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ORBIFOLDS WITH LOWER RICCI CURVATURE BOUNDS

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ABSTRACT. We show that the first betti number $b_1(O) = \dim H_1(O, \mathbb{R})$ of a compact Riemannian orbifold O with Ricci curvature $\mathrm{Ric}(O) \geq -(n-1)k$ and diameter $\mathrm{diam}(O) \leq D$ is bounded above by a constant $c(n, kD^2) \geq 0$, depending only on dimension, curvature and diameter. In the case when the orbifold has nonnegative Ricci curvature, we show that the $b_1(O)$ is bounded above by the dimension $\dim O$, and that if, in addition, $b_1(O) = \dim O$, then O is a flat torus T^n.

INTRODUCTION

In 1946, S. Bochner [Bo] proved that if M is a compact n–dimensional Riemannian manifold with nonnegative Ricci curvature, then the first betti number $b_1(M) = \dim H_1(M, \mathbb{R}) \leq \dim M$. Moreover, the universal cover $\tilde{M}$ of M is the product $N \times \mathbb{R}^{b_1(M)}$. The question naturally arises as to which manifolds admit maximal first betti number. The answer is that M must be an n–dimensional flat torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$. The original proof is based on the study of harmonic 1–forms and uses the Weitzenböck formula $\Delta\alpha = D^*D\alpha + \mathrm{Ric}\ \alpha$ which expresses the Hodge Laplacian of a 1–form α in terms of its connection Laplacian and Ricci curvature. See [Be]. Later, Gromov [GLP] and Gallot [G] showed that there is a constant $c(n, kD^2) \geq 0$ such that the class of n–dimensional compact Riemannian manifolds with Ricci curvature $\mathrm{Ric}(M) \geq -(n-1)k$ and diameter $\mathrm{diam}(M) \leq D$ have first betti number $b_1(M) \leq c(n, kD^2)$. Moreover, $\lim_{kD^2 \to 0^+} c(n, kD^2) = c(n, 0) = \dim(M)$.

In this paper we generalize these results to the class of Riemannian orbifolds. Specifically, we prove the following

Theorem 1. *Let O be a compact n–dimensional Riemannian orbifold whose Ricci curvature $\mathrm{Ric}(O) \geq -(n-1)k$ and diameter $\mathrm{diam}(O) \leq D$. Then there is a constant $c(n, kD^2) \geq 0$ depending only on dimension, curvature and diameter such that*

(i) $b_1(O) \leq c(n, kD^2)$.
(ii) $c(n, kD^2) = 0$ *for* $k < 0$.
(iii) $\lim_{kD^2 \to 0^+} c(n, kD^2) = c(n, 0) = n$.

Remark 2. For orbifolds there is a notion of the orbifold fundamental group $\pi_1^{\mathrm{orb}}(O)$, and thus one can define a first orbifold betti number as the rank of the abelianization of $\pi_1^{\mathrm{orb}}(O)$. It turns out, however, that this orbifold betti number agrees with

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the standard one (Proposition 9), and hence there is no loss in generality in stating our theorems in terms of the standard betti number $b_1(O)$. These matters will be discussed more thoroughly in the next section.

We have the following immediate corollary of Theorem 1:

Corollary 3. *Let O be a compact n–dimensional orbifold whose Ricci curvature* $\mathrm{Ric}(O) \geq -(n-1)k$ *and diameter* $\mathrm{diam}(O) \leq D$. *There exists an $\varepsilon = \varepsilon(n) > 0$, such that if $kD^2 < \varepsilon$, then $b_1(O) \leq n$.*

As a consequence of Theorem 1 and Proposition 9 we will be able to generalize Bochner's Theorem to Riemannian orbifolds. Note that Theorem 4 (i) and (ii) are immediate corollaries of Theorem 1.

Theorem 4. *Let O be a compact n–dimensional Riemannian orbifold.*

(i) *If the Ricci curvature is strictly positive, then $b_1(O) = 0$.*
(ii) *If the Ricci curvature is nonnegative, then $b_1(O) \leq n$.*
(iii) *If the Ricci curvature is nonnegative and if $b_1(O) = n$, then O is a good orbifold. In fact, O is isometric to an n–dimensional flat torus.*

Remark 5. One might expect that in the orbifold situation the equality case would only yield a flat orbifold of the form T^n/G where G is some finite group acting isometrically (with possible fixed points) on T^n. The fact is, however, that the maximality of the first betti number rules out the possibility of singularities.

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Orbifold preliminaries

The basic reference for orbifolds is [T]. The book [R, Chapter 13] is also a good reference for the notions that appear in this paper. The essential facts we need are that $\pi_1^{\mathrm{orb}}(O)$ is the group of (orbifold) homotopy classes of loops in O, and that associated to every orbifold O there is a connected universal covering orbifold $\tilde{O}$ with $\pi_1^{\mathrm{orb}}(\tilde{O}) = 0$ such that the elements of $\pi_1^{\mathrm{orb}}(O)$ are the deck transformations of $\tilde{O}$. The group $\pi_1^{\mathrm{orb}}(O)$ acts (properly) discontinuously and isometrically (but not necessarily freely) on $\tilde{O}$ and as orbifolds $O = \tilde{O}/\pi_1^{\mathrm{orb}}(O)$.

Definition 6. Let $\pi_1^{\mathrm{orb}}(O)$ be the orbifold fundamental group. We define the *first orbifold homology group* $H_1^{\mathrm{orb}}(O,\mathbb{Z})$ to be the abelianization of $\pi_1^{\mathrm{orb}}(O)$ and the *first orbifold betti number* to be $\dim\left(H_1^{\mathrm{orb}}(O,\mathbb{Z}) \otimes \mathbb{R}\right)$.

Example 7. Consider the standard 2–sphere $S^2 \subset \mathbb{R}^3$. Define a $\mathbb{Z}_p$–action on S^2 by rotation around the z–axis by an angle of $2\pi/p$. The quotient space $O = S^2/\mathbb{Z}_p$ is commonly referred to as a $\mathbb{Z}_p$–football. In this case, $\pi_1^{\mathrm{orb}}(O) = H_1^{\mathrm{orb}}(O,\mathbb{Z}) \cong \mathbb{Z}_p$, but $H_1(O,\mathbb{Z}) = 0$. Note, however, that $b_1^{\mathrm{orb}}(O) = b_1(O) = 0$.

Remark 8. Although, as Example 7 illustrates, $H_1^{\mathrm{orb}}(O,\mathbb{Z})$ is in general not equal to $H_1(O,\mathbb{Z})$, it is always true that $b_1^{\mathrm{orb}}(O) = b_1(O)$. We prove this in Proposition 9.

Note that by the construction of the orbifold fundamental group (for example, see [HD]) there is an epimorphism $\varphi : \pi_1^{\mathrm{orb}}(O) \twoheadrightarrow \pi_1(O)$ from the orbifold fundamental group to the (standard) fundamental group. This gives rise to the following

commutative diagram:

$$\begin{array}{ccccc} \pi_1^{\mathrm{orb}}(O) & \xrightarrow{\mathrm{Ab}} & H_1^{\mathrm{orb}}(O,\mathbb{Z}) & \xrightarrow{\otimes\mathbb{R}} & H_1^{\mathrm{orb}}(O,\mathbb{Z})\otimes\mathbb{R} = H_1^{\mathrm{orb}}(O,\mathbb{R}) \\ \varphi\downarrow & & \tilde{\varphi}\downarrow & & \hat{\varphi}\downarrow \\ \pi_1(O) & \xrightarrow{\mathrm{Ab}} & H_1(O,\mathbb{Z}) & \xrightarrow{\otimes\mathbb{R}} & H_1(O,\mathbb{Z})\otimes\mathbb{R} = H_1(O,\mathbb{R}) \end{array}$$

where $\mathrm{Ab} : G \to G/[G,G]$ is the abelianization of a group G, and $\tilde{\varphi} = \mathrm{Ab}\circ\varphi\circ\mathrm{Ab}^{-1}$. It's easy to see that $\tilde{\varphi}$ is well–defined and that all arrows are epimorphisms.

We thus conclude the following

Proposition 9. *Let O be a Riemannian orbifold. Then the first orbifold homology group with real coefficients $H_1^{\mathrm{orb}}(O,\mathbb{R}) = H_1(O,\mathbb{R})$, and hence the first orbifold betti number $b_1^{\mathrm{orb}}(O) = b_1(O)$.*

Proof. Let $\hat{\alpha} \in \operatorname{Ker}\hat{\varphi}$. If $\hat{\alpha} \neq 0$, then any preimage $\tilde{\alpha} \in H_1^{\mathrm{orb}}(O,\mathbb{Z})$ has infinite order in $H_1^{\mathrm{orb}}(O,\mathbb{Z})$. Similarly, let $\alpha \in \pi_1^{\mathrm{orb}}(O)$ be any preimage of $\tilde{\alpha}$. Commutativity of the diagram implies that $\tilde{\varphi}(\tilde{\alpha})$ has finite order. Then $\tilde{\varphi}(\tilde{\alpha}^m) = 0$, which implies that $\varphi(\alpha^m) \in [\pi_1(O),\pi_1(O)]$, and thus $\alpha^m \in [\pi_1^{\mathrm{orb}}(O),\pi_1^{\mathrm{orb}}(O)]$. But then $[\mathrm{Ab}(\alpha)]^m = 0$, which implies that $\tilde{\alpha}$ has finite order, which is a contradiction. Thus, $\hat{\alpha} = 0$ and $\hat{\varphi}$ is an isomorphism. This completes the proof. □

Growth of the orbifold fundamental group

Let G be a finitely generated group and let $S = \{g_1,\dots,g_k\}$ be a system of generators. Each element g of G can be represented by a word $g_{i_1}^{p_1} g_{i_2}^{p_2}\cdots g_{i_\ell}^{p_\ell}$ and the number $|p_1| + \cdots + |p_\ell|$ is called the length of the word. The norm $\|g\|_{\mathrm{word}}$ (relative to S) is defined as the minimal length of words representing g. For any $t \in \mathbb{Z}^+$, the number of elements of G which can be represented by words whose length is $\leq t$ will be denoted $\varphi_S(t)$; that is, $\varphi_S(t)$ is the number of elements with $\|g\|_{\mathrm{word}} \leq t$. A group G is said to have *polynomial growth of degree* $\leq n$ if for some system of generators S there is a constant $c > 0$ such that $\varphi_S(t) \leq c\cdot t^n$. It is not hard to see that this definition is independent of the choice of generators S. See [Z].

J. Milnor [M] observed that if M is an n–dimensional compact Riemannian manifold with nonnegative Ricci curvature, then the fundamental group $\pi_1(M)$ has polynomial growth of degree $\leq n$. We should point out, however, that A. Schwarz [S] was the first to discover the relationship between the growth of the fundamental group and the volume growth of balls in the universal cover $\tilde{M}$. Essentially, the same argument yields

Proposition 10. *Let O be a compact n–dimensional Riemannian orbifold with nonnegative Ricci curvature. Then the orbifold fundamental group $\pi_1^{\mathrm{orb}}(O)$ has polynomial growth of degree $\leq n$.*

Proof. $\pi_1^{\mathrm{orb}}(O)$ acts isometrically and (properly) discontinuously on the universal orbifold cover $\tilde{O}$. Let $p \in \tilde{O}$ be a nonsingular point of $\tilde{O}$. There exists $r > 0$ such that for $g \in \pi_1^{\mathrm{orb}}(O)$ the balls $B(g(p),r)$ are pairwise disjoint. Let $S = \{g_1,\dots,g_k\}$ be a system of generators of $\pi_1^{\mathrm{orb}}(O)$. Let

$$L = \max_i d(p, g_i(p)).$$

If $g \in \pi_1^{\mathrm{orb}}(O)$ can be represented as a word of length $\leq t$ with respect to the g_i's, then $g = g_{i_1}^{p_1} \cdots g_{i_\ell}^{p_\ell}$, $1 \leq \ell \leq t$, with $p_i = \pm 1$. Note that

$$\begin{aligned} d(p, g(p)) &= d(p, g_{i_1}^{p_1} \cdots g_{i_\ell}^{p_\ell}(p)) \\ &\leq d(p, g_{i_1}^{p_1}(p)) + d(g_{i_1}^{p_1}(p), g_{i_1}^{p_1} \cdots g_{i_\ell}^{p_\ell}(p)) \\ &\leq L + d(p, g_{i_2}^{p_2} \cdots g_{i_\ell}^{p_\ell}(p)) \quad \text{since the } g_i\text{'s are isometries} \\ &\leq \cdots \leq \underbrace{L + L + \cdots + L}_{t \text{ times}} = L \cdot t. \end{aligned}$$

Taking all such g's, we obtain $\varphi_S(t)$ disjoint balls $B(g(p), r)$ such that $B(g(p), r) \subset B(p, Lt + r)$. Thus,

$$\varphi_S(t) \cdot \operatorname{Vol} B(p, r) \leq \operatorname{Vol} B(p, Lt + r)$$

which implies that

$$\varphi_S(t) \leq \frac{\operatorname{Vol} B(p, Lt + r)}{\operatorname{Vol} B(p, r)} \leq \frac{(Lt + r)^n}{r^n}.$$

The last inequality follows from the relative volume comparison theorem for orbifolds [B]. Hence we conclude that $\pi_1^{\mathrm{orb}}(O)$ has polynomial growth of degree $\leq n$. This completes the proof. □

Rigidity of orbifolds of maximal first Betti number

In this section we prove Theorem 4 (iii). We now assume that O has maximal first betti number $b_1(O) = n$. By Proposition 10, the orbifold fundamental group $\pi_1^{\mathrm{orb}}(O)$ has polynomial growth exactly $= n$.

By the orbifold generalization of the Cheeger–Gromoll splitting theorem [BZ] we know that the universal orbifold cover $\tilde{O}$ splits as an isometric product $N \times \mathbb{R}^k$, where N is a *compact* orbifold with nonnegative Ricci curvature. Also, the isometry group splits, namely, $\operatorname{Isom}(\tilde{O}) = \operatorname{Isom}(N) \times \operatorname{Isom}(\mathbb{R}^k)$. Now $\pi_1^{\mathrm{orb}}(O) \subset \operatorname{Isom}(\tilde{O})$ acts properly discontinuously on $\tilde{O}$. As a result, we have the exact sequence

$$1 \longrightarrow F \longrightarrow \pi_1^{\mathrm{orb}}(O) \xrightarrow{\mathrm{pr}_2} C \longrightarrow 1$$

where F is finite and C is a crystallographic group acting on $\mathbb{R}^k$, and where pr_2 denotes the homomorphism $\mathrm{pr}_2 : \pi_1^{\mathrm{orb}}(O) \to \operatorname{Isom}(\mathbb{R}^k)$ with kernel F. See [BZ]. It then follows that C must have polynomial growth of degree $= n$. By applying (standard) relative volume comparison it follows that $k = n$; that is, C must act on $\mathbb{R}^n$. This implies that $\tilde{O} = \mathbb{R}^n$ and thus O is the (good) flat orbifold $\tilde{O}/C$, and $C = \pi_1^{\mathrm{orb}}(O)$.

To finish the proof we use the Bieberbach theorems on crystallographic groups. Let C^* be the normal subgroup of pure translations of C. The Bieberbach theorems [W, Theorem 3.2.9] imply that any crystallographic group C satisfies the exact sequence

$$1 \longrightarrow (\mathbb{Z}^n \cong C^*) \longrightarrow C = \pi_1^{\mathrm{orb}}(O) \longrightarrow H \longrightarrow 1$$

where $H \cong C/C^*$ is a finite group.

Recall the following argument in the case that C acts freely (that is, C is a Bieberbach group). In this case, O is a compact flat manifold and $C = \pi_1(O)$. $\pi_1(O)$ is torsion free by [W, Theorem 3.1.3]. We claim in fact that $\pi_1(O)$ is abelian. Hence, $\pi_1(O) = \mathbb{Z}^n$, and by the Bieberbach theorems [W, Theorem 3.3.1], O must be a flat torus.

To show that $\pi_1(O)$ is abelian we proceed as follows. Suppose not. Then by abelianizing $\pi_1(O)$ some nontrivial element, say a, gets sent to 1. Now the exact sequence implies that any element $a \in \pi_1(O)$ satisfies $a^k \in \mathbb{Z}^n$ where we regard $\mathbb{Z}^n$ as a subgroup of $\pi_1(O)$. Since $\pi_1(O)$ is torsion free, $a^k \neq 1$, and a^k gets sent to 1 under abelianization. Thus, $\dim H_1(O, \mathbb{R}) < n$, contradicting the assumption of maximality of the first betti number of O. To see the last statement, just observe that since any element of $\pi_1(O)$ has a power which is in $\mathbb{Z}^n$, after abelianization and tensoring with $\mathbb{R}$, one finds that any element in $H_1(O, \mathbb{R})$ is a word in the (images) of the generators of $\mathbb{Z}^n$. These images obviously generate a group of rank $\leq n$.

Now consider the case when C is only assumed to be crystallographic. That is, C may have fixed points and thus $\mathbb{R}^n/C$ is an orbifold with possible singularities. Let T^n denote the n–dimensional torus $\mathbb{R}^n/C^*$. Then O is the orbifold $T^n/(C/C^*)$. Let φ denote the quotient map $\varphi : T^n \to O$. It is not hard to see [Br, Corollary II.6.5] that the map $\varphi_* : H_1(T^n, \mathbb{R}) \to H_1(O, \mathbb{R})$ is onto. By hypothesis $b_1(O) = \dim H_1(O, \mathbb{R}) = n$ which implies that φ_* is an isomorphism. We show that if C contains an element which has a fixed point, then the map φ_* has nontrivial kernel. giving a contradiction.

Let $\gamma \in C$ be an element with fixed point p. Then $\gamma = (A, \tau)$ with $A \in \mathbb{O}(n)$, $A \neq I$ and $\tau \in \mathbb{R}^n$. γ acts on $\mathbb{R}^n$ by $\gamma(x) = Ax + \tau$. Choose a minimal set of generators $\{a_1, \ldots, a_n\}$ for C^*. Then by [W, Theorem 3.2.1], relative to these generators A has integral entries. That is, A preserves the lattice C^*. Translate the origin of $\mathbb{R}^n$ to p. Note that γ preserves the translated lattice $p + C^*$. Let $\alpha_i(t)$ denote the line segment $p + ta_i$, $t \in [0, 1]$. Since $A \neq I$, there exists a segment α_i such that $\gamma(\alpha_i) \neq \alpha_i$. That is,

$$\gamma(\alpha_i) = p + tw(a_1, \ldots, a_n) = \beta(t) \neq \alpha_i(t)$$

where $w = A(a_i)$ denotes a word in the generators a_j (with integer coefficients). Regarding γ as an element of C/C^*, we can think of γ as acting on the torus T^n and the α_j as generators of $H_1(T^n, \mathbb{R})$. But then $\varphi_*(\alpha_i - \beta) = 0$, which shows that φ_* has nontrivial kernel, and we have our desired contradiction.

THE FIRST BETTI NUMBER ESTIMATES

We now prove Theorem 1. For the proof of part (ii), we use the fact that strictly positive Ricci curvature implies that O is compact. Since the universal orbifold cover satisfies the same curvature restrictions as O, we conclude that the universal orbifold cover $\tilde{O}$ is compact and thus $\pi_1^{\text{orb}}(O)$ is finite, and $b_1(O) = 0$. To see that strictly positive Ricci curvature implies compactness of O, assume for simplicity that $\text{Ric}(O) \geq (n-1)$. We show that $\text{diam}(O) \leq \pi$. Suppose not and choose points p', q' with $d(p', q') > \pi + \varepsilon$. We can choose nonsingular points p, q such that $d(p, p') < \varepsilon/3$ and $d(q, q') < \varepsilon/3$, since the singular set is nowhere dense. Then $d(p, q) > \pi + \varepsilon/3$. Let γ be a minimal unit speed geodesic joining p to q. Then γ does not intersect the singular set. See [B]. Thus $\gamma(\pi)$ is inside the cut locus of p and hence the distance function from p, $d(p, \cdot)$ is smooth at $\gamma(\pi)$. By the Laplacian comparison theorem for orbifolds [BZ] we have that $\Delta d(p, \cdot) \leq (n-1)\cot d(p, \cdot)$. Letting $d(p, \cdot) \to \pi$ from the left implies that $\Delta d(p, \cdot) \leq -\infty$, which contradicts smoothness of $d(p, \cdot)$ at $\gamma(\pi)$.

The proof of part (i) follows closely the proof of the standard case given in [Z]. The first observation to make is that it is sufficient to show that there is a

finite (orbifold) cover $\hat{O} \to O$ such that $\pi_1^{\mathrm{orb}}(\hat{O})$ has at most $c(n, kD^2)$ generators. To see this, suppose that $\{h_1, \dots, h_k\}$ generate $\pi_1^{\mathrm{orb}}(\hat{O})$. Then the index $[\pi_1^{\mathrm{orb}}(O) : \pi_1^{\mathrm{orb}}(\hat{O})] = m < \infty$, so there are $\{g_1, \dots, g_m\}$ such that $g_i^m \in \pi_1^{\mathrm{orb}}(\hat{O})$ and $\pi_1^{\mathrm{orb}}(O) = \bigcup_{i=1}^m g_i \pi_1^{\mathrm{orb}}(\hat{O})$. As before, let $\mathrm{Ab} : G \to G/[G,G]$ be the abelianization of a group G, and consider the composition:

$$\pi_1^{\mathrm{orb}}(O) \xrightarrow{\mathrm{Ab}} H_1^{\mathrm{orb}}(O, \mathbb{Z}) \xrightarrow{\otimes \mathbb{R}} H_1^{\mathrm{orb}}(O, \mathbb{R})$$

and let $f = (\otimes \mathbb{R}) \circ \mathrm{Ab}$ denote this composition. Since $\{h_1, \dots, h_k, g_1, \dots, g_m\}$ generates $\pi_1^{\mathrm{orb}}(O)$, the set $\{\mathrm{Ab}(h_i), \mathrm{Ab}(g_j)\}$ generates $H_1^{\mathrm{orb}}(O, \mathbb{Z})$. Now note that $m \cdot f(g_j) = f(g_j^m) = f(h)$ for some $h \in \pi_1^{\mathrm{orb}}(\hat{O})$. Thus $\{f(h_i)\}$ generate $H_1^{\mathrm{orb}}(O, \mathbb{R})$, and hence to bound $b_1(O)$ it is sufficient to bound the number of generators of $\pi_1^{\mathrm{orb}}(\hat{O})$.

Let $\tilde{\pi} : \tilde{O} \to O$ denote the universal orbifold cover. Choose a nonsingular point $p \in O$, and choose $\tilde{p} \in \tilde{O}$ with $\tilde{\pi}(\tilde{p}) = p$. Fix $\varepsilon > 0$. Denote $\|g\| = d(\tilde{p}, g(\tilde{p}))$. Take a maximal set $\{g_1, \dots, g_m\}$ of $\pi_1^{\mathrm{orb}}(O)$ such that

$$\|g_i\| \le 2D + \varepsilon, \text{ and } \|g_i^{-1} g_j\| \ge \varepsilon, \ i \ne j.$$

Let Γ be the subgroup of $\pi_1^{\mathrm{orb}}(O)$ generated by $\{g_i\}$ and let $\hat{\pi} : \hat{O} \to O$ be the orbifold covering of O with $\pi_1^{\mathrm{orb}}(\hat{O}) = \Gamma$. To show that $\hat{\pi}$ is a finite cover we show that $\mathrm{diam}(\hat{O}) \le 2D + 2\varepsilon$. Let $\hat{p} \in \hat{O}$ be such that $\hat{\pi}(\hat{p}) = p$. Then $\hat{p}$ is nonsingular. If $\mathrm{diam}(\hat{O}) > 2D + 2\varepsilon$, then there is a point $\hat{q} \in \hat{O}$ such that $d_{\hat{O}}(\hat{p}, \hat{q}) = D + \varepsilon$. Since $d_O(p, \hat{\pi}(\hat{q})) \le \mathrm{diam}(O) = D$, there is a deck transformation $\alpha \in \pi_1^{\mathrm{orb}}(O) - \Gamma$, such that $d_{\hat{O}}(\hat{q}, \overline{\pi}(\alpha \tilde{p})) \le D$, where $\overline{\pi} : \tilde{O} \to \hat{O}$ is the covering projection. Then

$$d_{\hat{O}}(\hat{p}, \overline{\pi}(\alpha \tilde{p})) \ge d_{\hat{O}}(\hat{p}, \hat{q}) - d_{\hat{O}}(\hat{q}, \overline{\pi}(\alpha \tilde{p})) \ge \varepsilon,$$

$$d_{\hat{O}}(\hat{p}, \overline{\pi}(\alpha \tilde{p})) \le d_{\hat{O}}(\hat{p}, \hat{q}) + d_{\hat{O}}(\hat{q}, \overline{\pi}(\alpha \tilde{p})) \le 2D + \varepsilon.$$

Now note that there is a $\beta \in \pi_1^{\mathrm{orb}}(\hat{O}) = \Gamma$ such that $d_{\tilde{O}}(\Gamma \tilde{p}, \alpha \tilde{p}) = d_{\tilde{O}}(\beta \tilde{p}, \alpha \tilde{p})$, so

$$\|\beta^{-1} \alpha\| = d_{\tilde{O}}(\tilde{p}, \beta^{-1} \alpha \tilde{p}) = d_{\tilde{O}}(\beta \tilde{p}, \alpha \tilde{p}) = d_{\tilde{O}}(\Gamma \tilde{p}, \alpha \tilde{p})$$

$$= d_{\hat{O}}(\hat{p}, \overline{\pi}(\alpha \tilde{p})) \le 2D + \varepsilon.$$

Also, for any $g \in \pi_1^{\mathrm{orb}}(\hat{O}) = \Gamma$,

$$\|g^{-1} \beta^{-1} \alpha\| = d_{\tilde{O}}(\tilde{p}, g^{-1} \beta^{-1} \alpha \tilde{p}) = d_{\tilde{O}}(\beta g \tilde{p}, \alpha \tilde{p}) \ge d_{\tilde{O}}(\Gamma \tilde{p}, \alpha \tilde{p})$$

$$= d_{\hat{O}}(\hat{p}, \overline{\pi}(\alpha \tilde{p})) \ge \varepsilon.$$

So by maximality of Γ, $\beta^{-1} \alpha$ should be in Γ. However, α is not in Γ by construction and β is; therefore we have a contradiction, and we conclude that $\mathrm{diam}(\hat{O}) \le 2D + 2\varepsilon$.

Recall that $\{g_1, \dots, g_m\}$ is a set of generators of Γ. We now give a bound on the the number of generators m in terms of dimension, curvature, and diameter. We denote by $B(\tilde{p}, r)$ the metric r–ball in $\tilde{O}$ centered at $\tilde{p}$ and $\overline{B}(r)$, the metric r–ball in the simply connected space form of curvature $-k$.

Since

$$\varepsilon \le \|g_i^{-1} g_j\| = d_{\tilde{O}}(g_i^{-1} g_j \tilde{p}, \tilde{p}) = d_{\tilde{O}}(g_j \tilde{p}, g_i \tilde{p})$$

the balls $B(g_i\tilde{p}, \frac{1}{2}\varepsilon)$ are pairwise disjoint. Moreover, we have that $\bigcup_{i=1}^{m} B(g_i\tilde{p}, \frac{1}{2}\varepsilon) \subset B(\tilde{p}, 2D + \frac{3}{2}\varepsilon)$. Thus,

$$m = \frac{\operatorname{Vol}\left(\bigcup_{i=1}^{m} B(g_i\tilde{p}, \frac{1}{2}\varepsilon)\right)}{\operatorname{Vol} B(\tilde{p}, \frac{1}{2}\varepsilon)} \leq \frac{\operatorname{Vol} B(\tilde{p}, 2D + \frac{3}{2}\varepsilon)}{\operatorname{Vol} B(\tilde{p}, \frac{1}{2}\varepsilon)}$$

$$\leq \frac{\operatorname{Vol} \overline{B}(2D + \frac{3}{2}\varepsilon)}{\operatorname{Vol} \overline{B}(\frac{1}{2}\varepsilon)} \leq \frac{\operatorname{Vol} \overline{B}(5D)}{\operatorname{Vol} \overline{B}(D)} = c(n, kD^2)$$

where in the third to last inequality we have used the relative volume comparison for orbifolds [B] and in the second to last we have chosen $\varepsilon = 2D$. This completes the proof of part (i) of Theorem 1.

We now proceed to the proof of part (iii) of Theorem 1. We are going to modify the previous proof by choosing longer loops to generate $H_1^{\mathrm{orb}}(O, \mathbb{R})$. Again let $f : \pi_1^{\mathrm{orb}}(O) \to H_1^{\mathrm{orb}}(O, \mathbb{R})$ be the map defined in the proof of part (i), and let $\{g_1, \ldots, g_{b_1(O)}\}$ be a basis for $H_1^{\mathrm{orb}}(O, \mathbb{R})$ chosen as above with $\varepsilon = 2D$. Denote by Γ the subgroup of $\pi_1^{\mathrm{orb}}(O)$ generated by the g_i's. If Γ contains an element $g \neq 1$ with $\|g\| < 2D$, then $f(g) \neq 0$ and $f(g)$ has infinite order. Thus, there exists an $r > 0$ such that $2D \leq \|g^r\| \leq 4D$. Write $f(g) = a_1 f(g_1) + \cdots + a_{b_1(O)} f(g_{b_1(O)}) \neq 0$. Without loss of generality we may assume that $a_1 \neq 0$. Let Γ_1 be the group generated by $\{g^r, g_2, \ldots, g_{b_1(O)}\}$. Then clearly, the set $\{f(g^r), \ldots, f(g_{b_1(O)})\}$ is still a basis for $H_1^{\mathrm{orb}}(O, \mathbb{R})$.

Also, $g \notin \Gamma_1$. To see this suppose that

$$g = c_1 \cdot (g^r)^{k_1} \cdot c_2 \cdot (g^r)^{k_2} \cdots c_\ell \cdot (g^r)^{k_\ell} \cdot c_{\ell+1}$$

where the c_i's are words in $\{g_2, \ldots, g_{b_1(O)}\}$. Then

$$f(g) = f(c_1) + f(g^{rk_1}) + \cdots + f(c_\ell) + f(g^{rk_\ell}) + f(c_{\ell+1})$$

$$= f(c_1 c_2 \cdots c_{\ell+1}) + r(k_1 + \cdots + k_\ell) f(g).$$

Since $r \geq 2$, we can rewrite the expression above as

$$f(g) = \frac{1}{1 - r(k_1 + \cdots + k_\ell)} f(c_1 c_2 \cdots c_{\ell+1}) = a_2 f(g_2) + \cdots + a_{b_1(O)} f(g_{b_1(O)})$$

which contradicts the assumption that $a_1 \neq 0$.

Since $\pi_1^{\mathrm{orb}}(O)$ contains only finitely many g with $\|g\| < 2D$ ($\pi_1^{\mathrm{orb}}(O)$ acts discontinuously and $\tilde{p}$ is nonsingular), after finitely many such replacements we produce a new set $\{h_1, \ldots, h_{b_1(O)}\}$ of elements of $\pi_1^{\mathrm{orb}}(O)$ so that $\{f(h_1), \ldots, f(h_{b_1(O)})\}$ form a basis of $H_1^{\mathrm{orb}}(O, \mathbb{R})$, and $2D \leq \|h_i\| \leq 4D$. If $\Gamma' \subset \Gamma$ is the subgroup generated by the h_i's, then for any element $h \in \Gamma' - \{1\}$, we have $\|h\| \geq 2D$. Let

$$S(t) = \left\{ h \in \Gamma' \;\middle|\; \|h\|_{\mathrm{word}} \leq t \right\}.$$

If $h, h' \in \Gamma'$, with $h \neq h'$, then $\|h^{-1}h'\| \geq 2D$, which implies that the balls $B(h\tilde{p}, D), h \in \Gamma'$, are pairwise disjoint. If $b_1(O) \geq b$, then it is not hard to see that $\operatorname{card}(S(t)) \geq \left(\frac{t}{b}\right)^b$ where $\operatorname{card}(S(t))$ denotes the cardinality of $S(t)$. Since

$$\bigcup_{h \in S(t)} B(h\tilde{p}, D) \subset B(\tilde{p}, 4Dt + D)$$

we have

$$\operatorname{Vol} B(\tilde{p}, 4Dt + D) \geq \operatorname{Vol}\left(\bigcup_{h \in S(t)} B(h\tilde{p}, D)\right) = \operatorname{card}(S(t)) \cdot \operatorname{Vol} B(\tilde{p}, D)$$
$$\geq \left(\frac{t}{b}\right)^b \operatorname{Vol} B(\tilde{p}, D).$$

By applying relative volume comparison for orbifolds again, we see that

$$\left(\frac{t}{b}\right)^b \leq \frac{\operatorname{Vol} B(\tilde{p}, 4Dt + D)}{\operatorname{Vol} B(\tilde{p}, D)} \leq \frac{\operatorname{Vol} \overline{B}(4Dt + D)}{\operatorname{Vol} \overline{B}(D)}$$
$$= \frac{\int_0^{4Dt+D} \left(k^{-1/2} \sinh s\sqrt{k}\right)^{n-1} ds}{\int_0^{D} \left(k^{-1/2} \sinh s\sqrt{k}\right)^{n-1} ds} = \frac{\int_0^{(4t+1)D\sqrt{k}} \sinh^{n-1} s\, ds}{\int_0^{D\sqrt{k}} \sinh^{n-1} s\, ds}$$
$$= (4t+1)^n + \cdots \leq 5^n t^n \quad \text{if } kD^2 \text{ is small enough.}$$

Thus, if $b_1(O) \geq (n+1)$, then we set $b = (n+1)$ above and fix $t > 5^n \cdot (n+1)^{n+1}$. Now choosing kD^2 sufficiently small gives a contradiction, for we would have $\left(\frac{t}{n+1}\right)^{n+1} \leq 5^n t^n$ which implies that $t \leq 5^n \cdot (n+1)^{n+1}$. Thus $b_1(O) \leq n$. This completes the proof. □

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