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ORIGINAL ARTICLE

Opinion formation in social networks: a time-variant and non-linear model

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Abstract This paper develops a discrete-time, non-linear, and time-variant model of opinion formation in a social network with global interactions to investigate the relationship between the final consensus belief and the set of agents' initial opinions. The model uses a novel and considerably intuitive updating rule, according to which the weight placed by an agent on another one's opinion in each period decreases continuously with the distance between their beliefs in the previous period. In this context, the first part of our analysis proves that agents' beliefs converge and reach a consensus over time (under a fairly general set of conditions). For the two-agent case, it is then shown that the consensus belief is the simple arithmetic mean of the initial opinions. When there are three agents in the network, the combined use of computational and analytical methods reveals a relatively more complex polynomial relationship between long-run and initial beliefs. In particular, our results for the three-agent case imply that the deviation of the limiting belief from the corresponding average of the initial beliefs can be expressed as a third degree polynomial function incorporating the pairwise differences of agents' starting beliefs.

Keywords Social networks · Opinion dynamics · Global interactions · Consensus

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Introduction

The structure of social networks has a central role in the process of opinion formation and information transmission among interacting agents (nodes). The network under consideration might be either small (such as a committee of scientists who exchange pieces of information to form their beliefs about the global warming effect) or large (such as a group of voters who discuss on the relative merits of a politician or a group of consumers who are trying to evaluate the quality of a new product). In all such cases, the topological properties of the underlying network critically affect the temporal evolution of beliefs and the associated behavioral patterns. Most related studies investigate the conditions under which agents' beliefs about the value of an underlying parameter converge and reach a consensus over time. In this paper, we suggest a new rule governing the process of belief updating to build a non-linear and non-stationary model of opinion dynamics. In this context, we use both analytical and computational methods to examine the question of how the limiting consensus belief varies with the initial opinion vector.

In general, there are two main approaches to modeling the process of opinion formation in social networks. The first approach assumes that individual agents are fully rational and can process all available information in a sophisticated manner to update their beliefs according to Bayes' rule [1–3]. This kind of Bayesian updating makes the problem of drawing inferences about one's neighbors rather complex and quickly becomes intractable even in simple networks. Furthermore, a series of laboratory experiments conducted by Choi et al. [4,5] have shown that individuals' behavioral patterns substantially deviate from strategies that would be followed by fully rational (Bayesian) agents. As a result, a second widely used approach to modeling opinion dynamics has built on DeGroot's [6] seminal work which relies on the assumption that agents are boundedly rational. According to this formulation, individual nodes use some simple rule-of-thumb to update their beliefs in each period as a function of their neighbors' beliefs in the previous period [7–9]. This paper also lies within the framework of bounded rationality but suggests a novel updating rule which stipulates that the weight placed by an agent on another one's opinion is a continuously decreasing function of the distance between their beliefs in the previous period.

The classical DeGroot model considers a set of myopic agents $N = \{1, \ldots, n\}$ that are connected in a possibly directed and weighted network. Each agent $i \in N$ starts with an initial belief ($p_i(0) \in [0, 1]$) about an underlying state of the world. The network of interactions is captured by a non-negative row-stochastic matrix T, such that

$$\sum_{j=1}^{n} T_{ij} = 1, \quad \forall i \in N. \tag{1}$$

Each element T_{ij} of this matrix represents the weight agent i places on j's current opinion while forming her next-period belief. The updating rule stipulates that each individual forms her beliefs in each period as a weighted average (i.e., as a linear combination) of her neighbors' beliefs in the previous period according to the following equation:

$$p_i(t+1) = \sum_{j=1}^{n} T_{ij} \cdot p_j(t). \tag{2}$$

Since the interaction matrix remains constant over time, this is a linear and stationary model that derives necessary and sufficient conditions for convergence and consensus of beliefs in the long-run. An interesting variation of the model assumes that agents have some degree of persistence on their initial beliefs but maintains the properties of linearity and stationarity [10,11].

On the contrary, [12,13] extend the seminal model by assuming that the elements of the interaction matrix are time-dependent. In this non-stationary (but still linear) framework, it is shown that consensus can still be reached provided that the weights remain sufficiently positive over time (i.e., they do not tend to zero too fast). Of course, the most general form of the opinion dynamics model involves a non-linear and non-stationary updating rule obtained by assuming that the interaction matrix depends on the opinion vector $\mathbf{p}(t)$ itself in each period: $T = T(\mathbf{p}(t))$. To address the questions of convergence and consensus in this setting, one must distinguish between the cases of local and global interactions. Local interaction models often rely on a particularly tractable kind of non-linearity involving bounded confidence among agents in the network. This means that each individual i only considers the set of agents whose opinions differ from her own less than a confidence level d_i. This subset of the overall population constitutes i's confidence set. The updating rule in the bounded confidence model assumes that each node puts equal weight on the opinion of all agents who belong to her confidence set. Under the assumption of a uniform confidence level across all agents, it can be shown that beliefs will converge in finite time and a pattern of fragmentation will eventually prevail [7,14–16]. Since the set of conditions for reaching a consensus in the overall society depends on the number of agents, the mathematical analysis becomes difficult for arbitrary network sizes and the investigation of the bounded confidence model usually proceeds by a series of computer simulations [16–19].

A second class of non-linear (and non-stationary) opinion formation models considers the case of global interactions, where each agent forms her opinion in period $t + 1$ by considering (i.e., by compromising with) the opinion of potentially all other agents in period t. in this case, each node's confidence set coincides with the overall population. In fact, a model of local interactions becomes global by sufficiently increasing the confidence level of each individual. Under the assumption of putting equal weight on others' beliefs, each agent's opinion in period $t + 1$ will simply be given by the arithmetic mean of all individual opinions in period t. This case is not really interesting, since it implies that an overall consensus will already be obtained at $t = 1$. For this reason, Krause [20] considers a more general updating rule stipulating that each individual's current belief lies between the lowest and highest beliefs held within the population in the previous period. The use of such a compromising rule implies that a consensus will eventually be reached in the long-run. However, the relationship between the limiting consensus belief and the initial opinion vector can be rather complex (depending on the specific form of the updating rule) and remains an open question for research.

This paper remains within the context of global interactions to introduce a simple and intuitive updating rule according to which the weight placed by an agent on another one's opinion decreases continuously with the distance between their beliefs in the previous period. Our main objective is to jointly show convergence and consensus by utilizing the conceptual framework of dynamical systems as well as to study how the consensus belief depends on the initial conditions in this context. For the case of two agents, we analytically find that the limiting belief is the simple arithmetic mean of the initial opinions. When there are three agents in the network, a similar analysis of convergence and consensus is followed by the use of both computational and analytical methods to derive the implicit relationship between the consensus belief and the vector of the initial opinions. Therefore, our analysis implies the existence of an implicit functional relationship between the limiting consensus belief and the set of agents' initial opinions.

Our findings can be compared to the set of results reached by Pan [21]. She also proves convergence and consensus using an updating rule which stipulates that the weight put by an agent on another one's beliefs in each period is inversely related to the distance between their beliefs in the previous period. However, she adopts a rather ad hoc formulation concerning the diagonal elements of the interaction matrix, according to which the level of trust placed by an agent on herself is not based on distances but only indirectly changes over time due to a normalization term ensuring the row-stochasticity. This formulation implies that each agent puts zero weight on herself and equal weight on other agents in the limit. We treat this result as a paradox in the sense that the aforementioned limiting behavior of the social interaction matrix T is not in accordance with its gradual updating rule at any time instance t. In particular, each agent neither assigns an equal amount of relative influence to any other agent in the network nor evaluates her own belief as having zero relative importance ($T_{ij}(t) \neq \frac{1}{n-1}$, $\forall t$ and $T_{ii}(t) \neq 0$, $\forall t$).

Our own treatment of the updating process modifies this rather paradoxical result by revealing a wider spectrum of limiting behavior for the social interaction matrix which does not reproduce the suspiciously simple convergence pattern predicted by Pan. Specifically, we show that as $t \to \infty$, the limiting degree of influence assigned by an agent to any other agent in a network with more than two nodes is non-uniform and the amount of relative importance assigned on one's own belief is different than zero. Moreover, we show that the actual form of the limiting consensus belief $\mathbf{p}(t)$ as $t \to \infty$ depends on the initial opinion vector $\mathbf{p}(0)$ according to a non-trivial, complex polynomial relationship. Therefore, the most important diversification of our model relates to the limiting behavior of the social interaction matrix. However, our approach suggests a more general opinion formation model with a common updating rule for all (non-diagonal and diagonal) entries of the social interaction matrix $T(t)$, which are updated according to the pairwise distances of agents' beliefs at any given time instance t. In addition, our model does not rely on the arbitrary definition of an initial social interaction matrix $T(0)$ which persists throughout the opinion updating process. Finally, the relative degree of importance is modeled as a continuous decreasing function of the distance in such a way that no particular treatment (e.g., through the use of an arbitrary parameter $\underline{d}$, as in Pan's model) is necessary to avoid a division by zero when a given pair of agents happens to share the same belief.

The rest of this paper is organized as follows. Section 2 introduces our non-linear and time-variant opinion formation model. Section 3 studies the two-agent network and Sect. 4 deals with the case where there are three agents in the network. Finally, Sect. 5 concludes the paper and suggests directions for future research.

Modeling framework

Consider a set of agents, $N = \{1, \ldots, n\}$, who are connected in a weighted and possibly directed network. Each agent $i \in N$ starts with an initial belief $p_i(0) \in [0, 1]$ about an underlying state of world, so that the vector of initial beliefs for all network agents is given by $\mathbf{p}(0) = [p_1(0), \ldots, p_n(0)]^T \in M_{n\times 1}$. In general, agent i's belief in period t is given by $p_i(t)$, so that the corresponding complete vector of beliefs in period t is denoted as $\mathbf{p}(t) = [p_1(t), \ldots, p_n(t)]^T \in M_{n\times 1}$. The network is captured by a non-negative row-stochastic social interaction matrix $\mathbf{T} \in M_{n\times n}$. Each element T_{ij} of this matrix represents the weight agent i places on j's current belief in forming her own next-period belief.

This paper introduces an alternative opinion updating rule that extends the original bounded confidence model proposed by Hegselmann and Krause [22]. Our primary objective lies upon the incorporation of a time-variant functional form for the social interaction matrix, exhibiting properties that enforce the following directives:

- Global network interactions should be considered when a particular agent's belief is updated;
- Continuous, time-varying connection weights should be utilized for each pair of interacting agents;
- A non-uniform confidence distribution should be employed on the agents' beliefs that are taken into consideration by a particular agent when forming her next-period belief;
- Higher connection weights should be assigned on pairs of agents whose private beliefs present lower deviation.

The original bounded confidence model rests on the simplifying assumption that each individual $i \in N$ pays attention exclusively on a *restricted* confidence set of agents, whose beliefs fall within a predefined distance range from her own private belief. Specifically, this set is parameterized by a threshold value d_i, as shown in Eq. 3:

$$S_i(t; d_i) = \{k \in N : |p_i(t) - p_k(t)| < d_i\}, \qquad (3)$$

so that the corresponding opinion updating rule becomes:

$$p_i(t+1) = \sum_{j \in S_i(t;d_i)} \frac{1}{|S_i(t; d_i)|} \cdot p_j(t). \qquad (4)$$

Thus, agents are characterized by a sort of distrust against external information that significantly deviates from their own opinions, leading them to completely ignore these particular belief signals. The other fundamental property of this model relates to the fact that each agent assigns the same weight on all agents pertaining to her confidence set, without employing a scalable trust schema that incorporates different levels of confidence. Therefore, the model proposed by Hegselmann and Krause can be summarized by the functional form of the time-varying social interaction matrix given below:

$$T_{ij}(t) = \begin{cases} \frac{1}{|S_i(t;di)|}, & \text{if } |p_i(t) - p_j(t)| < d_i; \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

The model proposed in this paper is also based on the assumption that the time-varying form of the social interaction matrix depends on the opinion vector itself, i.e., $\mathbf{T} = \mathbf{T}(\mathbf{p}(t))$. This assumption gives rise to a non-linear and non-stationary opinion updating rule of the form:

$$p_i(t+1) = \sum_{j \in N} \lambda_j^{(i)}(\mathbf{p}(t)) \cdot p_j(t), \quad (6)$$

where $\lambda_j^{(i)}(\mathbf{p}(t))$ denotes the time-dependent interaction strength between agents i and j, such that

$$\sum_{j=1}^{n} \lambda_j^{(i)}(\mathbf{p}(t)) = 1, \quad \forall i \in N, \quad \forall t \geq 0. \quad (7)$$

Our approach constitutes a major differentiation from the original bounded confidence model, since we consider global interactions amongst the agents pertaining to the network. Each agent updates her private belief by taking into consideration information signals from all other agents of the network, including her own opinion. This fact is encoded within Eq. 6, where summation is taken over the complete set N, implying a fully connected (and directional) topology for the underlying network. The most important extension, however, concerns the utilization of a non-uniform, weighting scheme, determining the magnitude of the connection strength for any given pair of interacting agents. This scheme entails that the weighting terms $\lambda_j^{(i)}(t)$'s should be provided by a continuous function of the form $g(\mathbf{p}(t))$ that focuses on the ith and jth elements of the opinion vector in period t, so that

$$\lambda_j^{(i)}(t) = g(p_i(t), p_j(t)). \quad (8)$$

Our intention is to formulate a social network interaction scheme stemming from the idea that agents with similar beliefs will exert a higher level of influence on each other. This, in turn, signifies the incorporation of a notion of homophily within the network, since an agent evaluates as more credible opinions that are closer to hers. Therefore, the analytical form of the function appearing on the right-hand side of Eq. 8 could be given by a monotonically decreasing function of the distance $p_i(t) - p_j(t)$. Amongst the admissible functions are those implementing mappings of the following form:

$$g : [0, 1] \times [0, 1] \rightarrow [0, 1] \quad (9)$$

such that

$$g(x, y) = \phi(x - y) = \phi(u) \quad (10)$$

where

$$\phi(u) = 1 - |u| \quad (11)$$

or

$$\phi(u) = 1 - u^2. \quad (12)$$

However, functional forms described by Eqs. 11 and 12 do not satisfy the requirement defined by Eq. 7 which guarantees the row-stochasticity of matrix $T(\mathbf{p}(t))$, $\forall t \geq 0$. Therefore, an extra regularization term is needed, so that the function $g(p_i(t), p_j(t))$ will be replaced by:

$$\tilde{g}(\mathbf{p}(t)) = \frac{g(p_i(t), p_j(t))}{\sum_{j=1}^{n} g(p_i(t), p_j(t))}, \quad (13)$$

thus taking into consideration the pairwise differences of beliefs between agent i and all other agents within the network. Having in mind that this paper uses the functional form defined in Eq. 12, the resulting opinion updating rule will be given by:

$$p_i(t+1) = \sum_{j \in N} \frac{1 - [p_i(t) - p_j(t)]^2}{n - \sum_{j \in N} [p_i(t) - p_j(t)]^2} p_j(t). \quad (14)$$

By taking into consideration Eq. 14, it is easy to deduce that the non-linear and time-varying form of the social interaction matrix employed by our model will be as follows:

$$T_{ij}(\mathbf{p}(t)) = \frac{1 - [p_i(t) - p_j(t)]^2}{n - \sum_{j \in N \setminus \{i\}} [p_i(t) - p_j(t)]^2}. \quad (15)$$

It is very important to note that the normalization factor, incorporated to sustain the row-stochasticity of the social interaction matrix $\mathbf{T}$, does not affect the monotonicity of the weighting function that quantifies the interaction strength between agents i and j with respect to the distance of beliefs $d_{ij} = (p_i - p_j)^2$. Letting aside the time-dependence of the model, we may write that the weighting term T_{ij} can be expressed as a function of d_{ij} in the following way:

$$T_{ij}(d_{ij}) = \frac{1-d_{ij}}{n-d_{ij}-\sum_{k\neq i,j} d_{ik}}, \quad i,j\in N, \quad j\neq i. \tag{16}$$

According to Eq. 16, it is easy to deduce that

$$\frac{\partial T_{ij}(d_{ij})}{\partial d_{ij}} = \frac{-(n-1)+\sum_{k\neq i,j} d_{ik}}{\left(n-d_{ij}-\sum_{k\neq i,j} d_{ik}\right)^2} < 0, \quad i,j\in N, \quad j\neq i. \tag{17}$$

since $0 \leq \sum_{k\neq i,j} d_{ik} \leq n-2$. Therefore, T_{ij} is a strictly decreasing function of d_{ij}, assigning higher confidence values on nodes whose beliefs exhibit lower deviation from the current belief of a given node i.

The two-agent case

In this section, we are interested in the limiting behavior of the smallest possible, non-trivial, network of agents whose beliefs are updated according to Eq. 14. Specifically, we focus on the stability analysis of the emerging discrete-time non-linear dynamical system when $n=2$, which provides significant insights concerning the social networks' ability to convergence and reach a consensus.

Convergence and consensus analysis

The general form of the updating equations for our model, when $n=2$, takes the following form:

$$F_x(x,y) = \lambda_x(x)\cdot x + \lambda_x(y)\cdot y \tag{18}$$
$$F_y(x,y) = \lambda_y(x)\cdot x + \lambda_y(y)\cdot y \tag{19}$$

where $x, y \in [0,1]$ are the current node beliefs and $F_x(x,y)$, $F_y(x,y)$ are the functions assigning the next-period node beliefs. The weighting terms for each node are given be the following set of equations:

$$\lambda_x(s) = \frac{g(x,s)}{g(x,x)+g(x,y)}, \tag{20}$$
$$\lambda_y(s) = \frac{g(y,s)}{g(y,y)+g(y,x)}, \tag{21}$$

with $s\in\{x,y\}$ satisfying that

$$\lambda_x(x)+\lambda_x(y) = 1 \tag{22}$$
$$\lambda_y(x)+\lambda_y(y) = 1. \tag{23}$$

Moreover, Eqs. 18 and 19 can be rewritten in the following form by taking into consideration Eqs. 22 and 23:

$$F_x(x,y) = (1-\lambda_x(y))\cdot x + \lambda_x(y)\cdot y \tag{24}$$
$$F_y(x,y) = \lambda_y(x)\cdot x + (1-\lambda_y(x))\cdot y \tag{25}$$

which yields that

$$F_x(x,y) = x + (y-x)\cdot\lambda_x(y) \tag{26}$$
$$F_y(x,y) = y + (x-y)\cdot\lambda_y(x) \tag{27}$$

By incorporating the time parameter within Eqs. 26 and 27, they may be reexpressed in the following form:

$$F_x(x_t,y_t) = x_t + (y_t-x_t)\cdot\lambda_x(y_t) \tag{28}$$
$$F_y(x_t,y_t) = y_t + (x_t-y_t)\cdot\lambda_y(x_t) \tag{29}$$

such that

$$x_{t+1} = F_x(x_t,y_t) \tag{30}$$
$$y_{t+1} = F_y(x_t,y_t) \tag{31}$$

where $t\in\mathbb{N}_+^*$. Letting

$$\delta_t = x_t - y_t \tag{32}$$

and by taking into consideration Eqs. 30 and 31, Eqs. 28 and 29 can be reformulated as:

$$x_{t+1} = x_t - \delta_t\cdot\lambda_x(y_t) \tag{33}$$
$$y_{t+1} = y_t + \delta_t\cdot\lambda_y(x_t). \tag{34}$$

Having in mind Eq. 12, it is easy to deduce that the weighting terms associated with each node will be given by the following equation:

$$\lambda_x(y_t) = \lambda_y(x_t) = \frac{1-(x_t-y_t)^2}{2-(x_t-y_t)^2} = \frac{1-\delta_t^2}{2-\delta_t^2} \tag{35}$$

which finally yields that Eqs.33 and 34 may be rewritten as follows:

$$x_{t+1} = x_t - \delta_t\cdot\frac{1-\delta_t^2}{2-\delta_t^2} \tag{36}$$
$$y_{t+1} = y_t + \delta_t\cdot\frac{1-\delta_t^2}{2-\delta_t^2}. \tag{37}$$

By performing pairwise substraction between Eqs.36 and 37, we obtain that:

$$x_{t+1}-y_{t+1} = (x_t-y_t) - 2\cdot\delta_t\cdot\frac{1-\delta_t^2}{2-\delta_t^2} \tag{38}$$

which can be written in the following form:

$$\delta_{t+1} = \delta_t - 2\cdot\delta_t\cdot\frac{1-\delta_t^2}{2-\delta_t^2} \tag{39}$$

finally, yielding

$$\delta_{t+1} = \frac{\delta_t^3}{2 - \delta_t^2}. \tag{40}$$

Equation 40 describes the time evolution of a discrete-time non-linear dynamical system, by defining the recursive relation that controls the sequence of differences between the network agents' beliefs. Exploration of such non-linear difference equations, in the sense of determining and characterizing their solutions in terms of stability, can be conducted according to [23]. Specifically, the steady states of this dynamical system can be obtained by setting: $\delta_{t+1} = \delta_t = \overline{\delta}$, which is equivalent to determining the solutions of $\overline{\delta}^3 - \overline{\delta} = 0$. Therefore, the set of equilibrium points for the aforementioned dynamical system will be given by $\Delta_s = \{\overline{\delta}_1, \overline{\delta}_2, \overline{\delta}_3\}$, where $\overline{\delta}_1 = 0$, $\overline{\delta}_2 = 1$, and $\overline{\delta}_3 = -1$. The stability status for each one of the obtained solutions, $\overline{\delta}_i,\ i \in [3]$, can be determined by computing the values of $\frac{dh}{d\delta}(\overline{\delta}_i)$, where $h(\delta) = \frac{\delta^3}{2-\delta^2}$ and $\frac{dh}{d\delta} = \frac{\delta^2 \cdot (6-\delta^2)}{(2-\delta^2)^2}$, with stable points being those that correspond to solutions satisfying $\left|\frac{dh}{d\delta}(\overline{\delta}_i)\right| < 1$. Having in mind that $\left|\frac{dh}{d\delta}(\overline{\delta}_2)\right| = \left|\frac{dh}{d\delta}(\overline{\delta}_3)\right| = 5 > 1$ and that $\left|\frac{dh}{d\delta}(\overline{\delta}_1)\right| = 0 < 1$, it is easy to deduce that the discrete-time non-linear dynamical system under investigation possesses a unique stable fixed point when $\delta_t = 0$.

The previous stability analysis results might be better understood by considering the underlying phase-space of differences between agents' beliefs. In this context, the most significant facts concerning the dynamical system arising from our model can be conceived by studying the behavior of the following quantity:

$$\Delta\delta_t = \delta_{t+1} - \delta_t \tag{41}$$

as a function of the current difference of nodes' beliefs δ_t. Interestingly, this quantity may be interpreted as a velocity parameter v, since $v = v(\delta_t) = \frac{\Delta\delta_t}{\Delta_t} = \frac{\Delta\delta_t}{(t+1)-t} = \Delta\delta_t$, which finally yields:

$$\Delta\delta_t = h(\delta_t) - \delta_t = \frac{2(\delta_t^3 - \delta_t)}{2 - \delta_t^2}, \quad \delta_t \in [-1, 1]. \tag{42}$$

The graphical representation of the quantity $\Delta\delta_t$ as a function of δ_t appears in Fig. 1. The fixed points of the dynamical system are indicated as circles on the horizontal line, corresponding to states within the considered phase-space when the velocity equals zero ($\Delta\delta_t = 0$). In other words, such points in the phase-space characterize states of the dynamical system that when reached, no further evolution of the system over time is possible. The unstable fixed points are marked as unfilled circles at the points where $\delta_t = -1$ or $\delta_t = +1$, while the stable one (*map sink* or *attractor* within the terminology of dynamical systems) is depicted as a filled circle at the point where $\delta_t = 0$. Moreover, by taking into consideration Eqs. 26 and 27, it is easy to derive that when $\delta_{t^*} \in \Delta_s$, then $x_{t+1} = x_t,\ \forall t \geq t^*$ and $y_{t+1} = y_t,\ \forall t \geq t^*$. This implies that once the dynamical system enters one of the identified steady states for some time $t^* \geq 0$, the corresponding agents' beliefs cease to evolve over time as well. Specifically, given that $x_{t^*}, y_{t^*} \in [0, 1]$, the unstable fixed points for $\delta_{t^*} = -1$ and $\delta_{t^*} = +1$ emerge exclusively from the situations where $(x_{t^*}, y_{t^*}) = (0, 1)$ and $(x_{t^*}, y_{t^*}) = (1, 0)$. Furthermore, the underlying nodes' beliefs for the case where $\delta_{t^*} = 0$ are, such that $x_{t^*} = y_{t^*}$.

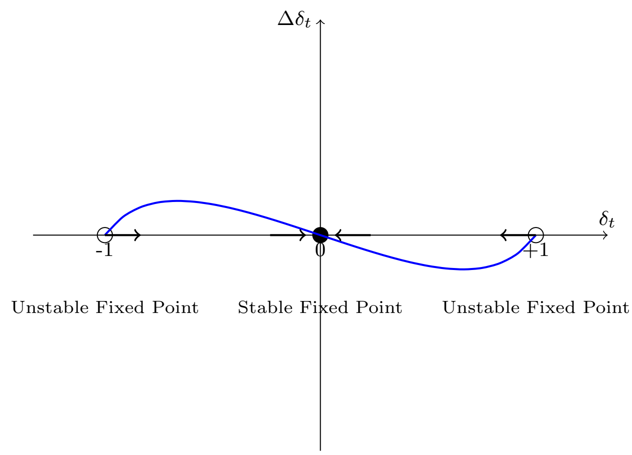

Fig. 1 Stability analysis for $n = 2$

The most important fact, however, concerning the stable fixed point at $\delta_t = 0$ is that all possible configurations of the initial nodes' beliefs will converge to that point excluding the cases for which $\delta_0 = -1$ and $\delta_0 = +1$, as indicated by the arrows drawn in Fig. 1 pointing towards that starting point of the horizontal axis. This is indicative of the fact that even slight perturbations of these initial states will lead the system to evolve towards its unique stable fixed point, which is also represented in Fig. 1 by the arrows pointing to the right and to the left of the points at -1 and $+1$ on the horizontal axis, respectively. Therefore, we may write that

$$\lim_{t\to\infty} \delta_t = \delta_\infty = \begin{cases} -1, & \delta_0 = -1; \\ 0, & -1 < \delta_0 < +1; \\ +1, & \delta_0 = 1. \end{cases} \tag{43}$$

which is equivalent to:

$$\lim_{t\to\infty} (x_t, y_t) = (x_\infty, y_\infty) = \begin{cases} (0, 1), & (x_0, y_0) = (0, 1); \\ (p^*, p^*), & (x_0, y_0) \in [0, 1]^2 \backslash \{(0, 1), (1, 0)\}; \\ (1, 0), & (x_0, y_0) = (1, 0). \end{cases} \tag{44}$$

where $p^* \in [0, 1]$. The previous analysis shows convergence in the network for the various initial configurations of nodes' beliefs, given that the adopted updating rule is the one described in Eq.14. In addition, we have shown that under certain initial conditions, nodes' limiting beliefs lead to a consensus. In this case, the updating rule specified in Eq. 15 immediately implies that influence weights are uniformly distributed among all agents in the limit, that is

$$T_{ij}(p^*) = \frac{1}{n}, \quad \forall i, j \in N \tag{45}$$

Limiting consensus beliefs

Having determined the initial conditions under which the learning process leads the network to a consensus for $n = 2$, we will proceed by extracting the functional relation that associates the network agents' starting beliefs $(x_0, y_0) \in P_{cons}$ with the corresponding consensual belief p^*, where

$$P_{cons} = \left\{(x_0, y_0) \in [0, 1]^2 : (x_0, y_0) \neq (0, 1) \wedge (x_0, y_0) \neq (1, 0)\right\}. \tag{46}$$

This task may be undertaken by considering the reformulation of Eqs. 36 and 37 appearing below:

$$x_{t+1} - x_t = -\delta_t \cdot \frac{1 - \delta_t^2}{2 - \delta_t^2} \tag{47}$$

$$y_{t+1} - y_t = +\delta_t \cdot \frac{1 - \delta_t^2}{2 - \delta_t^2} \tag{48}$$

which leads to the following set of equations:

$$\delta_t^x = -\delta_t \cdot \frac{1 - \delta_t^2}{2 - \delta_t^2} \tag{49}$$

$$\delta_t^y = +\delta_t \cdot \frac{1 - \delta_t^2}{2 - \delta_t^2} \tag{50}$$

by letting

$$\delta_t^x = x_{t+1} - x_t \tag{51}$$

$$\delta_t^y = y_{t+1} - y_t. \tag{52}$$

From Eqs. 49 and 50, and given that the sequence of δ_t's converges to zero ($\delta_t \to 0, \forall \delta_0 \in (-1, +1)$), it is easy to derive that $\forall (x_0, y_0) \in P_{\text{cons}}$:

$$\lim_{t\to\infty} \delta_t^x = \delta_\infty^x = 0 < \infty \tag{53}$$

$$\lim_{t\to\infty} \delta_t^y = \delta_\infty^y = 0 < \infty. \tag{54}$$

Given Eqs. 51 and 52, and the existence of the above limits, summation over time validates the following formulation for the limiting nodes' beliefs:

$$x_\infty = p^* = x_0 + \sum_{t=0}^{\infty} \delta_t^x \tag{55}$$

$$y_\infty = p^* = y_0 + \sum_{t=0}^{\infty} \delta_t^y \tag{56}$$

Furthermore, by performing pairwise addition between the pairs of Eqs. 55, 56 and 49, 50, we obtain that

$$2p^* = x_0 + y_0 + \sum_{t=0}^{\infty} (\delta_t^x + \delta_t^y) \tag{57}$$

and

$$\delta_t^x + \delta_t^y = 0, \quad \forall t \geq 0, \tag{58}$$

respectively. Finally, the combination of Eqs. 57 and 58 yields:

$$p^* = \frac{x_0 + y_0}{2} \tag{59}$$

indicating that the society's limiting consensus belief is equal to the arithmetic mean of the corresponding initial beliefs given that $(x_0, y_0) \in P_{\text{cons}}$.

The three-agent case

In this section, we focus our attention on the limiting behavior of a society with three agents, whose beliefs are updated according to the proposed model defined in Eq. 14. Once again, investigation concerning the convergence and consensus of the network instance is conducted through the stability analysis of the emergent discrete-time non-linear dynamical system when $n = 3$.

Convergence and consensus analysis

The general form of the updating equations for our model, when $n = 3$, takes the following form:

$$F_x(x, y, z) = \mu_x(x) \cdot x + \mu_x(y) \cdot y + \mu_x(z) \cdot z \tag{60}$$

$$F_y(x, y, z) = \mu_y(x) \cdot x + \mu_y(y) \cdot y + \mu_y(z) \cdot z \tag{61}$$

$$F_z(x, y, z) = \mu_z(x) \cdot x + \mu_z(y) \cdot y + \mu_z(z) \cdot z \tag{62}$$

where $x, y, z \in [0, 1]$ are the current node beliefs and $F_x(x, y, z)$, $F_y(x, y, z)$, and $F_z(x, y, z)$, respectively, are the functions assigning the next-period node beliefs. The weighting terms for each node are given by the following set of equations:

$$\mu_x(s) = \frac{g(x,s)}{g(x,x)+g(x,y)+g(x,z)} \tag{63}$$

$$\mu_y(s) = \frac{g(y,s)}{g(y,x)+g(y,y)+g(y,z)} \tag{64}$$

$$\mu_z(s) = \frac{g(z,s)}{g(z,x)+g(z,y)+g(z,z)} \tag{65}$$

with $s \in \{x, y, z\}$ satisfying that:

$$\mu_x(x)+\mu_x(y)+\mu_x(z) = 1 \tag{66}$$

$$\mu_y(x)+\mu_y(y)+\mu_y(z) = 1 \tag{67}$$

$$\mu_z(x)+\mu_z(y)+\mu_z(z) = 1. \tag{68}$$

Moreover, Eqs. 60, 61, and 62 can be rewritten in the following form by taking into consideration Eqs. 66, 67, and 68:

$$F_x(x,y,z) = [1-\mu_x(y)-\mu_x(z)]\cdot x + \mu_x(y)\cdot y + \mu_x(z)\cdot z \tag{69}$$

$$F_y(x,y,z) = \mu_y(x)\cdot x + [1-\mu_y(x)-\mu_y(z)]\cdot y + \mu_y(z)\cdot z \tag{70}$$

$$F_z(x,y,z) = \mu_z(x)\cdot x + \mu_z(y)\cdot y + [1-\mu_z(x)-\mu_z(y)]\cdot z \tag{71}$$

leading to the following set of equations:

$$F_x(x,y,z) = x + (y-x)\cdot\mu_x(y) + (z-x)\cdot\mu_x(z) \tag{72}$$

$$F_y(x,y,z) = y + (x-y)\cdot\mu_y(x) + (z-y)\cdot\mu_y(z) \tag{73}$$

$$F_z(x,y,z) = z + (x-z)\cdot\mu_z(x) + (y-z)\cdot\mu_z(y) \tag{74}$$

By incorporating the time parameter within Eqs. 72, 73 and 74, they may be rewritten in the following form:

$$F_x(x_t,y_t,z_t) = x_t + (y_t-x_t)\cdot\mu_x(y_t) + (z_t-x_t)\cdot\mu_x(z_t) \tag{75}$$

$$F_y(x_t,y_t,z_t) = y_t + (x_t-y_t)\cdot\mu_y(x_t) + (z_t-y_t)\cdot\mu_y(z_t) \tag{76}$$

$$F_z(x_t,y_t,z_t) = z_t + (x_t-z_t)\cdot\mu_z(x_t) + (y_t-z_t)\cdot\mu_z(y_t) \tag{77}$$

such that

$$x_{t+1} = F_x(x_t,y_t,z_t) \tag{78}$$

$$y_{t+1} = F_y(x_t,y_t,z_t) \tag{79}$$

$$z_{t+1} = F_z(x_t,y_t,z_t) \tag{80}$$

where $t \in \mathbb{N}_+^*$. Letting

$$\delta_{xy}(t) = x_t - y_t \tag{81}$$

$$\delta_{xz}(t) = x_t - z_t \tag{82}$$

$$\delta_{yz}(t) = y_t - z_t \tag{83}$$

and by taking into consideration Eqs. 78, 79 and 80, Eqs. 75, 76 and 77 can be reformulated as:

$$x_{t+1} = x_t - \delta_{xy}(t)\cdot\mu_x(y_t) - \delta_{xz}(t)\cdot\mu_x(z_t) \tag{84}$$

$$y_{t+1} = y_t + \delta_{xy}(t)\cdot\mu_y(x_t) - \delta_{yz}(t)\cdot\mu_y(z_t) \tag{85}$$

$$z_{t+1} = z_t + \delta_{xz}(t)\cdot\mu_z(x_t) + \delta_{yz}(t)\cdot\mu_z(y_t). \tag{86}$$

Having in mind Eqs. 63, 64 and 65, it is easy to deduce that the time evolution of the emergent discrete-time non-linear dynamical system will be given by the following system of non-linear difference equations:

$$x_{t+1} = x_t - \delta_{xy}(t)\cdot\frac{1-\delta_{xy}^2(t)}{3-\delta_{xy}^2(t)-\delta_{xz}^2(t)} - \delta_{xz}(t)\cdot\frac{1-\delta_{xz}^2(t)}{3-\delta_{xy}^2(t)-\delta_{xz}^2(t)} \tag{87}$$

$$y_{t+1} = y_t + \delta_{xy}(t)\cdot\frac{1-\delta_{xy}^2(t)}{3-\delta_{xy}^2(t)-\delta_{yz}^2(t)} - \delta_{yz}(t)\cdot\frac{1-\delta_{yz}^2(t)}{3-\delta_{xy}^2(t)-\delta_{yz}^2(t)} \tag{88}$$

$$z_{t+1} = z_t + \delta_{xz}(t)\cdot\frac{1-\delta_{xz}^2(t)}{3-\delta_{xz}^2(t)-\delta_{yz}^2(t)} + \delta_{yz}(t)\cdot\frac{1-\delta_{yz}^2(t)}{3-\delta_{xz}^2(t)-\delta_{yz}^2(t)}. \tag{89}$$

The set of Eqs. 87, 88, and 89 defines the recursive relations that control the sequences of agents' beliefs as functions of their corresponding pairwise differences over time. Once again, characterizing the above system of non-linear difference equations according to the stability of its steady states can be performed as shown in [23]. Particularly, the process of stability analysis could be facilitated by considering that $\delta_{xy}(t) = u_t$, $\delta_{yz}(t) = v_t$ and $\delta_{xz}(t) = s_t$. Eqs. 87, 88, and 89 can now be rewritten in the following form:

$$x_{t+1} = x_t + f_x(u_t, s_t) \tag{90}$$

$$y_{t+1} = y_t + f_y(u_t, v_t) \tag{91}$$

$$z_{t+1} = z_t + f_z(v_t, s_t) \tag{92}$$

such that

$$f_x(u_t, s_t) = \frac{-u_t\cdot(1-u_t^2) - s_t\cdot(1-s_t^2)}{3-u_t^2-s_t^2} \tag{93}$$

$$f_y(u_t, v_t) = \frac{u_t \cdot (1-u_t^2) - v_t \cdot (1-v_t^2)}{3-u_t^2-v_t^2} \quad (94)$$

$$f_z(v_t, s_t) = \frac{v_t \cdot (1-v_t^2) + s_t \cdot (1-s_t^2)}{3-v_t^2-s_t^2}. \quad (95)$$

Consequently, the steady states of this dynamical system can be obtained by setting $x_{t+1} = x_t = \overline{x}$, $y_{t+1} = y_t = \overline{y}$, and $z_{t+1} = z_t = \overline{z}$, such that $\overline{u} = \overline{x} - \overline{y}$, $\overline{v} = \overline{y} - \overline{z}$, and $\overline{s} = \overline{x} - \overline{z}$. In this context, the aforementioned equilibrium points correspond to the solutions of the following system of non-linear equations:

$$f_x(\overline{u}, \overline{s}) = 0 \quad (96)$$
$$f_y(\overline{u}, \overline{v}) = 0 \quad (97)$$
$$f_z(\overline{v}, \overline{s}) = 0 \quad (98)$$

which is equivalent to

$$\overline{u} \cdot (1-\overline{u}^2) + \overline{s} \cdot (1-\overline{s}^2) = 0 \quad (99)$$
$$\overline{u} \cdot (1-\overline{u}^2) - \overline{v} \cdot (1-\overline{v}^2) = 0 \quad (100)$$
$$\overline{s} \cdot (1-\overline{s}^2) + \overline{v} \cdot (1-\overline{v}^2) = 0 \quad (101)$$

arising from Eqs. 90, 91, and 92. Taking into consideration that $\delta_{xy}(t) + \delta_{yz}(t) = \delta_{xz}(t),\ \forall t \in \mathbb{N}_+^*$, which may be translated to $u_t + v_t = s_t,\ \forall t \in \mathbb{N}_+^*$, it is easy to deduce that the same relation holds for the equilibrium points $\overline{u}$, $\overline{v}$ and $\overline{s}$, such that

$$\overline{u} + \overline{v} = \overline{s}. \quad (102)$$

Hence, the system of non-linear equations describing the steady states of the dynamical system under investigation may be reduced to

$$\overline{u} \cdot (1-\overline{u}^2) + (\overline{u}+\overline{v}) \cdot (1-(\overline{u}+\overline{v})^2) = 0 \quad (103)$$
$$\overline{v} \cdot (1-\overline{v}^2) + (\overline{u}+\overline{v}) \cdot (1-(\overline{u}+\overline{v})^2) = 0 \quad (104)$$

subject to the following constraint:

$$-1 \leq \overline{u} + \overline{v} \leq +1. \quad (105)$$

Performing pairwise addition between Eqs. 103 and 104 yields that

$$(\overline{u}+\overline{v}) - (\overline{u}^3+\overline{v}^3) = -2 \cdot (\overline{u}+\overline{v}) \cdot (1-(\overline{u}+\overline{v})^2) \quad (106)$$

which, in turn, may be rewritten as:

$$(\overline{u}+\overline{v}) \cdot [1-(\overline{u}+\overline{v})^2 + 3 \cdot \overline{u} \cdot \overline{v}] = -2 \cdot (\overline{u}+\overline{v}) \cdot [1-(\overline{u}+\overline{v})^2]. \quad (107)$$

The solutions of Eq. 107 are those given by setting $\overline{u} + \overline{v} = 0$ or $1 - (\overline{u}+\overline{v})^2 + \overline{u} \cdot \overline{v} = 0$, when $\overline{u} + \overline{v} \neq 0$, that simultaneously satisfy the constraint defined in inequality Eq. 105. Therefore, the final set of steady states will be given by $\Delta_s = \{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7\}$, where $\delta_1 = (0, 0, 0)$, $\delta_2 = (0, +1, +1)$, $\delta_3 = (+1, -1, 0)$, $\delta_4 = (+1, 0, +1)$, $\delta_5 = (-1, 0, -1)$, $\delta_6 = (-1, +1, 0)$, and $\delta_7 = (0, -1, -1)$, with each $\delta_i,\ i \in [7]$ being a triplet of the form $(\overline{u}_i, \overline{v}_i, \overline{s}_i)$.

The stability status for each one of the obtained steady states in $\Delta_s = \{(\overline{u}_i, \overline{v}_i, \overline{s}_i),\ i \in [7]\}$ can be assessed by considering the equivalent two-dimensional dynamical system of the pairwise differences between the agents' beliefs:

$$u_{t+1} = u_t + f_{xy}(u_t, v_t) \quad (108)$$
$$v_{t+1} = v_t + f_{yz}(u_t, v_t) \quad (109)$$

which emerges by performing pairwise subtraction between Eqs. 90, 91 and Eqs. 91, 92, yielding:

$$f_{xy}(u_t, v_t) = f_x(u_t, u_t + v_t) - f_y(u_t, v_t) \quad (110)$$
$$f_{yz}(u_t, v_t) = f_y(u_t, v_t) - f_z(v_t, u_t + v_t) \quad (111)$$

It is important to note that the reduction of the original three-dimensional dynamical system defined by the set of Eqs. 78, 79, and 80, to the two-dimensional one given by the pair of Eqs. 108 and 109, is based on the fact that the recurrent relation which provides the value of s_{t+1} as a function of s_t is no longer required, since $u_t + v_t = s_t,\ \forall t \in \mathbb{N}_+^*$. Under these conditions, we may finally write that

$$u_{t+1} = F_u(u_t, v_t) \quad (112)$$
$$v_{t+1} = F_v(u_t, v_t) \quad (113)$$

where

$$F_u(u_t, v_t) = u_t + f_{xy}(u_t, v_t) \quad (114)$$
$$F_v(u_t, v_t) = v_t + f_{yz}(u_t, v_t). \quad (115)$$

In this setting, steady states $(\overline{u}_i, \overline{v}_i)$ are those corresponding to eigenvalues λ_i of the matrix

$$A_i = \begin{bmatrix} \frac{\partial F_u}{\partial u}|_{(\overline{u}_i, \overline{v}_i)} & \frac{\partial F_u}{\partial v}|_{(\overline{u}_i, \overline{v}_i)} \\ \frac{\partial F_v}{\partial u}|_{(\overline{u}_i, \overline{v}_i)} & \frac{\partial F_v}{\partial v}|_{(\overline{u}_i, \overline{v}_i)} \end{bmatrix} \quad (116)$$

with magnitude smaller than 1. Table 1 presents the obtained eigenvalues for each pair of steady states $(\overline{u}_i, \overline{v}_i),\ i \in [7]$, implying that the dynamical system under investigation possesses a unique stable fixed point for $(\overline{u}, \overline{v}) = (0, 0)$, which is also an attractor in the phase-space defined by the pairwise differences of nodes' beliefs. More importantly, this fact indicates that unless the network of agents is initiated with one of the states within the set $\hat{\Delta}_s = \Delta_s \backslash \{\delta_1\}$, the society will

Table 1 Stability analysis for $n = 3$

Steady states		Eigen values	
$\overline{u}$	$\overline{v}$	λ_1	λ_2
0	0	0	0
0	1	1	6
1	−1	6	1
1	0	1	6
−1	0	1	6
−1	1	6	1
0	−1	1	6

reach a consensus, since $\overline{u} = 0$ and $\overline{v} = 0$ yield that $\overline{x} = \overline{y}$ and $\overline{y} = \overline{z}$, such that $\overline{x} = \overline{y} = \overline{z} = p^*$.

Hence, by performing an analysis similar to the one discussed in Sect. 3.1, it is easy to deduce that

$$\lim_{t\to\infty}(u_t, v_t) = (u_\infty, v_\infty) = \begin{cases} (0,+1), & (u_0, v_0) = (0,+1); \\ (-1,+1), & (u_0, v_0) = (-1,+1); \\ (-1,0), & (u_0, v_0) = (-1,0); \\ (0,0), & (u_0, v_0) \in [-1,+1]^2 : -1 < u_0 + v_0 < +1); \\ (0,-1), & (u_0, v_0) = (0,-1); \\ (+1,-1), & (u_0, v_0) = (+1,-1); \\ (+1,0), & (u_0, v_0) = (+1,0). \end{cases} \tag{117}$$

which is equivalent to:

$$\lim_{t\to\infty}(x_t, y_t, z_t) = (x_\infty, y_\infty, z_\infty) = \begin{cases} (+1,+1,0), & (x_0, y_0, z_0) = (+1,+1,0); \\ (0,+1,0), & (x_0, y_0, z_0) = (0,+1,0); \\ (0,+1,+1), & (x_0, y_0, z_0) = (0,+1,+1); \\ (p^*, p^*, p^*), & (x_0, y_0, z_0) \in \hat{P}_{cons}; \\ (0,0,+1), & (x_0, y_0, z_0) = (0,0,+1); \\ (+1,0,+1), & (x_0, y_0, z_0) = (+1,0,+1); \\ (+1,0,0), & (x_0, y_0, z_0) = (+1,0,0). \end{cases} \tag{118}$$

where $p^* \in [0, 1]$ and

$$\hat{P}_{cons} = \{(x_0, y_0, z_0) \in [0,1]^3 : (x_0, y_0, z_0) \notin \{(+1,+1,0), (0,+1,0), (0,+1,+1), (0,0,+1), (+1,0,+1), (+1,0,0)\}\}. \tag{119}$$

The previous analysis shows convergence in the network for the various initial configurations of its nodes' beliefs, given that the adopted updating rule is the one described in Eq. 14. In addition, we have shown that under certain initial conditions, nodes' limiting beliefs lead to a consensus. This, in turn, also implies that the influence weights are uniformly distributed among all agents in the limit according to Eq. 45.

Limiting consensus beliefs

Having specified the conditions under which the learning procedure leads the network to a consensus for $n = 3$, we will proceed by sketching the functional relation that associates the starting nodes' beliefs $(x_0, y_0, z_0) \in \hat{P}_{cons}$ with the corresponding consensual belief p^*.

This task may be undertaken by considering the following recombination of Eqs. 87, 88, and 89:

$$Q_x(t) \cdot \delta_x(t) + Q_y(t) \cdot \delta_y(t) + Q_z(t) \cdot \delta_z(t) = 0, \ \forall t \in \mathbb{N}_+^* \tag{120}$$

where:

$$Q_x(t) = 3 - \delta_{xy}^2(t) - \delta_{xz}^2(t) \tag{121}$$

$$Q_y(t) = 3 - \delta_{xy}^2(t) - \delta_{yz}^2(t) \tag{122}$$

$$Q_z(t) = 3 - \delta_{xz}^2(t) - \delta_{yz}^2(t) \tag{123}$$

and

$$\delta_x(t) = x_{t+1} - x_t \tag{124}$$

$$\delta_y(t) = y_{t+1} - y_t \tag{125}$$

$$\delta_z(t) = z_{t+1} - z_t. \tag{126}$$

In particular, expressing p^* as a function of x_0, y_0, z_0 can be conducted by applying the reasoning described in Sect. 3.2, given that Eq. 120 is initially expanded in the following form:

$$\begin{aligned} &\delta_{xy}^2(t) \cdot [\delta_x(t) + \delta_y(t)] + \delta_{yz}^2(t) \cdot [\delta_y(t) + \delta_z(t)] \\ &\quad + \delta_{xz}^2(t) \cdot [\delta_x(t) + \delta_z(t)] \\ &\quad = 3 \cdot [\delta_x(t) + \delta_y(t) + \delta_z(t)], \quad \forall t \in \mathbb{N}_+^* \end{aligned} \tag{127}$$

and by subsequently taking the infinite sum over both hand sides of Eq. 127.

Approximate solutions for $x(t)$, $y(t)$, $z(t)$

The utilization of Eq. 127 in the context designated in the end of the previous subsection requires the adaptation of a particular functional form for the expressions of $x(t)$, $y(t)$, and $z(t)$, that describe the time evolution of the agents' beliefs. To this end, we conducted a series of computational experiments with different initial conditions $(x_0, y_0, z_0) \in \hat{P}_{\text{cons}}$, where we tracked the time evolution of beliefs for each agent within the network for a range of 100 time steps, that is $t \in \{0, \ldots, 99\}$. Considering a wider time axis was not required, since, in practice, the nodes' beliefs remained constant after at most the 7th time step. This, however, does not indicate convergence in a finite number of time steps, but is

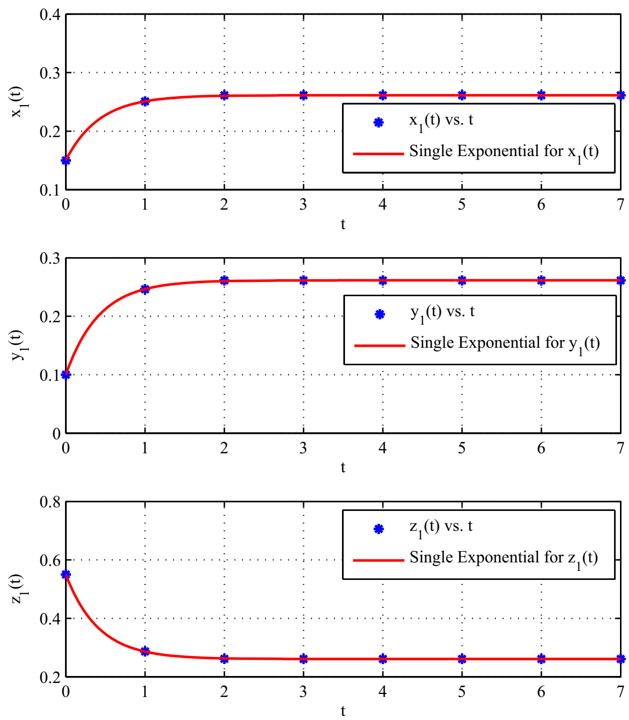

Fig. 2 Single exponential fits for $x_1(t)$, $y_1(t)$, and $z_1(t)$

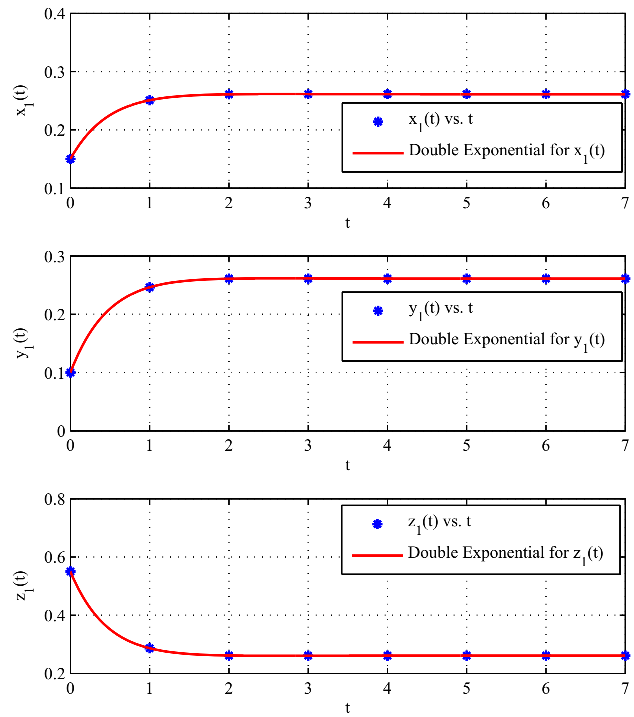

Fig. 3 Double exponential fits for $x_1(t)$, $y_1(t)$, and $z_1(t)$

rather a consequence of the limited representational capacity of a computer. Moreover, the initial points (x_0, y_0, z_0) were sampled from the unit cube $[0, 1]^3$ of possible initial beliefs, so that they were exclusively lying within the set $\hat{P}_{\text{cons}}$. This task was accomplished by sampling the $[0.1, 0.9]$ interval for each initial belief with a step size of 0.05. Therefore, we considered a total number of 4913 different scenarios of the initial conditions, forming, each time, the actual vectors $\mathbf{x} = [x_0, \ldots, x_{99}]$, $\mathbf{y} = [y_0, \ldots, y_{99}]$ and $\mathbf{z} = [z_0, \ldots, z_{99}]$ that store the trajectory of beliefs for each agent pertaining to the social network. By plotting the obtained trajectories as depicted in Figs. 2 and 3, we hypothesized that the required expressions should be strictly increasing or decreasing exponential functions of time with a negative exponential factor, so that convergence is guaranteed as $t \to \infty$. Therefore, the time-dependent functions of $x(t)$, $y(t)$, and $z(t)$ could be given by a single exponential function of the form:

$$A(t) = \alpha_1 \cdot e^{\lambda_1^{\alpha} \cdot t} + \alpha_0 \tag{128}$$

or by a more complex, double exponential function of the form:

$$B(t) = \beta_1 \cdot e^{\lambda_1^{\beta} \cdot t} + \beta_2 \cdot e^{\lambda_2^{\beta} \cdot t} + \beta_0 \tag{129}$$

where $\lambda_1, \lambda_2 < 0$.

To validate our hypotheses, the parameters involved in Eqs. 128 and 129 were estimated by performing non-linear curve fitting on each of the trajectory vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, where the time parameter was utilized as the free variable. The goodness of fit for both models and for each different scenario of the initial conditions was measured in terms of the root-mean-square error (RMSE) and R^2 (RSQUARE). A typical curve fitting example for the triplet of trajectories $\mathbf{x}_1$, $\mathbf{y}_1$ and $\mathbf{z}_1$ appears in Figs. 2 and 3, with $(x_0, y_0, z_0) = (0.15, 0.10, 0.55)$, where the associated goodness of fit and model parameters statistics are presented in Tables 2 and 3 for the single exponential model and in Tables 4 and 5 for the double exponential model. It is easy to deduce that despite the fact that the double exponential fitting curve achieves a lower RMSE and R^2 equal to 1, the corresponding 95% confidence intervals' ranges (shown within parentheses) for its internal parameters are significantly wider than their single exponential model counterparts. Therefore, the single exponential model provides a more accurate model of the data. The same conclusion may be drawn when considering the overall assessment of the utilized models, as measured by the mean and variance of RMSE and R^2 for the curve fitting tasks over all triplets within the sampled space of the initial

Table 2 Single exponential fit for $x_1(t)$, $y_1(t)$, and $z_1(t)$

Statistical measures	$x_1(t)$	$y_1(t)$	$z_1(t)$
RMSE	0.0003916	0.0005626	0.0009010
Rsquare	0.999	0.9999	0.999

Table 3 Single exponential fit parameters for $x_1(t)$, $y_1(t)$, and $z_1(t)$

Node beliefs functions	$x_1(t)$	$y_1(t)$	$z_1(t)$
α_0	0.2614 (0.2609, 0.2618)	0.2614 (0.2608, 0.2620)	0.2609 (0.2599, 0.2618)
α_1	−0.1114 (−0.1125, −0.1103)	−0.1614 (−0.1630, −0.1599)	0.2892 (0.2867, −0.2917)
λ_1^α	−2.374 (−2.268, −2.479)	−2.378 (−2.273, −2.483)	−2.433 (−2.334, −2.533)

Table 4 Double exponential fit for $x_1(t)$, $y_1(t)$, and $z_1(t)$

Statistical measures	$x_1(t)$	$y_1(t)$	$z_1(t)$
RMSE	0.0001734	0.0002472	0.0003845
Rsquare	1.0000	1.0000	1.0000

Table 6 Single exponential fit overall statistics

Node beliefs functions	$x(t)$	$y(t)$	$z(t)$
RMSE mean	0.000368386	0.000368386	0.000368386
RMSE variance	0.000000495	0.000000495	0.000000495
Rsquare mean	0.9993	0.9992	0.9993
Rsquare variance	0.000005193	0.000005590	0.000003516

Table 7 Single exponential fit 95% confidence interval range statistics

Fit parameters	Range mean	Range variance
α_0	0.037131686	0.000906785
α_1	0.001504342	0.000007971
λ_1^α	0.000150795	0.000000080

Table 8 Double exponential fit overall statistics

Node beliefs functions	$x(t)$	$y(t)$	$z(t)$
RMSE mean	0.000239287	0.000240794	0.000240050
RMSE variance	0.000000294	0.000000298	0.000000294
Rsquare mean	0.9996	0.9996	0.9996
Rsquare variance	0.00000145	0.00000156	0.00000148

conditions. Specifically, Tables 6 and 8 summarize the goodness of fit measures for the two models, where once again the double exponential models exhibits lower RMSE and higher R^2. However, taking into consideration the information contained in Tables 7 and 9, it is easy to see that the best candidate model for describing the time evolution of nodes' beliefs is the single exponential one given by Eq. 128.

It is important to note that the functional forms given by Eqs. 128 and 129 also cover the special case arising when initial beliefs $(x_0, y_0, z_0) \in \hat{P}_{\text{cons}}$ happen to be successive terms of an arithmetic progression. In that case, the opinion of the node whose initial belief lies in the middle of the sequence will remain constant over time. This exceptional case can be incorporated in Eqs. 128 and 129 by letting $(\alpha_1 = 0)$ and $(\beta_1 = 0, \beta_2 = 0)$ for this specific node, as shown in Figs. 4 and 5, for the triplet of trajectories $\mathbf{x}_2$, $\mathbf{y}_2$ and $\mathbf{z}_2$ with $(x_0, y_0, z_0) = (0.4, 0.6, 0.8)$.

Refining approximate solutions for $x(t)$, $y(t)$, $z(t)$

In the light of previously discussed computational experiments and taking into consideration the fact that we seek the functional forms of $x(t), y(t), z(t)$ when the corresponding initial conditions $x(0), y(0), z(0) = (x_0, y_0, z_0)$ lie within the set $\hat{P}_{\text{cons}}$, we can refine our hypotheses as follows:

$$x(t) = a_x \cdot e^{\lambda_x \cdot t} + p^* \tag{130}$$

$$y(t) = a_y \cdot e^{\lambda_y \cdot t} + p^* \tag{131}$$

$$z(t) = a_z \cdot e^{\lambda_z \cdot t} + p^* \tag{132}$$

where $\lambda_x, \lambda_y, \lambda_z < 0$. The functional forms given by Eqs. 130, 131, and 132 guarantee that

$$\lim_{t\to\infty} x(t) = p^* \tag{133}$$

$$\lim_{t\to\infty} y(t) = p^* \tag{134}$$

Table 5 Double exponential fit parameters for $x_1(t)$, $y_1(t)$, and $z_1(t)$

Node beliefs functions	$x_1(t)$	$y_1(t)$	$z_1(t)$
β_0	0.2611 (0.2608, 0.2615)	0.2611 (0.2606, 0.2616)	0.2614 (0.2606, 0.2622)
β_1	−0.4078 (−55.23, 54.41)	−1.075 (−600, 597.9)	−0.3297 (−18.28, 17.62)
β_2	0.2967 (−54.52, 55.11)	0.9137 (−598.1, 599.9)	0.6183 (−17.34, 18.57)
λ_1^β	−1.783 (−13.15, 9.585)	−1.751 (−24.87, 21.37)	−1.599 (−8.81, 5.611)
λ_2^β	−1.627 (−14.94, 11.68)	−1.671 (−26.73, 23.38)	−1.908 (−7.186, 3.37)

Table 9 Double exponential fit 95% confidence interval range statistics

Fit parameters	Range mean	Range variance
β_0	0.013×10^3	0.00007×10^9
β_1	4.567×10^3	9.71979×10^9
β_2	0.065×10^3	0.00079×10^9
λ_1^β	0.032×10^3	0.00183×10^9
λ_2^β	4.585×10^3	9.72008×10^9

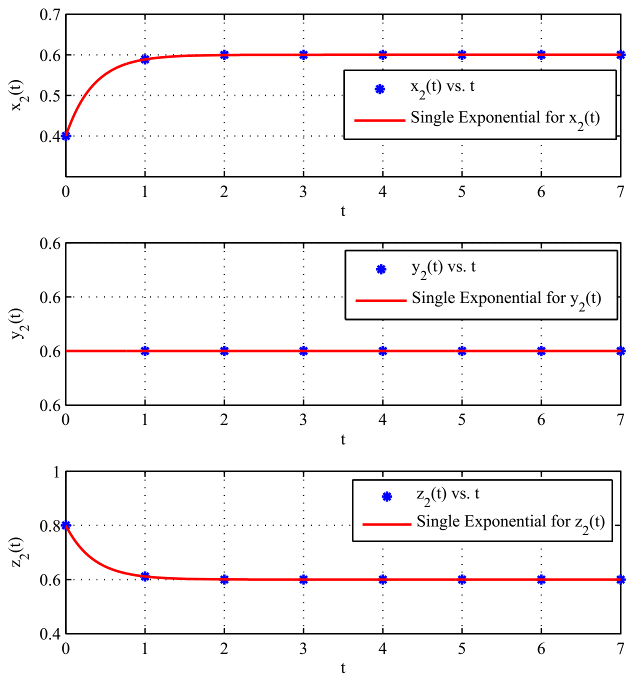

Fig. 4 Single exponential fits for $x_2(t)$, $y_2(t)$, and $z_2(t)$

$$\lim_{t\to\infty} z(t) = p^*. \tag{135}$$

Moreover, given that $x(0), y(0), z(0) = (x_0, y_0, z_0)$, we can write that

$$\alpha_x = x_0 - p^* \tag{136}$$
$$\alpha_y = y_0 - p^* \tag{137}$$
$$\alpha_z = z_0 - p^*. \tag{138}$$

By utilizing the updated functional forms defined in Eqs. 130, 131 and 132, we can also derive that:

$$\delta_x(t) = \alpha_x \cdot {r_x}^t \cdot (r_x - 1) \tag{139}$$
$$\delta_y(t) = \alpha_y \cdot {r_y}^t \cdot (r_y - 1) \tag{140}$$
$$\delta_z(t) = \alpha_z \cdot {r_z}^t \cdot (r_z - 1) \tag{141}$$

and

$${\delta_{xy}}^2(t) = (\alpha_x \cdot {r_x}^t - \alpha_y \cdot {r_y}^t)^2 \tag{142}$$

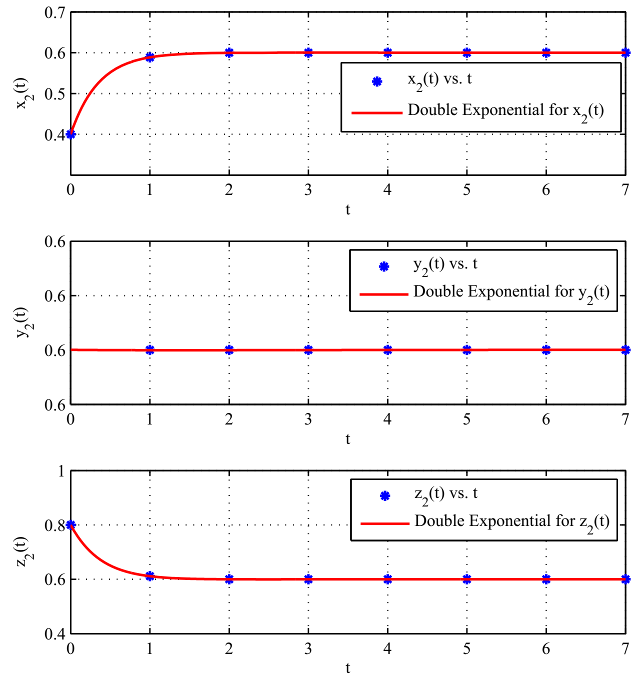

Fig. 5 Double exponential fits for $x_2(t)$, $y_2(t)$, and $z_2(t)$

$${\delta_{yz}}^2(t) = (\alpha_y \cdot {r_y}^t - \alpha_z \cdot {r_z}^t)^2 \tag{143}$$
$${\delta_{xz}}^2(t) = (\alpha_x \cdot {r_x}^t - \alpha_z \cdot {r_z}^t)^2 \tag{144}$$

where $r_x = e^{\lambda_x}$, $r_y = e^{\lambda_y}$ and $r_z = e^{\lambda_z}$, such that $0 < r_x, r_y, r_z < 1$. Taking the infinite sum over both sides of Eqs. 139, 140 and 141 yields that

$$\sum_{t=0}^{\infty} \delta_x(t) = \sum_{t=0}^{\infty} \alpha_x \cdot {r_x}^t \cdot (r_x - 1) = -\alpha_x = p^* - x_0 \tag{145}$$

$$\sum_{t=0}^{\infty} \delta_y(t) = \sum_{t=0}^{\infty} \alpha_y \cdot {r_y}^t \cdot (r_y - 1) = -\alpha_y = p^* - y_0 \tag{146}$$

$$\sum_{t=0}^{\infty} \delta_z(t) = \sum_{t=0}^{\infty} \alpha_z \cdot {r_z}^t \cdot (r_z - 1) = -\alpha_z = p^* - z_0 \tag{147}$$

which is really the case, since

$$\sum_{t=0}^{\infty} \delta_x(t) = \Delta X = x_\infty - x_0 \tag{148}$$
$$\sum_{t=0}^{\infty} \delta_y(t) = \Delta Y = y_\infty - y_0 \tag{149}$$
$$\sum_{t=0}^{\infty} \delta_y(t) = \Delta Z = z_\infty - y_0. \tag{150}$$

Approximate expressions for p^*

Having in mind the derivations presenting in Sect. 4.2.1 and by taking the infinite sum over both hand sides of Eq. 127, one may arrive to the following equation:

$$\begin{aligned}&\frac{2}{3}R_x\cdot\alpha_x^3+\frac{2}{3}R_y\cdot\alpha_z^3+\frac{2}{3}R_z\cdot\alpha_z^3\\&+\frac{1}{3}R_{xy}\cdot\alpha_x^2\cdot\alpha_y+\frac{1}{3}R_{yx}\cdot\alpha_y^2\cdot\alpha_x\\&+\frac{1}{3}R_{yx}\cdot\alpha_y^2\cdot\alpha_z+\frac{1}{3}R_{zy}\cdot\alpha_z^2\cdot\alpha_y\\&+\frac{1}{3}R_{xz}\cdot\alpha_x^2\cdot\alpha_z+\frac{1}{3}R_{zx}\cdot\alpha_z^2\cdot\alpha_x\\&+\alpha_x+\alpha_y+\alpha_z=0\end{aligned}\tag{151}$$

where

$$R_p=\frac{r_p-1}{1-r_p^3},\quad p\in\{x,y,z\}\tag{152}$$

and

$$R_{pq}=\frac{r_q-2r_p+1}{1-r_p^2r_q},\quad p,q\in\{x,y,z\}:p\neq q.\tag{153}$$

Equation 151 implies that the quantities p^*, x_0, y_0, and z_0 are connected by a complex third degree polynomial $F(p^*, x_0, y_0, z_0)$ in the following way:

$$F(p^*,x_0,y_0,z_0)=0\tag{154}$$

such that

$$\begin{aligned}&F(p^*,x_0,y_0,z_0)\\&\quad=\sum_{k+l+m+n=3}f_{k,l,m,n}(p^*)^kx_0^ly_0^mz_0^n+(x_0-p^*)\\&\qquad+(y_0-p^*)+(z_0-p^*)\end{aligned}\tag{155}$$

where $f_{k,l,m,n}$ are coefficients of the polynomial that can be determined by expanding the original functional form. Moreover, by letting $p_{\mathrm{mean}}=\frac{x_0+y_0+z_0}{3}$, we may also obtain the following formulation:

$$p^*-p_{\mathrm{mean}}=\frac{1}{3}\sum_{k+l+m+n=3}f_{k,l,m,n}(p^*)^kx_0^ly_0^mz_0^n\tag{156}$$

which describes the deviation of the final consensual belief from the corresponding mean.

The quantity p^*-p_{mean} may also be expressed as a function of the initial nodes' beliefs differences, $u_0=x_0-y_0$ and $v_0=y_0-z_0$, given by the following third degree polynomial:

$$p^*-p_{\mathrm{mean}}=\sum_{0\leq k+l\leq 3}P_{kl}u_0^kv_0^l\tag{157}$$

where $P_{kl},\ 0\leq k+l\leq 3$ are the coefficients of the polynomial. This time, however, the values of P_{kl} were determined by running a curve fitting routine on the trajectory data generated within the experimentation framework described in Sect. 4.2.1. The associated curve fitting task result is depicted in Fig. 6, while the corresponding goodness of fit measures and coefficient values are presented in Tables 10 and 11. Once again, it is observed that the most significant coefficients P_{kl} are those for which $k+l=3$.

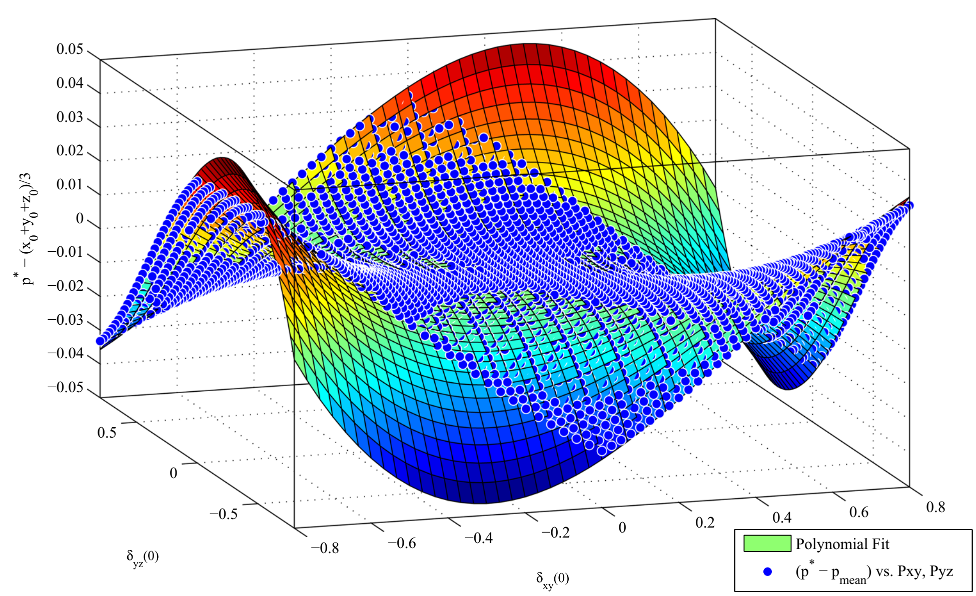

Fig. 6 Third degree two-dimensional polynomial fit

Table 10 Third degree polynomial fit statistics

RMSE	2.4001×10^{-5}
Rsquare	0.9991

Table 11 Third degree polynomial fit parameters

Coefficients	Value (95% confidence interval)
P_{00}	-1.408×10^{-5} $(-2.135 \times 10^{-5}, -6.804 \times 10^{-6})$
P_{10}	-3.175×10^{-5} $(-6.475 \times 10^{-5}, 1.252 \times 10^{-6})$
P_{01}	3.175×10^{-5} $(-1.252 \times 10^{-5}, 6.475 \times 10^{-6})$
P_{20}	-4.451×10^{-5} $(-8.273 \times 10^{-5}, -6.303 \times 10^{-6})$
P_{11}	-0.0001117 $(-0.0001686, -5.468 \times 10^{-5})$
P_{02}	-4.451×10^{-5} $(-8.273 \times 10^{-5}, -6.303 \times 10^{-6})$
P_{30}	-0.6981 $(-0.06991, -0.06970)$
P_{21}	-0.1048 $(-0.1050, -0.1046)$
P_{12}	0.1048 $(0.1046, 0.1050)$
P_{03}	0.6981 $(0.06970, 0.06991)$

Conclusions and directions for further research

This paper suggests a discrete-time, non-linear, and time-variant model of opinion formation in a social network with global interactions. The analysis remains within the general context of bounded rationality but extends the previous modeling approaches (including the classical DeGroot model and the widely applied bounded confidence model) by introducing an intuitive updating rule which assumes that the weight placed by an agent on another one's opinion in each period continuously decreases with the distance between their beliefs in the previous period. In this framework, we first study the fundamental questions of convergence and consensus and then we investigate the relationship between the society's consensus belief and the set of agents' initial opinions. We have proved convergence and consensus for the two-agent case and we have analytically shown that the consensus belief is simply the arithmetic mean of agents' initial opinions. A similar approach has been followed to show convergence and consensus in a network with three agents. In this case, a series of computational experiments has been conducted to find an approximate exponential solution for the evolution of nodes' beliefs over time. Finally, this approximate exponential function has been appropriately used to derive the relationship between the limiting consensus belief and the vector of the initial opinions. We close our paper by raising a number of suggestions for future research. First, the analytical approach for the three-agent network might benefit from turning the opinion formation model into a continuous-time one. This transformation would imply that the characterization of the consensus belief requires solving a non-linear system of differential equations and the solution process might be facilitated by the use of the Adomian decomposition method [21]. Second, our analysis and computational results should be extended to networks with more than three agents. We have already concluded (by studying the two-agent and three-agent cases) that the relationship between the consensus belief and initial conditions depends crucially on the size of the network. However, the characterization of this relationship for an arbitrary number of agents still remains an open question. Third, our model might be extended to account for local interactions by assuming a finite confidence level for each agent in the network. In this setting, Hegselmann and Krause [24] have shown that opinion fragmentation is obtained in the limit for a fairly general set of updating rules, whereas an overall consensus is eventually reached if there is a large enough confidence level shared by all agents—i.e., if the confidence level exceeds a critical level or consensus brink. Here, one might study how the consensus brink and the limiting belief depend on the initial conditions. Since consensus might only be reached asymptotically but not in finite time, another open question concerns the speed at which beliefs converge to their equilibrium value. This is an important issue from a practical point of view, since a slow speed of convergence would imply the persistence of disagreement for a substantial amount of time. Therefore, the relationship between the rate of convergence and the topological properties of the network (such as the degree of homophily or the overall link density) should also be investigated. This kind of analysis has been conducted by Golub and Jackson [25] in the context of the classical DeGroot model but should be extended to more general updating rules. Finally, a valuable contribution would be to study the properties (e.g., the number and the relative size) of final clusters for the case where consensus is not achieved and opinion fragmentation is sustained in the limit. These questions are left for future research.

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