Adjacencies in Words

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Based on two inversion formulas for enumerating words in the free monoid by adjacencies, we present a new approach to a class of permutation problems having Eulerian-type generating functions. We also show that a specialization of one of the inversion formulas gives Diekert's lifting to the free monoid of an inversion theorem due to Cartier and Foata.

1. INTRODUCTION

There are a number of powerful theories of inversion [9, 10, 13, 16] for dealing with combinatorial objects having generating functions of *Eulerian-type*

$$\frac{1}{1 + \sum_{n \geq 1}(-1)^n(1-t)^{n-1}c_n z^n}.$$

Using two such inversion formulas, we present new derivations of Stanley's [13] generating functions for generalized q-Eulerian and q-Euler polynomials on r-tuples of permutations. We further indicate how one of the inversion formulas gives Diekert's [5] lifting to the free monoid of an inversion theorem of Cartier and Foata [4]. The inversion theorems we use enumerate words in the free monoid by adjacencies.

An *alphabet* X is a non-empty set whose elements are referred to as *letters*. A finite sequence (possibly empty) $w = x_1 x_2 \cdots x_n$ of n letters is said to be a *word* of *length n*. The empty word will be denoted 1. The set of all words formed with letters in X along with the concatenation product is known as the *free monoid* generated by X and is denoted by X^*. We let X^+ be the set of words having positive length.

From X, we construct the *adjacency* alphabet $A = \{a_{xy} : (x, y) \in X \times X\}$. The *adjacency monomial* and the *sieve polynomial* for $w = x_1 x_2 \cdots x_n \in X^*$ of length $n \geq 2$ are defined respectively as $a(w) = a_{x_1 x_2} a_{x_2 x_3} \cdots a_{x_{n-1} x_n}$ and $\bar{a}(w) = (a_{x_1 x_2} - 1)(a_{x_2 x_3} - 1) \cdots (a_{x_{n-1} x_n} - 1)$. For $0 \leq n \leq 1$, we set $a(w) = \bar{a}(w) = 1$. In $Z[A] \ll X \gg$, the algebra of formal series of words in X^* with coefficients from the commutative ring of polynomials in A having integer coefficients, the following inversion formulas hold:

THEOREM 1. *According to adjacencies, the words in X^* are generated by*

$$\sum_{w \in X^*} a(w) w = \left(1 - \sum_{w \in X^+} \bar{a}(w) w\right)^{-1}. \tag{1}$$

THEOREM 2. *For non-empty subsets $U, V \subseteq X$, the words according to adjacencies in $U^*V = \{uv : u \in U^*, v \in V\}$ are generated by*

$$\sum_{w \in U^*V} a(w) w = \left(1 - \sum_{w \in U^+} \bar{a}(w) w\right)^{-1} \left(\sum_{w \in U^*V} \bar{a}(w) w\right). \tag{2}$$

Theorem 1 may be deduced from Stanley's [14, p. 266] synthesis of an inversion formula on clusters due to Goulden and Jackson [10, p. 131] with a related result of Zeilberger's [16] that enumerates words by mistakes. Theorem 2 bears comparison to (but is not equivalent to either) Viennot's [15] formula that counts heaps of pieces with restricted maximal elements and with a theorem of Goulden and Jackson [10, p. 238] for strings with distinguished final string. Proofs of Theorems 1 and 2 are deferred to Section 6. In passing, we mention that Hutchinson and Wilf [11] have given a closed formula for counting words by adjacencies.

The applications we give rely on the fact that setting $a_{xy} = 1$ eliminates all words containing *xy* as a factor from the right-hand sides of (1) and (2).

For instance, suppose that $X = \{x, y, z\}$. Set $a_{xx} = a$, $a_{xy} = b$, and the remaining $a_{ij} = 1$. Theorem 1 yields

$$\sum_{w \in \{x,y,z\}^*} a(w)w$$

$$= \frac{1}{1 - y - z - \sum_{n \geq 1}(a-1)^{n-1}x^n - \sum_{n \geq 1}(a-1)^{n-1}(b-1)x^n y}$$

$$= (1 + x - ax)(1 - ax - y - z + (a-b)xy + (a-1)xz)^{-1}.$$

2. A KEY BIJECTION

In applying Theorems 1 and 2 to the enumeration of permutations, we make repeated use of a bijection that associates a pair (σ, λ), where σ is a permutation and λ is a partition, to a finite sequence w of non-negative integers. Let $N = \{0, 1, 2, \ldots\}$ and N^n be the set of words of length n in N^*. The *rise set*, *rise number*, *inversion number*, and *norm* of $w = i_1 i_2 \cdots i_n \in N^n$ are respectively defined to be

$$\operatorname{Ris} w = \{k : 1 \leq k < n, i_k \leq i_{k+1}\}, \qquad \operatorname{ris} w = |\operatorname{Ris} w|,$$

$$\operatorname{inv} w = |\{(k, m) : 1 \leq k < m \leq n, i_k > i_m\}|, \qquad \|w\| = i_1 + \cdots + i_n.$$

The set of non-decreasing words in N^n (i.e., partitions with at most n parts) will be denoted by P_n. A permutation σ in the symmetric group S_n on $\{1, 2, \ldots, n\}$ will be viewed as the word $\sigma(1)\sigma(2)\cdots\sigma(n)$. The key bijection used in Sections 3 and 4 may be described as follows.

LEMMA 1. *For $n \geq 1$, there exists a bijection f_n: $S_n \times P_n \to N^n$ such that* $\operatorname{Ris} \sigma = \operatorname{Ris} w$ *and* $\operatorname{inv} \sigma + \|\lambda\| = \|w\|$ *whenever* $f_n(\sigma, \lambda) = w$.

Proof. First, for $\sigma \in S_n$ and $1 \leq k \leq n$, let c_k be the cardinality of the set $\{j : k + 1 \leq j \leq n, \sigma(k) > \sigma(j)\}$. The number c_k counts the inversions in σ due to $\sigma(k)$. The word $c = c_1 c_2 \cdots c_n$ is known as the *Lehmer code* [12] of σ. Note that $\operatorname{inv} \sigma = c_1 + \cdots + c_n = \|c\|$ and that $\operatorname{Ris} \sigma = \operatorname{Ris} c$. As an illustration, the Lehmer code of $\sigma = 5\,1\,3\,4\,2 \in S_5$ is $c = 4\,0\,1\,1\,0$. Also, $\operatorname{inv} \sigma = 6 = \|c\|$ and $\operatorname{Ris} \sigma = \{2, 3\} = \operatorname{Ris} c$.

Next, for $(\sigma, \lambda) = (\sigma(1)\sigma(2)\cdots\sigma(n), \lambda_1\lambda_2\cdots\lambda_n) \in S_n \times P_n$, define $f_n(\sigma, \lambda)$ to be the word $w = i_1 i_2 \cdots i_n \in N^n$, where $i_k = c_k + \lambda_{\sigma(k)}$ for $1 \leq k \leq n$. When $f_n(\sigma, \lambda) = w$, we clearly have the properties

$$k \in \operatorname{Ris} \sigma \text{ iff } c_k + \lambda_{\sigma(k)} \leq c_{k+1} + \lambda_{\sigma(k+1)} \text{ iff } k \in \operatorname{Ris} w,$$

$$\operatorname{inv} \sigma + \|\lambda\| = c_1 + \cdots + c_n + \lambda_1 + \cdots + \lambda_n = \|w\|.$$

For example, the map f_5 sends the pair $(\sigma, \lambda) = (5\,1\,3\,4\,2, 1\,1\,1\,1\,2) \in S_5 \times P_5$ to the word $w = 6\,1\,2\,2\,1 \in N^5$. Note that $\mathrm{Ris}\,\sigma = \{2, 3\} = \mathrm{Ris}\,w$ and that $\mathrm{inv}\,\sigma + \|\lambda\| = 6 + 6 = \|w\|$.

The inverse of f_n may be realized by applying the *insertion-shift* bijection presented in [6] to the word w to obtain (σ^{-1}, λ). The description of f_n given above was suggested by Foata (personal communication).

3. q-EULERIAN POLYNOMIALS

As the first application of Theorem 1, we derive a generating function for the sequence

$$A_n(t, q) = \sum_{\sigma \in S_n} t^{\mathrm{ris}\,\sigma} q^{\mathrm{inv}\,\sigma}.$$

The polynomial $A_n(t, 1)$ is the nth *Eulerian polynomial.* We further obtain the generating function for Stanley's [13] generalized q-Eulerian polynomials on r-tuples of permutations.

The first step in obtaining a generating function for the distribution of (ris, inv) on S_n is to appropriately define the adjacency monomial and sieve polynomial for the alphabet N. Toward this end, we set $a_{ij} = t$ if $i \le j$ and $a_{ij} = 1$ otherwise. For $w = i_1 i_2 \cdots i_n$, note that $a(w) = t^{\mathrm{ris}\,w}$ and that

$$\bar{a}(w) = \begin{cases} (t-1)^{n-1} & \text{if } i_1 \le i_2 \le \cdots \le i_n \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1 reduces to

$$\sum_{w \in N^*} t^{\mathrm{ris}\,w} w = \frac{1}{1 - \sum_{n \ge 1} (t-1)^{n-1} \sum_{0 \le i_1 \le i_2 \le \cdots \le i_n} i_1 i_2 \cdots i_n}. \tag{3}$$

Next, we assign the weight $W(i) = zq^i$ to each $i \in N$ and extend W to a multiplicative homomorphism on N^*. Let $(q; q)_0 = 1$ and, for $n \ge 1$, set $(q; q)_n = (1-q)(1-q^2) \cdots (1-q^n)$. Then, Lemma 1 and (3) justify the calculation

$$\sum_{n \ge 0} \frac{A_n(t, q) z^n}{(q; q)_n} = \sum_{n \ge 0} z^n \sum_{(\sigma, \lambda) \in S_n \times P_n} t^{\mathrm{ris}\,\sigma} q^{\mathrm{inv}\,\sigma + \|\lambda\|} = \sum_{w \in N^*} t^{\mathrm{ris}\,w} W(w)$$

$$= \frac{1}{1 - \sum_{n \ge 1} (t-1)^{n-1} z^n \sum_{0 \le i_1 \le \cdots \le i_n} q^{i_1 + \cdots + i_n}}$$

$$= \frac{1}{1 - \sum_{n \ge 1} (t-1)^{n-1} z^n/(q;q)_n}$$

$$= \frac{1-t}{e(-z(1-t),q) - t}, \tag{4}$$

where $e(z,q) = \sum_{n \ge 0} z^n/(q;q)_n$ is a well-known q-analog of e^z.

The *common rise number* of an r-tuple $(\sigma_1, \sigma_2, \ldots, \sigma_r)$ of permutations in $S_n^r = S_n \times \cdots \times S_n$ is defined to be $\operatorname{cris}(\sigma_1, \sigma_2, \ldots, \sigma_r) = |\bigcap_{j=1}^r \operatorname{Ris} \sigma_j|$. The argument in (4) is readily adapted to deriving Stanley's [13] generating function for the polynomials

$$A_{n,r}(t, q_1, q_2, \ldots, q_r) = \sum_{(\sigma_1, \sigma_2, \ldots, \sigma_r) \in S_n^r} t^{\operatorname{cris}(\sigma_1, \sigma_2, \ldots, \sigma_r)} q_1^{\operatorname{inv} \sigma_1} q_2^{\operatorname{inv} \sigma_2} \cdots q_r^{\operatorname{inv} \sigma_r}. \tag{5}$$

We sketch the details for $r = 2$ and then state the general result.

For letters $\mathbf{i} = (i_1, i_2)$ and $\mathbf{j} = (j_1, j_2)$ in the alphabet $N \times N$, we define

$$a_{\mathbf{ij}} = \begin{cases} t & \text{if } i_1 \le i_2 \text{ and } j_1 \le j_2 \\ 1 & \text{otherwise.} \end{cases}$$

For $(v, w) = (i_1 i_2 \cdots i_n, j_1 j_2 \cdots j_n) \in (N \times N)^*$, we have $a(v,w) = t^{\operatorname{cris}(v,w)}$, where $\operatorname{cris}(v,w) = |\operatorname{Ris} v \cap \operatorname{Ris} w|$. Also,

$$\bar{a}(v,w) = \begin{cases} (t-1)^{n-1} & \text{if } i_1 \le i_2 \le \cdots \le i_n \text{ and } j_1 \le j_2 \le \cdots \le j_n \\ 0 & \text{otherwise.} \end{cases}$$

The map of Lemma 1 applied component-wise to $(S_n \times P_n) \times (S_n \times P_n)$,

$$f_n \times f_n(\sigma_1, \lambda; \sigma_2, \mu) = (f_n(\sigma_1, \lambda), f_n(\sigma_2, \mu)) = (v, w),$$

is a bijection to $N^n \times N^n$ with $\operatorname{cris}(\sigma_1, \sigma_2) = \operatorname{cris}(v,w)$, $\operatorname{inv} \sigma_1 + \|\lambda\| = \|v\|$, and $\operatorname{inv} \sigma_2 + \|\mu\| = \|w\|$. Repeating (4) with appropriate modifications gives

$$\sum_{n \ge 0} \frac{A_{n,2}(t, q_1, q_2) z^n}{(q_1;q_1)_n (q_2;q_2)_n} = \frac{1-t}{J(z(1-t), q_1, q_2) - t},$$

where $J(z, q_1, q_2) = \sum_{n \ge 0} (-1)^n z^n/(q_1;q_1)_n (q_2;q_2)_n$ is a bibasic Bessel function. We note that replacing z by $z(1-q_1)(1-q_2)$ and letting $q_1, q_2 \to 1^-$ give the original result of Carlitz, Scoville, and Vaughan [3] that initiated the study of statistics on r-tuples of permutations.

If we let $\mathbf{q} = (q_1, q_2, \dots, q_r)$ and $(\mathbf{q};\mathbf{q})_{n,r} = (q_1;q_1)_n(q_2;q_2)_n \cdots (q_r;q_r)_n$, it follows in general that

THEOREM 3 (Stanley). *For $r \geq 1$, the sequence $\{A_{n,r}(t,\mathbf{q})\}_{n\geq 0}$ is generated by*

$$\sum_{n\geq 0} \frac{A_{n,r}(t,\mathbf{q})z^n}{(\mathbf{q};\mathbf{q})_{n,r}} = \frac{1-t}{F_r(z(1-t),\mathbf{q})-t},$$

where $F_r(z,\mathbf{q}) = \sum_{n\geq 0}(-1)^n z^n/(\mathbf{q};\mathbf{q})_{n,r}$.

Further consideration of statistics on r-tuples of permutations is given in [7, 8]. In [7], we extend the technique of Carlitz *et al.* [3] and present recurrence relationships that refine Theorem 3. We also discuss several related distributions. In [8], we obtain a stronger version of Theorem 3 by using Theorems 1 and 2 in combination with a map that carries more information than does the bijection of Lemma 1.

4. q-EULER POLYNOMIALS

André [1] shows that if E_n is the number of up–down alternating permutations in S_n (that is, $\sigma \in S_n$ such that $\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \cdots$), then

$$\sum_{n\geq 0} \frac{E_n z^n}{n!} = \frac{1+\sin z}{\cos z}. \tag{6}$$

The number E_n is known as the nth *Euler number.*

We now apply Theorems 1 and 2 to the more general problem of counting the set of *odd–up permutations*

$$\mathscr{O}_n = \{\sigma \in S_n : \sigma(1) < \sigma(2), \sigma(3) < \sigma(4), \dots\}$$

by inversion number and by the *number of even indexed rises*

$$\operatorname{ris}_2 \sigma = |\{k \in \operatorname{Ris}\sigma : k \text{ is even}\}|.$$

Toward this end, let

$$E_n(t,q) = \sum_{\sigma\in\mathscr{O}_n} t^{\operatorname{ris}_2\sigma} q^{\operatorname{inv}\sigma}.$$

Note that $E_n(0,1) = E_n$. The analysis is split into two cases: n odd and n even. We only present the odd case, which requires use of Theorem 2.

Let $U = \{\mathbf{i} = i_1 i_2 : i_1, i_2 \in N \text{ with } i_1 \leq i_2\}$, $V = N$, and X be the union of U and V. For $\mathbf{i} = i_1 i_2$, $\mathbf{j} = j_1 j_2 \in U$, and $k \in V$, we set

$$a_{\mathbf{ij}} = \begin{cases} t & \text{if } i_2 \leq j_1 \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad a_{\mathbf{i}k} = \begin{cases} t & \text{if } i_2 \leq k \\ 1 & \text{otherwise.} \end{cases}$$

Viewing a word $w \in U^*V$ as being in N^*, let $\operatorname{ris}_2 w$ denote the number of rises in w having even index. Theorem 2 implies that

$$\sum_{w \in U^*V} t^{\operatorname{ris}_2 w} w = \frac{\sum_{m \geq 0}(t-1)^m \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_{2m+1}} i_1 i_2 \cdots i_{2m+1}}{1 - \sum_{m \geq 1}(t-1)^{m-1} \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_{2m}} i_1 i_2 \cdots i_{2m}}. \quad (7)$$

Again set $W(i) = zq^i$ for $i \in N$ and multiplicatively extend W to N^*. Let $U^m V = \{uv : u \in U^* \text{ is of length } m, v \in V\}$. From Lemma 1, the bijection $f_{2m+1}: \mathscr{O}_{2m+1} \times P_{2m+1} \to U^m V$ satisfies the properties $\operatorname{ris}_2 \sigma = \operatorname{ris}_2 w$ and $\operatorname{inv} \sigma + \|\lambda\| = \|w\|$ whenever $f_{2m+1}(\sigma, \lambda) = w$. It then follows from (7) that

$$\begin{aligned}
\sum_{m \geq 0} \frac{E_{2m+1}(t,q) z^{2m+1}}{(q;q)_{2m+1}}
&= \sum_{m \geq 0} z^{2m+1} \sum_{(\sigma,\lambda) \in \mathscr{O}_{2m+1} \times P_{2m+1}} t^{\operatorname{ris}_2 \sigma} q^{\operatorname{inv} \sigma + \|\lambda\|} \\
&= \sum_{w \in U^*V} t^{\operatorname{ris}_2 w} W(w) \\
&= \frac{\sum_{m \geq 0}(t-1)^m z^{2m+1} \sum_{0 \leq i_1 \leq \cdots \leq i_{2m+1}} q^{i_1 + \cdots + i_{2m+1}}}{1 - \sum_{m \geq 1}(t-1)^{m-1} z^{2m} \sum_{0 \leq i_1 \leq \cdots \leq i_{2m}} q^{i_1 + \cdots + i_{2m}}} \\
&= \frac{\sum_{m \geq 0}(t-1)^m z^{2m+1}/(q;q)_{2m+1}}{1 - \sum_{m \geq 1}(t-1)^{m-1} z^{2m}/(q;q)_{2m}} \\
&= \frac{(1-t)^{1/2} \sin_q(z\sqrt{1-t})}{\cos_q(z\sqrt{1-t}) - t},
\end{aligned} \quad (8)$$

where $\cos_q z = \sum_{n \geq 0}(-1)^n z^{2n}/(q;q)_{2n}$ and $\sin_q z = \sum_{n \geq 0}(-1)^n z^{2n+1}/(q;q)_{2n+1}$. As the even case is essentially contained in the analysis above, we have

$$\sum_{n \geq 0} \frac{E_n(t,q) z^n}{(q;q)_n} = \frac{(1-t)\left(1 + (1-t)^{-1/2} \sin_q(z\sqrt{1-t})\right)}{\cos_q(z\sqrt{1-t}) - t}.$$

Setting $t = 0$, replacing z by $z(1-q)$, and letting $q \to 1^-$ give (6).

Generalization to r-tuples of *m-permutations* is relatively straightforward. Let $S_{n,m}$ denote the set of $\sigma \in S_n$ satisfying the property that $\sigma(k) > \sigma(k+1)$ implies k is a multiple of m. Note that $S_{n,2} = \mathscr{O}_n$. For $(\sigma_1, \sigma_2, \ldots, \sigma_r) \in S^r_{n,m}$, define $\operatorname{cris}_m(\sigma_1, \sigma_2, \ldots, \sigma_r)$ to be the number of $k \in \bigcap_{j=1}^{r} \operatorname{Ris} \sigma_j$ such that k is a multiple of m. Combining the ideas behind Theorem 3 and (8) gives

THEOREM 4. *For $m, r \geq 1$, the sequence of polynomials*

$$E_{n,m,r}(t, \mathbf{q}) = \sum_{(\sigma_1, \sigma_2, \ldots, \sigma_r) \in S^r_{n,m}} t^{\operatorname{cris}_m(\sigma_1, \sigma_2, \ldots, \sigma_r)} q_1^{\operatorname{inv} \sigma_1} q_2^{\operatorname{inv} \sigma_2} \cdots q_r^{\operatorname{inv} \sigma_r}$$

is generated by

$$\sum_{n \geq 0} \frac{E_{n,m,r}(t, \mathbf{q}) z^n}{(\mathbf{q}; \mathbf{q})_{n,r}} = \frac{(1-t)\left(1 + \sum_{\rho=1}^{m-1} (1-t)^{-\rho/m} \Phi_{m,\rho,r}\left(z \sqrt[m]{1-t}, \mathbf{q}\right)\right)}{\Phi_{m,0,r}\left(z \sqrt[m]{1-t}, \mathbf{q}\right) - t},$$

where $\Phi_{m,\rho,r}(z, \mathbf{q}) = \sum_{\nu \geq 0} (-1)^\nu z^{\nu m + \rho} / (\mathbf{q}; \mathbf{q})_{\nu m + \rho, r}$.

Theorem 4 is essentially due to Stanley [13]. Note that $E_{n,1,r}(t, \mathbf{q})$ is equal to the generalized q-Eulerian polynomial defined in (5). Thus, taking $m = 1$ in Theorem 4 gives Theorem 3 as a corollary. We further remark that $\Phi_{m,\rho,1}(z, q)$ is a q-Olivier function. When $r = 1$ and $t = s = 0$, replacing z by $z(1-q)$ and letting $q \to 1^-$ give the initial result of Carlitz [2] on m-permutations.

5. FROM THE TRACE TO THE FREE MONOID

As the final application, we use Theorem 1 to obtain Diekert's [5, pp. 96–99] lifting to the free monoid of an inversion formula due to Cartier and Foata [4] from a partially commutative monoid (or trace monoid) in which the defining binary relation admits a transitive orientation.

Let θ be an irreflexive symmetric binary relation on X. Define $\equiv_\theta$ to be the binary relation (*induced by* θ) on X^* consisting of the set of pairs (w, v) of words such that there is a sequence $w = w_0, w_1, \ldots, w_m = v$, where each w_i is obtained by transposing a pair of letters in w_{i-1} that are consecutive and contained in θ. For instance, if $X = \{x, y, z\}$ and $\theta = \{(x, y), (y, x)\}$, then the sequence $zyyx, zyxy, zxyy$ implies that $zyyx \equiv_\theta zxyy$.

Clearly, $\equiv_\theta$ is an equivalence relation on X^*. The quotient of X^* by $\equiv_\theta$ gives the *partially commutative monoid* induced by θ and is denoted by $M(X, \theta)$. The equivalence class $\hat{w}$ of $w \in X^*$ is referred to as the *trace* of w.

A word $w = x_1 x_2 \cdots x_n \in X^*$ is said to be a *basic monomial* if $x_i \theta x_j$ for all $i \neq j$. A trace $\hat{w}$ is said to be *θ-trivial* if any one of its representatives is a basic monomial. If one lets $\mathcal{T}^+(X, \theta)$ be the set of θ-trivial traces, the inversion formula of Cartier and Foata reads as follows.

THEOREM 5 (Cartier and Foata). *For θ an irreflexive symmetric binary relation on X, the traces in $M(X, \theta)$ are generated by*

$$\sum_{\hat{w} \in M(X,\theta)} \hat{w} = \frac{1}{1 + \sum_{\hat{t} \in \mathcal{T}^+(X,\theta)} (-1)^{l(\hat{t})} \hat{t}},$$

where $l(\hat{t})$ denotes the length of any representative of $\hat{t}$.

A natural question to ask is whether $\hat{w}$ and $\hat{t}$ can be replaced by some canonical representatives so that Theorem 5 remains true as a formula in the free monoid X^*. As resolved by Diekert [5], such canonical representatives exist if and only if θ admits a transitive orientation.

To be precise, a subset $\vec{\theta}$ of θ is said to be an *orientation* of θ if θ is a disjoint union of $\vec{\theta}$ and $\{(x, y) : (y, x) \in \vec{\theta}\}$. The set of $t = t_1 t_2 \cdots t_n \in X^*$ satisfying $t_1 \vec{\theta} t_2 \vec{\theta} \cdots \vec{\theta} t_n$ is denoted by $T^+(X, \vec{\theta})$. Note that $T^+(X, \vec{\theta})$ is a set of representatives for the θ-trivial traces $\mathcal{T}^+(X, \theta)$ whenever $\vec{\theta}$ is transitive. A word $w = x_1 x_2 \cdots x_n \in X^*$ is said to have a *$\vec{\theta}$-adjacency* in position k if $x_k \vec{\theta} x_{k+1}$. We denote the number of $\vec{\theta}$-adjacencies of w by $\vec{\theta}\mathrm{adj}\, w$. Although Diekert did not explicitly introduce the notion of a $\vec{\theta}$-adjacency, his lifting theorem may be paraphrased as follows.

THEOREM 6 (Diekert). *Let θ be an irreflexive symmetric binary relation on X and let $\vec{\theta}$ be an orientation of θ. Then, $\vec{\theta}$ is transitive if and only if there exists a complete set W of representatives for the traces of $M(X, \theta)$ such that*

$$\sum_{w \in W} w = \frac{1}{1 + \sum_{t \in T^+(X, \vec{\theta})} (-1)^{l(t)} t}.$$

Moreover, $W = \{w \in X^* : \vec{\theta}\mathrm{adj}\, w = 0\}$.

To see how Theorem 1 intervenes in the matter, suppose that $\vec{\theta}$ is an orientation of θ (not necessarily transitive for now). If for $x, y \in X$ we set $a_{xy} = a$ when $x \vec{\theta} y$ and $a_{xy} = 1$ otherwise, then Theorem 1 reduces to

$$\sum_{w \in X^*} a^{\vec{\theta}\mathrm{adj}\, w} w = \frac{1}{1 + \sum_{t \in T^+(X, \vec{\theta})} (-1)^n (1 - a)^{l(t)-1} t}. \tag{9}$$

When $\vec{\theta}$ is transitive, setting $a = 0$ in (9) gives the lifting of Theorem 5 to the free monoid as stated in Diekert's theorem. We close this section with two examples.

TRANSITIVE EXAMPLE. Let $X = \{x, y, z\}$ with $\theta = \{(x, y), (y, x), (x, z), (z, x)\}$. Among other possibilities, $\vec{\theta} = \{(y, x), (z, x)\}$ is a transitive orientation of θ. The $\vec{\theta}$-adjacencies of a word correspond to factors yx and zx. Note that $T^+(X, \vec{\theta}) = \{x, y, z, yx, zx\}$ is a complete set of representatives for the θ-trivial traces $\mathcal{T}^+(X, \theta)$. Also, the only word in

$$\widehat{xzyxy} = \left\{\begin{matrix} xzxyy, & xzyxy, & xzyyx, & xxzyy, & zxxyy, \\ zxyxy, & zxyyx, & zyxxy, & zyxyx, & zyyxx \end{matrix}\right\}$$

having no $\vec{\theta}$-adjacencies is $xxzyy$. From (9), we have

$$\sum_{w \in \{x, y, z\}^*} a^{\vec{\theta}\mathrm{adj}\, w} w = \frac{1}{1 - (x + y + z) + (1 - a)(yx + zx)}.$$

Setting $a = 0$ gives an identity that can be viewed as having been lifted from the trace monoid as in Theorem 6.

NON-TRANSITIVE EXAMPLE. Let X and θ be as in the previous example. The orientation $\vec{\theta} = \{(y, x), (x, z)\}$ is not transitive. Observe that the word yxz in $T^+(X, \vec{\theta}) = \{x, y, z, yx, xz, yxz\}$ is not a θ-trivial trace. Also, $\widehat{yxz} = \{yxz, xyz, yzx\}$ contains two words having no $\vec{\theta}$-adjacencies. Nevertheless, (9) implies

$$\sum_{w \in \{x, y, z\}^*} a^{\vec{\theta}\mathrm{adj}\, w} w = \frac{1}{1 - (x + y + z) + (1 - a)(yx + xz) - (1 - a)^2 yxz}.$$

6. PROOFS FOR THEOREMS 1 AND 2

To establish Theorem 1, we begin by noting that (1) is equivalent to

$$\sum_{w \in X^*} a(w) w - \sum_{w \in X^+} \Big(\sum_{w = uv,\, v \neq 1} a(u) \bar{a}(v) \Big) w = 1.$$

Thus, by equating coefficients, it suffices to show that

$$a(w) = \sum_{w = uv,\, v \neq 1} a(u) \bar{a}(v) \tag{10}$$

for all $w \in X^+$. We proceed by induction on the length $l(w)$ of w. For $l(w) = 1$, (10) is trivially true. Suppose $l(w) \geq 2$. Then w factorizes as $w = w_1 xy$, where $w_1 \in X^*$ and $x, y \in X$. Assuming (10) holds for words of length smaller than w, it follows that

$$\begin{aligned} a(w) &= a_{xy} a(w_1 x) = a(w_1 x) + (a_{xy} - 1) a(w_1 x) \\ &= a(w_1 x)\bar{a}(y) + \bar{a}(xy) \sum_{w_1 x = u v_1 x} a(u)\bar{a}(v_1 x) \\ &= a(w_1 x)\bar{a}(y) + \sum_{w_1 xy = u v_1 xy} a(u)\bar{a}(v_1 xy) \\ &= \sum_{w = uv,\, v \neq 1} a(u)\bar{a}(v), \end{aligned}$$

and the proof is complete.

We use an alternate approach to prove Theorem 2. Let W denote the left-hand side of (2) and define

$$W_{n+1} = \sum_w a(w) w,$$

where the sum is over words $w = x_1 x_2 \cdots x_{n+1} \in U^* V$ of length $(n+1)$. Note that $W = \sum_{n \geq 0} W_{n+1}$. Since $\bar{a}(x_1) = 1$ and $\bar{a}(x_1 x_2) = a_{x_1 x_2} - 1$, it is a triviality that

$$W_{n+1} = \sum_w \bar{a}(x_1) a(x_2, \ldots, x_{n+1}) w + \sum_w \bar{a}(x_1 x_2) a(x_2 \cdots x_{n+1}) w$$

for $n \geq 1$. Similarly, the second sum on the above right may be split as

$$\sum_w \bar{a}(x_1 x_2) a(x_3 \cdots x_{n+1}) w + \sum_w \bar{a}(x_1 x_2 x_3) a(x_3 \cdots x_{n+1}) w$$

so that

$$\begin{aligned} W_{n+1} = &\sum_{k=1}^{2} \sum_w \bar{a}(x_1 \cdots x_k) a(x_{k+1} \cdots x_{n+1}) w \\ &+ \sum_w \bar{a}(x_1 x_2 x_3) a(x_3 \cdots x_{n+1}) w. \end{aligned}$$

Iterating the above argument and then factoring give

$$W_{n+1} = \sum_{k=1}^{n} \sum_{w} \bar{a}(x_1, \dots, x_k) a(x_{k+1} \cdots x_{n+1}) w + \sum_{w} \bar{a}(w) w$$
$$= \sum_{k=1}^{n} \left(\sum_{u} \bar{a}(u) u \right) W_{n+1-k} + \sum_{w} \bar{a}(w) w,$$

where the sum to the immediate left of W_{n+1-k} is over words $u = x_1 \cdots x_k \in U^+$ of length k. As the above recurrence relationship for W_{n+1} is valid for $n \geq 0$, it follows that

$$W = \left(\sum_{w \in U^+} \bar{a}(w) w \right) W + \sum_{w \in U^* V} \bar{a}(w) w,$$

which implies Theorem 2.

Either of the preceding arguments may be easily modified to give an inversion formula for words in X^* that end in a fixed word v. Without giving the details, we have

THEOREM 7. *According to adjacencies, words ending in a word* $v = b_1 b_2 \cdots b_m \in X^*$ *of length* m *are generated by*

$$\sum_{w \in X^*} a(wv) wv = \left(1 - \sum_{w \in X^+} \bar{a}(w) w \right)^{-1} \left(a(v) \sum_{w \in X^*} \bar{a}(wb_1) wv \right).$$

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