Bull. Math. Sci. (2013) 3:349–361
DOI 10.1007/s13373-013-0033-3

Bernstein type inequalities on star-like domains in $\mathbb{R}^d$ with application to norming sets

András Kroó

Received: 27 February 2013 / Accepted: 5 March 2013 / Published online: 2 April 2013

Abstract Let P_n^d be the space of real algebraic polynomials of d variables and degree at most n, $K \subset \mathbb{R}^d$ a compact set, $||p||_K := \sup_{\mathbf{x}\in K} |p(\mathbf{x})|$ the usual supremum norm on K. Let $\varphi_K(\mathbf{x}) := \inf\{\alpha > 0 : \mathbf{x}/\alpha \in K\}$ denote the Minkowski functional of K. In this note we shall prove that if K is a star-like domain with Lipα boundary, that is $\varphi_K(\mathbf{x})$ satisfies the Lipα condition, $0 < \alpha \le 1$ then the following Bernstein type inequality holds: for any $p \in P_n^d$, $\|p\|_K = 1$ and $\mathbf{x} \in \mathrm{Int}K$

$$|\nabla p|(\mathbf{x}) \le \frac{cn}{(1-\varphi_K(\mathbf{x}))^{\frac{1}{\alpha}-\frac{1}{2}}},$$

where $|\nabla p|$ stands for the Euclidean length of the gradient of p. Furthermore, if $1 < \alpha \le 2$ and K is a C^α star like-domain, that is $\nabla\varphi_K(\mathbf{x})$ has the Lip$(\alpha - 1)$ property, then the same inequality holds for the *tangential* derivatives of p. These new Bernstein type inequalities are applied for the study of cardinality of norming sets, or admissible meshes. The sequence of discrete sets $\mathbf{Y} = \{Y_n \subset K, n \in \mathbb{N}\}$ is called an *optimal admissible mesh* in K if there exist constants c_1, c_2 depending only on K such that

Communicated by S.K. Jain.

This paper was written during A. Kroó visit as **Chair of Excellence at the Carlos III University of Madrid**.

A. Kroó (✉)
Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary
e-mail: kroo.andras@renyi.mta.hu

A. Kroó
Department of Analysis, Budapest University of Technology and Economics, Budapest, Hungary

$$||p||_K \leq c_1||p||_{Y_n}, \quad p \in P_n^d, n \in \mathbb{N},$$

and $card(Y_n) \leq c_2 n^d, n \in \mathbb{N}$. It was proved earlier that optimal admissible meshes exist in C^2 star-like domains. In this paper it will be shown that $C^{2-\frac{2}{d}}$ smoothness also suffices for their existence.

Keywords Multivariate polynomials · Bernstein type inequalities · Norming sets · Optimal meshes · Star-like sets

Mathematics Subject Classification (2000) 41A17

1 Introduction

Consider the space P_n^d of real algebraic polynomials of d variables and degree at most n. Let $K \subset \mathbb{R}^d, d \geq 2$, be any compact set and $\|p\|_K := \sup_{\mathbf{x}\in K} |p(\mathbf{x})|$ the usual supremum norm on K. Denote by $D_{\mathbf{u}}p$ the directional derivative of p in the direction $\mathbf{u} \in S^{d-1} := \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| = 1\}$, and $|\nabla p| := \max_{\mathbf{u}\in S^{d-1}} |D_{\mathbf{u}}p|$ the Euclidean length of its gradient. The classical Bernstein problem consists in estimating $|\nabla p|(\mathbf{x})$ for a given $p \in P_n^d$, $\|p\|_K = 1$ and $\mathbf{x} \in \mathrm{Int}K$. Typically, this estimate is given in terms of the degree n of the polynomials and the distance of point $\mathbf{x} \in \mathrm{Int}K$ to the boundary ∂K of the compact K. This problem goes back to Bernstein [3] who showed that when $d = 1$ and $K = [a, b]$ we have the sharp estimate

$$|p'(x)| \leq \frac{n}{\sqrt{(x-a)(b-x)}}\|p\|_{[a,b]}, \quad x \in (a, b). \tag{1}$$

Let

$$\varphi_K(\mathbf{x}) := \inf\{\alpha > 0 : \mathbf{x}/\alpha \in K\} \tag{2}$$

denote the usual Minkowski functional of K. In case when $K \subset \mathbb{R}^d$ is a $\mathbf{0}$-symmetric convex body Sarantopoulos [13] established a complete analogue of (1)

$$|D_{\mathbf{u}}p|(\mathbf{x}) \leq \frac{n\varphi_K(\mathbf{u})}{\sqrt{1-\varphi_K(\mathbf{x})^2}}\|p\|_K, \quad \mathbf{x} \in \mathrm{Int}K,\ \mathbf{u} \in S^{d-1}. \tag{3}$$

The above inequality was also independently verified by Baran [2]. Thus for Bernstein type inequalities on $\mathbf{0}$-symmetric convex bodies $\sqrt{1-\varphi_K(\mathbf{x})^2} \sim \sqrt{1-\varphi_K(\mathbf{x})}$ is the proper measure of distance from the given point to the boundary of the set. The problem of finding the correct measure of distance for Bernstein type inequalities on general compact sets in $\mathbb{R}^d$ was studied by Baran [1] and Totik [14] using potential theoretic methods. In [1,14] this distance is given in terms of the so called equilibrium measure of the set, however this equilibrium measure can be rarely found explicitly.

In this note we shall provide an *explicit asymptotically sharp* measure of distance in Bernstein type inequalities for *star-like domains*. This will be accomplished for both general and *tangential* Bernstein type inequalities. We shall also apply these new inequalities in order to generalize some estimates for cardinality of the *norming sets*.

It should be noted that the star like property of the domains is used in this note for the sake of convenience, the methods applied can be extended to more general domains. A detailed examination of the proofs of the Bernstein type inequalities given by Theorems 1 and 2 reveals that they are based on embedding certain l_p balls into the domain, and technically this can be accomplished easier for a star-like domain. In particular, all results of the paper also hold for finite unions of star-like domains.

2 New results

Let $K \subset \mathbb{R}^d$ be a compact *star-like set* with respect to the origin, that is $\mathbf{0} \in \operatorname{Int} K$ and for every $\mathbf{x} \in K$ we have that $[\mathbf{0}, \mathbf{x}) \subset \operatorname{Int} K$. Note that the study of Bernstein type inequalities is shift independent hence assuming that the center of the star is in the origin does not restrict the generality of our considerations. Then the Minkowski functional $\varphi_K(\mathbf{x})$ of K given by (2) is a *homogeneous functional* satisfying the properties $\varphi_K(\mathbf{x}) < 1, \mathbf{x} \in \operatorname{Int} K, \varphi_K(\mathbf{x}) = 1, \mathbf{x} \in \partial K$ and $\varphi_K(\mathbf{x}) > 1$ when $\mathbf{x}$ is not in K. Thus the quantity $1 - \varphi_K(\mathbf{x})$ measures the distance from the point to the boundary of the domain. Our goal is to show that in Bernstein type inequalities on star-like domains the proper measure of distance to the boundary is of the form $(1 - \varphi_K(\mathbf{x}))^\beta$ with some $\beta > 0$ which depends on the geometry of the boundary. In order to identify this parameter β let us introduce the Lip property of the boundary.

Definition 1 Let us say that the star-like set $K \subset \mathbb{R}^d$ has Lipα, $0 < \alpha \le 1$ boundary if for some $M > 0$ depending on K we have

$$|\varphi_K(\mathbf{x}) - \varphi_K(\mathbf{x} + \mathbf{h})| \le M|\mathbf{h}|^\alpha, \quad |\mathbf{h}| \le 1, \ \mathbf{x} \in S^{d-1}. \tag{4}$$

With the above definition we have the next

Theorem 1 *If $K \subset \mathbb{R}^d$ is a star-like domain with Lipα, $0 < \alpha \le 1$ boundary then for any $p \in P_n^d$, $\|p\|_K = 1$ and $\mathbf{x} \in \operatorname{Int} K$*

$$|\nabla p|(\mathbf{x}) \le \frac{c_K n}{(1 - \varphi_K(\mathbf{x}))^{\frac{1}{\alpha} - \frac{1}{2}}}, \tag{5}$$

where $c_K > 0$ depends only on K.

It is interesting to note in the above estimate that the measure of distance to the boundary of the set given by the quantity $(1 - \varphi_K(\mathbf{x}))^{\frac{1}{\alpha} - \frac{1}{2}}$ depends on the geometry of the set. It will be pointed out below that in general this is the sharp quantity measuring the distance to the boundary of Lipα, $0 < \alpha \le 1$ domain. In the Lip 1 case this quantity reduces to $\sqrt{1 - \varphi_K(\mathbf{x})}$ which is essentially what is implied by (3) in case when K is convex.

Clearly, (5) yields an asymptotically optimal estimate in case when $\alpha = 1$ (or in particular K is convex). Nevertheless, a further improvement in the Bernstein inequality can be achieved when only *tangential derivatives* of polynomials are considered.

This approach was investigated in [9] were tangential Markov type inequalities were studied for smooth star-like domains.

Let us assume that the Minkowski functional is continuously differentiable on $\mathbb{R}^d \setminus \{\mathbf{0}\}$ and denote by $\nabla\varphi_K$ its gradient. For any $\mathbf{x} \in \partial K$ we have that $\nabla\varphi_K(\mathbf{x})$ gives the normal direction to the boundary at $\mathbf{x} \in \partial K$. In addition, let

$$T_K(\mathbf{x}) := \{\mathbf{u} \in S^{d-1} : \mathbf{u} \perp \nabla\varphi_K(\mathbf{x})\}$$

be the set of tangent unit vectors at $\mathbf{x} \in \partial K$. When the Minkowski functional is continuously differentiable on $\mathbb{R}^d \setminus \{\mathbf{0}\}$ we shall always assume in addition that the star-like domain K does not contain *critical points* on its boundary in the sense that tangent directions at any $\mathbf{x} \in \partial K$ are not collinear with $\mathbf{x}$.

Definition 2 Given $1 < \alpha \leq 2$, we say that the star-like domain $K \subset \mathbb{R}^d$ is C^α if for some $M > 0$ depending on K we have

$$|\nabla\varphi_K(\mathbf{x}) - \nabla\varphi_K(\mathbf{x} + \mathbf{h})| \leq M|\mathbf{h}|^{\alpha-1}, \quad \mathbf{x} \in S^{d-1}, \ |\mathbf{h}| \leq 1. \tag{6}$$

Theorem 2 *Let $K \subset \mathbb{R}^d$ be a C^α star-like domain with some $1 < \alpha \leq 2$. Then for any $p \in P_n^d$, $\|p\|_K = 1$, $\mathbf{x} \in \mathrm{Int}K \setminus \{\mathbf{0}\}$ and $\mathbf{u} \in T_K(\frac{\mathbf{x}}{\varphi_K(\mathbf{x})})$*

$$|D_{\mathbf{u}}p|(\mathbf{x}) \leq \frac{c_K n}{(1 - \varphi_K(\mathbf{x}))^{\frac{1}{\alpha} - \frac{1}{2}}}, \tag{7}$$

where $c_K > 0$ depends only on K.

Clearly, since $1 < \alpha \leq 2$ in (7) it provides a sharper Bernstein type inequality than (5), but only for tangential derivatives of the polynomial. It should be also noted that when $\alpha = 2$ (7) becomes a uniform Markov type estimate of order n.

Bernstein type inequalities are very useful, they are widely applied in various areas of analysis. One particular application that we intend to exhibit in this note is the study of the so called *norming sets* or *optimal meshes*, see e.g. [8] and [5], were these corresponding notions are introduced.

Definition 3 ([5]) A family of sets $\mathbf{Y} = \{Y_n \subset K, n \in \mathbb{N}\}$ is called an **admissible mesh** in K if there exist constants c_1, c_2 depending only on K such that

$$||p||_K \leq c_1||p||_{Y_n}, \quad p \in P_n^d, \ n \in \mathbb{N},$$

where the cardinality of Y_n grows at most polynomially, i.e. $card(Y_n) \leq c_2 n^m$, $n \in \mathbb{N}$, with some fixed $m \in \mathbb{N}$ depending only on K.

This definition of admissible meshes is similar to the notion of **norming sets** [8]. Admissible meshes are applied in various areas, for instance they are used for discrete least squares approximation, extracting discrete extremal sets of Fekete and Leja type, scattered data interpolation, etc. The study of admissible meshes has received lately a considerable attention, see for instance [4,6,12] were various applications and algorithms for the construction of admissible meshes is discussed.

Since $\dim P_n^d \sim n^d$ we clearly must have $m \geq d$ in the above definition (assuming, of course that no polynomial vanishes on K). Naturally, in optimal case we aim for a mesh with asymptotically minimal number of points in it, that is we would like to have $m = d$ in Definition 3. This leads to the following

Definition 4 We shall say that an admissible mesh (norming set) $\mathbf{Y} = \{Y_n \subset K, n \in \mathbb{N}\}$ in $K \subset \mathbb{R}^d$ is **optimal** if $card(Y_n) \leq cn^d, n \in \mathbb{N}$, with some $c > 0$ depending only on K.

The basic question in this respect consists in describing those sets $K \subset \mathbb{R}^d$ which possess optimal admissible meshes. Finding exact geometric properties characterizing sets with optimal admissible meshes appears to be a rather difficult problem. It was shown recently in [10] that any C^2 star-like domain possesses an optimal mesh. Applying the tangential Bernstein inequality (7) of Theorem 2 we can extend this result to C^α star-like domains with $2 > \alpha > 2 - \frac{2}{d}$. This is a substantial decrease in required smoothness of the star-like domain, especially for low dimensions.

Theorem 3 *Let $1 \leq \alpha \leq 2$. Assume that $K \subset \mathbb{R}^d$ is a star-like domain which is C^α if $1 < \alpha \leq 2$ and has Lip1 boundary if $\alpha = 1$. Then K possesses an admissible mesh $\mathbf{Y} = \{Y_n \subset K, n \in \mathbb{N}\}$ with $card(Y_n) = O(n^d)$ if $\alpha > 2 - \frac{2}{d}$; $card(Y_n) = O(n^d \log n)$ if $\alpha = 2 - \frac{2}{d}$ and $card(Y_n) = O(n^{\frac{2d-2}{\alpha}})$ if $\alpha < 2 - \frac{2}{d}$ $(n \in \mathbb{N})$.*

Hence by the above theorem when $\alpha > 2 - \frac{2}{d}$ the C^α star-like domains possess optimal meshes. This extends an earlier result given in [10], where the existence of optimal meshes was verified for C^2 star-like domains. Furthermore, when $\alpha = 2 - \frac{2}{d}$ the meshes are near optimal with only an extra log factor present. Finally, when $1 \leq \alpha < 2 - \frac{2}{d}$ the cardinality $O(n^{\frac{2d-2}{\alpha}})$ of admissible meshes provided by Theorem 3 is essentially better than the bound of order $O(n^{2d/\alpha})$ which can be deduced by a routine application of Markov type inequality $\|Dp\|_K = O(n^{2/\alpha})$, $\|p\|_K = 1$ (see [5] for details). In addition, when $\alpha = 1$ only estimates $card(Y_n) = O(n^{2d-2}), d > 2$ and $card(Y_n) = O(n^2 \log n), d = 2$ are applicable in the above theorem. In [10] these estimates for admissible meshes were given for the case of convex bodies in $K \subset \mathbb{R}^d$, Theorem 3 presents an extension of these bounds to the Lip1 star-like domains.

It should be also noted that since the method of proof of Theorem 3 is totally constructive it automatically provides an algorithm for explicit construction of optimal meshes in star-like domains.

3 Bernstein type inequalities for star-like domains

In this section we shall verify Theorems 1 and 2 which give Bernstein type bounds for star-like domains.

Proof of Theorem 1 Let $K \subset \mathbb{R}^d$ be a star-like domain with Lipα, $0 < \alpha \leq 1$ boundary, and consider any $p \in P_n^d$, $\|p\|_K = 1$ and $\mathbf{x} \in \mathrm{Int}K$. Since $\mathbf{0} \in \mathrm{Int}K$ a ball centered at the origin of some radius $c > 0$ is contained in K. We may assume without loss of generality that $|\mathbf{x}| > c/2$ since otherwise the point $\mathbf{x}$ is separated away from the boundary and the statement of the theorem easily follows from (3). Consider the univariate polynomial $g(t) := p(t\mathbf{x}), t \in \mathbb{R}$. Clearly, $t\mathbf{x} \in K$ for any $-c/|\mathbf{x}| \le t \le 1/\varphi_K(\mathbf{x})$. Thus $|g(t)| \le 1$ whenever $-c/|\mathbf{x}| \le t \le 1/\varphi_K(\mathbf{x})$. Hence using the univariate Bernstein inequality (1) for $g \in P_n^1$ with $a = -c/|\mathbf{x}|, b = 1/\varphi_K(\mathbf{x})$ easily yields setting $\mathbf{w} := \frac{\mathbf{x}}{|\mathbf{x}|}$

$$|\mathbf{x}||D_{\mathbf{w}}p|(\mathbf{x}) = |\langle \mathbf{x}, \nabla p(\mathbf{x})\rangle| = |g'(1)| \le \frac{n}{\sqrt{(1 + c/|\mathbf{x}|)(1/\varphi_K(\mathbf{x}) - 1)}}.$$

Since K contains a ball centered at the origin of radius $c > 0$ it follows that $c\varphi_K(\mathbf{x}) \le |\mathbf{x}|$ i.e., we obtain from the previous estimate that

$$|D_{\mathbf{w}}p|(\mathbf{x}) \le \frac{n}{\sqrt{(|\mathbf{x}| + c)(1 - \varphi_K(\mathbf{x}))}}\sqrt{\frac{\varphi_K(\mathbf{x})}{|\mathbf{x}|}} \le \frac{n}{c\sqrt{(1 - \varphi_K(\mathbf{x}))}} \tag{8}$$

This provides a bound for the derivative in radial direction which is clearly stronger than (5).

Now we proceed to the more difficult part of estimating the derivatives in any direction $\mathbf{u} \in S^{d-1}$ orthogonal to $\mathbf{x}$. Clearly we can restrict our considerations to the 2-dimensional plane containing $\mathbf{u}$ and $\mathbf{x}$, so without loss of generality we may assume that $d = 2, \mathbf{u} = (1, 0), \varphi_K(\mathbf{x}) = A < 1$ and $\mathbf{x} = (0, A)$ i.e., $(0, 1) = \mathbf{x}/\varphi_K(\mathbf{x}) \in \partial K, A > c/2$. Let $B > 0$ be the largest real number for which $[\mathbf{x}, \mathbf{z}] \subset K, \mathbf{z} := (B, A)$. Hence $\mathbf{z} \in \partial K$, i.e., $\varphi_K(\mathbf{z}) = 1$. Setting now $\mathbf{h} := (-B, 0)$ and using the Lipα property of K we obtain by (4)

$$1 - A = |\varphi_K(\mathbf{x}) - \varphi_K(\mathbf{z})| = |\varphi_K(\mathbf{z} + \mathbf{h}) - \varphi_K(\mathbf{z})| \le MB^\alpha.$$

This last relation immediately implies that

$$D := \{(x, y) \in \mathbb{R}^2 : 0 \le y \le 1 - M|x|^\alpha\} \subset K. \tag{9}$$

Now we shall inscribe into domain D a proper line passing through the point $\mathbf{x} = (0, A)$. Consider the line $L := \{y = A - M^{\frac{1}{\alpha}}(1 - A)^{\frac{\alpha-1}{\alpha}}x\}$. Since $A > c/2$ it follows by a routine calculation that

$$A - M^{\frac{1}{\alpha}}(1 - A)^{\frac{\alpha-1}{\alpha}}x \le 1 - Mx^\alpha$$

whenever $0 \le x \le x_1 := \frac{c}{2}M^{-\frac{1}{\alpha}}(1 - A)^{\frac{1-\alpha}{\alpha}}$, i.e., the line L passes inside $D \subset K$ for $0 \le x \le x_1$. Moreover, it is easy to see that this line intersects the curve $y = 1 - M(-x)^\alpha$ at the point $x_2 := -\eta(1 - A)^{\frac{1}{\alpha}}$ where $\eta > 0$ is the unique solution of the equation

$$1 = M\eta^\alpha + M^{\frac{1}{\alpha}}\eta.$$

Hence the line L is contained in the domain $D \subset K$ for $x \in [x_2, x_1]$. Now for the univariate polynomial $q(x) := p(x, A - M^{\frac{1}{\alpha}}(1-A)^{\frac{\alpha-1}{\alpha}}x)$ we have $|q(x)| \le 1$, $x \in [x_2, x_1]$. Hence applying again the univariate Bernstein inequality (1) with $a = x_2 = -\eta(1-A)^{\frac{1}{\alpha}}$, $b = x_1 = \frac{c}{2}M^{-\frac{1}{\alpha}}(1-A)^{\frac{1-\alpha}{\alpha}}$ and recalling that $A = \varphi_K(\mathbf{x})$ yields

$$|q'(0)| \le \frac{n}{\sqrt{-x_2x_1}} \le \frac{c_K n}{(1-\varphi_K(\mathbf{x}))^{\frac{1}{\alpha}-\frac{1}{2}}}, \tag{10}$$

where c_K is a positive constant depending only on K. Furthermore,

$$|q'(0)| = \frac{\partial p}{\partial x}(\mathbf{x}) - M^{\frac{1}{\alpha}}(1-A)^{\frac{\alpha-1}{\alpha}}\frac{\partial p}{\partial y}(\mathbf{x}), \tag{11}$$

where by (8)

$$\left|\frac{\partial p}{\partial y}(\mathbf{x})\right| \le \frac{n}{c\sqrt{1-\varphi_K(\mathbf{x})}}. \tag{12}$$

Finally, combining relations (10)–(12) we arrive at

$$\begin{aligned}\left|\frac{\partial p}{\partial x}(\mathbf{x})\right| &\le |q'(0)| + \frac{n}{c\sqrt{1-\varphi_K(\mathbf{x})}}M^{\frac{1}{\alpha}}(1-\varphi_K(\mathbf{x}))^{\frac{\alpha-1}{\alpha}}\\ &\le \frac{c_K n}{(1-\varphi_K(\mathbf{x}))^{\frac{1}{\alpha}-\frac{1}{2}}} + \frac{M^{\frac{1}{\alpha}}n}{c(1-\varphi_K(\mathbf{x}))^{\frac{1}{\alpha}-\frac{1}{2}}} = \frac{c_K^* n}{(1-\varphi_K(\mathbf{x}))^{\frac{1}{\alpha}-\frac{1}{2}}}.\end{aligned}$$

This completes the proof of Theorem 1. □

Remark 1 Bernstein type inequality (5) implies a corresponding uniform Markov type estimate for star-like domains with Lipα boundary. Indeed, it is well known that univariate polynomials of degree at most n bounded by 1 on $[-1, 1]$ are also bounded by an absolute constant on the larger interval $[-1-1/n^2, 1+1/n^2]$, this easily follows, for instance, from the classical Remez inequality. Thus in case when $K \subset \mathbb{R}^d$ is a Lipα star-like domain a bound that holds for a $p \in P_n^d$ when $\varphi_K(\mathbf{x}) \le 1 - 1/n^2$ will also hold with another constant for every $\mathbf{x} \in K$. Since (5) implies a bound of magnitude $cn^{\frac{2}{\alpha}}$ if we put $\varphi_K(\mathbf{x}) = 1-1/n^2$ it follows that $\|Dp\|_K = O(n^{\frac{2}{\alpha}})$ whenever $p \in P_n^d$, $\|p\|_K = 1$. This Markov type estimate can be found in various papers (see e.g. [11] for corresponding references). The fact that a Markov type inequality of order $O(n^{\frac{2}{\alpha}})$ is sharp in general for Lipα domains goes back to an earlier paper by Goetgheluck [7]. Therefore Bernstein type inequality (5) must be sharp, as well, in the sense that the quantity $(1-\varphi_K(\mathbf{x}))^{\frac{1}{\alpha}-\frac{1}{2}}$ in (5) can not be replaced by another function tending slower to 0 when $\mathbf{x}$ approaches the boundary of the domain, since this would clearly yield a better Markov type inequality.

Proof of Theorem 2 Let $K \subset \mathbb{R}^d$ be a C^α star-like domain with some $1 < \alpha \le 2$. Consider any $\mathbf{x} \in \mathrm{Int}K \setminus \{\mathbf{0}\}$. Since the star-like domain K does not contain saddle points on its boundary, that is $\mathbf{x}/|\mathbf{x}|$ can not be a tangent direction to the boundary of K at any $\frac{\mathbf{x}}{\varphi_K(\mathbf{x})} \in \partial K$, a standard compactness argument yields that the angles between $\mathbf{x}$ and the tangents to ∂K at $\frac{\mathbf{x}}{\varphi_K(\mathbf{x})}$ are uniformly bounded away from zero. Hence without loss of generality we may assume that for the given $\mathbf{x} \in \mathrm{Int}K\backslash\{\mathbf{0}\}$ any tangent $\mathbf{u}$ to ∂K at $\frac{\mathbf{x}}{\varphi_K(\mathbf{x})}$ is orthogonal to $\mathbf{x}$ (this can be always achieved by a uniformly bounded linear transformation of the space). In addition, in order to estimate $|D_{\mathbf{u}}p|(\mathbf{x})$ it suffices to work in the 2-dimensional plane containing $\mathbf{u}$ and $\mathbf{x}$.

Thus summarizing, we can assume that $d = 2, \mathbf{x} := (1-h, 0), h = 1 - \varphi_K(\mathbf{x})$, $(1, 0) \in \partial K$ and $\mathbf{u} = (0, 1)$ is the tangent to ∂K at $(1, 0)$. Now choose an arbitrary $\mathbf{z} = (A, 0) \in \mathrm{Int}K, 1 > A > 0$. Then $\varphi_K(\mathbf{z}) = A$. Let $B > 0$ be the largest real for which with $\mathbf{y} := (A, B)$ we have $[\mathbf{z}, \mathbf{y}] \subset K$. Thus $\mathbf{y} \in \partial K$. Furthermore

$$1 - A = \varphi_K(\mathbf{y}) - \varphi_K(\mathbf{z}) = \frac{\partial}{\partial y}\varphi_K(A, \xi)B \le c_1 B, \quad \xi \in [0, B].$$

Here and in what follows we denote by c_j positive constants depending only on K. Since $\mathbf{u} = (0, 1)$ is the tangent to ∂K at $(1, 0)$ we also have that $\frac{\partial}{\partial y}\varphi_K(1, 0) = 0$. This and the previous relation together with the Lip$(\alpha - 1)$ property of $\nabla\varphi_K$ [see (6)] yield

$$1 - A = \frac{\partial}{\partial y}\varphi_K(A, \xi)B - \frac{\partial}{\partial y}\varphi_K(1, 0)B \le c_2 B(1 - A + \xi)^{\alpha-1} \le c_3 B^{\alpha}.$$

This means that

$$D := \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1 - c_3|y|^{\alpha}\} \subset K. \tag{13}$$

We shall choose a proper $C > 1$ such that

$$Q := \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1 - h - Cy^2\} \subset D \subset K. \tag{14}$$

In view of (13) this will be accomplished provided that $c_3|y|^{\alpha} - Cy^2 \le h$ whenever $|y| \le (\frac{1-h}{C})^{1/2} < 1$. Clearly, if $\alpha = 2$ we simply can set $C = c_3$. So consider the case when $1 < \alpha < 2$. A routine calculation shows that

$$\max_{y\in[0,1]} (y^{\alpha} - Cy^2) = \beta C^{\frac{\alpha}{\alpha-2}}$$

with some $0 < \beta < 1$ depending only on α. Hence we can set

$$C := c_3^{\frac{2}{\alpha}} \left(\frac{h}{\beta}\right)^{\frac{\alpha-2}{\alpha}} = c_4 h^{\frac{\alpha-2}{\alpha}} \tag{15}$$

and with this choice of C inclusions (14) will hold.

Consider now the univariate trigonometric polynomial of degree $\le n$ defined by

$$t(\phi) := p\left(\frac{1-h}{2}(1 + \cos\phi), \frac{1}{2}\sqrt{\frac{1-h}{C}}\sin\phi\right),$$

where $p \in P_n^2$, $\|p\|_K = 1$. Setting $x = \frac{1-h}{2}(1+\cos\phi)$, $y = \frac{1}{2}\sqrt{\frac{1-h}{C}}\sin\phi$ it follows that

$$\begin{aligned} 1-h-Cy^2-x &= 1-h-\frac{1-h}{4}\sin^2\phi-\frac{1-h}{2}(1+\cos\phi)\\ &= \frac{1-h}{4}(1+\cos^2\phi-2\cos\phi)\geq 0, \quad \phi\in[-\pi,\pi]. \end{aligned}$$

Thus in view of inclusions (14) and $\|p\|_K = 1$ we obtain that $|t(\phi)| \leq 1$ $\phi \in [-\pi, \pi]$.

Hence using the Bernstein inequality for trigonometric polynomials of degree $\leq n$ we arrive at

$$n \geq |t'(0)| = \frac{1}{2}\sqrt{\frac{1-h}{C}}\left|\frac{\partial p}{\partial y}(1-h, 0)\right|,$$

i.e., also using (15)

$$\left|\frac{\partial p}{\partial y}(\mathbf{x})\right| \leq \frac{c_5 n h^{\frac{\alpha-2}{2\alpha}}}{\sqrt{1-h}}.$$

Recall that K contains a ball centered at the origin and radius c, thus the statement of the theorem holds trivially if $|\mathbf{x}| < c/2$ i.e., $1-h < c/2$. On the other hand when $1-h \geq c/2$ we obtain from the last estimate that

$$\left|\frac{\partial p}{\partial y}(\mathbf{x})\right| \leq \frac{c_6 n}{h^{\frac{1}{\alpha}-\frac{1}{2}}}.$$

Since $h = 1-\varphi_K(\mathbf{x})$ this completes the proof of the theorem. □

Remark 2 Again similar to the Remark 1 given after the proof of Theorem 1 the Bernstein type inequality of Theorem 2 yields a uniform Markov type upper bound for tangential derivatives of polynomials on C^α star-like domains when $1 < \alpha \leq 2$. Namely it implies that for $p \in P_n^d$, $\|p\|_K = 1$

$$|D_{\mathbf{u}}p(\mathbf{x})| = O(n^{\frac{2}{\alpha}}), \quad \mathbf{u} \in T_K(\mathbf{x}),$$

uniformly for $\mathbf{x} \in \partial K$. Since the above tangential Markov type inequality is known to be sharp, in general (see [9] for details), it follows that the tangential Bernstein type inequality given by Theorem 2 also provides the sharp measure of distance to the boundary of the domain.

Proof of Theorem 3 First let us point out how the Bernstein type inequalities of Theorems 1 and 2 will be used in the proof. Given two points $\mathbf{x}, \mathbf{y} \in K_\rho := \{\mathbf{z} \in K : \varphi_K(\mathbf{z}) = \rho\}, 0 < \rho < 1$ and any $p \in P_n^d, \|p\|_K = 1$ we shall need a proper estimate for $|p(\mathbf{x}) - p(\mathbf{y})|$ in terms of $|\mathbf{x}-\mathbf{y}|$. Clearly we can pass to the 2-dimensional plane containing the origin and $\mathbf{x}, \mathbf{y}$, i.e., we may assume

$d = 2$ and consider the polar coordinate representation for K_ρ given by the polar curves $K_\rho = (r(t)\cos t, r(t)\sin t), t \in [0, 2\pi]$ where the radial functions $r(t)$ depend on ρ and are Lip1 ($\alpha = 1$) or $C^\alpha(1 < \alpha \leq 2)$, respectively. Then setting $Q(t) := p(r(t)\cos t, r(t)\sin t)$ easily yields that with some positive constant c_K depending only on K

$$|p(\mathbf{x}) - p(\mathbf{y})| = |Q(t_1) - Q(t_2)| = \left| \int_{t_1}^{t_2} Q'(t)\, dt \right| \leq c_K |\mathbf{x} - \mathbf{y}| D_T p$$

where $D_T p$ denotes the largest magnitude of tangential derivative (when $1 < \alpha \leq 2$) or the largest gradient (when $\alpha = 1$) of p on the level curve K_ρ, respectively. But in both of these cases Theorems 1 and 2 give an upper bound of magnitude $\frac{n}{(1-\rho)^{\frac{1}{\alpha}-\frac{1}{2}}}$ for these derivatives. Hence we obtain from the above estimate that

$$|p(\mathbf{x}) - p(\mathbf{y})| \leq c_K \frac{n|\mathbf{x} - \mathbf{y}|}{(1-\rho)^{\frac{1}{\alpha}-\frac{1}{2}}}, \quad \mathbf{x}, \mathbf{y} \in K_\rho. \tag{16}$$

As before, let c be the radius of the ball centered at the origin which is contained in K. Set $a = c^{5/2}/50$ and choose an integer q such that $1 - c/2 \leq a^2 q^2 \leq 1 - c/4$ (such an integer obviously exists). Set $m := qn$ and

$$\rho_j := 1 - \frac{a^2 j^2}{n^2}; \quad K_j := K_{\rho_j} = \{\mathbf{x} \in \mathbb{R}^d : \varphi_K(\mathbf{x}) = \rho_j\},$$
$$1 \leq j \leq m = qn, \ n \in \mathbb{N}. \tag{17}$$

We may assume without loss of generality that K is also contained in the unit ball centered at the origin, i.e.,

$$c\varphi_K(\mathbf{x}) \leq |\mathbf{x}| \leq \varphi_K(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d.$$

Consider first $\mathbf{x} \in K$ such that $\rho_1 \geq \varphi_K(\mathbf{x}) \geq \rho_m = 1 - a^2 q^2$ that is $\rho_{j+1} \leq \varphi_K(\mathbf{x}) \leq \rho_j$ for some $1 \leq j \leq m-1$. Set $t := \frac{\varphi_K(\mathbf{x})}{\rho_{j+1}}$, $\mathbf{y} := \mathbf{x}/t$. Then $\mathbf{y} \in K_{j+1}$, $\mathbf{x} = t\mathbf{y}$, $t \geq 1$. Hence using that $c/2 \geq \rho_m = 1 - a^2 q^2 \geq c/4$ we have

$$|\mathbf{x} - \mathbf{y}| = (t-1)|\mathbf{y}| \leq t - 1 \leq \frac{\rho_j - \rho_{j+1}}{\rho_{j+1}} \leq \frac{12a^2 j}{cn^2};$$
$$\left| \frac{\mathbf{x}}{\varphi_K(\mathbf{x})} - \mathbf{x} \right| = \frac{|\mathbf{x}|(1-\rho_j)}{\varphi_K(\mathbf{x})} \geq c(1-\rho_j).$$

By these estimates we have for any $p \in P_n^d$, $\|p\|_K = 1$ using the univariate Bernstein inequality (1) for $p(r\mathbf{x})$, $0 \leq r \leq 1//\varphi_K(\mathbf{x})$

$$|p(\mathbf{x}) - p(\mathbf{y})| \leq \frac{n|\mathbf{x} - \mathbf{y}|}{\sqrt{|\mathbf{y}||\mathbf{x}/\varphi_K(\mathbf{x}) - \mathbf{x}|}} \leq \frac{12a^2 j}{c^2 n \sqrt{(1-\rho_j)\rho_{j+1}}} \leq \frac{24a}{c^{5/2}} \leq \frac{1}{2}. \tag{18}$$

If $\rho_1 \leq \varphi_K(\mathbf{x})$ we can use the univariate Markov inequality

$$\|q'\|_{[a,b]} \leq \frac{2n^2}{b-a}\|q\|_{[a,b]}, \quad q \in P_n^1$$

yielding for $p(r\mathbf{x}), 0 \leq r \leq 1//\varphi_K(\mathbf{x})$ with some $\mathbf{y} = t\mathbf{x} \in K_1$

$$|p(\mathbf{x}) - p(\mathbf{y})| \leq 2n^2 \frac{\varphi_K(\mathbf{x})}{|\mathbf{x}|}|\mathbf{x} - \mathbf{y}| \leq \frac{2n^2}{c}\frac{a^2}{\rho_1 n^2} \leq \frac{1}{2},$$

which is the same as (18).

Thus for every $\mathbf{x} \in K$ such that $1 - a^2q^2 \leq \varphi_K(\mathbf{x})$ there exists some $\mathbf{y} \in K_j$, $1 \leq j \leq n$ for which (18) holds.

Let us denote again by c_j positive constants depending only on K.

In what follows for the sets A, B in $\mathbb{R}^d$ we shall denote by

$$d(A, B) := \sup_{\mathbf{x}\in A} \inf_{\mathbf{y}\in B} \mid \mathbf{x} - \mathbf{y} \mid$$

the density of the set B in the set A. Evidently, given $h > 0$ we can choose a discrete set $\Omega \subset S^{d-1}$ so that $d(\Omega, S^{d-1}) \leq h$ and card$\Omega \leq c(d)h^{1-d}$ with some constant $c(d)$ depending only on d. Also note that since the boundary of K is Lip1 for any $\mathbf{x}, \mathbf{y} \in \partial K$ with $\mid \frac{\mathbf{x}}{|\mathbf{x}|} - \frac{\mathbf{y}}{|\mathbf{y}|} \mid \leq h$ we have $|\mathbf{x} - \mathbf{y}| \leq c_2 h$.

Based on these remarks and utilizing the constant c_K from (16) we can set

$$h_j := \frac{1}{4c_K n}\left(\frac{a^2 j^2}{n^2}\right)^{\frac{1}{\alpha}-\frac{1}{2}}, \quad 1 \leq j \leq m \tag{19}$$

and choose $N_j \leq c_1 h_j^{1-d}$ points $\{\mathbf{y}_{i,j} \in K_j, 1 \leq i \leq N_j\} := Y_{n,j}$ on K_j so that

$$d(K_j, Y_{n,j}) \leq h_j, \quad 1 \leq j \leq m. \tag{20}$$

Set $K^* := \{\mathbf{x} \in \mathbb{R}^d : \varphi_K(\mathbf{x}) \leq 1 - a^2q^2\}$. Since for $\mathbf{x} \in K^*$ we have that

$$|\mathbf{x}| \leq \varphi_K(\mathbf{x}) \leq 1 - a^2q^2 \leq c/2$$

it follows that K^* is contained in the ball of radius $c/2$ centered at the origin. Recalling that ball of radius c centered at the origin is contained in K we obtain using (3) for this ball

$$|p(\mathbf{x}) - p(\mathbf{y})| \leq \frac{2n}{c}|\mathbf{x} - \mathbf{y}|, \quad \mathbf{x}, \mathbf{y} \in K^*. \tag{21}$$

Furthermore, we can choose a discrete set $Y_n^* \subset K^*$ with card$(Y_n^*) \leq c_2 n^d$ so that

$$d(K^*, Y_n^*) \leq \frac{c}{4n}. \tag{22}$$

Set now $\mathbf{Y}_n := \cup_{1\le j\le m} Y_{n,j} \cup Y_n^*$. Using (19) the cardinality of this set can be estimated as follows

$$\mathrm{card}\mathbf{Y}_n \le c_5 \left(n^d + n^{\frac{2(d-1)}{\alpha}} \sum_{j=1}^{m} j^{(d-1)(1-2/\alpha)} \right).$$

If $\alpha > 2 - \frac{2}{d}$ the the sum above is of magnitude $n^{d-\frac{2(d-1)}{\alpha}}$ yielding that $\mathrm{card}(\mathbf{Y}_n) = O(n^d)$. When $\alpha = 2-\frac{2}{d}$ the above sum is of magnitude $\log n$ and we obtain $\mathrm{card}(\mathbf{Y}_n) = O(n^d \log n)$. Finally, if $\alpha < 2 - \frac{2}{d}$ the sum is convergent and $d < \frac{2(d-1)}{\alpha}$ so we end up with $\mathrm{card}(\mathbf{Y}_n) = O(n^{\frac{2(d-1)}{\alpha}})$. Thus the cardinality of $\mathbf{Y}_n$ is of magnitude stated in Theorem 3.

It remains now to show that $\mathbf{Y}_n$ is an admissible mesh. Choose any $p \in P_n^d$, $\|p\|_K = 1$ and $\mathbf{x} \in K, p(\mathbf{x}) = 1$. Assume first that $\mathbf{x} \in K^*$. Then by (22) we can find $\mathbf{y} \in Y_n^* \subset K^*$, $|\,\mathbf{x}-\mathbf{y}\,| \le \frac{c}{4n}$. Thus using (21) we obtain

$$1 - p(\mathbf{y}) = p(\mathbf{x}) - p(\mathbf{y}) \le \frac{2n}{c} |\mathbf{x}-\mathbf{y}| \le \frac{1}{2},$$

i.e., $||p||_{\mathbf{Y}_n} \ge 1/2$ in this case.

Now let $\varphi_K(\mathbf{x}) \ge 1 - a^2 q^2$. Then as it was shown above there exists some $\mathbf{y} \in K_j, 1 \le j \le n$ for which (18) holds. Moreover, since $p(\mathbf{x}) = 1$ we have $|p(\mathbf{y})| \ge 1/2$. Furthermore, by (19) and (20) we can find $\mathbf{y}_{i,j} \in Y_{n,j} \subset K_j$ such that

$$|\mathbf{y} - \mathbf{y}_{i,j}| \le h_j = \frac{1}{4c_K n} \left(\frac{a^2 j^2}{n^2} \right)^{\frac{1}{\alpha}-\frac{1}{2}}.$$

Thus using again (16) with $\rho = \rho_j$ defined by (17) we obtain

$$|p(\mathbf{y}) - p(\mathbf{y}_{i,j})| \le c_K |\mathbf{y} - \mathbf{y}_{i,j}| \frac{n}{\left(\frac{a^2 j^2}{n^2}\right)^{\frac{1}{\alpha}-\frac{1}{2}}} \le \frac{1}{4}.$$

Since $|p(\mathbf{y})| \ge 1/2$ the last estimate implies that $|p(\mathbf{y}_{i,j})| \ge 1/4$ which means that $||p||_{\mathbf{Y}_n} \ge 1/4$ in this case, as well.

The proof of Theorem 3 is completed. □

References

1. Baran, M.: Complex equilibrium measure and Bernstein type inequality for compact sets in $\mathbb{R}^n$. Proc. Am. Math. Soc. **123**, 485–494 (1995)

2. Baran, M.: Bernstein type theorems for compact sets in $\mathbb{R}^n$ revisited. J. Approx. Theory **79**, 190–198 (1994)
3. Bernstein, S.N.: Sur l'ordre de la meilleure approximation des fonctions continues par les polynômes de degré donné. Mem. Cl. Sci. Acad. R. Belg. **4**, 1–103 (1912)
4. Bos, L., De Marchi, S., Sommariva, A., Vianello, M.: Weakly admissible meshes and discrete extremal sets. Numer. Math. Theory Methods Appl. **4**, 1–12 (2011)
5. Calvi, J.P., Levenberg, N.: Uniform approximation by discrete least squares polynomials. J. Approx. Theory **152**, 82–100 (2008)
6. De Marchi, S., Marchioro, M., Sommariva, A.: Polynomial approximation and cubature at approximate Fekete and Leja points of the cylinder. Appl. Math. Comput. **218**, 10617–10629 (2012)
7. Goetgheluck, P.: Inegalité de Markov dans les ensembles effilés. J. Approx. Theory **30**, 149–154 (1980)
8. Jetter, K., Stöckler, J., Ward, J.D.: Error estimates for scattered data interpolation. Math. Comput. **68**, 733–747 (1999)
9. Kroó, A.: Markov type inequalities for surface gradients of multivariate polynomials. J. Approx. Theory **118**, 235–245 (2002)
10. Kroó, A.: On optimal polynomial meshes. J. Approx. Theory **163**, 1107–1124 (2011)
11. Kroó, A., Szabados, J.: Markov–Bernstein type inequalities for multivariate polynomials on sets with cusps. J. Approx. Theory **99**, 134–152 (1999)
12. Piazzon, F., Vianello, M.: Analytic transformation of admissible meshes. East J. Approx. **16**, 313–322 (2010)
13. Sarantopoulos, Y.: Bounds on the derivatives of polynomials on Banach spaces. Math. Proc. Cambr. Philos. Soc. **110**, 307–312 (1991)
14. Totik, V.: Polynomial inverse images and polynomial inequalities. Acta Math. **187**, 139–160 (2001)