n) is contained in T,
- (p_n) is contained in the complement of T.

In the former case we consider a point q in the complement of T close to p. Let us consider the μ-geodesic segment c joining q to p and the segments c_n joining q to p_n. Clearly for every $t \in (0, 1]$ the points $c_n(t)$ and $c(t)$ are time related and their Lorentzian distance converges to the distance between p_n and p as $t \to 1$. On the other hand, since $\psi(c_n(t))$, $\psi(c(t))$ converges respectively to $\psi(p_n)$, $\psi(p)$ as $t \to 1$, we can conclude that

$$d(\psi(p_n), \psi(p)) = d(p_n, p)$$

where d denotes the Lorentzian distance along T. This equation implies that $\psi(p_n) \to \psi(p)$ as $n \to +\infty$.

Let us suppose now that the points p_n are contained in the complement of T. We can take r_+ and r_- such that $p_n \in I^+_\mu(r_-) \cap I^-_\mu(r_+)$ for $n \geq n_0$. Thus the same argument used to define $\psi(p)$ shows that $(\psi(p_n))$ is contained in some compact subset of $S \times \mathbb{R}$. To conclude it is sufficient to prove that if $\psi(p_n) \to x$ then $x = \psi(p)$. Clearly $x \in T$. Moreover either x coincides with $\psi(p)$ or there is a piece-wise geodesic segment in T connecting x to $\psi(p)$. Since the length of this geodesic should be equal to the limit of $d(p_n, p)$, that is 0, we conclude that $x = \psi(p)$.

Eventually we have to check that the map ψ is an isometry at p. Let us note that ψ induces a map

$$\psi^\# : \Sigma_p \to \Sigma'_{\psi(p)} \ ,$$

where Σ_p and $\Sigma'_{\psi(p)}$ are respectively the link of p with respect to μ and the link of $\psi(p)$ with respect to ν. Simply, if c is the tangent vector of a geodesic arc c at p, we define $\psi^\#(v) = w$ where w is the tangent vector to the arc $\psi \circ c$ at $\psi(p)$. Notice that ψ is an isometry around p if and only if $\psi^\#$ is a HS-isomorphism.

Clearly $\psi^\#$ is bijective and is an isomorphism of HS surfaces outside the singular locus. On the other hand, the singularities are contained in the hyperbolic regions, which are the metrics completions of their regular parts. Hence the bijection $\psi^\#$ is an extension of the isometry between the regular parts, therefore a HS-isomorphism. □

3. The holonomy map on the space of admissible spacetimes

3.1. **Holonomies of singular AdS-spacetimes around the singular locus.** Let (M, T) be a an admissible AdS structure on $S_g \times \mathbb{R}$. Recall that M_{reg} is the regular part $M \setminus T$. We consider the holonomy

$$h : \pi_1(M_{reg}) \to SO(2, 2) \ .$$

Fix a collision point, p, of M, and let Σ_p be the link of the point p. Notice that the inclusion map $\iota : \Sigma_p \to M$ produces an inclusion of groups well-defined up to conjugation

$$\iota_\star : \pi_1(\Sigma_{p,reg}) \to \pi_1(M_{reg}) \ ,$$

where $\Sigma_{p,reg}$ is the regular part of Σ_p.

In this section we will investigate the behavior of the restriction of h to $\pi_1(\Sigma_{p,reg})$.

Lemma 3.1. *If γ is a meridian loop in $\pi_1(M_{reg})$ around a particle e — a loop going once around e — then $h(\gamma)$ is a rotation of angle θ_e around a time-like geodesic of AdS_3.*

If p is a collision point, then there is $x_0 \in AdS_3$ which is fixed by $h(\gamma)$ for any $\gamma \in \pi_1(\Sigma_{p,reg})$.

Proof. In the first case a regular neighborhood of γ embeds in the local model of the particle whose holonomy is a rotation of angle θ_e.

In the second case, notice that a neighborhood of the point p is isometric to the AdS cone of the HS surface Σ_p. So in order to prove the statement it is sufficient to check that the holonomy of the AdS cone of a HS surface fixes a point in AdS_3.

Now fix any point $x_0 \in AdS_3$ and identify $T_{x_0}AdS_3$ with the Minkowski space $\mathbb{R}^{2,1}$. In this way the HS-sphere is identified with the space of geodesic rays starting from $\tilde{x}_0$. If $c_0 : \tilde{\Sigma}_p \to HS_2$ is the developing map of Σ_p, then the developing map of the AdS cone of Σ_p is the map

$$dev : \tilde{\Sigma}_p \times \mathbb{R} \to AdS_3$$

such that if x is not a photon of $\tilde{\Sigma}_p$ then $dev(x,t) = c_0(x)[t]$, where $c_0(x)[\bullet]$ is the arc-length parameterization of the segment $c_0(x)$.

In particular, if γ is an element of $\pi_1(\Sigma_{p,reg})$ then the holonomy of the AdS cone around γ is the transformation of $SO(2,2)$ which fixes x_0 such that its differential at x_0 coincides with the holonomy of Σ_p (as HS-surface) around γ. □

3.2. **The holonomy map.** After the preliminary material in the previous section, we now turn to the statement and proof of Theorem 3.3. We first define the notion of "admissible" holonomy representations.

Definition 3.2. *We say that a representation*

$$h : \pi_1(M_{reg}) \to SO(2,2)$$

is admissible if

- *for any meridian loop γ in M_{reg} around an edge e of T $h(\gamma)$ is a rotation around a time-like geodesic in AdS_3 of angle θ_e,*
- *for any vertex $p \in T$, there is a point $x_0 \in AdS_3$ fixed by $h(\gamma)$ for any $\gamma \in \pi_1(\Sigma_{p,reg})$.*

We denote by $\mathcal{R}(g,T,\theta)$ the space of admissible representations up to conjugacy.

By Lemma 3.1 the holonomy of any structure of $\mathcal{U}(g,T,\theta)$ lies in $\mathcal{R}(g,T,\theta)$. We prove now that $\mathcal{U}(g,T,\theta)$ is locally homeomorphic to $\mathcal{R}(g,T,\theta)$.

Theorem 3.3. *The holonomy map*

$$\mathcal{U}(g,T,\theta) \to \mathcal{R}(g,T,\theta)$$

is a local homeomorphism.

To prove this proposition we will use the following well-known fact about (G,X)-structures on compact manifolds with boundary, see Theorem 1.7.1 in [CEG86]. Let us recall that a collar of a manifold N with boundary is a neighborhood of the boundary homeomorphic to $\partial N \times [0,1)$.

Lemma 3.4. *Let N be a smooth compact manifold with boundary and let $N' \subset N$ be a submanifold such that $N \setminus N'$ is a collar of N.*

- *Given a (G,X)-structure M on N let $hol(M) : \pi_1(N) \to G$ be the corresponding holonomy (that is defined up to conjugacy). Then, the holonomy map from the space of (G,X)-structures on N to the space of representations of $\pi_1(N)$ into G (up to conjugacy)*

  $$M \mapsto hol(M)$$

  is an open map.
- *Let M_0 be a (G,X)-structure on N and denote by M_0' the restriction of M_0 to N'. There is a neighborhood $\mathcal{U}$ of M_0 in the set of (G,X)-structures on N and a neighbourhood $\mathcal{V}$ of M_0' in the set of (G,X)-structures on N' such that*

(1) *If $M \in \mathcal{U}$ and $M' \in \mathcal{V}$ share the same holonomy, there is an embedding as (G, X)-manifolds*
$$M' \hookrightarrow M.$$
homotopic to the inclusion $N' \hookrightarrow N$.
(2) *For every $M' \in \mathcal{V}$ there is M in $\mathcal{U}$ such that $hol(M) = hol(M')$.*

First we prove Theorem 3.3 assuming just one collision in M. Let p_0 be the collision point of M and Σ_0 be the link of p_0 in M (that is a HS-surface). Denote by $\Sigma_{0,reg}$ the regular part of Σ_0 and by $G_0 < \pi_1(S \times \mathbb{R} \setminus T)$ the fundamental group of $\Sigma_{0,reg}$.

Given any representation $h \in \mathcal{R}$, let us denote by x_0 the point fixed by $h(G_0)$ and by $Lh : G_0 \to SO(2,1) = SO(T_{x_0}AdS_3)$ the action of G_0 at the tangent space of x_0. Notice that, identifying $SO(T_{x_0}AdS_3)$ with $SO(2,1)$, the conjugacy class of Lh only depend of the conjugacy class of h. Moreover the map sending h to Lh is a continuous map between $\mathcal{R}$ and the space of conjugacy classes of representations of G_0 into $SO(2,1)$.

Lemma 3.5. *There is a neighborhood $\mathcal{U}_0$ of Σ_0 in the space of HS-surfaces homeomorphic to Σ_0 such that the holonomy map on $\mathcal{U}_0$ is injective.*

Moreover, there is a neighbourhood $\mathcal{V}$ of h in $\mathcal{R}(g, T, \theta)$ such that for every $h' \in \mathcal{V}$ there is an HS-surface in $\mathcal{U}_0$, say $\Sigma(h')$, such that the holonomy of $\Sigma(h')$ is conjugate to $Lh' : G_0 \to SO(2,1)$.

Proof. Around each cone point q_i of Σ_0 take small disks
$$\Delta_1(i) \supset \Delta_2(i)$$
Let now Σ, Σ' be two HS-surfaces close to Σ_0 sharing the same holonomy. By Lemma 3.4, up to choosing $\mathcal{U}_0$ sufficiently small, there is an isometric embedding
$$f : (\Sigma \setminus \bigcup \Delta_2(i)) \to \Sigma'.$$
Moreover, $\Delta_1(i)$ equipped with the structure induced by Σ embeds in Σ' (this because the holonomy locally determines the HS-structure near the singular points of HS-surfaces). It is not difficult to see that such an inclusion coincides with f on $\Delta_1(i) \setminus \Delta_2(i)$ (basically this depends on the fact that an isometry of a hyperbolic annulus into a disk containing a cone point is unique up to rotations). Thus gluing those maps we obtain an isometry between Σ and Σ'.

To prove the last part of the statement, let us consider for each cone point a smaller disk $\Delta_3(i) \subset \Delta_2(i)$. Let $U = \Sigma_0 \setminus \bigcup \Delta_3(i)$. Clearly we can find a neighbourhood $\mathcal{V}$ of h such that if $h' \in \mathcal{V}$ then there is a structure U' on U close to the original one with holonomy Lh'. On the other hand it is clear that there exists a structure, say $\Delta'_1(i)$, on $\Delta_1(i)$ with cone singularity with holonomy given by Lh' and close to the original structure. By Lemma 3.4, if h' is sufficiently close to h, then $\Delta_2(i) \setminus \Delta_3(i)$ equipped with the structure given by U' embeds in $\Delta'_1(i)$. Moreover $\partial\Delta_2(i)$ bounds in $\Delta'_1(i)$ a disk $\Delta(i)$ containing the cone point. Thus we can glue the $\Delta_1(i)$ to U' to obtain the HS-surface with holonomy Lh'. □

Let $C(h')$ be the AdS cone on $\Sigma(h')$. By construction, the holonomy of $C(h')$ is conjugated to $h'|_{G_0}$.

Consider now two space-like surfaces S_1, S_2 in M orthogonal to the singular locus that are disjoint and such that $p_0 \in I^+(S_1) \cap I^-(S_2)$. Let $M_0 = I^+(S_1) \cap I^-(S_2)$. Clearly S_1 is the past boundary of M_0 and S_2 is its future boundary.

Take the neighborhood $\mathcal{V}$ of h in $\mathcal{R}$ given by Lemma 3.5 and, for $h' \in \mathcal{V}$, consider the AdS cone $C(h')$ constructed above. The following is a simple application of Lemma 3.4, we leave the proof to the reader.

Corollary 3.6. *Let N_0 be the AdS-manifold with boundary obtained by cutting a regular neighborhood of T from M_0, and let U be a collar of ∂N_0 in N_0. If N' is a slight deformation of the AdS-structure on N_0 with holonomy h' then U, with the AdS-structure induced by N', embeds in $C(h')$.*

Now up to shrinking $\mathcal{V}$ we may suppose that:

- For any $h' \in \mathcal{V}$, there is a deformation of the AdS structure on N_0, say $N_0(h')$, with holonomy h'.
- If $U(h')$ denotes the AdS structure induced by $N_0(h')$ on U, then $U(h')$ isometrically embeds in $C(h')$.
- The image of the boundary of $N_0(h')$ through this embedding is the frontier of a regular neighborhood B of the singular locus in $C(h')$.
- The image of $U(h')$ is disjoint from B

The spacetime obtained by gluing B to $N_0(h')$, by identifying the boundary of $N_0(h')$ with the frontier of B, is a spacetime with collisions with holonomy h'. Its maximal extension, say $M(h')$, is a m-spacetime with holonomy h'.

To conclude we have to prove that if h' is sufficiently close to h, then $M(h')$ is unique in a neighborhood of M_0.

In fact, it is sufficient to show that any given m-spacetime M' with holonomy h' close to M must contain a spacetime $M'_0 \subset M'$ containing the collisions which embeds isometrically in $M(h')$. We can assume M'_0 close to M_0 (this precisely means that M'_0 is obtained by deforming slightly the metric on M_0).

Take small neighborhoods $U_2 \subset U_1$ of the singular locus in M'_0. By Lemma 3.4 $M'_0 \setminus U_2$ embeds in $M(h')$. By the uniqueness of the HS-surface with holonomy h', U_1 embeds in $M(h')$ as well.

It is not difficult to check that there exists a unique isometric embedding $U_1 \cap U_2 \to M(h')$, so the embeddings $U_1 \hookrightarrow M(h')$ and $M'_0 \setminus U_2 \hookrightarrow M(h')$ coincide on the intersection. So they can be combined to an embedding $M'_0 \hookrightarrow M(h')$.

This concludes the proof of Theorem 3.3 when only one interaction occurs. The following lemma allows to conclude in the general case by an inductive argument.

Lemma 3.7. *Let S be a space-like surface of M, and let M_-, M_+ be the past and the future of S in M. Suppose that for a small deformation h' of the holonomy h of M there are two spacetimes with collisions $M'_- \cong M_-$ and $M'_+ \cong M_+$ such that the holonomy of $M'_\pm$ is equal to $h'|_{\pi_1(M_\pm)}$. Then there is a spacetime M' close to M containing both M'_- and M'_+.*

Proof. Let $N(h)$ denote the maximal GH structure with particles on $S \times \mathbb{R}$ whose holonomy is $h|_{\pi_1(S_{reg})}$. There is a neighborhood of S in M which embeds in $N(h)$. We can suppose that $S \subset M$ is sent to $S \times \{0\}$ through this embedding.

Now let $U_\pm$ be a collar of S in $M_\pm$ such that the image of U_- in $N(h)$ is $S \times [-\epsilon, 0]$ and the image of U_+ is $S \times [0, \epsilon]$ for some $\epsilon > 0$.

If h' is sufficiently close to h, then there is an isometric embedding of $U_\pm$ (considered as subset of $M'_\pm$) into $N(h')$

$$i_\pm : U_\pm \hookrightarrow N(h')$$

such that the image of U_- is contained in $S \times [-2\epsilon, \epsilon/3]$ and contains $S \times \{-\epsilon/2\}$, and that the image of U_+ is contained in $S \times [-\epsilon/3, 2\epsilon]$ and contains $S \times \{\epsilon/2\}$. Thus we can glue $M'_\pm$ and $S \times [-\epsilon/2, \epsilon/2]$ by identifying $p \in U_\pm \cap i_\pm^{-1}(S \times [-\epsilon/2, \epsilon/2])$ with its image. The spacetime we obtain, say M', clearly contains M'_- and M'_+. ☐

Remark 3.8. To prove that there is a unique m-spacetime in a neighborhood of M with holonomy h', we again use an inductive argument. Suppose we can find in any small neighborhood of M two m-spacetimes M' and M'' with holonomy h'. We fix a space-like surface S in M such that both the future and the past of S, say $M_\pm$, contain some collision points. Let $U \subset V$ be regular neighborhoods of S in M with space-like boundaries. We can consider collars $U' \subset V'$ in M' and $U'' \subset V''$ in M'' such that

- $U' \cup U''$ and $V' \cup V''$ are close to U and V respectively,
- they do not contain any collision,
- they have space-like boundary.

Applying the inductive hypothesis on the connected regions of the complement of U' in M' and U'' in M' we have that for h' sufficiently close to h there is an isometric embedding

$$\psi : M' \setminus U' \to M''$$

such that $\psi(\partial U')$ is contained in V''.

Now consider the isometric embeddings

$$u' : V' \to N(h') \qquad u'' : V'' \to N(h')$$

where $N(h')$ is the GHMC structure on $S \times \mathbb{R}$ whose holonomy is $h'|_{\pi_1(S_{reg})}$. Notice that the maps u' and $u'' \circ \psi$ provide two isometric embeddings

$$V' \setminus U' \to N(h')$$

so they must coincide (we are using the fact that the inclusion of a GH spacetime with particles in its maximal extension is uniquely determined).

Finally we can extend ψ on the whole M' by setting on V'

$$\psi = (u'')^{-1} \circ u' .$$

4. The left and right metrics on space-like slices of good spacetimes

The main goal of this section is to construct, for each space-like slice containing no particle collision, two hyperbolic metrics with cone singularities on a surface. It is the sequence of those pairs of hyperbolic metrics (or, more precisely, the graph of those pairs of hyperbolic metrics) which provide a complete description of a spacetime with interacting particles, as seen in Theorem 6.1.

4.1. **The left and right connections.** The constructions of the left and right hyperbolic metrics, below, can be understood in a fairly simple manner through two flat linear connections on the tangent bundle of an AdS 3-manifold. In this first part we consider an AdS manifold M, which could for instance be the regular part of an AdS manifold with particles.

Definition 4.1. *Let M be an AdS manifold and ∇ be its Levi-Civita connection. On M we consider two linear connections defined by*

$$D^l_v u = \nabla_v u + u \times v \ , \ D^r_v u = \nabla_v u - u \times v \ ,$$

where $\times$ is the cross-product in AdS_3 — it can be defined by $(v \times y)^* = *(v^* \wedge y^*)$, where v^* is the 1-form dual to v for the AdS metric and $*$ is the Hodge star operator.

Lemma 4.2. *D^l and D^r are flat connections compatible with the AdS-metric.*

Proof. The fact that D^l and D^r are compatible with the metric easily follows from the property of the cross-product.

Since the cross product is flat with respect to the Levi-Civita connection, there is a simple relation between the curvature R^l of D^l to the curvature R of ∇, that can be proved by a direct computation. In fact we have

$$R^l(v,w)u = R(v,w)u + v \times (w \times u) - w \times (v \times u) \ .$$

A basic point is that the Riemann curvature tensor of a Lorentzian space form of constant curvature K can be easily expressed in terms of the cross product. Indeed if v, w, u are tangent vectors in M we have

$$R(v, w)u = K(v \times w) \times u \ . \tag{1}$$

So we get

$$R^l(v, w)u = u \times (v \times w) + v \times (w \times u) + w \times (u \times v) = 0$$

where the last identity holds for the Jacobi identity for the cross product. □

Remark 4.3. For a 3-dimensional Riemannian space form, formula (1) holds with the opposite sign. For this reason the construction above applied to the Riemannian setting produces two flat connections on the unit tangent bundle of a 3-dimensional spherical manifold.

This phenomenon is closely related to the fact that the isometry group of the three dimensional sphere S^3 (as well as the isometry group of AdS_3) has a natural product structure.

Definition 4.4. *We call $T^{1,t}M$ the bundle of positively directed unit time-like vectors on M.*

Notice that if V is a unit time-like vector field, then $D^l_x V$ and $D^r_x V$ are orthogonal to V at any point. In particular they belong to the tangent space of $T^{1,t}M$.

In this section we want to relate the holonomies of connections D^l and D^r — that are representations $\pi_1(M) \to SO_0(2, 1)$ — to the holonomy of the AdS-structure on M, that is a representation $\pi_1(M) \to SO(2, 2)$.

First we prove that the holonomy of the model space AdS_3 is trivial.

Lemma 4.5. *The holonomy of D^l and D^r on AdS_3 is trivial.*

Proof. Since the fundamental group of AdS_3 is generated by the geodesic curve $\gamma(t) = (\cos t, \sin t, 0, 0)$, it is sufficient to compute the linear maps

$$h^l(\gamma) : T_{\gamma(0)}AdS_3 \ni v \mapsto V^l(2\pi) \in T_{\gamma(0)}AdS_3$$

where $V^l(t)$ is the D^l-parallel field along γ with initial condition $V^l(0) = v$.

If $v = \dot\gamma(0)$, it is easy to see that $V^l(t) = \dot\gamma(t)$, so $h^l(\gamma)(v) = v$.

If v is orthogonal to $\dot\gamma$, denote by V the ∇-parallel field extending v. Then we easily see that

$$V^l(t) = \cos(t)V(t) \ - \ \sin(t)V(t) \times \dot\gamma \ .$$

Since $V(2\pi) = v$, we obtain that $h^l(\gamma)(v) = v$.

By linearity we conclude that $h^l(\gamma)$ is the identity. Similarly we can prove that $h^r(\gamma)$ is the identity. □

Let us fix a base point $x_0 \in AdS_3$ and consider the maps

$$\tau^l(x), \tau^r(x) : T_{x_0}AdS_3 \to T_x AdS_3$$

obtained by using parallel transport with respect to D^l and D^r along any curve joining x_0 to x. By Lemma 4.5, these maps are well-defined.

We identify once for all $T_{x_0}AdS_3$ with Minkowski space, and $O(T_{x_0}AdS_3)$ with $O(2, 1)$.

Given any isometry $g \in SO(2, 2)$, we can consider the linear transformations of $T_{x_0}M$ obtained by composing the differential map $dg(x_0) : T_{x_0}AdS_3 \to T_{g(x_0)}AdS_3$ by the inverse of the parallel transports $\tau_l(g(x_0)), \tau_r(g(x_0)) : T_{x_0}AdS_3 \to T_{g(x_0)}AdS_3$. Namely

$$g_l = \tau_l(g(x_0))^{-1} \circ dg(x_0) \ , \ g_r = \tau_r(g(x_0))^{-1} \circ dg(x_0) \ .$$

Notice that g_l and g_r are elements of $SO_0(2, 1)$.

Lemma 4.6. *The map*

$$I : SO_0(2,2) \ni g \mapsto (g_l, g_r) \in SO_0(2,1) \times SO_0(2,1) \tag{2}$$

is a surjective homomorphism and its kernel is the $\mathbb{Z}/2\mathbb{Z}$*-subgroup generated by the antipodal map.*

Proof. Notice that $\tau_l(hg(x_0)) = \tau_l' \circ \tau_l(h(x_0))$ where τ_l' is the parallel transport $T_{h(x_0)}M \to T_{hg(x_0)}M$. Since D^l is preserved by isometries of AdS_3 we deduce that $\tau_l' = dh(g(x_0)) \circ \tau_l(g(x_0)) \circ (dh(x_0))^{-1}$. From these formulas we easily get that I is a homomorphism.

Given a Killing vector field $X \in \mathfrak{so}(2,2)$ we have that

$$dI_{id}(X) = (D^l X(x_0), D^r X(x_0)) \ .$$

(Notice that $D^l X$ and $D^r X$ are skew-symmetric operators of $T_{x_0}M$.) This formula easily shows that dI_{id} is an isomorphism. We conclude that I is a covering map. Since the center of $SO_0(2,1)$ is trivial, $\ker I$ is the center of $SO_0(2,2)$, that is, the group generated by the antipodal map. □

Remark 4.7. Mess [Mes07] described this map $\mathrm{SO}_0(2,2) \to PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$ in a different way, using the double ruling of the projective quadric $C = \{[x] \in \mathbb{P}(\mathbb{R}^{2,2}) | \langle x, x\rangle = 0\}$.

Lemma 4.8. *Through the identification between* $SO_0(2,2)$ *with* $SO_0(2,1) \times SO_0(2,1)$*, the stabilizer in* $SO_0(2,2)$ *of a point* $x \in AdS_3$ *corresponds to a subgroup of* $SO_0(2,1) \times SO_0(2,1)$ *conjugated to the diagonal subgroup.*

Proof. It is sufficient to prove the statement in the case where $x = x_0$. In that case it is clear by definition that if g fixes x_0, then $g_l = g_r = dg(x_0)^{-1}$. So the stabilizer of x_0 is contained in the diagonal subgroup. Since those groups have the same dimension, they must coincide. □

Let us fix $p_0 \in M$. For any loop γ centered at p_0 let us denote by $h_l(\gamma), h_r(\gamma) \in SO_0(T_{x_0}M) \cong SO_0(2,1)$ the holonomy along γ^{-1} with respect to D^l and D^r. The reason why we consider the parallel transport along the inverse of γ is that in this way $h_\bullet(\gamma\delta) = h_\bullet(\gamma)h_\bullet(\delta)$.

Since D^l and D^r are flat, $h_l(\gamma), h_r(\gamma)$ only depend on the homotopy class of γ. In particular two holonomy representations $h_l, h_r : \pi_1(M) \to SO_0(2,1)$ are associated to D^l and D^r. Though the construction depends on the choice of a point p_0, those representations are well-defined up to conjugation.

Lemma 4.9. *Up to conjugation we have that*

$$I \circ h = (h_l, h_r) \ .$$

Proof. Let us fix a universal covering map $\pi : \tilde{M} \to M$, a base point $p_0 \in M$, and a point $\tilde{p}_0 \in \pi^{-1}(p_0)$. Without loss of generality we may suppose that the developing map sends $\tilde{p}_0$ to x_0.

Let $\gamma : [0,1] \to M$ be a closed loop in M such that $\gamma(0) = \gamma(1) = p_0$. Consider the lift $\overline{\gamma}$ of γ to $\tilde{M}$ with starting point $\tilde{p}_0$ and denote by $L = L(\gamma)$ the covering automorphism such that $L(\tilde{p}_0) = \overline{\gamma}(1)$. Since $\pi \circ L = \pi$ we have

$$d\pi \circ dL = d\pi \ . \tag{3}$$

Let $g = h(\gamma)$ and $\bar{\tau}_l$ be the D^l-parallel transport along $\overline{\gamma}^{-1}$ we have the following commutative diagram

$$\begin{array}{ccccc} T_{p_0}M & = & T_{p_0}M & \xrightarrow{h_l(\gamma)} & T_{p_0}M \\ {\scriptstyle d\pi}\uparrow & & {\scriptstyle d\pi}\uparrow & & {\scriptstyle d\pi}\uparrow \\ T_{\tilde{p}_0}\tilde{M} & \xrightarrow{dL} & T_{L(\tilde{p}_0)}\tilde{M} & \xrightarrow{\bar{\tau}_l} & T_{\tilde{p}_0}\tilde{M} \\ {\scriptstyle d(dev)}\downarrow & & {\scriptstyle d(dev)}\downarrow & & {\scriptstyle d(dev)}\downarrow \\ T_{x_0}AdS_3 & \xrightarrow{dg} & T_{g(x_0)}AdS_3 & \xrightarrow{\tau_l(g(x_0))^{-1}} & T_{x_0}AdS_3 \end{array} \quad . \tag{4}$$

Indeed, the commutativity of the squares on the upper row is easy to check. The commutativity of the second square of the second lower row relies on the fact that dev sends D^l-parallel vector field on $\tilde{M}$ to D^l-parallel vector field on AdS_3. Finally the commutativity of the first square of the lower row follows from the fact that, by definition of holonomy, $g \circ dev = dev \circ L$.

By diagram (4), identifying $T_{p_0}M$ with $T_{x_0}AdS_3$ through the map $d\pi(\tilde{p}_0) \circ (d(dev)(\tilde{p}_0))^{-1} : T_{x_0}AdS_3 \to T_{p_0}M$, we have that $h_l(\gamma) = \tau_l(g(x_0))^{-1} \circ dg(x_0) = g_l$ and analogously $h_r(\gamma) = \tau(g(x_0))^{-1} \circ dg(x_0) = g_r$. □

4.2. **The left and right metrics.** Every smooth curve $V(t) = (x(t), v(t))$ in TM can be regarded as a vector field along the curve $x(t) = \pi(V(t))$, so we can consider its covariant derivative with respect to D^l and D^r.

There is a splitting of $T(TM)$ associated with the connections D^l and D^r. Namely we have

$$T(TM) = T^V(TM) \oplus H^l = T^V(TM) \oplus H^r$$

where

- $T^V(TM)$ is the vertical tangent space, that is the tangent space to the fiber (it is independent of the connection),
- H^l and H^r are the horizontal spaces (depending on the connection): a vector $\xi \in T_{(p_0,v_0)}(TM)$ lies in H^l (resp. H^r) if and only if there exists a D^l-parallel (resp. D^r-parallel) curve $V(t) = (x(t), v(t))$ with $\dot{V}(0) = \xi$.

Each of these splittings provides a linear projection

$$P^l, P^r : T_{(p,v)}(TM) \to T^V_{(p,v)}(TM) = T_pM$$

and we easily see that $P^l(\xi) = \frac{D^l V}{dt}(0)$ whereas $P^r(\xi) = \frac{D^r V}{dt}(0)$ where $V(t) = (x(t), v(t))$ is any curve in TM such that $\dot{V}(0) = \xi$.

Notice that if $(x, v) \in T^{1,t}M$ (cf. Definition 4.4) and $\xi \in T_{(x,v)}(T^{1,t}M)$, we can construct the curve $V(t) = (x(t), v(t))$ so that $\langle v(t), v(t)\rangle = -1$. Since D^l and D^r are compatible with the metric, we get that $P^l(\xi)$ and $P^r(\xi)$ are orthogonal to v in T_xM, so either they are 0 or they are space-like.

Definition 4.10. *We call M_l and M_r the two degenerate metrics (everywhere of rank 2) defined on $T^{1,t}M$ as follows:*

$$M_l(\xi) = ||P^l(\xi)||^2 \,, \;\; M_r(\xi) = ||P^r(\xi)||^2 \,.$$

By construction, M_l and M_r are symmetric quadratic forms on the tangent space of $T^{1,t}M$, and they are semi-positive, of rank 2 at every point.

We will derive a more concrete expressions of metrics M_l and M_r that will be useful in the sequel. We use a natural identification based on the Levi-Civita connection ∇ of M:

$$\forall (x,v)\in T^{1,t}M\,,\quad T_{(x,v)}(T^{1,t}M)\simeq T_xM\times v^\perp\subset T_xM\times T_xM\ .$$

In this identification, given $v'\in v^\perp$, the vector $(0,v')$, considered as a vector in $T(T^{1,t}M)$, corresponds to a "vertical" vector, fixing x and moving v according to v'. And, given $x'\in T_xM$, the vector $(x',0)$, considered as a vector in $T(T^{1,t}M)$, corresponds to a "horizontal" vector, moving x according to x' while doing a parallel transport of v (for the connection ∇).

Notice that $P^l(0,v')=P^r(0,v')=v'$: indeed by definition there exists a curve $V(t)=(x,v(t))$ in T_xM whose derivative in 0 is $(0,v')$, and we easily see that its covariant derivative (for any connection), coincides with v'.

On the other hand, given a vector $(x',0)$ in $T_{(x,v)}TM$, it can be extended to a curve $V(t)=(x(t),v(t))$ which is ∇-parallel and such that $\dot x(0)=x'$. So we have

$$\frac{D^l}{dt}V(t)=\frac{DV}{dt}+v(t)\times\dot x(t)\,,$$

and we conclude that $P^l(x',0)=v\times x'$. Analogously $P^r(x',0)=-v\times x'$. So we conclude that

$$M_l((x',v'),(x',v'))=\|v'+v\times x'\|^2\ ,\ M_r((x',v'),(x',v'))=\|v'-v\times x'\|^2\ .$$

Lemma 4.11. *With those definitions:*

- *M_l and M_r vanish on the integral curves of the geodesic flow of M.*
- *M_l and M_r are invariant under the geodesic flow of M.*

Proof. We denote by $\phi_\bullet : TM\to TM$ the geodesic flow on TM. Let us notice that the geodesic equation of the connection ∇ coincides with the geodesic equation of D^l and D^r. Thus $\phi_\bullet$ can be regarded as the geodesic flow of the connection D^l (D^r) as well.

Since the orbits of the geodesic flow are tangent to the horizontal space H^l (resp. H^r), M_l and M_r vanish on the direction tangent to the geodesic flow.

Given a point $(x,v)\in TM$ and $\xi\in T(TM)$, let us consider any curve $V(t)=(x(t),v(t))$ such that $\xi=\dot V(0)$. Putting $W(s,t)=\phi_s(V(t))=(y(s,t),w(s,t))$, by definition we have that $y(\bullet,t)$ is a geodesic for any fixed t and that $\frac{\partial y}{\partial s}(s,t)=w(s,t)$. In particular $\frac{D^lW}{ds}=0$.

By definition,

$$\frac{d}{ds}M_l(d\phi_s(\xi))=\frac{d}{ds}\left\langle\frac{DW}{dt},\frac{DW}{dt}\right\rangle(s,0)\,.$$

On the other hand, since D^l is flat we have

$$\frac{d}{ds}\left\langle\frac{DW}{dt},\frac{DW}{dt}\right\rangle=2\left\langle\frac{D}{dt}\frac{DW}{ds},\frac{DW}{dt}\right\rangle=0$$

and this shows that $M_l(d\phi_s(\xi))$ is constant. □

Definition 4.12. *We denote by $G(M)$ the space of time-like maximal geodesics in M, and by m_l and m_r the degenerate metrics on $G(M)$ induced by M_l and M_r, respectively.*

Lemma 4.13. *$(G(M),m_l\oplus m_r)$ is locally isometric to $\mathbb{H}^2\times\mathbb{H}^2$.*

Proof. Since the statement is local, we may suppose that M is simply connected. Let us fix a point $x_0\in M$. Notice that the set of time-like unit-vector at x_0, say $T^{1,t}_{x_0}M$, is a space-like surface in $T_{x_0}M$ which is isometric to the hyperbolic plane. We isometrically identify $T^{1,t}_{x_0}M$ with $\mathbb{H}^2$.

Let us consider the maps $\phi^l,\phi^r:T^{1,t}M\to T^{1,t}_{x_0}M=\mathbb{H}^2$ defined by using the parallel transport for D^l and D^r.

We claim that $M_l = (\phi^l)^*(g_{\mathbb{H}})$ and $M_r = (\phi^r)^*(g_{\mathbb{H}})$.

To check the claim, let $V(t) = (x(t), v(t))$ for $t \in (-\epsilon, \epsilon)$ be any curve in TM with $V(0) = (x, v)$ and $\dot{V}(0) = \xi$. There is a homotopy $\sigma(t, s) : (-\epsilon, \epsilon) \times [0, 1] \to M$ such that $\sigma(t, 0) = x(t)$ and $\sigma(t, 1) = x_0$. The field V can be uniquely extended to a field W on σ so that $\frac{D^l W}{ds} = 0$.

By definition we have that $W(t, 1) = \phi^l(V(t))$, so $d\phi^l(\xi) = \frac{D^l W}{dt}(0, 1)$. In particular we have

$$\langle d\phi^l(\xi), d\phi^l(\xi)\rangle = \left\langle \frac{D^l W}{dt}(0,1), \frac{D^l W}{dt}(0,1)\right\rangle .$$

On the other hand, since D^l is flat we have

$$\frac{d}{ds}\left\langle \frac{D^l W}{dt}, \frac{D^l W}{dt}\right\rangle = 2\left\langle \frac{D^l}{dt}\frac{D^l W}{ds}, \frac{D^l W}{dt}\right\rangle = 0\,,$$

so we deduce that

$$\langle d\phi(\xi), d\phi(\xi)\rangle = \left\langle \frac{D^l W}{dt}(0,0), \frac{D^l W}{dt}(0,0)\right\rangle = \left\langle \frac{D^l V}{dt}(0), \frac{D^l V}{dt}(0)\right\rangle = ||P^l(\xi)||^2\,.$$

We can obtain in the same way that the pull-back of the hyperbolic metric through ϕ^r is M^r. In particular, considering the map $\phi(v) = (\phi^l(v), \phi^r(v)) \in \mathbb{H}^2 \times \mathbb{H}^2$, $M_l \oplus M_r$ is the pull-back of the sum of hyperbolic metrics through ϕ.

Notice that the orbits of the geodesic flow are horizontal for both D^l and D^r (this because geodesics of ∇ coincide with geodesics of D^l and D^r), so we deduce that ϕ is constant on the orbits of geodesic flow, so it induces a map

$$\bar{\phi} : G(M) \to \mathbb{H}^2 \times \mathbb{H}^2$$

and we have that $\bar{\phi}^*(g_{\mathbb{H}} \oplus g_{\mathbb{H}}) = m_l \oplus m_r$.

In order to conclude we should prove that $\bar{\phi}$ is a local diffeomorphism. This is equivalent to showing that $m_l \oplus m_r$ is non-degenerate. Since the tangent space of $G(M)$ is the quotient of the tangent space of $T^{1,t}M$ along the line tangent to the orbit of the geodesic flow, it is sufficient to prove that vectors $\xi \in T_{(x,v)}(T^{1,t}M)$ such that $M_l(\xi) = M_r(\xi) = 0$ are tangent to the orbit of the geodesic flow.

Taking such a $\xi = (x', v')$, we deduce that $v' + v \times x' = v' - v \times x' = 0$, so $v' = 0$ and $v \times x' = 0$. Thus $\xi = (x', 0)$ with x' parallel to v, and this is the condition to be tangent to the geodesic flow. □

Remark 4.14. The proof of Lemma 4.13 shows that in the general case the developing map of $m_l \oplus m_r$ is the map

$$\bar{\phi} : \tilde{G}(M) = G(\tilde{M}) \to \mathbb{H}^2 \times \mathbb{H}^2$$

described above. Its holonomy is given by the pairs of representations (h_l, h_r) (up to the identification of $\pi_1(G(M))$ with $\pi_1(M)$).

Note that there is another possible way to obtain the same hyperbolic metrics m_l and m_r, using the identification of $\mathbb{H}^2 \times \mathbb{H}^2$ with $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})/O(2) \times O(2)$. We do not elaborate on this point here since it appears more convenient to use local considerations.

4.3. **Transverse vector fields and associated hyperbolic metrics.** The construction of the left and right hyperbolic surfaces associated to an AdS 3-manifold is based on the use of a special class of surfaces, endowed with a unit time-like vector field behaving well enough, in particular with respect to the singularities.

Let us first consider the case without particle:

Definition 4.15. *Let M be any smooth AdS manifold. Let $S \subset M$ be a space-like surface, and let V be a field of time-like unit vectors defined along S. It is* **transverse** *if for all $x \in S$, the maps $v \mapsto D^l_v V$ and $v \mapsto D^r_v V$ have rank 2.*

It is not essential to suppose that S is space-like, and the weaker topological assumption that S is isotopic in M to a space-like surface would be sufficient. The definition is restricted to space-like surface for simplicity.

Definition 4.16. *We still assume that M is a regular AdS spacetime. Let $S \subset M$ be a space-like surface, and let V be a transverse vector field on S. Let $\delta : S \to G(M)$ be the map sending a point $x \in S$ to the time-like geodesic parallel to V at x. We call $\mu_l := \delta^* m_l$ and $\mu_r := \delta^* m_r$.*

Notice that the field V can be regarded as a map $S \to T^{1,t}M$, and we have $\mu_l = V^*(M_l)$ and $\mu_r = V^*(M_r)$. In particular we easily see that

$$\mu_l(v,v) = ||D^l_v V||^2 \ , \ \mu_r(v,v) = ||D^r_v V||^2 \ .$$

So the metrics μ_l and μ_r are not degenerate.

If $\tau_l, \tau_r : T^{1,t}\tilde{M} \to T^{1,t}_{x_0}\tilde{M} = \mathbb{H}^2$ are the maps obtained by parallel transport for D^l and D^r on the universal covering, as in Lemma 4.13, we have that the developing map of μ_l is the map $dev_l(x) = \tau_l(\tilde{V}(x))$ where $\tilde{V}$ is the lift of V on $\tilde{S} \subset \tilde{M}$. Analogously $dev_r(x) = \tau_r(\tilde{V}(x))$ is a developing map for μ_r.

Now consider the case where M contains some particles, and denote by M_{reg} the smooth part of M. Let S be a space-like surface meeting the particles orthogonally and let V be a transverse vector field on $S_{reg} = M_{reg} \cap S$. The field V defines two hyperbolic metrics μ_l and μ_r on S_{reg} with holonomy h_l and h_r respectively. However in general the behavior of the metrics around the particles can be very degenerate. We say that V is a transverse vector field on S if it satisfies the following conditions, which ensure that μ_l and μ_r are hyperbolic metrics with cone singularities around the particles.

Definition 4.17. *Let T be a particle in M and p be the intersection point of T with S. We consider a neighborhood W of p that is obtained by glueing a wedge $\hat{W} \subset AdS_3$ of angle θ as explained in* [BBS11, 3.7.1]. *The intersection $\Delta = S \cap W$ corresponds to a surface $\hat{\Delta}$ on $\hat{W}$, with p corresponding to a point $\hat{p}$ on $\hat{\Delta}$. The surface S is smooth around p if $\hat{\Delta}$ can be extended to a smooth surface in a neighborhood of $\hat{p}$.*

The vector field V is transverse around a particle T if the following conditions are satisfied:

- *V extends to a unit vector field in p tangent to T.*
- *The induced vector field $\hat{V}$ on $\hat{W}$ extends to a smooth vector field in a neighborhood of $\hat{W}$ in AdS_3,*
- *The rank of $D^l\hat{V}$ and $D^r\hat{V}$ at $\hat{p}$ is 2.*

First let us exhibit a large class of vector fields that satisfy the conditions of this definition.

Lemma 4.18. *If the cone angle around the particle is $\theta \in (0, 2\pi), \theta \neq \pi$ and S is a smooth surface orthogonal to the particle, its unit normal vector field satisfies the conditions of Definition 4.17 at p.*

Proof. The first two conditions are easily verified. Let us check the third condition.

Using the fact that $\hat{V}$ is the pull-back of a vector field on W, at $\hat{p}$ we have that

$$\nabla_{R(v)}\hat{V} = R\nabla_v \hat{V} \ ,$$

where $R : T_{\hat{p}}AdS_3 \to T_{\hat{p}}AdS_3$ is the rotation of angle θ with axis the line tangent to T, and v is a vector orthogonal to T and tangent to the boundary of W. If $\theta \neq \pi$ this implies that $\nabla\hat{V}(\hat{p}) = \lambda I + \mu J$ where J is the rotation of $\pi/2$ around the line tangent to T.

On the other hand, if V is the normal field of S, we have that $\nabla\hat{V}$ is a self-adjoint operator on TS. Thus we have that the skew-symmetric part of $\nabla\hat{V}$ must vanishes at p. It follows that $\nabla\hat{V}(\hat{p}) = \lambda I$.

In particular, since the transformation $v \mapsto \hat{V}(p) \times v$ coincides with J, we deduce that

$$D^l\hat{V}(\hat{p}) = \lambda I + J \ , \qquad D^r\hat{V}(\hat{p}) = \lambda I - J \ ,$$

and the third condition in the definition follows. □

Remark 4.19. When the cone angle is π, the same conclusion follows provided that S is convex around the particle.

Proposition 4.20. *Let M be a space-time with particles and S be a closed smooth surface orthogonal to the particles. If V is a transverse field on S, then μ_l and μ_r are hyperbolic metrics with cone singularity. Moreover if p is the intersection point of S with a particle of angle θ, then p is a cone point for both μ_l and μ_r of the same angle.*

Proof. The metrics μ_l and μ_r are defined on the smooth part $S_{reg} = S \cap M_{reg}$ and are hyperbolic by Lemma 4.13.

Since V is smooth at any particle, it is easy to check that D^lV and D^rV are uniformly bounded operators of TS_{reg}. This implies that $\mu_\bullet$ is bi-Lipschitz to the first fundamental form. In particular the completion of $(S_{reg}, \mu_\bullet)$ is canonically identified with S.

Let p be the intersection point of S with a particle. A neighborhood W of p in M is obtained by glueing the boundary of a wedge $\hat{W}$ of angle θ in AdS_3. Let $\Delta = S \cap W$ and $\hat{\Delta}$ the corresponding surface in $\hat{W}$.

By hypothesis, $\hat{\Delta}$ is a sector of a smooth surface Σ around $\hat{p}$ in AdS_3 orthogonal to the edge of $\hat{W}$, and $\hat{V}$ can be extended to a smooth vector field on Σ.

In particular the metrics $\hat{\mu}_\bullet$ on $\hat{\Delta}$ extend to smooth hyperbolic metrics on Σ and $(\Delta, \mu_\bullet)$ is obtained by gluing the boundary of $(\hat{\Delta}, \hat{\mu}_\bullet)$ by a rotation around $\hat{p}$.

Let us consider in $T_{\hat{p}}AdS_3$ the sector P of vectors tangent to curves contained in $\hat{\Delta}$. It is clearly a sector of angle θ for the AdS metric. If we show that P is a sector of angle θ also for $\mu_\bullet$, then the result will easily follows.

Notice that if $\theta = \pi$, then P is a half-plane, so the angle is π for any metric. If $\theta \neq \pi$, as in Lemma 4.18, we have that $D_\bullet V$ is a conformal transformation at $\hat{p}$. Since $\mu_l(\bullet, \bullet) = \langle D^l_\bullet V, D^l_\bullet V\rangle$ we see that the angle of P with respect to μ_l is still θ (and analogously for μ_r). □

Note. The reason this paper is limited to manifolds with massive particles — rather than more generally with interacting singularities as in [BBS11] — is that we do not at the moment have good analogs of those surfaces with transverse vector fields when other singularities, e.g. tachyons, are present.

4.4. **A special case: good surfaces.**

The previous construction admits a simple special case, when the time-like vector field is orthogonal to the surface (which then has to be space-like).

Definition 4.21. *Let M be an AdS manifold with interacting particles. Let S be a smooth space-like surface. S is a good surface if:*

- *it does not contain any interaction point,*
- *it is orthogonal to the particles,*
- *the curvature of the induced metric is negative,*
- *the intersections of S with the particles of angle π are locally convex.*

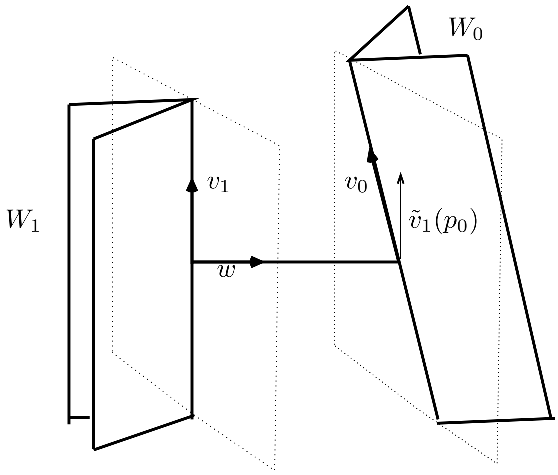

FIGURE 2. A simple example with no transverse vector field.

Note that, given a good surface S, one can consider the equidistant surfaces S_r at distance r on both side. For r small enough (for instance, if S has principal curvature at most 1, when $r \in (-\pi/4, \pi/4)$), S_r is a smooth surface, and it is also good. So from one good surface one gets a foliation of a neighborhood by good surfaces.

The key property of good surfaces is that their unit normal vector field is a transverse vector field, according to the definition given above. This simplifies the picture since the left and right metrics are defined only in terms of the surface, without reference to a vector field. However the construction of a good surface seems to be quite delicate in some cases, so that working with a more general surface along with a transverse vector field is simpler.

Lemma 4.22. *Let S be a good surface, let u be the unit normal vector field on S, then u is a transverse vector field.*

Proof. Let $x \in S_{reg}$ and let $v \in T_xS$. By definition,

$$D^l_v u = \nabla_v u + u \times v = -Bv + Jv \ ,$$

$$D^r_v u = \nabla_v u - u \times v = -Bv - Jv \ ,$$

where B is the shape operator of S and J is the complex structure of the induced metric on S. If S is a good surface then its induced metric has curvature $K < 0$. But $\det(-B \pm J) = \det(B) + 1 = -K$, so that $D^l_v u$ and $D^r_v u$ never vanish for $v \neq 0$. This means precisely that u is a transverse vector field along S_{reg}.

Now the lemma follows from Lemma 4.18 and Remark 4.19. □

Example. Let s_0 be a space-like segment in AdS_3 of length $l > 0$. Let d_0, d_1 be disjoint time-like lines containing the endpoints of s_0 and orthogonal to s_0, chosen so that the angle between the (time-like) plane P_0 containing s_0, d_0 and the (time-like) plane P_1 containing s_0 and d_1 is equal to some $\theta \in \mathbb{R}$. Let W_0 (resp. W_1) be wedges with axis d_0 (resp. d_1) not intersecting s_0 or d_1 (resp d_0) (see Figure 2).

Let M_θ be the space obtained from $AdS_3 \setminus W_0 \cup W_1$ by gluing isometrically the two half-planes in the boundary of W_0 (resp. W_1), and let $M_{ex} := M_\theta$ for $\theta = l$. We will see that M_{ex} does not contain any good surface, or even any surface with a transverse vector field.

Let x_0 be the end-point of s_0 contained in d_0. Let us identify $SO_0(2,1)$ with the isometry group of $T_{x_0}AdS_3$. By definition the holonomy g_0 of a loop of M around d_0 is a rotation of axis d_0. So $(g_0)_l = (g_0)_r$ are elliptic transformations with fixed point the vector v_0 tangent to d_0

On the other hand, if g_1 is the holonomy of a loop around d_1, $(g_1)_l$ and $(g_1)_r$ are elliptic transformations with fixed point the vectors $\tau_l(v_1)$ and $\tau_r(v_1)$ respectively, where v_1 is the direction tangent to d_1 at the end-point $p_1 \in d_1 \cap s_0$. Now if $\tilde{v}_1$ is the ∇-parallel field on s_0 extending v_1 on s_0 and w is the unit vector field tangent to s_0 and pointing to p_0 we have by our assumption

$$v_0 = \cosh(\theta)\tilde{v}_1(p_0) \pm \sinh(\theta)w \times \tilde{v}_1(p_0)$$

(where the sign $\pm$ depends on the way d_1 is turned with respect to d_0).

On the other hand, the D^l and D^r-parallel extensions of v_1 along s_0 are respectively

$$\tau_l(v_1)(p) = \cosh(s)\tilde{v}_1(p) + \sinh(s)w \times \tilde{v}_1(p)\,, \qquad \tau_l(v_1)(p) = \cosh(s)\tilde{v}_1(p) + \sinh(s)w \times \tilde{v}_1(p)$$

where s is the distance on s_0 between p and p_1.

In particular, assuming $\theta = l$, either $\tau_l(v_1)$ or $\tau_r(v_1)$ coincides with v_0. This implies that either the left or the right holonomy is an elementary representation, so it cannot be the holonomy of a hyperbolic disk with two cone singularities.

Notice that if $\theta < l$, M_θ does contain a space-like surface with a transverse vector field (we leave the construction to the interested reader) but with a left hyperbolic metric, say $\mu_l(\theta)$, which has two cone singularities which "collide" as $\theta \to l$. (This can be seen easily by taking a surface which contains s_0.) If M_{ex} admitted a surface with a transverse vector field, it could have only one cone singularity (as it is seen by considering the limit $M_\theta \to M_{ex}$), this is impossible.

Note that M_{ex} is obviously not globally hyperbolic, and it contains no closed space-like surface, it was chosen for its simplicity.

4.5. **Changing the transverse vector field and the space-like slice.** In this section we will consider the same framework as in the previous section. In particular we fix an AdS manifold M with particles, a closed space-like surface in M and a transverse vector field V on S. We will investigate how the metrics $\mu_\bullet$ change when deforming the surface S and the vector field. The first basic result is that the class of isotopy of $\mu_\bullet$ is independent of S and V. More precisely, if S and S' are two different surfaces in M we will prove that there exist isometries $\phi_\bullet : (S, \mu_\bullet) \to (S', \mu'_\bullet)$ such that the induced map $\phi_* : \pi_1(S_{reg}) \to \pi_1(S'_{reg})$ makes the following diagram commutative (up to conjugation)

$$\begin{array}{ccc} \pi_1(S_{reg}) & \xrightarrow{i_*} & \pi_1(M_{reg}) \\ \phi_* \big\downarrow & & Id \big\downarrow \\ \pi_1(S'_{reg}) & \xrightarrow{i'_*} & \pi_1(M_{reg}) \end{array} .$$

Notice that the commutativity of the diagram determines the isotopy class of ϕ. More geometrically ϕ is isotopic to any map $S \to S'$ obtained by following the flow of any time-like field tangent to the particles.

Lemma 4.23. *Given S, μ_l and μ_r do not depend (up to isotopy) on the choice of the transverse vector field V. Moreover, μ_l and μ_r do not change (again up to isotopy) if S is replaced by another surfaces isotopic to it.*

The proof uses a basic statement on hyperbolic surfaces with cone singularities. Although this result might be well known, we provide a proof for completeness.

Lemma 4.24. *A closed hyperbolic surface with cone singularities of angle less than 2π is uniquely determined by its holonomy.*

Let S be a closed surface with marked points $x_1, \cdots, x_n$, let $\theta_1, \cdots, \theta_n \in (0, 2\pi)$, and let μ_0, μ_1 be two hyperbolic metrics on S with cone singularities of angles θ_i (resp. θ'_i) on the x_i,

$1 \leq i \leq n$. We suppose that μ_0 and μ_1 have the same holonomy, and will prove that μ_0 is isotopic to μ_1.

The holonomy of a short curve around a singular point x_i is a rotation of angle the cone angle at x_i. Since μ_0 and μ_1 have the same holonomy at x_i, we see already that $\theta_i = \theta'_i$ for all $1 \leq i \leq n$.

Sublemma 4.25. *There exists a triangulation T_0 of S with vertices equal to the x_i, with as edges segments that are geodesic for μ_0.*

Proof. A basic point on metrics with cone singularity with angle less than 2π is that any two cone points are connected by a minimizing geodesic which does not pass through any other cone singularity. Given a curve γ_0 between two singular points x_i and x_j, there is a unique curve γ_1 between x_i and x_j which can be deformed to γ_0 in the complement of the x_k, and which has minimal length among all such curves. However γ_1 can go through some of the x_k.

It follows that there is a graph Γ embedded in S with vertices x_i having edges segments with are geodesic for μ_0.

Cutting S along Γ, we get a surface $\hat{S}$ with piecewise geodesic boundary such that the total angle at any vertex of the boundary is in $(0, 2\pi)$. Any loop c in $\hat{S}$ centered at some vertex of $\partial\hat{S}$ can be deformed to a geodesic curve (that may possibly contain some segment of the boundary). In particular, cutting $\hat{S}$ along geodesic curves which are not isotopic to the boundary, we eventually get a decomposition of S in hyperbolic disks $D_1, \ldots, D_n$ which have piecewise geodesic boundary, such that vertices of ∂D_i correspond to cone points of S.

To conclude the proof it is sufficient to show that each D_i admits a geodesic triangulation with vertices corresponding to the vertices of its boundary. We use an inductive argument on the number of vertices of ∂D_i.

Take any vertex p of D_i and consider the set $\mathcal{A}$ of end-points of maximal geodesic segments embedded in D_i starting at p. By maximality, $\mathcal{A}$ is a subset of ∂D_i. We distinguish two cases. First we suppose that $\mathcal{A}$ contains a vertex q of ∂D_i which is not adjacent to p. In this case, cutting D_i along the maximal segment joining p to q, we decompose D_i into 2 disks with less vertices. In the other case, any segment starting from p and contained in the interior of D_i must meet a single edge of D_i. This means that there exists a geodesic triangle in D_i with a vertex at p whose edges contains the edges of ∂D_i adjacent to p. In this case there is a segment in the interior of D_i joining the vertices adjacent to p. Thus we can cut D_i into a triangle T' contained in T and another disk D' with less vertices. □

Note that the argument given in the previous proof for the existence of a triangulation could be replaced by another argument, based on Voronoi diagrams, which is somewhat simpler in the setting considered here. The reason why we favored the slightly more involved argument used here is that we will repeat the same argument below in the slightly different setting of surfaces with (convex) boundary. The type of argument used here works directly for surfaces with boundary, while the argument based on Voronoi diagrams is less directly applicable there.

Proof of Lemma 4.24. We define a *3-disk* in S as a disk in S containing no marked point in its interior and exactly three marked points in its boundary. Those disks are considered up to homotopy of S fixing the marked points x_i. Let D be such a disk, containing in its boundary the marked points x_i, x_j, x_k. Considering the restriction to D of the developing map of the regular part of (S, μ_0) we associate to x_i, x_j and x_k a triple (x'_i, x'_j, x'_k) of points in $\mathbb{H}^2$, defined up to global isometry of $\mathbb{H}^2$, as well as a disk D' containing x'_i, x'_j and x'_k in its boundary (D' is defined up to homotopy in $\mathbb{H}^2$ fixing x'_i, x'_j and x'_k).

Notice that D' and (x'_i, x'_j, x'_k) are uniquely determined by the holonomy of μ_0 only, because the holonomy determines the cone angles at x'_i, x'_j, x'_k (through the holonomies of loops around these points) and the distance between x'_i and x'_j (through the trace of the holonomy of the boundary of a small neighborhood of the segment of ∂D between x_i and x_j) and similarly for x'_j and x'_k and for x'_k and x'_i.

We say that the 4-tuple (D', x'_i, x'_j, x'_k) is *realizable* if D' can be deformed to a triangle (with geodesic boundary) with vertices x'_i, x'_j and x'_k, without displacing x'_i, x'_j and x'_k. Clearly if (x_i, x_j, x_k) are the vertices of a triangle D of a geodesic triangulation T of (S, μ_0), then (D', x'_i, x'_j, x'_k) is realizable. But conversely, if, for any face D of a triangulation T with vertices x_i, x_j and x_k, (D', x'_i, x'_j, x'_k) is realizable, then considering the developing map of the metric shows that T can be realized as a geodesic triangulation.

Since the condition for a 3-disk to be realizable depends only on the holonomy, the geodesic triangulation T_0 of (S, μ_0) constructed in Sublemma 4.25 also corresponds to a geodesic triangulation of (S, μ_1). Moreover the length of the edges is the same for the two metrics, because we have seen that the length of an edge is determined by the holonomy. So μ_1 is isotopic to μ_0. □

Proof of Lemma 4.23. For the first point consider another transverse vector field V' on S, and let μ'_l, μ'_r be the hyperbolic metrics defined on S by the choice of V' as a transverse vector field. Let γ be a closed curve on the complement of the singular points in S. The holonomy of μ'_l (resp. μ'_r) on γ is equal to the holonomy of D^l (resp. D^r) acting on the hyperbolic plane, identified with the space of oriented time-like unit vectors at a point of S. So μ_l and μ'_l (resp. μ_r and μ'_r) have the same holonomy, so that they are isotopic by Lemma 4.24.

The same argument can be used to prove the second part of the lemma. Let γ_1 be a closed curve on S_1 which does not intersect the singular set of M, and let γ_2 be a closed curve on S_2 which is isotopic to γ_1 in the regular set of M. The holonomy of M on γ_1, $h(\gamma_1)$, is equal to the holonomy of M on γ_2, $h(\gamma_2)$. But $h = (h_l, h_r)$ by Lemma 4.9, and h_l, h_r are the holonomy representations of the left and right hyperbolic metrics on S_1 and on S_2 by Lemma 4.9. Therefore, (S_1, μ_l) has the same holonomy of (S_2, μ_l), and (S_1, μ_r) has the same holonomy as (S_2, μ_r). The result therefore follows by Lemma 4.24. □

Note that a weaker version of this proposition is proved as [KS07, Lemma 5.16] by a different argument. The notations μ_l, μ_r used here are the same as in [BS09], while the same metrics appeared in [KS07] under the notations $I^*_\pm$. Those metrics already appeared, although implicitly only, in Mess' paper [Mes07]. As we have mentioned in Section 1.2, this paper considers globally hyperbolic AdS manifolds, which are the quotient of a maximal convex subset Ω of AdS_3 by a surface group Γ acting by isometries on Ω. The identification of $SO_0(2, 2)$ with $PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$ then determines two representations of Γ in $PSL(2, \mathbb{R})$ with maximal Euler number, so that they define hyperbolic metrics. It is proved in [KS07] that those two hyperbolic metrics correspond precisely to the left and right metrics considered here.

Remark 4.26. In general the isometries $\phi_l : (S, \mu_l) \to (S', \mu'_l)$ and $\phi_r : (S, \mu_r) \to (S', \mu'_r)$ are different. This implies that the pair (μ_l, μ_r) is not uniquely determined up to isotopy (acting on both the factors).

In the remaining part of this section we will show that any transverse unit vector field V on S can be extended to a unit vector field on a neighborhood Ω of S such that

- it is tangent to the particle,
- it is transverse on any space-like surface S' contained in Ω,
- the map $\phi : S \to S'$ obtained by following the orbits of V is an isometry for both μ_l and μ_r.

In fact there exists $\epsilon > 0$ such that the map

$$F : S \times (-\epsilon, \epsilon) \ni (p, t) \mapsto \exp_p(tV(p)) \in AdS_3 .$$

is well-defined and it is a diffeomorphism onto some neighborhood Ω of S in AdS_3. Notice that if p is the intersection point of S with some particle, $F(p, t)$ lies on the particle for every t. Moreover, by the assumption on V it is easy to check that F is a diffeomorphism around the particles.

Clearly the map F induces on Ω a foliation by time-like geodesics parallel to V, so we can consider the induced map

$$\hat{\delta} : \Omega \to G(M)$$

and the bilinear forms $\hat{\mu}_l = \hat{\delta}^*(m_l)$, $\hat{\mu}_r = \hat{\delta}^*(m_r)$.

Since $\hat{\delta}(F(p, t)) = \delta(p)$, where $\delta : S \to G(M)$ is the map defined in Definition 4.16, we have that $F^*(\hat{\mu}_\bullet) = \pi_S^*(\mu_\bullet)$ where $\pi_S : S \times (-\epsilon, \epsilon) \to S$ is the projection.

In particular, $\hat{\mu}_\bullet$ is non-degenerate on every plane that is not tangent to $\hat{V}$. If S' is any space-like surface, $\hat{V}$ is transverse to it and we have that the induced metrics $\mu'_\bullet = \hat{\mu}_\bullet|_{S'}$. Finally, the map $S' \to S$ sending $q \in S'$ to the intersection point of the geodesic leaf through q with S turns out to be an isometry for both μ'_l, μ_l and μ'_r and μ_r.

4.6. **Left and right metrics on the future of a collision point.** We consider now the case where S is a space-like surface with a transverse vector field in a AdS spacetime which contains a unique collision point p. Without loss of generality we suppose p in the past of S. Clearly $I^+(p) \cap S$ is a disk D with k singular points where k is the number of particles starting from p.

Definition 4.27. *A connected open subset U of a hyperbolic surface S is convex if any path c contained in U can be deformed to a geodesic path in U keeping the endpoints of c fixed.*

The goal of this section is to prove the following proposition

Proposition 4.28. *There are convex disks D_l, D_r isotopic to D in S such that (D_l, μ_l) is isometric to (D_r, μ_r).*

(Let us stress that here the isotopies are supposed not to displace the cone points.)

First we show that the statement is true for the holonomies.

Lemma 4.29. *The holonomies of μ_l and μ_r restricted to $\pi_1(D_{reg})$ are conjugated.*

Proof. We use the fact that the holonomies of μ_l and μ_r are the left and right factors of the holonomy of the AdS structure, as stated in Lemma 4.9.

So we have to show that the restriction of the holonomy of h to $\pi_1(D_{reg})$ is conjugated to a diagonal representation into $SO_0(2, 2) = SO_0(2, 1) \times SO_0(2, 1)$.

If Σ is the link of the collision point, the inclusion $\pi_1(D_{\text{reg}}) \to \pi_1(M_{\text{reg}})$ can be factored as the composition $\pi_1(D_{reg}) \to \pi_1(\Sigma_{reg}) \to \pi_1(M_{reg})$. So it is sufficient to prove that the restriction o h to $\pi_1(\Sigma_{reg})$ is conjugated to a diagonal representation.

On the other hand, this clearly follows since the holonomy h restricted to $\pi_1(\Sigma_{reg})$ fixes a point (see Lemma 4.8). □

Notice that Lemma 4.29 is not sufficient to conclude the proof of Proposition 4.28, since we have to point out concrete disks D_l and D_r such that (D_l, μ_l) and (D_r, μ_l) are isometric. To that aim we will use the same triangulation argument as in Lemma 4.24.

The main difference is that in this case D_l and D_r have boundary, so we need to select them carefully: the key point in the proof of Lemma 4.24 is that pairs of points are joint by a minimizing geodesic. So in order to apply the same argument we need the convexity of D_l and D_r.

Proof. Take a sequence D_n of disks isotopic to D such that the μ_l-length of ∂D_n converges to the infimum of the μ_l-lengths of boundary curves of disks isotopic to D.

∂D_n converges to a μ_l-geodesic graph Γ with vertices at cone points of D. In fact, locally around each point x of Γ we see two regions Ω_1 and Ω_2 in $S \setminus \Gamma$. By the minimizing property we easily see that if Ω_i is in the limit of $S \setminus D_n$, then the angle contained in Ω_i with vertex at x is bigger than π. Since singularities are supposed to have angles in $(0, 2\pi)$ we easily see that one of the following possibilities occurs:

- D contains only one cone point, and Γ coincides with it.
- D contains exactly two cone points and Γ is a segment with vertices at cone points.
- D contains more than 2 cone points and Γ is a circle bounding a convex disk D'_l such that the regular neighborhoods of D'_l are isotopic to D.

In the first case it is sufficient to define D_l, D_r to be disks of radius ϵ around the cone point for μ_l and μ_r respectively.

In the second case, notice that the μ_l-length of Γ, say a, is determined by the holonomy of μ_l on D_{reg}. Since the holonomies of μ_l and μ_r on D_{reg} coincide Γ can be deformed to an arc Γ' which is μ_r geodesic and such that the μ_r-length of Γ' is also a. Then a μ_l-regular neighborhood of Γ and a μ_r-regular neighborhood of Γ' are isometric.

In the third case, we can construct a μ_l-geodesic triangulation of D'_l as in Proposition 4.24. The shape of this triangulation just depend on the holonomy of the disk, and this implies that each triangle of this triangulation can be deformed to a μ_r-triangle.

This implies that there exists an isometric embedding $(D'_l, \mu_l) \to (S, \mu_r)$ which is isotopic to the inclusion $D \to S$. Thickening a bit D'_l we get a convex disk D_l isotopic to D such that (D'_l, μ_l) admits an isometric embedding (isotopic to the identity) into (S, μ_r). □

Remark 4.30. In general the intersection of two convex disks is not connected. On the other hand, the proof of Proposition 4.28 shows that any convex disk isotopic to D must contain D'_l. This implies that if D_1, D_2 are convex disks isotopic to D, then $D_1 \cap D_2$ contains a convex disk isotopic to D.

Example. If only two particles, p_1 and p_2, collide, the corresponding cone points are at the same distance in the left and right hyperbolic metric of Ω; more precisely, there are two segments of the same length, one in the left and one in the right hyperbolic metric of Ω, joining the cone points corresponding to p_1 and to p_2. Moreover the length of those segments is equal to the "angle" between p_1 and p_2 at c, i.e., to the distance between the corresponding points in the link of c.

5. Surgeries at collisions

We now wish to understand how the left and right hyperbolic metrics change when a collision occurs.

5.1. **Good spacial slices.**

The first step in understanding AdS manifolds with colliding particles is to define more easily understandable pieces.

Definition 5.1. *Let M be an AdS manifold with colliding particles. A* **spacial slice** *in M is a subset Ω such that*

- *there exists a closed surfaces S with marked points $x_1, \cdots, x_n$ and a homeomorphism $\phi : S \times [0, 1] \to \Omega$,*
- *ϕ sends $\{x_1, \cdots, x_n\} \times [0, 1]$ to the singular set of Ω,*
- *$\phi(S \times \{0\})$ and $\phi(S \times \{1\})$ are space-like surfaces.*

Ω is a **good** *spacial slice if in addition*

- *it contains a space-like surface with a transverse vector field.*

Hence, we have constructed at the end of Section 4.5 a good spacial slice in the neighborhood of any Cauchy surface equipped with a transverse vector field. Observe that spacial slices do not contain interactions.

It is useful to note that Lemma 4.23, along with its proof, applies also to surfaces with boundary, with a transverse vector field, embedded in a good spacial slice. Such surfaces determine the holonomy of the restriction of the left and right metrics to surfaces with boundary, as explained in the following remark. The proof is a direct consequence of the arguments used in Section 4, and more specifically in the proof of Lemma 4.23.

Remark 5.2. Let Ω be a good spacial slice, let $D \subset \Omega$ be a space-like surface with boundary, and let u' be a transverse vector field on D. Then u' determines a left and a right hyperbolic metric, μ'_l, μ'_r on D, as for closed surfaces above. Moreover for any closed curve γ contained in D, the holonomies of μ'_l and μ'_r on γ are equal respectively to the left and right parts of the holonomy of γ in M.

5.2. **Surgeries on the left and right metrics.** In this section we consider in details how the left and right metrics change when a collision occurs. The first step is to define some simple notions of surgery on hyperbolic surfaces with cone singularities, and on pairs of such surfaces. We will later prove that those surgeries are exactly those that can happen on the left and right metrics of spacial slices of an AdS manifold with particles when a collision occurs.

5.2.1. *Surgery on hyperbolic surfaces.* The basic building block of the surgeries considered here is a simple operation where one replaces a disk, in a hyperbolic surface with cone singularities, by another disk with only one singularity.

Definition 5.3. *Let S_- and S_0 be two hyperbolic cone-surfaces, and let $D_- \subset S_-$ be homeomorphic to an open disk. We say that S_0 is obtained from S_- by* **collapsing** *D_- if*

- *there exists an isometric embedding $i : S_- \setminus D_- \to S_0$,*
- *$S_0 \setminus i(S_- \setminus D_-)$ is homeomorphic to an open disk and contains exactly one cone singularity s_0, of angle $\theta \in (0, 2\pi)$.*

We call s_0 the collapsed singularity of S_0.

Note that the geometry of the disk $S_0 \setminus i(S_- \setminus D_-)$ depends only on the geometry of D_-. In other terms, there is a hyperbolic disk $D_{0,-}$ with exactly one cone singularity, depending only on D_-, and an isometric embedding $j_- : D_{0,-} \to S_0$ such that $S_0 \setminus j(D_{0,-}) = S_- \setminus D_-$.

We now introduce the surgery on *pairs* of hyperbolic cone-surfaces, which corresponds — as it will be seen below — to what occurs to the left and right hyperbolic metrics of an AdS manifold with particles when a particle collision occurs. The basic idea is that a disk surgery is done on both S^l_- and S^r_-, collapsing the *same* disk D_- (up to isometry of course) and yielding the same disk D_+ (again up to isometry). However an additional condition is necessary, stating that the "relative position" of D_- and D_+ is the same on the left and on the right side.

We consider a surface S with a couple of hyperbolic cone metrics μ_l, μ_r and we suppose that (up to isotopy) they coincide in a neighborhood of a singular convex disk D. Moreover we will suppose that the holonomy of ∂D (for both μ_l and μ_r) is elliptic of angle $\theta \in (0, 2\pi)$ and that a collar neighborhood of ∂D admits an isometric embedding (for both μ_l and μ_r) $i : \partial D \to H_\theta$, where H_θ is the model of the singularity of angle θ. The image of those embeddings bound a disk D_0 in H_θ containing the singular point. We consider now the surface S_0 obtained by cutting from S the disk D and pasting the disk D_0 using i as glueing map. Notice that the metrics μ_l and μ_r glue to the metric μ of D_0, yielding two singular metrics which coincide on a disk D.

Definition 5.4. *We say that the triple (S_0, μ_l^0, μ_r^0) is obtained by (S, μ_l, μ_r) by collapsing D_-.*

Changing μ_l, μ_r by two different isotopies (but requiring that they coincide on some disk D' isotopic to D), we get another collapsed surface $S_0', {\mu_l'}^0, {\mu_r'}^0$. Clearly there are natural isometries

$$\phi_l : (S_0, \mu_l^0) \to (S_0', {\mu_l'}^0) \qquad \phi_r : (S_0, \mu_r^0) \to (S_0', {\mu_r'}^0)\ .$$

Though those isometries are in general different, the following lemma establishes that when D contains at least two cone points, ϕ_l and ϕ_r coincide in a neighborhood of the collapsed singularity.

Lemma 5.5. *If D contains at least two singular points, then ϕ_l and ϕ_r coincide in a neighborhood of the collapsed singularity of S_0.*

Notice that if D contains only a singular point, then the statement is false. In fact, the proof of Lemma 5.5 is based on the following simple fact which clearly depends on the fact that D contains at least two cone points.

Sublemma 5.6. *If D is a convex disk in a hyperbolic surface S with at least two cone singularities and if $\sigma : D \to S$ is an isometric immersion isotopic to the identity, then σ is the identity.*

Proof of Sublemma 5.6. Let p_1, p_2 cone points of D. The map σ fixes the two cone points, so it fixes any geodesic arc which joins p_1 to p_2 and this easily implies that it fixes every point. □

Proof of Lemma 5.5. Assuming $D' \subset D$ and $\mu_l = \mu_l'$, $\mu_r = \mu_r'$, the statement is clearly true.

Similarly, the statement is true also assuming there exists a diffeomorphism u of S isotopic to the identity such that $\mu_l' = u^*(\mu_l)$, $\mu_r' = u^*(\mu_l)$ and $D' = u^{-1}(D)$.

So it is sufficient to consider the case where $\mu_l' = \mu_l$, $\mu_r' = u^*(\mu_r)$ for some diffeomorphism u isotopic to the identity.

Notice that in this case the disk $u(D')$ is μ_r-convex. In particular there is a μ_r-convex disk Δ contained in $D \cap u(D')$ and isotopic to D. Notice that $\mu_r = \mu_l$ on Δ, so this disk is also μ_l-convex. The restriction of u^{-1} to Δ is an isometric embedding of $(\Delta, \mu_l = \mu_r) \to (D', \mu_l = \mu_r') \subset (S, \mu_l)$. By Sublemma 5.6,$u|_\Delta = Id$.

Now, let $(\hat{S}_0, \hat{\mu}_l, \hat{\mu}_r)$ be the surface obtained by collapsing Δ on (S, μ_l, μ_r) and let $(\hat{S}_0', \hat{\mu}_l, \hat{\mu}_r)$ be the surface obtained by collapsing Δ on (S, μ_l', μ_r'). Notice that the isometries $\hat{\phi}_l, \hat{\phi}_r : \hat{S}_0' \to \hat{S}_0$ extend respectively the identity and u on $S'' = S \setminus \Delta$. So they coincide on $\partial\Delta$ and this shows that they coincide on the disk $\Delta_0'' = \hat{S}_0 \setminus S''$.

On the other hand, the isometries $\psi_l, \psi_r : S_0 \to \hat{S}_0$ coincide in a neighborhood of the collapsed point (because $\Delta \subset D$) and similarly do the isometries $\psi_l', \psi_r' : S_0' \to \hat{S}_0'$.

The statement follows since $\phi_l = (\psi_l')^{-1} \circ \hat{\phi}_l \circ \psi_l$ and $\phi_r = (\psi_r')^{-1} \circ \hat{\phi}_r \circ \psi_r$. □

Definition 5.7. *Let S_- and S_+ be two surfaces, let μ_l^-, μ_r^- be hyperbolic cone metrics on S_- sharing the same singular locus σ_- and let μ_l^+, μ_r^+ be hyperbolic cone metrics on S_+ with singular locus σ_+. We say that (S_+, μ_l^+, μ_r^+) is obtained from (S_-, μ_l^-, μ_r^-) by a* **double surgery** *if up to changing those metrics in their isotopy classes, the following conditions are satisfied:*

1. *There are embedded singular disks $D_- \subset S_-$ and $D_+ \subset S_+$ such that μ_l^+ and μ_r^+ coincide in a neighborhood of D_+ and μ_l^- and μ_r^- coincide in a neighborhood of D_-.*
2. *The corresponding collapsed surfaces (S_0, μ_l^0, μ_r^0) and $(\hat{S}_0, \hat{\mu}_l^0, \hat{\mu}_r^0)$ are isometric, that is there are two isometries*

$$\phi_l : (S_0, \mu_l^0) \to (\hat{S}_0, \hat{\mu}_l^0) \qquad \phi_r : (S_0, \mu_r^0) \to (\hat{S}_0, \hat{\mu}_r^0)\ .$$

3. *ϕ_l and ϕ_r are homotopic and coincide in a neighborhood of the collapsed points.*

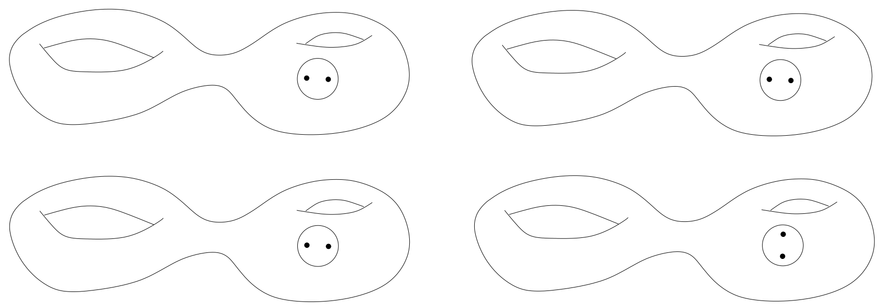

FIGURE 3. An example of two pairs of surfaces which satisfy all conditions but (3) in the definition of double surgery.

The disks D_- and D_+ are called the surgery disks, whereas the homotopy class of ϕ_l and ϕ_r is called the identification map.

Remark 5.8. Condition (3) in the above definition needs some explanation. Notice that since μ_l^0 coincides with μ_r^0 in a neighborhood of the collapsed point of S_0, and $\hat{\mu}_r^0$ coincides wih $\hat{\mu}_r^0$ in a neighborhood of the collapsed point of $\hat{S}_0$ we have that in general $\phi_r^{-1}\phi_l$ is an isometry in a neighborhood of the collapsed point, that is, there is a number θ such that $\phi_r^{-1}\phi_l$ is a rotation of angle θ. We require that $\theta = 0$. In the simple case where only one surgery is sufficient, this condition means that the *same* surgery transforms μ_l^- into μ_l^+ and μ_r^- into μ_r^+.

The following example shows a case where condition (3) is not satisfied (see Figure 3). Take a surface (S, μ) with two cone points such that the holonomy around the cone points is elliptic and there is a constant curvature circle c which bounds a disk D containing the cone points. Now take $S_- = S$, $\mu_l^-, \mu_r^- = \mu$, $S_+ = S$ and $\mu_l^+ = \mu$. Finally define μ_r^+ by twisting the disk D of angle θ_0 (this is possible since ∂D has constant curvature). More formally, the metric μ_r^+ is constructed as follows: let $\tau : S \setminus \mathring{D} \to S \setminus \mathring{D}$ be the identity outside a collar of ∂D and a rotation of angle θ on ∂D. Then, $\mu_r^+ = \tau^*(\mu)$ on $S \setminus D$ and μ on D.

In this case, if S_0 is the surface obtained from S_- by replacing D by a disk D_{θ_0} with only one cone point of angle θ_0, and $\hat{S}_0$ is the surface obtained by replacing D by D_{θ_0}, then ϕ_l is the identity map, whereas ϕ_r is τ outside $S \setminus D$ and is a rotation of angle θ on D.

Note however that if D_+ (or D_-) contains only one cone singularity, then condition (3) is always satisfied.

Heuristically, the fact that (S_+, μ_l^+, μ_r^+) and (S_-, μ_l^-, μ_r^-) are related by a double surgery means that there is a surface S_0, μ_l, μ_r and two singular disks D_- and D_+ such that S_+ is obtained by replacing a disk in S_0 containing a cone point by D_+ and S_- is obtained by replacing another disk containing the same cone point by D_-. It is tempting to simplify this definition, and to replace directly D_- by D_+ without going through the intermediate step of S_0 with only one singularity instead of either D_- or D_+. It appears however that it is not always possible to do this direct surgery — an example is described in Appendix A of a situation where a double surgery as defined here cannot be replaced by a simple surgery where one topological disk is replaced by another. Theorem 6.1 shows that the relevant notion when considering collisions of particles in AdS spacetimes is that of double surgery, rather than the simpler notion of simple surgery on both the left and right metrics.

5.2.2. *Setting and main statement.* We now consider a more precise setting. Let Ω be an AdS manifold with interacting particles, containing exactly one collision point p, which we suppose has positive mass. Suppose that p is the future endpoint of n particles $s_1, \cdots, s_n$ and the past endpoint of m particles $s'_1, \cdots, s'_m$. Let $\theta_1, \cdots, \theta_n$ be the cone singularities at the s_i, and let $\theta'_1, \cdots, \theta'_m$ be the cone singularities at the s'_j.

Suppose that Ω is the union of two good space-like slices Ω_- and Ω_+, such that:

- they have disjoint interior,
- the future boundary of Ω_- is equal to the past boundary of Ω_+,
- Ω_- contains $s_1, \cdots, s_n$ and Ω_+ contains $s'_1, \cdots, s'_m$.

We call S_- a space-like surface in Ω_- with a transverse vector field u_-, and S_+ a space-like surface in Ω_+ with a transverse vector field u_+. Let $\mu_l^\pm, \mu_r^\pm$ be the left and right hyperbolic metrics defined on $S_\pm$ by $u'_\pm$.

Proposition 5.9. *Under those conditions, the triple (S_+, μ_l^+, μ_r^+) is obtained from the triple (S_-, μ_l^-, μ_r^-) by a double surgery.*

The surgery disks are in the isotopy class of $D_+ = I^+(p) \cap S_+$ and $D_- = I^-(p) \cap S_-$, respectively, whereas the identification maps are in the isotopy class of the map $S_+ \setminus D_+ \to S_- \setminus D_-$ obtained by following any timelike flow sending ∂D_+ to ∂D_-.

Note: In the proof of this proposition and in the rest of the paper we will consider developing maps and holonomies of a singular manifold X. So we need to consider the universal covering and the fundamental groups of the regular part of X, whereas in general we will not be interested in the fundamental group and the universal covering of X. For this reason, from now on $\pi_1(X)$ and $\tilde{X}$ will denote respectively the fundamental group and the universal covering of the regular part of X.

Proof. By Proposition 4.28, up to changing μ_l^+ and μ_r^+ by an isotopy, we may suppose that they coincide around a disk D_+ containing the singular points $p_i = s_i \cap S_+$. Analogously we may suppose that μ_l^- and μ_r^- coincide on a disk D_- which contains the singular points $p'_i = s'_i \cap S_-$.

By the positivity of the mass of the collision point, the holonomy of ∂D_+ for $\mu_\bullet^+$ is elliptic of angle $\theta_0 \in (0, 2\pi)$. In particular there is an embedding of a neighborhood of ∂D_+ into the model space H_{θ_0} of the cone angle θ_0.

So we can consider the surface (S_0, μ_l^0, μ_r^0) obtained by collapsing D_+ on (S_+, μ_l^+, μ_r^+). Analogously let $(\hat{S}_0, \hat{\mu}_l^0, \hat{\mu}_r^0)$ be the surface obtained by collapsing D_- on (S_-, μ_l^-, μ_r^-).

Let us regard the fundamental group as the set of covering transformations on the universal cover. In particular, any lifting on the universal covering of the inclusions

$$(S_- \setminus D_-) \to S_- \to M \qquad (S_+ \setminus D_+) \to S_+ \to M$$

determines inclusions $\pi_1(S_- \setminus D_-) \to \pi_1(S_-) \to \pi_1(M)$ and $\pi_1(S_+ \setminus D_+) \to \pi_1(S_+) \to \pi_1(M)$.

Since $S_- \setminus D_-$ and $S_+ \setminus D_+$ are isotopic in M, we may fix those liftings

$$(\widetilde{S_- \setminus D_-}) \to \tilde{S}_- \to \tilde{M} \qquad (\widetilde{S_+ \setminus D_+}) \to \tilde{S}_+ \to \tilde{M}$$

so that $\pi_1(S_- \setminus D_-)$ is identified to $\pi_1(S_+ \setminus D_+)$ as subgroups of $\pi_1(M)$.

Finally since the inclusions $S_- \setminus D_- \to \hat{S}_0$ and $S_+ \setminus D_+ \to S_0$ are homotopy equivalence, we may fix identifications between $\pi_1(S_- \setminus D_-)$ and $\pi_1(\hat{S}_0)$ and $\pi_1(S_+ \setminus D_+)$ and $\pi_1(S_0)$. Notice that those identifications are unique up to conjugation and the choice of concrete ones is equivalent to choosing liftings $\widetilde{S_+ \setminus D_+} \to \tilde{S}_0$ and $\widetilde{S_- \setminus D_-} \to \tilde{\hat{S}}_0$ of the natural inclusions.

Notice that through these identifications, the holonomies of μ_l^0, μ_r^0 coincide with the holonomies of $\hat{\mu}_l^0, \hat{\mu}_r^0$, so, by Proposition 4.24, there exist isometries

$$\phi_l : (\hat{S}_0, \hat{\mu}_l^0) \to (S_0, \mu_l^0) \qquad \phi_r : (\hat{S}_0, \hat{\mu}_r^0) \to (S_0, \mu_r^0)$$

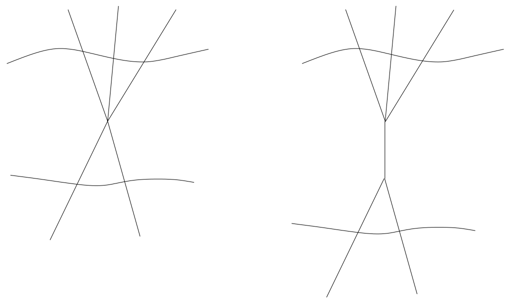

Figure 4. Two different interaction possibilities

which admit liftings to the universal covering $\tilde{\phi}_l, \tilde{\phi}_r : \tilde{\hat{S}}_0 \to \tilde{S}_0$ which act trivially on the fundamental groups

$$\tilde{\phi}_l \circ \gamma \circ (\tilde{\phi}_l)^{-1} = \tilde{\phi}_r \circ \gamma \circ (\tilde{\phi}_r)^{-1} = \gamma \ ,$$

where we are using the identification $\pi_1(S_0) = \pi_1(\hat{S}_0)$ fixed above.

Notice that ϕ_l and ϕ_r are isotopic, since they induce the same map on the fundamental groups.

In order to prove condition (3) in Definition 5.7, we also fix liftings $\tilde{D}_+ \to \tilde{S}_+$ and $\tilde{D}_- \to \tilde{S}_-$, so that $\tilde{D}_+ \cap \widetilde{S_+ \setminus D_+}$ and $\tilde{D}_- \cap \widetilde{S_- \setminus D_-}$ are the images of liftings c_+ and c_- of ∂D_+ and ∂D_-, respectively. We may moreover suppose that the stabilizers of c_+ and c_- in $\pi_1(M)$ are the same $\mathbb{Z}$-subgroup generated by γ_0.

We fix developing maps

$$dev_+^\bullet : (\tilde{S}_+, \mu_+^\bullet) \to \mathbb{H}^2$$

so that they coincides on $\tilde{D}_+$. Notice that the restriction of $dev_+^\bullet$ on $\widetilde{S_+ \setminus D_+}$ extends to a developing map $dev_0^\bullet : \tilde{S}_0 \to \mathbb{H}^2$.

Clearly, $\widehat{dev_0^\bullet} := dev_0^\bullet \circ \phi_\bullet$ is a developing map for $\hat{\mu}_\bullet^0$.

Finally let dev_-^l, dev_-^r be the developing maps of $\mu_-^\bullet$ which coincide with $\widehat{dev_0^\bullet}$ on $\widetilde{S_- \setminus D_-}$. Notice that on c_- the holonomy representations of the maps dev_-^l and dev_-^r differ exactly by the rotation $\tilde{R} \in PSL(2, \mathbb{R})$ which corresponds to $\phi_r \phi_l^{-1}$ around the cone point. In particular, if $\gamma \in \pi_1(D_-)$ we have

$$h_r(\gamma) = \tilde{R} h_l(\gamma) \tilde{R}^{-1} \ .$$

Notice that $\pi_1(\Omega)$ is the amalgamated product of $\pi_1(S_+)$ and $\pi_1(S_-)$ with the identification of $\pi_1(S_+ \setminus D_+)$ and $\pi_1(S_- \setminus D_-)$ described above.

The holonomies of the developing maps $\mathrm{dev}_\pm^\bullet$ glue to a pair of representations

$$(h_l, h_r) : \pi_1(\Omega) \to PSL_2(\mathbb{R})^2$$

which coincide with the holonomy representation of Ω.

If Σ is the link around the collision point, then $\pi_1(\Sigma)$ is the amalgamated product of $\pi_1(D_-)$ and $\pi_1(D_+)$. If $\gamma \in \pi_1(D_+)$ then $h_r(\gamma) = h_l(\gamma)$, whereas if $\gamma \in \pi_1(D_-)$ then $h_r(\gamma) = \tilde{R} h_l(\gamma) \tilde{R}^{-1}$.

Imposing that the restrictions of the representations h_l and h_r on $\pi_1(\Sigma)$ are the same, we obtain that $\tilde{R} = Id$. □

Remark 5.10. Let us consider the examples in Figure 4. In the example on the left, Proposition 5.9 indicates that the left and right metrics on the surface below and the left and right metrics on the surface above are related by a double surgery

In the example on the right, this is no longer true. On the other hand, conditions (1) and (2) in the definition of double surgery are still valid. The same argument used in the proof of Proposition 5.9 shows that in this case the map $\phi_l \circ \phi_r^{-1}$ is a rotation about the collapsed point of angle equal to the distance between the collision points.

5.3. **Transverse vector fields after a collision.** It might be interesting to remark that the description made in Proposition 5.9 of the surgery on the left and right hyperbolic metrics corresponding to a collision only holds – and actually only makes sense – if there is a space-like surface with a transverse vector field both before and after the collision. However the existence of such a surface before the collision does not ensure the existence of one after the collision, even for simple collisions.

A simple example of such a phenomenon can be obtained by an extension of the example given in Section 4.4 of an AdS space with two particles containing no space-like surface with a transverse vector field. Consider the space M_θ described in that example, with $\theta < l$, so that M_θ contains a space-like surface with a transverse vector field. This space has two cone singularities, d_0 and d_1, each containing one of the endpoints of s_0. It is now possible to perform on this space a simple surgery as described in [BBS11, Section 7.1] replacing the part of d_1 in the past of its intersection with s_0 by two cone singularities, say d_2 and d_3, intersecting at the endpoint of s_0. This can be done in such a way that the angle between the plane containing s_0 and d_2 and the plane containing s_0 and d_0, is equal to l. The argument given above for M_{ex} then shows that there is no space-like surface with a transverse vector field in a spacial slice before the collision.

5.4. **The graph of interactions.** The previous section contains a description of the kind of surgery on the left and right hyperbolic metrics corresponding to a collision of particles. Here a more global description is sought, and we will associate to an AdS manifold with colliding particles a graph describing the relation between the different spacial slices. In all this part we fix an AdS manifold with colliding particles, M.

Let Ω, Ω' be two spacial slices in M. They are **equivalent** if each space-like surface in Ω is isotopic to a space-like surface in Ω'. Note that this clearly defines an equivalence relation on the spacial slices in M.

Definition 5.11. *M is a* **good** *AdS manifold with colliding particles if any spacial slice in M is equivalent to a good spacial slice.*

Clearly if two good spacial slices are equivalent then their holonomies are the same, so that their left and right hyperbolic metrics are isotopic by Proposition 4.24.

Some of the examples constructed by [BBS11, Proposition 7.7] are indeed good AdS manifolds with colliding particles. (To obtain one such example, one can construct an AdS manifold with colliding particles by a surgery on a Fuchsian space with one particle, replacing a neighborhood of the particle by a tube where two particles collide to become two new particles, so that the two particles are almost parallel both before and after the collision.)

Definition 5.12. *Let Ω_- and Ω_+ be two spacial slices in M. They are* **adjacent** *if the union of the compact connected components of the complement of the interior of $\Omega_- \cup \Omega_+$ in M contains exactly one collision. We will say that Ω_- is* **anterior** *to Ω_+ if this collision is in the future of Ω_- and in the past of Ω_+.*

Note that this relation is compatible with the equivalence relation on the spacial slices: if Ω_- is adjacent to Ω_+ and Ω'_- (resp. Ω'_+) is equivalent to Ω_- (resp. Ω_+) then Ω'_- is adjacent to Ω'_+. Moreover if Ω_- is anterior to Ω_+ then Ω'_- is anterior to Ω'_+.

Definition 5.13. *The* **graph of spacial slices** *is the oriented graph associated to a good AdS manifold with colliding particles M in the following way.*

- *The vertices of G correspond to the equivalence classes of spacial slices in M.*
- *Given two vertices v_1, v_2 of G, there is an edge between v_1 and v_2 if the corresponding spacial slices are adjacent.*
- *This edge is oriented from v_1 to v_2 if the spacial slice corresponding to v_1 is anterior to the spacial slice corresponding to v_2.*

Remark 5.14. Notice that two different admissible AdS structures in $\mathcal{U}(g, T, \theta)$ may have different graph of spacial slices. On the other hand, it is clear that the graphs of spacial slices of spacetimes in a small neighborhood of some fixed space $M \in \mathcal{U}(g, T, \theta)$ naturally contain the graph of spacial slices of M.

Note that, for the constructions that follow and in particular for Theorem 6.1, it would be sufficient to require only, rather than Definition 5.11, that there is a path on the graph of spatial slices of M whose vertices are equivalent to good slices.

5.5. **The topological and geometric structure added to the graph of interactions.** Clearly the graph of spacial slice is not in general a tree – there might be several sequences of collisions leading from one spacial slice to another one. A simple example is given in Figure 5, where the graph of a manifold with colliding particles is shown together with a schematic picture of the collisions.

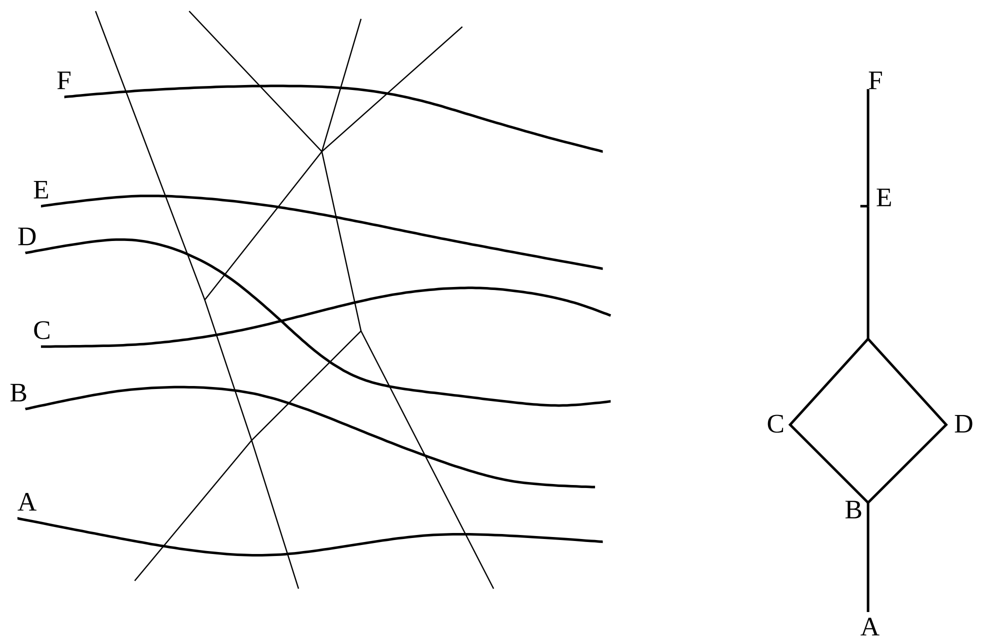

FIGURE 5. The graph of spacial slices.

The graph of spacial slices is clearly not sufficient to recover an AdS manifold with colliding particles, additional data are needed.

Definition 5.15. *A* **topological data** *associated to an oriented graph is the choice of:*

- *For each vertex v, of a closed surface S_v with n marked points $p_1, \cdots, p_n$, and a n-tuple $\theta_v = (\theta_1, \cdots, \theta_n) \in (0, 2\pi)^n$.*
- *For each oriented edge e with vertices e_- and e_+, of:*

(1) *a homotopy class of disks* $D_{e,+} \subset S_{e_+}$*, where the homotopies are in the complement of the marked points* p_i *(or equivalently they are homotopies of the complements of the* p_i *in* S_v*),*
(2) *a homotopy class of disks* $D_{e,-} \subset S_{e_-}$*, where again the homotopies fix the marked points,*
(3) *an isotopy class* i_e *of homeomorphisms from* $S_{e_-} \setminus D_{e,-}$ *to* $S_{e_+} \setminus D_{e,+}$ *sending the marked points to the marked points.*

Definition 5.16. *A* **geometric data** *associated to an oriented graph endowed with a topological data is the choice, for each vertex* v*, of two hyperbolic metrics* $\mu_l(v), \mu_r(v)$ *on* S_v*, with a cone singularity of angle* θ_i *at* p_i*, so that, for each edge* e *with endpoints* e_- *and* e_+*,* $(S_{e_+}, \mu_l(e_+), \mu_r(e_+))$ *is obtained from* $(S_{e_-}, \mu_l(e_-), \mu_r(e_-))$ *by a double surgery with surgery disks* D_{e_+} *and* D_{e_-} *respectively, as seen in Definition 5.7.*

Given a good AdS space with colliding particles M we can consider its graph of collisions Γ, there is a natural topological and geometric data associated to M on Γ. Given a vertex v of the graph of collisions Γ, it corresponds to a good spacial slice Ω_v in M, and we take as S_v a space-like surface in Ω_v. The marked points correspond to the intersections of S_v with the particles in S_v. By definition of a good spacial slice, S_v admits a transverse vector field, so one can define the left and right hyperbolic metrics $\mu_l(v)$ and $\mu_r(v)$ on S_v through Definition 4.16. The fact that those two metrics are well defined follows from Lemma 4.23.

Now consider an edge of Γ, that, is a collision between particles. Let e_- corresponds to the good spacial slice Ω_- in the past of the collision, and e_+ to the good spacial slice Ω_+ in the future of the collision. It follows from Proposition 5.9 that $(S_{e_+}, \mu_l(e_+), \mu_r(e_+))$ is obtained from $(S_{e_-}, \mu_l(e_-), \mu_r(e_-))$ by a double surgery.

6. From the geometric data to the structure

In this section we fix a maximal good AdS spacetime $M_0 \in \mathcal{U}(g, T, \theta)$ with collision and we consider the corresponding topological data X. Let $\mathcal{D}(X)$ the set of geometric data with topological data X. An element of $\mathcal{D}(X)$ is basically a collection of pairs of singular hyperbolic metrics $\mu_l(v)$ and $\mu_r(v)$ on S_v for every vertex of the graph of interaction, where the cone singularities are fixed by the topological data. Thus we can regard $\mathcal{D}(X)$ as a subset of the Cartesian product $\Pi_v \mathcal{T}(S_v, \theta_v)^2$, where $\mathcal{T}(S_v, \theta_v)$ denotes the Teichmüller space of singular hyperbolic metrics with cone angles $\theta_1 \dots \theta_n$. We will consider on $\mathcal{D}(X)$ the induced topology.

Notice that there is a neighborhood $\mathcal{U}$ of $M_0 \in \mathcal{U}(g, T, \theta)$ such that:

- The graph of spacial slice of spacetimes in $\mathcal{U}$ contains the graph of spacial slices of M_0.
- The topological data of M_0 coincides with the restriction of the topological data of any $M \in \mathcal{U}$ to the graph of M_0.
- Every vertex of the graph of M_0 corresponds to a good spacial slice for any structure of $\mathcal{U}$ (this because the transversality condition is open).

This defines a map
$$GD : \mathcal{U} \to \mathcal{D}(X)$$
sending any structure $M \in \mathcal{U}$ to the corresponding geometric data.

Theorem 6.1. *Up to shrinking* $\mathcal{U}$*, the map* GD *is injective and open.*

Proof. We will construct an open and injective map
$$H : \mathcal{D}(X) \to \mathcal{R}(g, T, \theta)$$
such that the holonomy map $hol : \mathcal{U}(g, T, \theta) \to \mathcal{R}(g, T, \theta)$ factors as $hol = H \circ GD$. The conclusion of the proof will follow from the existence of this map and from Theorem 3.3.

In order to construct the map H we give a description of the fundamental group of M by means of the topological data. We fix a path in the graph of M_0, say $v_1, \ldots, v_n$, joining the initial vertex to the final vertex, and consider the corresponding sequence of good space-like surfaces $S_1, \ldots, S_n$.

As in the proof of Proposition 5.9 we may fix lifting of the natural inclusions

$$\widetilde{S_k \setminus D_{k,-}} \to \tilde{S}_k, \qquad \widetilde{S_{k+1} \setminus D_{k,+}} \to \tilde{S}_{k+1}, \qquad \tilde{S}_k \to \tilde{M}$$

so that the corresponding inclusions of fundamental groups $\pi_1(S_k \setminus D_{k,\bullet}) < \pi_1(S_k) < \pi_1(M)$ make the following diagram commutative

$$\begin{array}{ccc} \pi_1(S_k \setminus D_{k,-}) & \xrightarrow{(i_{e_k})_*} & \pi_1(S_{k+1} \setminus D_{k,+}) \\ \downarrow & & \downarrow \\ \pi_1(M) & = & \pi_1(M) \end{array} \tag{5}$$

Notice that when the liftings $\tilde{S}_1 \setminus D_{1,-} \to \tilde{S}_1$ and $\tilde{S}_1 \to M$ are chosen, all the other liftings are fixed by the commutativity of (5).

An inductive argument, based on the van Kampen theorem, shows that the induced map

$$\pi_1(S_1) * \pi_1(S_2) * \ldots * \pi_1(S_n) \to \pi_1(M)$$

is surjective with kernel generated by elements $(i_{e_k})_*(\gamma)\gamma^{-1}$ for $\gamma \in \pi_1(S_k \setminus D_{k,-})$.

In particular given a geometric data $\mu = (\mu_l(v), \mu_r(v))$, we may fix the holonomy representations of the left and right metrics so that they determine a representation

$$H = H(\mu) : \pi_1(M) \to PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R}) \ .$$

More precisely we fix the holonomies of $\mu_l(v_1), \mu_r(v_1)$, say h^1_l, h^1_r, in their conjugacy classes. Then, we can fix recursively the holonomies of μ^k_l, μ^k_r in their conjugacy classes so that

$$h^k_\bullet(\gamma) = h^{k+1}_\bullet((i_{e_k})_*(\gamma))$$

for all $\gamma \in \pi_1(S_k \setminus D_{k,-})$. Notice that once $h^1_\bullet$ is fixed, all the other representations are uniquely determined, since the holonomy $S_v \setminus D$ is not elementary. In particular, though the representation H depends on some choices (including the isomorphism between $\pi_1(M)$ and the quotient of the free product of $\pi_1(S_i)$), its conjugacy class is well defined.

By definition $hol = H \circ GD$. So in order to conclude the proof we need to prove that

- H is injective;
- H takes value in $\mathcal{R}(g, T, \theta)$ (that is, in the space of admissible representations as in Definition 3.2);
- H is an open map.

The first point easily follows from Lemma 4.24, since the representation $H(\mu)$ contains all the holonomies of the metrics $\mu(v_i)$.

In order to prove that $H(\mu)$ is admissible, we need to check that its restriction to the fundamental group of the link of any collision point is conjugated to a diagonal representation. As we will see, this is essentially a consequence of property (3) of the definition of double surgery (as in Definition 5.7).

In fact, fix a collision point p, and suppose that S_k, S_{k+1} are the surfaces separated by p. Fix a lifting of the natural inclusions $\tilde{D}_{k,-} \to \tilde{S}_k$ so that:

- the intersection of the closure of $\tilde{D}_{k,-}$ and $\widetilde{S_k \setminus D_{k,-}}$ is not empty and it corresponds to a lifting $\tilde{\partial}_-$ of $\partial D_{k,-}$.

- Analogously the intersection of the closure of $\tilde{D}_{k,+}$ and $\widetilde{S_{k+1} \setminus D_{k,+}}$ corresponds to a lifting $\tilde{\partial}_+$ of $\partial D_{k,-}$.
- the stabilizers of $\tilde{\partial}_-$ and $\tilde{\partial}_+$ in $\pi_1(M)$ are the same $\mathbb{Z}$ subgroup generated by γ_0.

Notice that with these choices the fundamental group of the link Σ of p is generated by $\pi_1(\tilde{D}_{k,+})$ and $\pi_1(\tilde{D}_{k,-})$. Analogously if Ω is the union of adjacent spacial slices corresponding to v_k and v_{k+1}, its fundamental group in $\pi_1(M)$ is generated by $\pi_1(S_k)$ and $\pi_1(S_{k+1})$

Changing the metrics $\mu_l(v_k), \mu_r(v_k)$ and $\mu_l(v_{k+1}), \mu_r(v_{k+1})$ by some isotopy, we can require that they coincide on $D_{k,-}$ and $D_{k,+}$ respectively. Let $S_{c,-}, S_{c,+}$ be the surfaces obtained by collapsing respectively $D_{k,-}$ on $(S_k, \mu_l(v_k), \mu_r(v_k))$ and $D_{k,+}$ on $(S_{k+1}, \mu_l(v_{k+1}), \mu_r(v_{k+1}))$.

Finally choose liftings of the isometries $\phi_l, \phi_r : S_{c,-} \to S_{c,+}$, say $\tilde{\phi}_l, \tilde{\phi}_r$, so that

$$(\phi_\bullet)^{-1} \circ \gamma \circ \phi_\bullet = \gamma$$

for all $\gamma \in \pi_1(S_{c,-}) = \pi_1(S_k \setminus D_{k,-}) = \pi_1(S_{k+1} \setminus D_{k,+}) = \pi_1(S_{c,+})$.

We fix now

(1) developing maps $dev_l^-, dev_r^- : \tilde{S}_k \to \mathbb{H}^2$ of $\mu_l(v_k), \mu_r(v_k)$ so that they coincide on $\tilde{D}_{k,-}$,
(2) developing maps $d_l^-, d_r^- : \tilde{S}_{c,-} \to \mathbb{H}^2$ extending $dev_\bullet^-(v_k)$ on $\widetilde{S_k \setminus D_{k,-}}$,
(3) developing maps $d_l^+, d_r^+ : \tilde{S}_{c,+} \to \mathbb{H}^2$ defined as $d_\bullet^+ = d_\bullet^- \circ (\tilde{\phi}_\bullet)^{-1}$,
(4) developing maps $dev_l^+, dev_r^+ : \tilde{S}_{k+1} \to \mathbb{H}^2$ which extend $d_\bullet^+$ on $\widetilde{S_{k+1} \setminus D_{k,+}}$.

Notice that the holonomies of $dev_l^\pm, dev_r^\pm$ glue to a representation $H_\Omega : \pi_1(\Omega) \to PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$ which is conjugated to $H|_{\pi_1(\Omega)}$.

So it is sufficient to prove that $(H_\Omega)|_{\pi_1(\Sigma)}$ is a diagonal representations. Since we are assuming that dev_l^- and dev_r^- coincide on $\tilde{D}_{k,-}$, it is sufficient to prove that dev_l^+ and dev_r^+ coincide on $\tilde{D}_{k,+}$. Since $\mu_l(v_{k+1})$ coincides with $\mu_r(v_{k+1})$ on $D_{k,+}$, there exists $\tilde{R} \in PSL(2,\mathbb{R})$ such that

$$dev_r^+ = \tilde{R} \circ dev_l^+ \tag{6}$$

on $\tilde{D}_{k,+}$.

On the other hand, since $\partial D_{k,+}$ bounds a disk in $S_{c,+}$ where the left and right metrics coincide, by condition (3) of Definition 5.7, ϕ_l and ϕ_r coincide on $\partial D_{k,+}$ so $\tilde{\phi}_l$ and $\tilde{\phi}_r$ coincide on $\tilde{\partial}_+$. So d_l^+ and d_r^+ coincide on $\tilde{\partial}_+$. In particular $dev_r^+ = dev_l^+$ on $\tilde{\partial}_+$, and this with (6) implies that dev_r^+ and dev_l^+ coincide on the whole $\tilde{D}_{k,+}$. This finishes the proof that $H \in \mathcal{R}(g, T, \theta)$.

Finally we need to check that the map $H : \mathcal{D}(X) \to \mathcal{R}(g, T, \theta)$ is open. Given H' close to $H(\mu)$, the representation $H'|_{\pi_1(S_v)}$ is close to $H(\mu)|_{\pi_1(S_v)}$ so it is a pair of holonomies of hyperbolic structures with cone angles $\mu_l'(v), \mu_r'(v)$. Since the trace of the H'-image of peripheral elements of $\pi_1(S_v)$ is fixed, the cone angle at each point x_i is just $\theta(x_i)$. Now in order to conclude we need to check that if $e = [v_-, v_+]$ is an edge of the graph of the manifold M, $(S_{v_+}, \mu_l'(v_+), \mu_r'(v_+))$ is obtained by a double surgery on $(S_{v_-}, \mu_l'(v_-), \mu_r'(v_-))$ with surgery disks isotopic to $D_{v_-,-}$ and $D_{v_+,+}$ and identification maps isotopic to i_e.

This fact can be easily proved by the same argument used in the proof of Proposition 5.9. Let (S_0, μ_l^0, μ_r^0), and $(\hat{S}_0, \hat{\mu}_l^0, \hat{\mu}_r^0)$ be the surfaces obtained respectively from $(S_{v_+}, \mu_l'(v_+), \mu_r'(v_+))$ and $(S_{v_-}, \mu_l'(v_-), \mu_r'(v_-))$ by collapsing $D_{v_+,+}$ and $D_{v_-,-}$. Notice that the holonomy of $(S_0, \mu_\bullet^0)$ coincides with the holonomy of $(S_{v_+} \setminus D_{v_+,+}, \mu_\bullet(v_+))$ and analogously the holonomy of $(\hat{S}_0, \hat{\mu}_\bullet^0)$ coincides with the holonomy of $(S_{v_-} \setminus D_{v_-,-}, \mu_\bullet(v_-))$. Since $(S_{v_+} \setminus D_{v_+,+}, \mu_\bullet(v_+))$ is isotopic to $(S_{v_-} \setminus D_{v_-,-}, \mu_\bullet(v_-))$ in M we deduce that the holonomies of $(S_0, \mu_\bullet^0)$ and $(\hat{S}_0, \hat{\mu}_\bullet^0)$ are conjugate. Thus, by Lemma 4.24, the collapsed surfaces S_0 and $\hat{S}_0$ are isometric for both the left and right metrics.

The fact that H' restricted to the fundamental group of the link of the collision point between S_{v_-} and S_{v_+} is diagonal implies that the condition (3) in the definition of double surgery is satisfied. □

Appendix A. An example where a double surgery is needed

Let S_1 and S_2 be two hyperbolic surfaces with cone singularities of angles less than 2π and let D_1 and D_2 be two disks embedded in S_1 and S_2 respectively. We suppose that there is a diffeomorphism preserving cone points $f : S_1 \setminus D_1 \to S_2 \setminus D_2$ such that the holonomy of $S_1 \setminus D_1$ is conjugated to the representation obtained by composing the holonomy of $S_2 \setminus D_2$ with the map $f_* : \pi_1(S_1 \setminus D_1) \to \pi_1(S_2 \setminus D_2)$ induced by f. We will also assume in this appendix that the holonomy of ∂D_2 is elliptic of angle $\theta < 2\pi$.

In Section 5.2 we have shown that collapsing D_1 in S_1 and D_2 in S_2 yields the same surface (up to isometry). This means that there are two surgeries involved in the transformation from S_1 to S_2. First the disk D_1 is replaced by a disk P_1 containing only a cone point, yielding a surface S. Then another disk P_2 of S isotopic to P_1 is replaced by D_2. We could expect at first glance that a single surgery would be sufficient; for example, that choices of disks of surgery can be made so that P_1 and P_2 coincide. But in this section we will prove that this cannot always be the case. We will find a criterion to establish whether a single surgery is sufficient, and will then construct an example where this criterion fails.

Notice that the complement of any disk isotopic to $D_\bullet$ in $S_\bullet$ isometrically embeds in S. We consider the minimal convex disk Δ isotopic to $D_\bullet$ constructed in Proposition 4.28. (Notice that if $D_\bullet$ contains only 2 singular points, the Δ degenerates to a segment, but the argument below can be adapted.)

We still have that $S_\bullet \setminus \Delta_\bullet$ embeds in S. According to the following definition, the complement of its image is a polygon with center p which we denote by $P_\bullet$.

Definition A.1. *Let S be a hyperbolic surface with cone singularities, and p be a cone point on s. A polygon with center p in S, is an embedded disk P, such that*

- *p is in its interior;*
- *P is the union of hyperbolic triangles which have all a vertex at p.*

Proposition A.2. *S_2 is obtained by a single surgery on S_1 if and only if P_1 and P_2 are contained in a disk Π embedded in S.*

The if part is easy: $S \setminus \Pi$ embeds in $S_\bullet$ and its complement is a disk $\Delta'_\bullet$ isotopic to $D_\bullet$. Thus we can cut from S_1 the disk Δ'_1 and glue instead the disk Δ'_2. The surface that we obtain is obviously isometric to S_2.

The only if part easily follows from the following lemma.

Lemma A.3. *If S_2 is obtained by a single surgery on S_1, then the surgery can be done replacing a convex disk D'_1 whose boundary is piecewise geodesic with some vertices at cone points and some vertices in the smooth part (which can degenerate to segment) by a disk D'_2 with similar properties.*

Proof. We denote by $\mathcal{D}$ the set of disks embedded in S_1 isotopic to D_1, whose complement can be embedded in S_2 by an isometric map isotopic to f.

Take a sequence of disks $D_n \in \mathcal{D}$ which minimizes the length of the boundary. Take a parameterization $c_n : [0, 1] \to \partial D_n$. Up to taking a subsequence, c_n converges to a curve c_∞ and D_n converges to some subset D'_1. By the minimality it turns out that c_∞ is a piecewise geodesic curve with vertices at cone points. Moreover if $c_\infty(t_0)$ is a cone point, then the segments $[c_\infty(t_0) - \epsilon, c_\infty(t_0)]$ and $[c_\infty(t_0), c_\infty(t_0 + \epsilon)]$ form an angle bigger than π in $S \setminus D'_1$.

It follows that there are two cases: either c_∞ spans a segment with vertices at two cone points, or it is an embedded curve. In the first case D_1' coincides with the support of c_∞, whereas in the second case it is a convex disk bounded by c_∞.

In both cases $S_1 \setminus D_1'$ embeds in S_2. The complement D_2' of the embedding of $S_1 \setminus D_1'$ in S_2 is still a convex disk with piecewise geodesic or a segment. Clearly S_2 is obtained from S_1 by cutting D_1' and replacing D_2'. □

We can now prove that the condition of Proposition A.2 is necessary. Let D_1' and D_2' be as in Lemma A.3. The minimal disk $\Delta_\bullet$ is contained in $D_\bullet'$.

It follows that $S_1 \setminus D_1'$ and $S_2 \setminus D_2'$ are both isometric to a region S' of S, whose complement — say Π — is a polygon with vertex at p. Moreover S' is contained in $S \setminus P_1$ and $S \setminus P_2$, so Π contains both P_1 and P_2, and this concludes the proof of Proposition A.2.

In the remaining part of this section we will construct an example of two surfaces S_1 and S_2 satisfying the condition given at the beginning of the appendix but such that the disks P_1 and P_2 in S are not contained in a disk. This will show that S_2 cannot be obtained by a single surgery on S_1.

First we will construct a surface S containing only a single cone point, then we will find two convex polygons P_1 and P_2 around the cone point whose union is contained in no embedded disk. Finally, making a surgery on $P_\bullet$, we will construct two surfaces $S_\bullet$ such that the complement of the corresponding minimal disk is isometric to the complement of $S \setminus P_\bullet$.

Construction of S and $P_\bullet$. We consider in $\mathbb{H}^2$ a regular convex octagon Q such that the sum of its interior angles is $\theta \in (0, 2\pi)$. Gluing opposite sides of Q we obtain a hyperbolic surface S with a cone point corresponding to the vertices of Q whose cone angle is θ. Denote by $\pi : Q \to S$ the projection map.

Let l be the length of any edge of Q. Choosing $l' < l/2$, we can consider for every vertex p_i of Q the triangle T_i with two edges of length l' contained in the edges of Q at p. The union of those triangles projects to a convex polygon P_0 with center $\bar{p}$ in S. Notice that the angle of this polygon is equal to 2ϕ, where ϕ is the base angle of the triangles. We now add to P_0 a triangle Z_i with a vertex at the center of Q and opposite edge equal to an edge e_i of a triangle T_i. If l' is close to $l/2$ then the sum of the angle of Z_i at a vertex of e_i and 2ϕ is less than π, so $P_i = P_0 \cup Z_i$ is a convex polygon contained in S. Since $P_1 \cup P_2$ contains a non-trivial loop, it is not contained in any disk of S.

Construction of S_1 and S_2. Notice that P_1 is a polygon with center $\bar{p}$ in S. In fact $T_1 \cap Z_1$ is the union of two triangles with a vertex at $\bar{p}$. In particular P_1 can be decomposed as a union of triangles with vertex at $\bar{p}$. Each of these triangles has a boundary edge and two interior edges. There is one interior edge joining $\bar{p}$ to the center of Q whose length is u, whereas the length of all the other interior edges is l'.

Choose ϵ small. We can deform every triangle of the decomposition of P_1 without changing its boundary edge and shortening the other two edges of ϵ. Let us call P_1' the polygon obtained in this way. Replacing P_1 by P_1' we get a surface S_1 with a cone point at each vertex of P_1' and a central vertex.

Analogously we can obtain a surface S_2 by making a surgery on P_2. It is not difficult to construct a diffeomorphism $f : S_1 \setminus P_1' \to S_2 \setminus P_2'$ such that the following diagram is homotopically commutative.

To conclude it is sufficient to notice that $P_\bullet'$ is a minimal disk whose complement is isometric to $S \setminus P_\bullet$. Since there is no disk containing both P_1 and P_2, Proposition A.2 implies that S_2 cannot be obtained by a single surgery on S_1.

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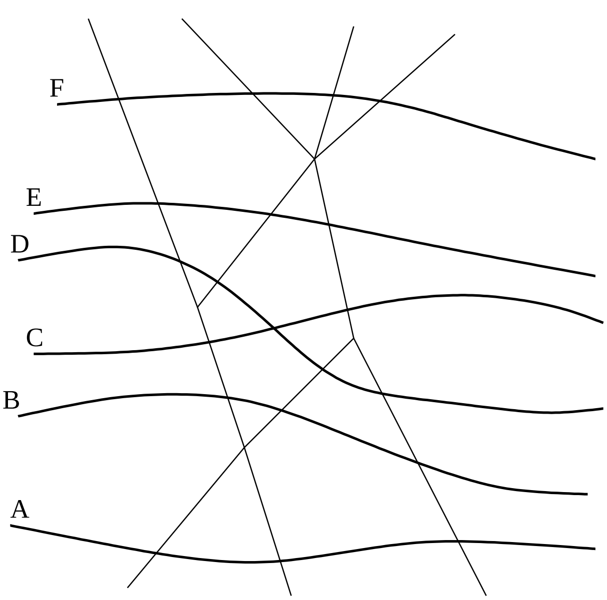

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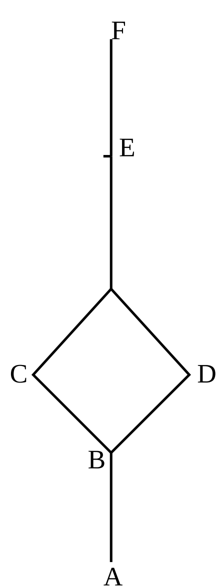

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