Nonstandard and standard compactifications of ordered topological spaces

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Abstract

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We construct the Nachbin ordered compactification and the ordered realcompactification, a notion defined in the paper, of a given ordered topological space as nonstandard ordered hulls. The maximal ideals in the algebras of the differences of monotone continuous functions are completely described. We give also a characterization of the class of completely regular ordered spaces which are closed subspaces of products of copies of the ordered real line, answering a question of T.H. Choe and Y.H. Hong. The methods used are topological (standard) and nonstandard.

Keywords: Ordered topological space, Nachbin ordered compactification, ordered realcompactification, maximal ideals, real maximal ideals, nonstandard extension, nonstandard ordered hull, saturation principle.

Introduction

The purpose of this paper is the study of two extensions of a completely regular ordered space $(X, T, \leqslant)$. On the one hand, the Nachbin ordered compactification

$\mathcal{N}(X, T, \leqslant)$ and on the other, the ordered realcompactification $\mathcal{R}(X, T, \leqslant)$, which we introduce and which coincides with the Hewitt realcompactification when the order is discrete.

Both of these constructions are obtained as "ordered nonstandard hulls". The Nachbin order compactification arises from the nonstandard extension ${}^*X$ of X and the order realcompactification from the set $\tilde{X}$ of prenearstandard points of ${}^*X$.

In the classical theory of rings of continuous functions, the points of the Stone-Čech compactification and of the Hewitt realcompactification are intimately connected with the maximal ideals of the rings of bounded continuous $C_b(X, \mathbb{R})$ and continuous $C(X, \mathbb{R})$ real-valued functions defined on X. In the ordered case let $C_b^{\uparrow}(X, \mathbb{R})$ and $C^{\uparrow}(X, \mathbb{R})$ denote the sets of functions from $C_b(X, \mathbb{R})$ and $C(X, \mathbb{R})$, respectively, which are monotone nondecreasing, in the sense that $f(x) \leqslant f(y)$ if $x \leqslant y$. Let $A_b(X, \mathbb{R})$ and $A(X, \mathbb{R})$ denote the smallest subrings of $C_b(X, \mathbb{R})$ and $C(X, \mathbb{R})$ which contain $C_b^{\uparrow}(X, \mathbb{R})$ and $C^{\uparrow}(X, \mathbb{R})$, respectively. We prove that the points of $\mathcal{N}(X, T, \leqslant)$ and $\mathcal{R}(X, T, \leqslant)$ completely describe the maximal ideals of $A_b(X, \mathbb{R})$ and $A(X, \mathbb{R})$, respectively. The maximal ideals of $A_b(X, \mathbb{R})$ and $A(X, \mathbb{R})$ are also related to nonstandard points in ${}^*X$ and $\tilde{X}$ and we give the exact relation in Proposition 4.3 and Corollary 4.6. For the nonordered case, which presents features essentially different from the ordered one, we mention the work of Dyre [3].

Our work also enables us to give a characterization of the completely regular ordered spaces which are closed subspaces of products of copies of $(\mathbb{R}, \tau, \leqslant)$, the real line with the usual topology and the usual order. This characterization answers a question raised by Choe and Hong [2].

The methods used in the paper are topological (standard) as well as nonstandard, in the belief that this interaction will prove fruitful.

For topology and the theory of rings of continuous functions, we refer to Gillman and Jerison [5] as well as Weir [12]. For ordered topological spaces and quasi-uniform spaces to Nachbin [9] and to Fletcher and Lindgren [4]. The nonstandard concepts and results can be found in Hurd and Loeb [8]. We emphasize that we use systematically the saturation principle and require a set of individuals S that contains both X and $\mathbb{R}$. The degree of saturation is the larger of $2^{2^{\aleph_0}}$ and $2^{2^{\operatorname{card} X}}$, in particular, any polysaturated nonstandard model of S will do.

1. Nonstandard compactification of ordered topological spaces

Let $(X, T, \leqslant)$ be an ordered topological space, i.e., a topological space with a binary relation $\leqslant$, reflexive, antisymmetric and transitive, whose graph is a closed subset of X (Nachbin [9]). Recall that the closedness of the graph implies that the space $(X, T, \leqslant)$ is Hausdorff. We represent the classes of continuous and bounded continuous real-valued functions defined on X by $C(X, \mathbb{R})$ and $C_b(X, \mathbb{R})$, respectively, and by $C^{\uparrow}(X, \mathbb{R})$ and $C_b^{\uparrow}(X, \mathbb{R})$ we will denote the monotone nondecreasing functions in $C(X, \mathbb{R})$ and $C_b(X, \mathbb{R})$, respectively.

Definition 1.1 (Nonstandard compactification). Let $\Phi^{\uparrow} \subseteq C^{\uparrow}(X, \mathbb{R})$ be a family of monotone nondecreasing continuous real-valued functions defined on X. We define the topological space with an additional binary relation $({}^*X, T, \leqslant)$ by:

(i) ${}^*X$ is the nonstandard extension of X and T is the standard topology in ${}^*X$ (also denoted by T) [11, Section 1] with basic open sets ${}^*G$, where G is an open set in (X, T).

(ii) Let $\alpha, \beta \in {}^*X$. Put $\alpha \sim \beta$ if ${}^*f(\alpha) \approx {}^*f(\beta)$ for all $f \in \Phi^{\uparrow}$ where $\approx$ is the infinitesimal relation in ${}^*\mathbb{R}$.

(iii) Let $\alpha, \beta \in {}^*X$. Put $\alpha \leqslant \beta$ if ${}^*f(\alpha) \,{}^*\!< {}^*f(\beta)$ or ${}^*f(\alpha) \approx {}^*f(\beta)$ for all $f \in \Phi^{\uparrow}$, where ${}^*\!<$ is the nonstandard extension of the usual order $<$ in $\mathbb{R}$ into ${}^*\mathbb{R}$.

The space $({}^*X, T, \leqslant)$ will be called the "$\Phi^{\uparrow}$-nonstandard compactification of $(X, T, \leqslant)$".

The terminology "nonstandard compactification" arises from the fact that $({}^*X, T)$ is a compact topological space containing (X, T) as a dense subspace [11, Proposition (1.5)].

Proposition 1.2. (i) *The relation* $\leqslant$ *is reflexive and transitive. For any* $\alpha, \beta \in {}^*X$, $(\alpha \leqslant \beta$ *and* $\beta \leqslant \alpha)$ *implies* $(\alpha \sim \beta)$.

(ii) "$\sim$" *is an equivalence relation on* ${}^*X$.

(iii) *If* $\alpha, \alpha', \beta, \beta' \in {}^*X$, *then* $(\alpha \leqslant \beta,\ \alpha \sim \alpha'$ *and* $\beta \sim \beta')$ *implies* $(\alpha' \leqslant \beta')$.

Proof. The proof is straightforward and will be omitted. □

Remark. Norice that the relations $\leqslant$ and ($\leqslant$ and $\not\sim$) in ${}^*X$ do not coincide, in general, with the nonstandard extensions ${}^*\!\leqslant$ and ${}^*\!<$ of $\leqslant$ and $<$, respectively. First of all $\leqslant$ is not an order relation in ${}^*X$ and ${}^*\!\leqslant$ is. Concerning ($\leqslant$ and $\not\sim$) and ${}^*\!<$, they both are strict order relations in ${}^*X$ but the following example shows that they do not coincide. Let $(X, T, \leqslant)$ be $(\mathbb{R}, \tau, \leqslant)$, the real line with the usual topology and usual order, and let $\alpha, \beta \in {}^*\mathbb{R}$, be two finite nonstandard numbers such that $\alpha \,{}^*\!< \beta$ and $\alpha \approx \beta$ (e.g. $\alpha = 0$ and β any positive infinitesimal). Then, by continuity of the functions in $\Phi^{\uparrow}$, we have ${}^*f(\alpha) \approx {}^*f(\beta)$ for all f in $\Phi^{\uparrow}$, which means that the numbers α and β are not in the relation given by ($\leqslant$ and $\not\sim$).

Definition 1.3 (Prenearstandard points). Let $\Phi^{\uparrow} \subseteq C^{\uparrow}(X, \mathbb{R})$. We define $\tilde{X} \subseteq {}^*X$ by

$$\tilde{X} = \{\alpha \in {}^*X \mid {}^*f(\alpha) \in {}^*\mathbb{R}_F \text{ for all } f \in \Phi^{\uparrow}\} \tag{1}$$

where ${}^*\mathbb{R}_F$ is the set of finite nonstandard real numbers. The points in $\tilde{X}$ will be called $\Phi^{\uparrow}$-*prenearstandard points.*

We shall require the following lemma.

Lemma 1.4. *Let* α *and* β *be points of* ${}^*X$ *such that* $\alpha \not\sim \beta$ *and* α *is* $C^{\uparrow}(X, \mathbb{R})$-*prenearstandard. Then there exists a continuous monotone nondecreasing function* $g:(X, T, \leqslant) \to (\mathbb{R}, \tau, \leqslant)$, $0 \leqslant g \leqslant 1$, *such that* $({}^*g(\alpha) = 0$ *and* ${}^*g(\beta) = 1)$ *or* $({}^*g(\alpha) = 1$ *and* ${}^*g(\beta) = 0)$.

$\mathcal{N}(X, T, \leqslant)$ and on the other, the ordered realcompactification $\mathcal{R}(X, T, \leqslant)$, which we introduce and which coincides with the Hewitt realcompactification when the order is discrete.

Both of these constructions are obtained as "ordered nonstandard hulls". The Nachbin order compactification arises from the nonstandard extension $^*X$ of X and the order realcompactification from the set $\tilde{X}$ of prenearstandard points of $^*X$.

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Definition 1.1 (Nonstandard compactification). Let $\Phi^{\uparrow} \subseteq C^{\uparrow}(X, \mathbb{R})$ be a family of monotone nondecreasing continuous real-valued functions defined on X. We define the topological space with an additional binary relation $({}^*X, T, \lessdot)$ by:

(i) ${}^*X$ is the nonstandard extension of X and T is the standard topology in ${}^*X$ (also denoted by T) [11, Section 1] with basic open sets ${}^*G$, where G is an open set in (X, T).

(ii) Let $\alpha, \beta \in {}^*X$. Put $\alpha \sim \beta$ if ${}^*f(\alpha) \approx {}^*f(\beta)$ for all $f \in \Phi^{\uparrow}$ where $\approx$ is the infinitesimal relation in ${}^*\mathbb{R}$.

(iii) Let $\alpha, \beta \in {}^*X$. Put $\alpha \lessdot \beta$ if ${}^*f(\alpha) \,{}^*\!< {}^*f(\beta)$ or ${}^*f(\alpha) \approx {}^*f(\beta)$ for all $f \in \Phi^{\uparrow}$, where ${}^*\!<$ is the nonstandard extension of the usual order $<$ in $\mathbb{R}$ into ${}^*\mathbb{R}$.

The space $({}^*X, T, \lessdot)$ will be called the "$\Phi^{\uparrow}$-nonstandard compactification of $(X, T, \leqslant)$".

The terminology "nonstandard compactification" arises from the fact that $({}^*X, T)$ is a compact topological space containing (X, T) as a dense subspace [11, Proposition (1.5)].

Proposition 1.2. (i) *The relation* $\lessdot$ *is reflexive and transitive. For any* $\alpha, \beta \in {}^*X$, *(*$\alpha \lessdot \beta$ *and* $\beta \lessdot \alpha$*) implies (*$\alpha \sim \beta$*).*

(ii) "$\sim$" *is an equivalence relation on* ${}^*X$.

(iii) *If* $\alpha, \alpha', \beta, \beta' \in {}^*X$, *then (*$\alpha \lessdot \beta$, $\alpha \sim \alpha'$ *and* $\beta \sim \beta'$*) implies (*$\alpha' \lessdot \beta'$*).*

Proof. The proof is straightforward and will be omitted. □

Remark. Norice that the relations $\lessdot$ and ($\lessdot$ and $\nsim$) in ${}^*X$ do not coincide, in general, with the nonstandard extensions ${}^*\!\leqslant$ and ${}^*\!<$ of $\leqslant$ and $<$, respectively. First of all $\lessdot$ is not an order relation in ${}^*X$ and ${}^*\!\leqslant$ is. Concerning ($\lessdot$ and $\nsim$) and ${}^*\!<$, they both are strict order relations in ${}^*X$ but the following example shows that they do not coincide. Let $(X, T, \leqslant)$ be $(\mathbb{R}, \tau, \leqslant)$, the real line with the usual topology and usual order, and let $\alpha, \beta \in {}^*\mathbb{R}$, be two finite nonstandard numbers such that $\alpha \,{}^*\!< \beta$ and $\alpha \approx \beta$ (e.g. $\alpha = 0$ and β any positive infinitesimal). Then, by continuity of the functions in $\Phi^{\uparrow}$, we have ${}^*f(\alpha) \approx {}^*f(\beta)$ for all f in $\Phi^{\uparrow}$, which means that the numbers α and β are not in the relation given by ($\lessdot$ and $\nsim$).

Definition 1.3 (Prenearstandard points). Let $\Phi^{\uparrow} \subseteq C^{\uparrow}(X, \mathbb{R})$. We define $\tilde{X} \subseteq {}^*X$ by

$$\tilde{X} = \{\alpha \in {}^*X \mid {}^*f(\alpha) \in {}^*\mathbb{R}_F \text{ for all } f \in \Phi^{\uparrow}\} \tag{1}$$

where ${}^*\mathbb{R}_F$ is the set of finite nonstandard real numbers. The points in $\tilde{X}$ will be called $\Phi^{\uparrow}$*-prenearstandard points.*

We shall require the following lemma.

Lemma 1.4. *Let* α *and* β *be points of* ${}^*X$ *such that* $\alpha \not\lessdot \beta$ *and* α *is* $C^{\uparrow}(X, \mathbb{R})$*-prenearstandard. Then there exists a continuous monotone nondecreasing function* $g:(X, T, \leqslant) \to (\mathbb{R}, \tau, \leqslant)$, $0 \leqslant g \leqslant 1$, *such that (*${}^*g(\alpha) = 0$ *and* ${}^*g(\beta) = 1$*) or (*${}^*g(\alpha) = 1$ *and* ${}^*g(\beta) = 0$*).*

Proof. Let ${}^*f(\alpha) \not\approx {}^*f(\beta)$ for some $f \in C^{\uparrow}(X, \mathbb{R})$ and assume ${}^*f(\alpha) \;{}^*\!> {}^*f(\beta)$. Since ${}^*f(\alpha)$ is finite, there are r, s in $\mathbb{R}$ such that ${}^*f(\alpha) \;{}^*\!> r > s \;{}^*\!> {}^*f(\beta)$. Let $\varphi : (\mathbb{R}, \tau, \leqslant) \to (\mathbb{R}, \tau, \leqslant)$ be such that $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ on $[r, \to)$ and $\varphi = 0$ on $(\leftarrow, s]$. Then for $g = \varphi \circ f$, we have ${}^*g(\alpha) = {}^*\varphi({}^*f(\alpha)) = 1$ (since ${}^*\varphi^{\leftarrow}[\{1\}] = {}^*(\varphi^{\leftarrow}[\{1\}]) \supseteq {}^*[r, \to) \ni {}^*f(\alpha)$). Also, ${}^*g(\beta) = 0$. If ${}^*f(\alpha) \;{}^*\!< {}^*f(\beta)$, then g can be found such that ${}^*g(\alpha) = 0$, ${}^*g(\beta) = 1$. The proof is complete. $\square$

We recall the following definition of Nachbin [9, p. 52]:

Definition 1.5. Let $(X, T, \leqslant)$ be an ordered topological space and let $\Phi^{\uparrow} \subseteq C^{\uparrow}(C, \mathbb{R})$. We say that $\Phi^{\uparrow}$ *distinguishes* the points and closed sets of $(X, T, \leqslant)$ if for any $x \in X$ and any closed set $F \subseteq X$ not containing x there exist two functions $f, g \in \Phi^{\uparrow}$ such that $0 \leqslant f \leqslant 1$, $0 \leqslant g \leqslant 1$, f is nondecreasing, g is nonincreasing, $f(x) = 1$, $g(x) = 1$ and

$$(f \wedge g)(y) = 0 \quad \text{for all } y \in F.$$

An ordered topological space $(X, T, \leqslant)$ is a *completely regular ordered space* if it admits a family $\Phi^{\uparrow} \subseteq C^{\uparrow}(X, \mathbb{R})$ which distinguishes points and closed sets of X.

The following result establishes a connection between the monads $\mu(x)$, $x \in X$, of the space $(X, T, \leqslant)$ (Hurd and Loeb [8, p. 111]) and the equivalence classes $q(x)$, $x \in X$, under the equivalence relation $\sim$ defined above.

Lemma 1.6. *$\mu(x) \subseteq q(x)$ for any $x \in X$. When the family $\Phi^{\uparrow}$ distinguishes points and closed sets in X, then $\mu(x) = q(x)$ for all $x \in X$.*

Proof. $\mu(x) \subseteq q(x)$ follows immediately from the continuity of the functions f in $\Phi^{\uparrow}$. Let $\Phi^{\uparrow}$ distinguish points and closed sets. Suppose that $\alpha \in q(x) - \mu(x)$ and let $\alpha \notin {}^*G$ for some open neighbourhood G of x. Let f and g be two functions which distinguish x and $X - G$ in the sense of Definition 1.5. We have ${}^*(f \wedge g)(\alpha) = 0$, so that ${}^*f(\alpha) = 0$ or ${}^*g(\alpha) = 0$. This contradicts $\alpha \in q(x)$. The proof is complete. $\square$

2. Nonstandard ordered hulls and the Nachbin ordered compactification

In this section we show that by identifying points of the nonstandard order compactification $({}^*X, T, \leqslant)$ we obtain the Nachbin ordered compactification introduced by Nachbin [9] and characterized by the following universal property: If $(X, T, \leqslant)$ is a completely regular ordered space which is a dense order subspace of a compact ordered space $(\bar{X}, \bar{T}, \bar{\leqslant})$, then every bounded $f : (X, T, \leqslant) \to (\mathbb{R}, \tau, \leqslant)$ admits a unique extension $\bar{f} : (\bar{X}, \bar{T}, \bar{\leqslant}) \to (\mathbb{R}, \tau, \leqslant)$. The Nachbin order compactification of $(X, T, \leqslant)$, denoted by $\mathcal{N}(X, T, \leqslant)$, has the above stated extension property [9]. By contrast, it should be noted that in our context the extension of a given function f is simply the restriction of the nonstandard extension ${}^*f$ to the equivalence classes arising from the identification.

Definition 2.1 (Nonstandard ordered hulls). Let $(X, T, \leqslant)$ be an ordered topological space, $\Phi^{\uparrow} \subseteq C^{\uparrow}(X, \mathbb{R})$ and $({}^*X, T, \leqslant)$ be its nonstandard $\Phi^{\uparrow}$-compactification (Definition 1.1).

(i) Let

$$\hat{X} = \tilde{X}/\sim \tag{2}$$

be the corresponding nonstandard $\Phi^{\uparrow}$-hull $\hat{T}$ the quotient topology and $\hat{\leqslant}$ the (order) relation in $\hat{X}$ defined by: if $a, b \in \hat{X}$, then $a \mathrel{\hat{\leqslant}} b$ if $\alpha \leqslant \beta$ for some $\alpha \in a$ and $\beta \in b$. The space

$$(\hat{X}, \hat{T}, \hat{\leqslant}) \tag{3}$$

will be called *the nonstandard $\Phi^{\uparrow}$-hull of $(X, T, \leqslant)$*.

(ii) For any $f \in \Phi^{\uparrow}$, we define $\hat{f}: \hat{X} \to \mathbb{R}$ by $\hat{f} \circ q = \mathrm{st} \circ {}^*f$ where q is the quotient mapping from $\tilde{X}$ onto $\hat{X}$ and "st" is the standard part mapping of ${}^*\mathbb{R}$.

Proposition 2.2 (Properties of $\hat{\leqslant}$). (i) *$\hat{\leqslant}$ is an order relation in $\hat{X}$ which is an extension of $\leqslant$ in X in the sense that if $x, y \in X$, then $(q(x) \mathrel{\hat{\leqslant}} q(y) \Leftrightarrow x \leqslant y)$.*

(ii) *The following formula for the graphs is valid*:

$$[\hat{\leqslant}] = (q \times q)[\leqslant]. \tag{4}$$

(iii) *Every $f \in \Phi^{\uparrow}$, $f: X \to \mathbb{R}$, has a unique continuous monotone nondecreasing extension $\hat{f}: \hat{X} \to \mathbb{R}$ (Definition 2.1(ii)) and*

$$f(x) = \hat{f}(q(x)), \quad x \in X. \tag{5}$$

(iii) (iv) *The graph $[\hat{\leqslant}]$* of *$\hat{\leqslant}$ is a closed subset of $\hat{X}$.*

Proof. (i) Follows immediately from Proposition 1.2. (ii) Formula (4) is a simple interpretation of the definition of $\hat{\leqslant}$. (iii) The continuity of $\hat{f}$ and formula (5) are proved in [11, Proposition (2.6)] and the property of $\hat{f}$ to be monotone and nondecreasing follows from the fact that ${}^*f$ (by the transfer principle (Hurd and Loeb [8])) and standard part mapping are so. (iv) Suppose that $(a, b) \notin [\hat{\leqslant}]$ for some $a, b \in \hat{X}$ and let $a = q(\alpha)$ and $b = q(\beta)$ for some $\alpha, \beta \in \tilde{X}$. That means $(\alpha, \beta) \notin [\leqslant]$, i.e., there exists a function $f \in \Phi^{\uparrow}$ such that ${}^*f(\alpha) \mathrel{{}^*>} {}^*f(\beta)$ and ${}^*f(\alpha) \not\approx {}^*f(\beta)$. Taking the standard part we obtain $\hat{f}(a) > \hat{f}(b)$. Let r be a real number between $\hat{f}(a)$ and $\hat{f}(b)$, i.e., $\hat{f}(b) < r < \hat{f}(a)$ and define

$$G = \hat{f}^{\leftarrow}[(r, \rightarrow)], \qquad H = \hat{f}^{\leftarrow}[(\leftarrow, r)].$$

The sets G, H are open in $\hat{X}$ since $\hat{f}$ is continuous and hence $G \times H$ is open in $\hat{X} \times \hat{X}$. We have, obviously, $(a, b) \in G \times H$ and $(G \times H) \cap [\hat{\leqslant}] = \emptyset$ since $\hat{f}$ is monotone nondecreasing. The proof is complete. $\square$

Proposition 2.3. *If $\Phi^{\uparrow} \subseteq C_b^{\uparrow}(X, \mathbb{R})$, then the graph $[\hat{\leqslant}]$ of $\hat{\leqslant}$ is $\hat{T} \times \hat{T}$-closed and $\hat{T} \times \hat{T}$-compact in $\hat{X} \times \hat{X}$.*

Proof. In this case $\hat{X} \times \hat{X}$ is compact [11, Proposition (2.11)]. □

Proposition 2.4. *Let $\Phi^{\uparrow} = C_b^{\uparrow}(X, \mathbb{R})$. Then the nonstandard hull $(\hat{X}, \hat{T}, \hat{\leqslant})$ of $(X, T, \leqslant)$ coincides with the Nachbin order compactification $\mathcal{N}(X, T, \leqslant)$ of $(X, T, \leqslant)$.*

Proof. We have $\tilde{X} = {}^*X$ in this case, $(\hat{X}, \hat{T})$ is Hausdorff and compact and contains a continuous image of X [11, Propositions (2.9) and (2.11)]. The statement follows directly from Proposition 2.2 and the universal property characterizing the Nachbin order compactification. The proof is complete. □

Example 2.5. (1) Consider $(\mathbb{R}, \tau, \leqslant)$ and let $\Phi^{\uparrow}$ consist of the single function $f(x) = \arctan x$. Then $\tilde{\mathbb{R}} = {}^*\mathbb{R}$ and $\hat{\mathbb{R}} = {}^*\mathbb{R}/\sim$ is the two-point compactification of the real line regarded as an ordered topological space.

(2) $(\mathbb{R}, \tau, \leqslant)$ and let $\Phi^{\uparrow}$ consist of all bounded monotone continuous real-valued functions on $(\mathbb{R}, \tau, \leqslant)$. The universal property characterizing the Nachbin ordered compactification $\mathcal{N}(\mathbb{R}, \tau, \leqslant)$ shows that $\mathcal{N}(\mathbb{R}, \tau, \leqslant)$ is the two-point ordered compactification of $(\mathbb{R}, \tau, \leqslant)$.

(3) When $\leqslant$ is the discrete order on X, then $\mathcal{N}(X, T, \leqslant) = \beta(X, T, \leqslant)$ where $\leqslant$ is the discrete order on the Stone–Čech compactification $\beta(X, T, \leqslant)$ of $(X, T, \leqslant)$ (Nachbin [9]).

(4) Consider $(\mathbb{R}, \tau, \leqslant)$. When $\Phi^{\uparrow}$ consists of all monotone continuous real-valued functions defined on X, then the corresponding nonstandard hull $\tilde{X}/\sim$ is simply $(\mathbb{R}, \tau, \leqslant)$, see Section 6.

3. The algebras of functions $A(X, \mathbb{R})$ and $A_b(X, \mathbb{R})$

In what follows, for the sake of simplicity and without loss of generality we shall restrict ourselves to completely regular ordered topological spaces $(X, T, \leqslant)$ (Definition 1.5), and also to $\Phi^{\uparrow}$ being $C_b^{\uparrow}(X, \mathbb{R})$ or $C^{\uparrow}(X, \mathbb{R})$. In this case X is a topological ordered subspace of $\hat{X}$ and formula (5) reduces to $\hat{f}(x) = f(x)$, $x \in X$ and the monads and the equivalence classes coincide for the standard points (Lemma 1.6).

The points of the Stone–Čech compactification specify uniquely the maximal ideals in $C_b(X, \mathbb{R})$ and $C(X, \mathbb{R})$ (Gillman and Jerison [5, Theorems (7.2) and (7.3)]). We shall show that the points of the Nachbin compactification $\mathcal{N}(X, T, \leqslant)$ and points of the nonstandard compactification $({}^*X, T, \leqslant)$ determine maximal ideals in certain algebras of functions $A_b(X, \mathbb{R})$ and $A(X, \mathbb{R})$, which are naturally associated with $C_b^{\uparrow}(X, \mathbb{R})$ and $C^{\uparrow}(X, \mathbb{R})$, respectively. In this section we shall present basic properties of the algebras $A_b(X, \mathbb{R})$ and $A(X, \mathbb{R})$ and in Section 4 we describe their maximal ideals.

Definition 3.1. (i) We denote by $A_b(X, \mathbb{R}) = C_b^{\updownarrow}(X, \mathbb{R}) - C_b^{\updownarrow}(X, \mathbb{R})$ the algebra of all continuous functions which can be represented as a difference of two functions from $C_b^{\updownarrow}(X, \mathbb{R})$.

(ii) By $A(X, \mathbb{R}) = C^{\uparrow}(X, \mathbb{R}) - C^{\uparrow}(X, \mathbb{R})$ will be denoted the algebra of all differences of functions in $C^{\uparrow}(X, \mathbb{R})$.

Proposition 3.2. (i) *$A(X, \mathbb{R})$ is the smallest subalgebra of $C(X, \mathbb{R})$ which contains $C^{\uparrow}(X, \mathbb{R})$. $A(X, \mathbb{R})$ is also a sublattice of $C(X, \mathbb{R})$, in the usual ordering of $C(X, \mathbb{R})$.*

(ii) *$A_b(X, \mathbb{R})$ is the smallest subalgebra of $C(X, \mathbb{R})$ and of $C_b(X, \mathbb{R})$ which contains $C_b^{\updownarrow}(X, \mathbb{R})$. $A_b(X, \mathbb{R})$ is also a sublattice of $C(X, \mathbb{R})$.*

Proof. (i) We first establish that $A(X, \mathbb{R})$ is a subalgebra and a sublattice of $C(X, \mathbb{R})$. It is clear that if f, g are in $C^{\uparrow}(X, \mathbb{R})$, then so are $f+g$, $f \vee g$ and $f \wedge g$. For the product in $A(X, \mathbb{R})$ observe that

$$f \cdot g = f \cdot (g \vee 0) - (-f) \cdot (g \wedge 0),$$

hence $f \cdot g \in A(X, \mathbb{R})$ for all $f, g \in A(X, \mathbb{R})$, since the product of a function in $C^{\uparrow}(X, \mathbb{R})$ and a nonnegative function in $C^{\uparrow}(X, \mathbb{R})$ is also in $C^{\uparrow}(X, \mathbb{R})$ and, furthermore, the product of a function in $-C^{\uparrow}(X, \mathbb{R})$ and a negative function in $C^{\uparrow}(X, \mathbb{R})$ is in $C^{\uparrow}(X, \mathbb{R})$. Finally, if f, g are in $C^{\uparrow}(X, \mathbb{R})$, then $|f_1 - f_2| \in A(X, \mathbb{R})$ since $|f_1 - f_2| = (f_1 \vee f_2) - (f_1 \wedge f_2)$. The result now follows since $f \vee g$ and $f \wedge g$ can be expressed algebraically in terms of $f+g$, $|f+g|$, $|f-g|$. (ii) is proved similarly. □

Lemma 3.3. *For any $f \in A_b(X, \mathbb{R})$ and any $c \in \mathbb{R}$ there exist functions f_1 and f_2 in $A_b(X, \mathbb{R})$ such that $f = f_1 - f_2$ and $f_1(x) \geqslant c$, and $f_2(x) \geqslant c$ for all $x \in X$.*

Proof. (i) Let $f = \varphi_1 - \varphi_2$ for some functions φ_1 and φ_2 in $C^{\uparrow}(X, \mathbb{R})$ and let b be a lower bound for both of them, i.e.,

$$\varphi_1(x) \geqslant b, \qquad \varphi_2(x) \geqslant b,$$

for all $x \in X$. Then the functions

$$f_1 = \varphi_1 - b + c, \qquad f_2 = \varphi_2 - b + c$$

are as required. □

Example 3.4. From the above result, with $c = 0$, it follows that

$$A_b(X, \mathbb{R}) = C_b^{\updownarrow +}(X, \mathbb{R}) - C_b^{\updownarrow +}(X, \mathbb{R}) \tag{6}$$

where $C_b^{\updownarrow +}(X, \mathbb{R})$ consists of all nonnegative functions in $C_b^{\updownarrow}(X, \mathbb{R})$. The following example shows that it is not always possible to express a monotone function as the difference of two nonnegative monotone functions, which, in particular, implies that $A^+(X, \mathbb{R})$, the algebra of functions $C^{\uparrow +}(X, \mathbb{R}) - C^{\uparrow +}(X, \mathbb{R})$, is a proper subalgebra of $A(X, \mathbb{R})$. For suppose

$$f : (\mathbb{R}, \tau, \leqslant) \to (\mathbb{R}, \tau, \leqslant)$$

is $f(x)=x$. If $f(x)=f_1(x)-f_2(x)$ where both f_1 and f_2 are nonnegative, and monotone nondecreasing, then

$$\lim_{x\to-\infty} f_1(x)=c_1, \qquad \lim_{x\to-\infty} f_2(x)=c_2$$

both exist in $\mathbb{R}$, so that

$$\lim_{x\to-\infty} (f_1(x)-f_2(x))=c_1-c_2\neq \lim_{x\to-\infty} f(x).$$

Lemma 3.5. *Let the function $f: X\to\mathbb{R}$ be bounded away from the zero, $|f(x)|\geqslant c>0$ for all $x\in X$ and some $c\in\mathbb{R}$, and let $1/f$ be its reciprocal function. Then:*

(i) $\quad f\in A(X,\mathbb{R}) \quad\Rightarrow\quad 1/f\in A(X,\mathbb{R})$,

(ii) $\quad f\in A_b(X,\mathbb{R}) \quad\Rightarrow\quad 1/f\in A_b(X,\mathbb{R})$.

Proof. (i) Let $f\in A(X,\mathbb{R})$, i.e., $f=f_1-f_2$ for some $f_1, f_2\in C^{\uparrow}(X,\mathbb{R})$. Obviously, we have $1/f=\varphi_1-\varphi_2$ where $\varphi_1=cf_2$ and $\varphi_2=cf_2-1/f$ and these functions are continuous. Then, consider the real functions

$$F_1(x,y)=cy, \quad F_2(x,y)=cy-\frac{1}{x-y}, \qquad x,y\in\mathbb{R},\ |x-y|>\frac{1}{\sqrt{c}}.$$

These functions are monotone nondecreasing with respect to both x and y and, obviously, we have

$$\varphi_1=F_1(f_1,f_2), \qquad \varphi_2=F_2(f_1,f_2).$$

So, φ_1, φ_2 are monotone nondecreasing, i.e., they belong to $C^{\uparrow}(X,\mathbb{R})$, as compositions of monotone nondecreasing functions. Hence $1/f\in A(X,\mathbb{R})$. Moreover, they are bounded whenever f_1 and f_2 are bounded which means that $f\in A_b(X,\mathbb{R})$ implies $1/f\in A_b(X,\mathbb{R})$. The proof is complete. □

As for the case when the order is discrete, we have:

Proposition 3.6. *If $\Phi^{\uparrow}$ is $C_b^{\uparrow}(X,\mathbb{R})$ or $C^{\uparrow}(X,\mathbb{R})$, then $\Phi^{\uparrow}-\Phi^{\uparrow}$ and $A(\hat{X},\mathbb{R})$ are isomorphic as algebras under the mapping $f\to\hat{f}$ where $\hat{X}$ is the corresponding nonstandard $\Phi^{\uparrow}$-hull of X, $\hat{f}$ is the extension of f on $\hat{X}$ (Proposition 2.2) and $A(\hat{X},\mathbb{R})=C^{\uparrow}(\hat{X},\mathbb{R})-C^{\uparrow}(\hat{X},\mathbb{R})$ where $C^{\uparrow}(\hat{X},\mathbb{R})$ consists of all continuous monotone nondecreasing real-valued functions defined on X.*

The proof is similar to the proof of [11, Proposition (2.14)] and will be omitted.

4. Maximal ideals of $A(X,\mathbb{R})$ and $A_b(X,\mathbb{R})$

In this section we shall describe the maximal ideals of $A(X,\mathbb{R})$ and $A_b(X,\mathbb{R})$ by means of the points of the Nachbin ordered compactification $\mathcal{N}(X,T,\leqslant)$ and the nonstandard compactification $({}^*X, T, \leqslant)$ of $(X, T, \leqslant)$. We shall present a unified description, whenever possible, of $A(X, \mathbb{R})$ and $A_b(X, \mathbb{R})$ and shall use $\boldsymbol{F}$ to denote one or the other of these algebras.

To indicate an essential difference between the ordered and nonordered (or discretely ordered) situations, consider a maximal ideal M of $C(X, \mathbb{R})$. The zero sets $\mathscr{Z}(f)$, $f \in M$, are nonempty, since otherwise $1/f \in C(X, \mathbb{R})$, so $1 = (1/f) \cdot f \in M$. In the ordered case the following example shows that $\mathscr{Z}(f)$ can be empty for some f in M when M is a maximal ideal of $A(X, \mathbb{R})$.

Example 4.1. Let $(X, T, \leqslant)$ be the set $[-1, 1] - \{0\}$ with the usual topology and the usual order. Observe that $f(-1) \leqslant f(x) \leqslant f(1)$ for every function f in $C^{\uparrow}(X, \mathbb{R})$, so that $A(X, \mathbb{R}) = A_b(X, \mathbb{R})$. The set I of multiples of f, where $f(x) = x$, forms a proper ideal of $A(X, \mathbb{R})$, since $1 = k(x) \cdot x$ gives

$$1 = \left(\lim_{x \to 0} k(x)\right) \cdot 0 = 0 \quad \left(\lim_{x \to 0} k(x) \text{ exists and is finite}\right).$$

Thus I can be included in a maximal ideal M of $A(X, \mathbb{R})$, nevertheless $\mathscr{Z}(f)$ is empty and f is in M.

We establish some basic properties of maximal ideals in $\boldsymbol{F}$.

Proposition 4.2. *Let M be a maximal ideal in $\boldsymbol{F}$, then the following properties hold:*

(i) $f \in M \;\Leftrightarrow\; |f| \in M$,

(ii) $f, g \in M \;\Rightarrow\; f \vee g \in M$ *and* $f \wedge g \in M$,

(iii) $f \in M \;\Rightarrow\; 0 \wedge f \vee 1 \in M$.

Proof. Only (iii) requires a proof. (ii) shows that $0 \vee f \in M$ when $f \in M$. So we may assume that f is nonnegative and show that $f \wedge 1 \in M$. We first show that $(f-1) \vee 0 \in M$, from which it follows that $f - ((f-1) \vee 0) \in M$, but

$$f - ((f-1) \vee 0) = -(((f-1) \vee 0) - f) = -((-1) \vee (-f)) = 1 \wedge f,$$

as required. If $(f-1) \vee 0 \notin M$, then for some $g \in \boldsymbol{F}$ and $h \in M$ we have

$$g(x)((f(x)-1) \vee 0) + h(x) = 1, \quad x \in X,$$

so that $h(x) = 1$ on $\{x \in X \mid |f(x)| \leqslant 1\}$, hence $|f| + |h| > 1$, which is impossible since $|f| + |h| \in M$ and M has no invertible element. □

We can now characterize maximal ideals of $\boldsymbol{F}$ using the nonstandard ordered compactification.

Proposition 4.3. *Let $M \subseteq \boldsymbol{F}$. Then M is a maximal ideal of $\boldsymbol{F}$ if and only if there exists a point $\alpha \in {}^*X$ such that*

$$M = \{f \in \boldsymbol{F} \mid {}^*f(\alpha)\, {}^*g(\alpha) \approx 0 \text{ for all } g \in \boldsymbol{F}\}. \tag{7}$$

Proof. Suppose M is given by (7). Obviously, M is an ideal of $\boldsymbol{F}$. To show that M is maximal, assume $f \in M' - M$ for some ideal M' of $\boldsymbol{F}$ such that $M \subset M'$. Then, there is g in $\boldsymbol{F}$ such that $^*f(\alpha)\, ^*g(\alpha) \not\approx 0$. By multiplying by -1, if necessary, we may take $^*f(\alpha)^*g(\alpha) \,^*\!\!> 0$, so that $^*f(\alpha)^*g(\alpha) \,^*\!\!> r > 0$ for some r in $\mathbb{R}$. Taking $g' = g/r$ and writing g for g', we have $^*f(\alpha)^*g(\alpha) \,^*\!\!> 1 + 2\delta$ for some $g \in \boldsymbol{F}$ and some $\delta \in \mathbb{R}$, $\delta > 0$. Then, there is a function $h \in \boldsymbol{F}$, $h : X \to [0, 1]$, such that $h = 1$ on $(fg)^{\leftarrow}[(\leftarrow, 1 + \delta]]$ and $h = 0$ on $H = (fg)^{\leftarrow}[[1 + 2\delta, \to)]$. For example,

$$h = \frac{(1+2\delta) - fg}{\delta} \wedge 1 \vee 0.$$

We have $^*h(\alpha) = 0$ since $\alpha \in {}^*H$. Hence $h \in M$. Also we have $|fg| + h \geq 1$, so by Lemma 3.5, $|fg| + h$ is invertible. Now, the representation

$$\frac{|f||g|}{|f||g| + h} + \frac{h}{|f||g| + h} = 1$$

shows that $1 \in M'$, since $|f||g|/(|f||g| + h) \in M'$ and $h/(|f||g| + h) \in M$, by Proposition 4.2(i). Hence, M is maximal. Conversely, suppose M is a maximal ideal of $\boldsymbol{F}$. Then for each $f \in M$, each $g \in \boldsymbol{F}$ and each $n \in \mathbb{N}$ the set

$$A_{f,g,n} = \{x \in X \mid |f(x)g(x)| < 1/n\} \tag{8}$$

is not empty. For suppose not, then $|f(x)g(x)| \geq 1/n$ for all $x \in X$ would imply that $1/fg \in \boldsymbol{F}$, by Lemma 3.5, which is impossible since $fg \in M$. Moreover, we have

$$A_{(fg)^2 + (f'g')^2, \max(m^2, n^2)} \subseteq A_{f,g,m} \cap A_{f',g',n}$$

so that the family (8) has the finite intersection property. By the saturation principle (Hurd and Loeb [6, p. 106]), there exists a point α such that

$$\alpha \in \bigcap \{^*A_{f,g,n} \mid f \in M,\ g \in \boldsymbol{F},\ n \in \mathbb{N}\},$$

i.e., $^*f(\alpha)^*g(\alpha) \approx 0$ for all $f \in M$ and all $g \in \boldsymbol{F}$. The proof is complete. □

Corollary 4.4. *For any maximal ideal M of $\boldsymbol{F}$ there exists a point $\alpha \in {}^*X$ such that*

$$M \subseteq \{f \in \boldsymbol{F} \mid {}^*f(\alpha) \approx 0\}. \tag{9}$$

Moreover, if $^*g(\alpha)$ is finite for all $g \in \boldsymbol{F}$, then equality in (9) holds.

Corollary 4.5. *The maximal ideals of $\boldsymbol{F}$ preserve the usual order in $\boldsymbol{F}$.*

Proof. We have to show that M is convex, i.e., $0 \leq f_1 \leq f \in M$ in $\boldsymbol{F}$ implies $f_1 \in M$ which follows immediately from (7). □

Corollary 4.6. *Let $M \subseteq A_b(X, \mathbb{R})$. Then, M is a maximal ideal of $A_b(X, \mathbb{R})$ if and only if there exists a point $\alpha \in {}^*X$ such that*

$$M = \{f \in A_b(X, \mathbb{R}) \mid {}^*f(\alpha) \approx 0\}. \tag{10}$$

Proof. In this case $F = A_b(X, \mathbb{R})$, so that ${}^*g(\alpha)$ is finite. □

The maximal ideals in $A_b(X, \mathbb{R})$ can be completely specified by the points of $\mathcal{N}(X, T, \leqslant)$ just as the maximal ideals in $C_b(X, \mathbb{R})$ are characterized by $\beta(X, T)$ in the discrete-order case (Gillman and Jerison [5, Theorem (7.2)]).

Proposition 4.7. *Let $M \subseteq A_b(X, \mathbb{R})$. Then M is a maximal ideal of $A_b(X, \mathbb{R})$ iff M is of the form*

$$M = \{f \in A_b(X, \mathbb{R}) \mid \hat{f}(a) = 0\} \tag{11}$$

for some point $a \in \mathcal{N}(X, T, \leqslant)$ (Proposition 2.4). The point a is uniquely determined by M.

Proof. Let M be defined by (11). The set

$$M_a = \{\hat{f} \in A(\hat{X}, \mathbb{R}) \mid \hat{f}(a) = 0\} \tag{12}$$

is, obviously, a maximal ideal of $A(\hat{X}, \mathbb{R})$ and, therefore, M is a maximal ideal of $A_b(X, \mathbb{R})$, since $A_b(X, \mathbb{R})$ and $A(\hat{X}, \mathbb{R})$ are isomorphic as rings, by Proposition 3.6, and M_a is the image of M under this isomorphism. Conversely, suppose $M \subset A_b(X, \mathbb{R})$ is a maximal ideal of $A_b(X, \mathbb{R})$. Then, by Corollary 4.6, M can be represented by (10) for some $\alpha \in {}^*X$. Hence, $\hat{f}(a) = 0$ for all $f \in M$ where $a = q(\alpha)$, i.e., $M \subseteq \{f \in A_b(X, \mathbb{R}) \mid \hat{f}(a) = 0\}$, which immediately implies the equality, by the maximality of M. To show that a is unique, suppose $b \in \hat{X}$, $a \neq b$, and let b generate the same ideal M, i.e.,

$$M = \{f \in A_b(X, \mathbb{R}) \mid \hat{f}(b) = 0\}.$$

So, we have either $a \not\leqslant b$ or $b \not\leqslant a$. Suppose $a \not\leqslant b$ (the case $b \not\leqslant a$ is treated similarly). Then, by Nachbin [9, Theorems 4 and 6], there exists a function $g \in C_b^{\updownarrow}(X, \mathbb{R})$ such that

$$\hat{g}(a) = 1 \quad \text{and} \quad \hat{g}(b) = 0.$$

Hence g, $1 - g \in M$ which is a contradiction. The proof is complete. □

Proposition 4.8. *Let $M \subset A(X, \mathbb{R})$. Then M is a real maximal ideal of $A(X, \mathbb{R})$ if and only if there exists a point $\alpha \in \tilde{X}$ such that*

$$M = \{f \in A(X, \mathbb{R}) \mid {}^*f(\alpha) \approx 0\} \tag{13}$$

where $\tilde{X}$ is the set of $C^{\uparrow}(X, \mathbb{R})$-prenearstandard points of ${}^*X$ (Definition 1.3).

Proof. Suppose M is a real maximal ideal of $A(X, \mathbb{R})$, i.e., $A(X, \mathbb{R})/M$ and $\mathbb{R}$ are isomorphic as fields. Then, by Corollary 4.4, there is a point $\alpha \in {}^*X$ for which ${}^*f(\alpha) \approx 0$, when $f \in M$. To show that $\alpha \in \tilde{X}$, consider any $g \in A(X, \mathbb{R})$. Since M is a *real* ideal, we have $g = c + f$ for some $c \in \mathbb{R}$ and some $f \in M$, so that ${}^*g(\alpha) \approx c$, i.e., $^*g(\alpha)$ is a finite number for all $g \in A(X, \mathbb{R})$. Hence, (13) holds, by Corollary 4.4. Conversely, suppose M is given by (13) for some $\alpha \in \tilde{X}$. Then, $^*g(\alpha)$ is finite for all $g \in A(X, \mathbb{R})$, so representation (13) is equivalent to (7), which means that M is a maximal ideal of $A(X, \mathbb{R})$. Then for any $g \in A(X, \mathbb{R})$ we have $g - c \in M$ where $c = \text{st}(^*g(\alpha))$. That means that M is a real maximal ideal. The proof is complete. □

Note. The characterization of maximal ideals M of $C(X, \mathbb{R})$ due to Gelfand and Kolmogorov (Gillman and Jerison [5, Theorem (7.3)])

$$f \in M^p \quad \Leftrightarrow \quad p \in \text{cl}_{\beta X}(\mathscr{Z}(f))$$

does not hold in the ordered case, for example, the maximal ideal M of $A(X, \mathbb{R})$ which contains $f(x) = x$, where $X = [-1, 1] - \{0\}$ with the usual topology and usual order, has $\mathscr{Z}(f) = \emptyset$, so $\text{cl}_{\beta X}(\mathscr{Z}(f)) = \emptyset$ (see Example 4.1).

5. Ordered realcompact spaces

In [2] Choe and Hong defined and studied the class of k-compact ordered spaces, where k is an infinite cardinal and observed that the $\aleph_1$-compact ordered spaces are not the closed subspaces of products of copies of $\mathbb{R}$, in contrast with the discrete-order case. They raised the question of characterizing these $\mathbb{R}$-compact spaces in the category of completely regular ordered spaces.

In this section we shall define in a natural way the class of ordered realcompact spaces and show that they are precisely the closed order subspaces of a product of copies of $(\mathbb{R}, \tau, \leqslant)$, thereby answering the question of Choe and Hong.

Definition 5.1. A completely regular ordered space $(X, T, \leqslant)$ (Definition 1.5) is an *ordered realcompact space* if every real maximal ideal M of $A(X, \mathbb{R})$ is fixed, in the sense that there is an x in X, such that $f \in M \Leftrightarrow f(x) = 0$.

Just as compact ordered spaces are order isomorphic to closed subspaces of the canonical product $\mathbb{R}^J$, where $J = C_b^{\uparrow}(X, \mathbb{R})$ (Nachbin [9]), the analogous result for ordered realcompact spaces is true. We first establish productivity and hereditary properties for these spaces. The method of proof is that of [10], where it is only necessary to verify that the functions that are constructed and used in the proof are, in fact, in $A(X, \mathbb{R})$.

Definition 5.2. Let $\boldsymbol{F}$ denote $A_b(X, \mathbb{R})$ or $A(X, \mathbb{R})$. An ordered completely regular space is an *ordered $\boldsymbol{F}$-compact space* if every maximal ideal M of $\boldsymbol{F}$ such that $\boldsymbol{F}/M$ is order isomorphic to $\mathbb{R}$, is fixed.

When $\boldsymbol{F}$ is $A(X, \mathbb{R})$, then the ordered $\boldsymbol{F}$-compact spaces are precisely the ordered realcompact spaces defined above. We obtain an algebraic characterization of ordered compact spaces when $\boldsymbol{F}$ is $A_b(X, \mathbb{R})$, in the case, as in the unordered case, all maximal ideals are real since the ordered field $A_b(X, \mathbb{R})/M$ is Archimedean.

Proposition 5.3. *Let $(X, T, \leqslant)$ be a completely regular ordered space and let* $\boldsymbol{F}$ *be* $A_b(X, \mathbb{R})$. *Then, $(X, T, \leqslant)$ is compact ordered if and only if it is* $\boldsymbol{F}$*-compact.*

Proof. Assume $(X, T, \leqslant)$ is compact ordered and let M be a maximal ideal of $A_b(X, \mathbb{R})$. Now M is a real maximal ideal and there is α in ${}^*X$ such that ${}^*f(\alpha) \approx 0 \Leftrightarrow f \in M$. Since (X, T) is compact and Hausdorff, there is a unique x such that $\alpha \in \mu(x)$, since

$${}^*X = \bigcup \{\mu(x) \mid x \in X\}.$$

By continuity, ${}^*f(\alpha) \approx f(x)$, so that $f(x) = 0$. Thus, $f \in M \Leftrightarrow f(x) = 0$. Conversely, suppose every (real) maximal ideal M of $A_b(X, \mathbb{R})$ is fixed. It remains to show that (X, T) is a compact space. Again, we use Robinson's criterion: consider $\alpha \in {}^*X$ and find $x \in X$ such that $\alpha \in \mu(x)$. Let $M = \{f \in A_b(X, \mathbb{R}) \mid {}^*f(\alpha) \approx 0\}$. Then M is a maximal ideal of $A_b(X, \mathbb{R})$ so there is $x \in X$ such that $f \in M \Leftrightarrow f(x) = 0$, i.e., ${}^*f(\alpha) \approx 0 \Leftrightarrow f(x) = 0$. For any function g in $C_b^{\uparrow}(X, \mathbb{R})$, we have, for $h = g - g(x)$, that $h(x) = 0$ and ${}^*h(\alpha) = {}^*g(\alpha) - g(x)$, so that ${}^*g(\alpha) \approx g(x)$. Hence, $\alpha \in q(x)$, the equivalence class of x determined by $\Phi^{\uparrow} = C_b^{\uparrow}(X, \mathbb{R})$. By Lemma 1.6, this equivalence class is $\mu(x)$. The proof is complete. □

The following results concern productivity and hereditary properties of $\boldsymbol{F}$-compact spaces. When $\boldsymbol{F}$ is $A_b(X, \mathbb{R})$ these results were established by Nachbin [9], for $A(X, \mathbb{R})$ the corresponding facts would require different proofs. Instead of presenting the proof for $A(X, \mathbb{R})$ only, we have chosen to unify the $\boldsymbol{F}$-compact cases, providing, in particular, new proofs for the results of Nachbin quoted above.

Proposition 5.4. *Products of order* $\boldsymbol{F}$*-compact spaces are ordered* $\boldsymbol{F}$*-compact.*

Proof. We shall omit the proof which is precisely that of [10, Theorem 2], once it is established that the functions g_{ij} are in $A_b(X, \mathbb{R})$. This is the content of the following lemma. □

Lemma 5.5. *Let $(X, T, \leqslant)$ be a completely regular ordered space and V an open set, $x \in V$. Then there is a function h in $A_b(X, \mathbb{R})$ such that $0 \leqslant h \leqslant 1$, $h(x) = 0$ and $h = 1$ on $X - V$.*

Proof. By definition, there are continuous monotone f, g, such that $0 \leqslant f \leqslant 1$, $0 \leqslant g \leqslant 1$, f is nondecreasing, g is nonincreasing, $f(a) = 0$, $g(a) = 0$ *and* $\sup\{f(x), g(x)\} = 1$ for all $x \in X - V$. Let $h = f \vee g$. It is clear that $h(a) = 0$ and $h = 1$ on $X - V$. Moreover, $f, g \in A(X, \mathbb{R})$, so $f \vee g \in A(X, \mathbb{R})$, by Proposition 3.2. □

Proposition 5.6. *Closed subspaces of ordered* $\boldsymbol{F}$*-compact spaces are ordered* $\boldsymbol{F}$*-compact.*

Proof. Let X_0 be a closed subspace of X. To see that X_0 is ordered $\boldsymbol{F}$-compact, let $\pi_0 : A(X_0, \mathbb{R}) \to \mathbb{R}$ be an order-preserving ring homomorphism. Then $\pi : A(X, \mathbb{R}) \to \mathbb{R}$

given by $\pi(f)=\pi_0(f_0)$, where f_0 is the restriction of f to X_0, is also an order-preserving ring homomorphism. By assumption, there is $x \in X$ such that $\pi(f)=f(x)$. It remains to show that x is in X_0. Suppose not. Then, by order complete regularity, there are continuous monotone functions f, g such that $0 \leq f \leq 1$, $0 \leq g \leq 1$, $f(x)=1$, $g(x)=1$ and $\inf\{f(a), g(a)\}=0$ for all a in $X - X_0$, f is nondecreasing and g nonincreasing. It is easy to verify that a monotone ring homomorphism from $A(X, \mathbb{R})$ to $\mathbb{R}$ is a lattice homomorphism, so that $\pi(f \vee g) = \pi(f) \vee \pi(g)$ for f, g in $A(X, \mathbb{R})$. Now, $\pi(1-f) = \pi_0((1-f) \mid X_0) = (1-f)(x_0) = 0$. Similarly, $\pi(1-g)=0$. However, $(1-f) \vee (1-g) = 1-(f \wedge g)$ has a restriction to X_0 which is identically 1, hence $\pi((1-f) \vee (1-g))=1$, which is impossible since $\pi((1-f) \vee (1-g)) = \pi(1-f) \vee \pi(1-g)=0$. □

Proposition 5.7. *Let $(X, T, \leq)$ be an ordered realcompact space, then $(X, T, \leq)$ is order isomorphic to a closed subspace of the canonical product $\mathbb{R}^J$, where J is $C^{\uparrow}(X, \mathbb{R})$.*

Proof. Let α be a point in $e[X]$, where e is the canonical map $e:(X, T, \leq) \to (\mathbb{R}, \tau, \leq)^J$. Define $\pi: F \to \mathbb{R}$ by $\pi(f) = \alpha_{f_1} - \alpha_{f_2}$, where $f = f_1 - f_2$, $f_i \in C^{\uparrow}(X, \mathbb{R})$. Note that π is well defined since $f_1 - f_2 = g_1 - g_2$, f_i, $g_i \in C^{\uparrow}(X, \mathbb{R})$, gives $f_1 + g_2 = g_1 + f_2$, so that $\pi(f_1 + g_2) = \alpha_{f_1 + g_2} = \alpha_{f_1} + \alpha_{g_2} = \pi(f_1) + \pi(g_2)$, similarly, $\pi(g_1 + f_2) = \pi(g_1) + \pi(f_2)$. Hence $\pi(f_1) - \pi(f_2) = \pi(g_1) - \pi(g_2)$. We have used the fact that if α is in the closure of $e[X]$, then, $\alpha_{f+g} = \alpha_f + \alpha_g$ for f, g in $C^{\uparrow}(X, \mathbb{R})$; it is also true that $\alpha_{f \cdot g} = \alpha_f \cdot \alpha_g$ and, for $f \geq 0$ we have $\alpha_f \geq 0$. Hence π is an order-preserving ring homomorphism from F to $(\mathbb{R}, \tau, \leq)$. By assumption, there is $x \in X$ such that $\pi(f)=0 \Leftrightarrow f(x)=0$. Then for f in $C^{\uparrow}(X, \mathbb{R})$ we have $\pi(f - \alpha_f)=0$, so that $f(x)=\alpha_f$, as required. □

Corollary 5.8. *Every compact ordered topological space is ordered realcompact.*

Note. A characterization of pairwise realcompactness was given in [1]. The argument can be used to establish a characterization of order realcompactness, that is easier to use than the algebraic characterization. We state the characterization and refer to Fletcher and Lindgren [4] for the theory of quasi-uniform spaces. As a consequence of Proposition 5.7 we have the following:

Proposition 5.9. *A completely regular ordered space $(X, T, \leq)$ is ordered realcompact if and only if the quasi-uniformity $C^{\uparrow}(X, \mathbb{R})$, induced by the function $f:(X, T, \leq) \to (\mathbb{R}, \tau, \leq)$, is complete.*

Proposition 5.10. *$(\mathbb{R}, \tau, \leq)$ is order isomorphic to a closed order subspace of the canonical product $(\mathbb{R}, \tau, \leq)^J$, where $J = C^{\uparrow}(\mathbb{R}, \mathbb{R})$, and $(\mathbb{R}, \tau, \leq)$ is order realcompact.*

Proof. Let $i: \mathbb{R} \to \mathbb{R}$ denote the identity map and $e:(\mathbb{R}, \tau, \leq) \to (\mathbb{R}, \tau, \leq)^J$, the embedding in the canonical product. Let $\alpha \in \mathrm{cl}(e[X])$. Put $x = \pi_i(\alpha)$. We show that

$\alpha = e(x)$. Let $f \in C^{\uparrow}(\mathbb{R}, \mathbb{R})$, since $\alpha \in \mathrm{cl}(e[X])$, we have $y_n \in \mathbb{R}$ such that $|f(y_n) - \alpha_f| < 1/n$ and $|x - y_n| < 1/n$. By continuity of f, we have $f(x) = \alpha_f$, so that $\alpha = e(x)$, as required. □

We state for completeness an immediate consequence of the above.

Proposition 5.11. *If $(X, T, \leqslant)$ is isomorphic to a closed subspace of a product of copies of $(\mathbb{R}, \tau, \leqslant)$, then it is order realcompact.*

6. Ordered realcompactification

We now show that every completely regular ordered space $(X, T, \leqslant)$ can be embedded as a dense ordered subspace of an ordered realcompact space $\mathcal{R}(X, T, \leqslant)$, denoted also by $\mathcal{R}X$, with the property that every monotone continuous function $f:(X, T, \leqslant) \to (\mathbb{R}, \tau, \leqslant)$ admits a unique extension to $\mathcal{R}(X, T, \leqslant)$. This universal property characterizes $\mathcal{R}(X, T, \leqslant)$.

$\mathcal{R}(X, T, \leqslant)$ will be obtained as a nonstandard ordered $C^{\uparrow}(X, \mathbb{R})$-hull of $(X, T, \leqslant)$.

Proposition 6.1. *The nonstandard ordered $C^{\uparrow}(X, \mathbb{R})$-hull $\mathcal{R}X = (\hat{X}, \hat{T}, \hat{\leqslant})$ (Definition 2.1) is a completely regular ordered space.*

Proof. Let $a \in \hat{X}$ and let $F \subseteq \hat{X}$ be a closed subset not containing a. Since $q^{\leftarrow}[F]$ is closed in $\tilde{X}$, there is a closed set K in ${}^*X$ such that $K \cap \tilde{X} = q^{\leftarrow}[F]$. Since $({}^*X, T)$ is compact, it follows that K is compact. Let $\alpha \in \tilde{X}$ be such that $q(\alpha) = a$. Clearly, $\alpha \notin K$ so that $\alpha \not\sim \beta$ for all $\beta \in K$. By Lemma 1.4, for any $\beta \in K$ there is $g_\beta : (X, T, \leqslant) \to (\mathbb{R}, \tau, \leqslant)$ such that $0 \leqslant g_\beta \leqslant 1$, ${}^*g_\beta(\alpha) = 0$ and ${}^*g_\beta(\beta) = 1$ or there is $h_\beta : (X, T, \leqslant) \to (\mathbb{R}, \tau, \leqslant)$ such that $0 \leqslant h_\beta \leqslant 1$, ${}^*h_\beta(\alpha) = 1$ and ${}^*h_\beta(\beta) = 0$. By compactness of K, there are finitely many points $\beta_1, \ldots, \beta_n, \beta_{n+1}, \ldots, \beta_m$ such that

$$K \subseteq \left(\bigcup_{j=1}^{n} {}^*g_{b_j}^{\leftarrow}[{}^*(\tfrac{3}{4}, 1]] \right) \cup \left(\bigcup_{j=n+1}^{m} {}^*h_{b_j}^{\leftarrow}[{}^*[0, \tfrac{1}{4})] \right).$$

Put $g = \sup\{\frac{4}{3} g_{\beta_j} \wedge 1 \mid 1 \leqslant j \leqslant n\}$, $h = \inf\{\frac{4}{3}(h_{\beta_j} - \frac{1}{4}) \vee 0 \mid n+1 \leqslant j \leqslant m\}$. Now $\bigcup_{j=1}^{n} {}^*g_{\beta_j}^{\leftarrow}[{}^*(\frac{3}{4}, 1]] = {}^*g^{\leftarrow}[\{1\}]$ and $\bigcup_{j=n+1}^{m} {}^*h_{\beta_j}^{\leftarrow}[{}^*[0, \frac{1}{4})] = {}^*h^{\leftarrow}[\{0\}]$, so $K \subseteq {}^*g^{\leftarrow}[\{1\}] \cup {}^*h^{\leftarrow}[\{0\}]$. Also, ${}^*g(\alpha) = 0$, ${}^*h(\alpha) = 1$. Hence, $k = 1 - g$ is monotone nonincreasing, h is monotone nondecreasing and $0 \leqslant k \leqslant 1$, $0 \leqslant h \leqslant 1$, ${}^*h(\alpha) = {}^*k(\alpha) = 1$ and $\inf\{{}^*h(\beta), {}^*k(\beta)\} = 0$ for all $\beta \in F = \tilde{X} \cap K$. Hence, $\hat{h}(q(\alpha)) = {}^*h(\alpha) = 1$, $\hat{k}(q(\alpha)) = {}^*k(\alpha) = 1$ and

$$\inf\{\hat{h}(q(\beta)), \hat{k}(q(\beta))\} = 0,$$

which means that $\mathcal{R}X = (\hat{X}, \hat{T}, \hat{\leqslant})$ is a completely regular ordered space. □

Note. This result could also have been established by first showing that $\mathcal{R}X$ is an ordered subspace of the Nachbin ordered compactification $\mathcal{N}X$. It would follow that $\mathcal{R}X$ is a completely regular ordered space (Nachbin [9, Theorem 7]).

The points of $\mathscr{R}X$ are intimately related to the algebraic structure of $A(X, \mathbb{R})$.

Proposition 6.2. *The real maximal ideals M of $A(X, \mathbb{R})$ and the points of the ordered realcompactification $\mathscr{R}X$ of X are in one-to-one correspondence given by*

$$M = \{f \in A(X, \mathbb{R}) \mid \hat{f}(a) = 0\}, \quad a \in \mathscr{R}X. \tag{14}$$

Proof. Let M be a real maximal ideal of $A(X, \mathbb{R})$. Then, (13) holds for some $\alpha \in \tilde{X}$ so we obtain $\hat{f}(a) = 0$ for $a = q(\alpha)$. Also $a \in \mathscr{R}X$ since $\alpha \in \tilde{X}$. Conversely, let M be defined by (14) for some $a \in \mathscr{R}X$. Then $\hat{f}(a) = 0$ is equivalent to ${}^*f(\alpha) \approx 0$ for any $\alpha \in a$ and hence, by Proposition 4.8, M is a real maximal ideal, since $\alpha \in \tilde{X}$. To show the uniqueness of a, suppose $b \in \mathscr{R}X$, $a \neq b$, and b determines the same real maximal ideal, i.e.,

$$M = \{f \in A(X, \mathbb{R}) \mid \hat{f}(b) = 0\}. \tag{15}$$

Since $a \neq b$ we have $a \not\leqslant b$ or $b \not\leqslant a$. Assume $a \not\leqslant b$. Since $(X, T, \leqslant)$ is a completely regular ordered space, hence, there are continuous functions f, $g: (X, T) \to (\mathbb{R}, \tau)$ such that f is monotone nondecreasing, g is monotone nonincreasing, $0 \leqslant f, g \leqslant 1$, and

$$0 \leqslant \hat{f} \leqslant 1, \qquad 0 \leqslant \hat{g} \leqslant 1,$$
$$\hat{f}(a) = 1, \qquad \hat{g}(a) = 1,$$
$$\inf\{\hat{f}(c), \hat{g}(c)\} = 0, \quad \text{for all } c \in \hat{X}, \text{ such that } c \leqslant b,$$

we have $(\hat{f} \wedge \hat{g})(b) = 0$, so $\hat{f} \wedge \hat{g} \in M$. Also $(\hat{f} \wedge \hat{g}) = 1$, so that $1 - (\hat{f} \wedge \hat{g}) \in M$. This is impossible, the proof is complete. □

Proposition 6.3. *The ordered realcompactification $\mathscr{R}(X, T, \leqslant) = (\hat{X}, \hat{T}, \hat{\leqslant})$ is an ordered realcompact topological space and every function $f: (X, T, \leqslant) \to (\mathbb{R}, \tau, \leqslant)$, $f \in C^{\uparrow}(X, \mathbb{R})$, has a unique continuous monotone nondecreasing extension to $\mathscr{R}(X, T, \leqslant)$.*

Proof. The existence and uniqueness of the extension $\hat{f}$ of f are proved in Proposition 2.2. Moreover, $A(X, \mathbb{R})$ and $A(\hat{X}, \mathbb{R})$ are isomorphic as rings, by Proposition 3.6, and hence, every real maximal ideal M of $A(\hat{X}, \mathbb{R})$ determines a real maximal ideal M_0 of $A(X, \mathbb{R})$ such that $M = \hat{M}_0$. By Proposition 6.2, $f \in M_0 \Leftrightarrow \hat{f}(a) = 0$, hence $f \in M \Leftrightarrow \hat{f}(a) = 0$, so that M is fixed, as required. □

From the categorical properties established above, it follows that the ordered realcompactification of $(X, T, \leqslant)$ with the extension property for continuous monotone real-valued functions is essentially unique (see, for example, Herrlich [6] and Herrlich and Strecker [7]). We shall give a direct argument for the sake of completeness.

Proposition 6.4. *If* $(X, T, \leqslant) \subseteq (\bar{X}, \bar{T}, \bar{\leqslant})$ *is such that* X *is a dense order subspace of* $\bar{X}$ *and every* $f:(X, T, \leqslant) \to (\mathbb{R}, \tau, \leqslant)$ *has an extension* $\bar{f}:(\bar{X}, \bar{T}, \bar{\leqslant}) \to (\mathbb{R}, \tau, \leqslant)$, *then every map to an ordered realcompact space* $F:(X, T, \leqslant) \to (Y, S, \preccurlyeq)$ *has an extension* $\bar{F}:(\bar{X}, \bar{T}, \bar{\leqslant}) \to (Y, S, \preccurlyeq)$.

Proof. $(Y, S, \preccurlyeq)$ is essentially an order subspace of the canonical product $(\mathbb{R}, \tau, \leqslant)^J$, $J = C^{\uparrow}(Y, \mathbb{R})$, by Proposition 5.7. We shall identify $(Y, S, \preccurlyeq)$ with its image in the product. Let $F:(X, T, \leqslant) \to (Y, S, \preccurlyeq)$ be given. For each f in $C^{\uparrow}(Y, \mathbb{R})$, we have $f \circ F:(X, T, \leqslant) \to (\mathbb{R}, \tau, \leqslant)$, so there is an extension $\overline{f \circ F}:(\bar{X}, \bar{T}, \bar{\leqslant}) \to (\mathbb{R}, \tau, \leqslant)$. These extensions give a mapping $\varphi:(X, T, \leqslant) \to (\mathbb{R}, \tau, \leqslant)^J$, with $\varphi | X$ equal to F on X. Now,

$$\varphi[X] = \varphi[\mathrm{cl}_T X] \subseteq \mathrm{cl}_T F[X] = Y,$$

since Y is closed in the product. The proof is complete. □

We can now prove the uniqueness of the order realcompactification.

Proposition 6.5. *Suppose* $(\bar{X}, \bar{T}, \bar{\leqslant})$ *is an ordered realcompact space containing* $(X, T, \leqslant)$ *as a dense order subspace. If every* $f:(X, T, \leqslant) \to (\mathbb{R}, \tau, \leqslant)$ *admits an extension* $\bar{f}:(\bar{X}, \bar{T}, \bar{\leqslant}) \to (\mathbb{R}, \tau, \leqslant)$, *then* $(\bar{X}, \bar{T}, \bar{\leqslant})$ *is order isomorphic to the ordered realcompactification* $(\hat{X}, \hat{T}, \hat{\leqslant})$ *of* $(X, T, \leqslant)$.

Proof. By the above, there is a continuous monotone nondecreasing map $\varphi:(\bar{X}, \bar{T}, \bar{\leqslant}) \to (\hat{X}, \hat{T}, \hat{\leqslant})$ such that $\varphi(x) = x$ for all x in X. There is also $\psi:(\hat{X}, \hat{T}, \hat{\leqslant}) \to (\bar{X}, \bar{T}, \bar{\leqslant})$ such that $\psi(x) = x$ for all x in X. Hence $\psi \circ \varphi | X$ and $\varphi \circ \psi | X$ both coincide with the identity mapping on X. Hence, $\psi \circ \varphi = \mathbf{1}_{\bar{X}}$, $\varphi \circ \psi = \mathbf{1}_{\hat{X}}$. □

Example 6.6. Let $\Delta = (x, y)$ be an open interval of $\mathbb{R}$. Then $\mathscr{R}(\Delta, \tau, \leqslant) = (\Delta, \tau, \leqslant)$ and hence, $(\Delta, \tau, \leqslant)$ is an ordered realcompact space. Indeed, applying the non-standard ordered-hull construction of the realcompactification for $\Phi = C^{\uparrow}(\Delta, \mathbb{R})$, we obtain

$$\tilde{\Delta} = \{\alpha \in {}^*\mathbb{R} \mid x \;{}^*\!\leqslant \alpha \;{}^*\!\leqslant y,\ \alpha \not\approx x,\ \alpha \not\approx y\},$$

so that, $\hat{\Delta} = \tilde{\Delta}/\sim$ is isomorphic to Δ, $\hat{\leqslant}$ reduces to the usual order $\leqslant$ in $\mathbb{R}$ and $\hat{\tau}$ to τ.

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