ON THE EXISTENCE OF INFINITELY MANY CLOSED GEODESICS ON ORBIFOLDS OF REVOLUTION

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ABSTRACT. Using the theory of geodesics on surfaces of revolution, we introduce the period function. We use this as our main tool in showing that any two-dimensional orbifold of revolution homeomorphic to S^2 must contain an infinite number of geometrically distinct closed geodesics. Since any such orbifold of revolution can be regarded as a topological two-sphere with metric singularities, we will have extended Bangert's theorem on the existence of infinitely many closed geodesics on any smooth Riemannian two-sphere. In addition, we give an example of a two-sphere cone-manifold of revolution which possesses a single closed geodesic, thus showing that Bangert's result does not hold in the wider class of closed surfaces with cone manifold structures.

1. INTRODUCTION

In this paper, we study closed geodesics on surfaces of revolution with certain types of metric singularities. In particular, we are interested in closed (compact, without boundary) surfaces of revolution that are Riemannian 2-orbifolds. Loosely speaking, an 2-orbifold is modeled locally by convex Riemannian surfaces modulo finite groups of isometries acting with possible fixed points. This means that a neighborhood of each point p of such an orbifold is isometric to a Riemannian quotient U_p/Γ_p where U_p is a convex Riemannian surface diffeomorphic to $\mathbb{R}^2$, and Γ_p is a finite group of isometries acting effectively on U_p. Every Riemannian surface is trivially an orbifold, with each Γ_p being the trivial group. The reader interested in more background on orbifolds should consult [4], Thurston's classic [16], or the more recent textbook [14]. For the purposes of this paper, however, we will only need to apply a simple explicit criterion to determine whether a closed surface of revolution is a 2-orbifold (see section 7).

The existence of closed geodesics on Riemannian manifolds has a long and storied past dating back to Poincaré [2]. It seems that not much has been done on the existence of closed geodesics in singular spaces. The existence of at least one closed geodesic on a compact 2-orbifold was shown in [7] and closed geodesics in orbifolds of higher dimensions have recently been studied in [10]. The paper [11] studies the issue of closed geodesics in spaces with incomplete metrics. The relevance here is that a complete Riemannian orbifold with singular set removed is a Riemannian manifold with incomplete metric and it is known [5] that closed geodesics in a

complete Riemannian orbifold may not pass through the singular set, unless they are entirely contained within it.

Here we are interested in the question of the existence of infinitely many closed geodesics. In [1], Bangert used the work of Franks [9] to show that every smooth Riemannian S^2 has infinitely many closed geodesics. For orbifolds with S^2 as the underlying topological space, the existence of an infinity of closed geodesics is an open question. In the general category of closed surfaces of revolution with singular points (which have underlying topological space S^2), one may construct examples with exactly one closed geodesic (see example 8.2), showing that ana-logue of Bangert's result is false in this category. We call such a surface *void*. A spherical 2-orbifold of revolution is a closed two-dimensional surface of revolution homeomorphic to S^2 that satisfies a certain special orbifold condition at its north and south poles. It is natural to ask whether void orbifolds of revolution exist. In resolving this question we extend Bangert's result by proving that

Theorem 1.1. *Every spherical 2-orbifold of revolution has infinitely many closed geodesics.*

Since we are dealing only with surfaces of revolution, our techniques are relatively elementary. We begin by recalling the basic theory about surfaces of revolution and their geodesics, most of which can be found in [8], [12], or [13].

2. Basic Theory

In what follows the term *smooth function* will refer to a function of class C^∞. In fact, C^2 is sufficient for our needs.

Definition 2.1. *Let $\alpha : [u_N, u_S] \to \mathbb{R}^2$ be a simple (no self intersections) smooth plane curve $\alpha(u) = (g(u), h(u))$ where g and h are smooth functions on the interval $[u_N, u_S]$, with $h \geq 0$, and $h(u) = 0$ if and only if $u = u_N$ or $u = u_S$. A spherical surface of revolution M is a surface embedded isometrically in $\mathbb{R}^3$ that admits a parametrization $\mathbf{x} : [u_N, u_S] \times \mathbb{R} \to M$ of the form*

$$\mathbf{x}(u, v) = (g(u), h(u) \cos v, h(u) \sin v),$$

That is, M is the surface of revolution obtained by rotating α about the x-axis. The curve α will be called the profile curve.

Note that a spherical surface of revolution M is necessarily homeomorphic to S^2 and that by definition the sets $N = \mathbf{x}(u_N, v)$ and $S = \mathbf{x}(u_S, v)$ for $v \in \mathbb{R}$ reduce to single points which will be referred to as the *north* and *south poles* of M. Metric singularities may only occur at these two points. M is smooth everywhere else. We also do not require that the function g be monotone. Throughout this paper all surfaces of revolution will be assumed spherical as in definition 2.1 even though much of the classical theory we review applies equally well to any surface of revolution.

Rotation about the x-axis in $\mathbb{R}^3$ descends to a natural S^1-action $S^1 \times M \to M$ on M by isometries:

$$(e^{i\theta}, (x, y, z)) \mapsto (x, y \cos \theta - z \sin \theta, y \sin \theta + z \cos \theta).$$

This action is free except at the north and south poles which remain fixed.

For a surface of revolution M, a simple computation gives the coefficients of the first fundamental form or metric tensor (subscripts denote partial derivatives):

$$E = \mathbf{x}_u \cdot \mathbf{x}_u = [g'(u)]^2 + [h'(u)]^2, \quad F = \mathbf{x}_u \cdot \mathbf{x}_v = 0 \quad \text{and} \quad G = \mathbf{x}_v \cdot \mathbf{x}_v = h^2(u),$$

so that the metric (away from any singular point) is

$$ds^2 = \left([g'(u)]^2 + [h'(u)]^2\right) du^2 + h^2(u) dv^2.$$

Note that the parametrization is orthogonal ($F = 0$) and that $E_v = G_v = 0$. Surfaces given by parametrizations with these properties are said to be *u-Clairaut*.

For any u-Clairaut surface, and hence any surface of revolution, the geodesic equations reduce to

$$(2.1) \qquad\qquad u'' + \frac{E_u}{2E}u'^2 - \frac{G_u}{2E}v'^2 = 0$$

$$(2.2) \qquad\qquad v'' + \frac{G_u}{G}u'v' = 0.$$

A curve $\gamma(t) = \mathbf{x}(u(t), v(t))$ on M is a geodesic if and only if the above equations are satisfied by the coordinate functions u and v of γ. Also, a geodesic satisfying these equations must be parametrized proportional to arc length and hence has constant speed. In particular, we may assume that γ has unit speed. That is, $\|\gamma'\| = Eu'^2 + Gv'^2 \equiv 1$. The existence and uniqueness theorem for solutions of ordinary differential equations implies that, given a point in p in M and a vector ξ in T_pM, the tangent plane to M at p, there is a unique geodesic γ satisfying $\gamma(0) = p$ and $\gamma'(0) = \xi$.

We now recall two important classes of geodesics on surfaces of revolution.

Example 2.2. A unit speed curve $\gamma(t) = \mathbf{x}(u(t), v(t))$ with $v(t) \equiv v_0$, a constant, is a u-parameter curve or *meridional arc*. Such curves are always geodesics. To see this, note that $v' = v'' \equiv 0$, so equation (2.2) is satisfied trivially. The unit speed relation is, in this case, $Eu'^2 = 1$, so $u'^2 = 1/E$. Differentiating each side and dividing by $2u'$ gives

$$u'' = -\frac{E_u u' + E_v v'}{2u'E^2} = -\frac{E_u}{2E^2} = -\frac{E_u}{2E}u'^2,$$

which is equivalent to (2.1), since $v' \equiv 0$. We will use the term *meridian* for those meridional arcs that join N to S.

Example 2.3. A unit speed curve $\gamma(t) = \mathbf{x}(u(t), v(t))$ with $u(t) \equiv u_0 \in (u_N, u_S)$, a constant, is a v-parameter curve or *parallel arc*. For a parallel arc we have $u' = u'' \equiv 0$ and the unit speed relation $Gv'^2 = 1$. Differentiating the unit speed relation yields $v'' = -\frac{1}{2}G_v/G^2 = 0$ since $G = h(u)^2$ depends only on u. Thus (2.2) is satisfied. Equation (2.1) reduces to $G_u v'^2 = 0$. Hence parallel arcs are geodesic precisely when $G_u(u_0) = 0$, or equivalently, $h'(u_0) = 0$. We will use the term *parallel* for those parallel arcs which are entire circles.

For the remainder of the paper we will assume all geodesics come with unit speed parametrizations.

The main classical tool used to get qualitative information about geodesics on surfaces of revolution is the *Clairaut relation*, which we present now. Let $\gamma(t)$ be a geodesic on M. Then $\gamma' = u'\mathbf{x}_u + v'\mathbf{x}_v$. If there exists t_0 with $v'(t_0) = 0$, then the uniqueness of geodesics implies that γ must be a meridional arc as γ' is parallel to

$\mathbf{x}_u$ at t_0. As a result v' cannot change sign, and we may assume, without loss of generality that $v'(t) \geq 0$. In fact, $v'(t) > 0$ unless γ is a meridional arc.

Let $\varphi_\gamma(t) = \angle(\gamma', \mathbf{x}_u)$ be the angle between γ' and $\mathbf{x}_u$ at time t. Since the surface parametrization $\mathbf{x}$ and γ are smooth, $\varphi_\gamma(t)$ is a smooth function that takes its values in the interval $[0, \pi]$. From the discussion above we see that $\varphi_\gamma(t) \in (0, \pi)$ for all t if and only if γ is not a meridional arc. Now consider the quantity $c_\gamma = Gv'$ along a geodesic γ. Then

$$c'_\gamma = Gv'' + (G_u u' + G_v v')v' = Gv'' + G_u u' v' = 0$$

where the second equality follows since $G_v = 0$ and the last equality follows from the second geodesic equation (2.2). Thus the quantity c_γ is constant along geodesic paths. Comparing the two expressions for $\gamma' \cdot \mathbf{x}_v$:

$$\gamma' \cdot \mathbf{x}_v = (u'\mathbf{x}_u + v'\mathbf{x}_v) \cdot \mathbf{x}_v = Gv' = c_\gamma$$

$$\gamma' \cdot \mathbf{x}_v = \|\gamma'\| \, \|\mathbf{x}_v\| \cos\left(\frac{\pi}{2} - \varphi_\gamma\right) = \sqrt{G} \sin \varphi_\gamma$$

yields the *Clairaut relation*:

Proposition 2.4. *If $c_\gamma(t) = G(t)v'(t)$ along a geodesic $\gamma(t)$, then the quantity*

$$(2.3) \qquad c_\gamma(t) = \sqrt{G(t)} \sin \varphi_\gamma(t) = h(u(t)) \sin \varphi_\gamma(t)$$

is constant.

The constant c_γ is called the *slant* of γ. Since $0 \leq \sin \varphi_\gamma(t) \leq 1$ for all t we must have that $h(u(t)) \geq c_\gamma$ for all t. That is, γ is must lie entirely in the region of the surface M where $h(u) \geq c_\gamma$.

Corollary 2.5. *For a spherical surface of revolution, a geodesic γ with an endpoint at either pole must be a meridional arc.*

Proof. Since γ has an endpoint at a pole assume for concreteness that $\gamma(a) = N$ and that γ is defined over an interval $[a, b]$. Let $t_n \to a$ be a sequence of real numbers $t_n \in (a, b)$ converging to a. Then $h(u(t_n)) \to h(u(a)) = 0$, whence $c_\gamma(t_n) \to 0$. By proposition 2.4, $c_\gamma(t) \equiv 0$, which implies that $\sin \varphi_\gamma(t) \equiv 0$. Thus, γ is a meridional arc. $\square$

Corollary 2.6. *If γ is not a meridional arc, then γ cannot pass through a pole of M. Thus, non-meridional geodesics γ have a unique extension to a unit speed geodesic $\hat{\gamma} : \mathbb{R} \to M$.*

Proof. This follows from corollary 2.5 and the existence and uniqueness theorem for geodesics. $\square$

3. Qualitative Theory and a Classification of Geodesics

In light of corollary 2.6 we will now assume that all non-meridional geodesics will be defined on $\mathbb{R}$, and meridional arcs are extended to meridians.

Motivated by the Clairaut relation we define, for $c > 0$, the super-level sets

$$M^c = \{p \in M \mid \text{if } p = \mathbf{x}(u, v), \text{then } h(u) > c\}$$

Points of M^c will be referred to as points of M with $h(u) > c$ for convenience. M^c may have several connected components $M^c_{(j)}$, but it is always true that the boundary of any such component $\partial M^c_{(j)} = \rho_0 \cup \rho_1$ where ρ_i are parallels $\rho_i : t \to$

$\mathbf{x}(u_i, t)$ with $h(u_i) = c$ for $i = 0, 1$. Note also that if c is a local minimum value of $h(u)$ then $c > 0$ and the closure $M_{(j)}^c$ is a proper subset of $\{p \in M \mid \text{if } p = \mathbf{x}(u, v), \text{then } h(u) \geq c\}$.

The qualitative behavior of non-meridional geodesics given next is key to our analysis.

Proposition 3.1. *Suppose a geodesic $\gamma(t) = \mathbf{x}(u(t), v(t))$ on a spherical surface of revolution M is tangent at $t = t_0$ to a non-geodesic parallel $\rho_0 : t \to \mathbf{x}(u_0, t)$ of M. Then γ is constrained to lie in the connected component M_γ of M^{c_γ} which contains γ. The boundary $\partial M_\gamma = \rho_0 \cup \rho_1$ where $\rho_1 : t \to \mathbf{x}(u_1, t)$ is a parallel of M with $h(u_1) = c_\gamma$. Moreover, γ either oscillates between the parallels ρ_i intersecting them tangentially or γ spirals asymptotically to ρ_1 which is necessarily a geodesic parallel.*

Proof. If γ is tangent at $t = t_0$ to a non-geodesic parallel ρ_0, then $u(t_0) = u_0$ and $\gamma'(t_0)$ is parallel to $\mathbf{x}_v$. Thus, $\varphi_\gamma(t_0) = \pi/2$ which implies $c_\gamma = h(u(t_0)) = h(u_0)$. Thus, by the Clairaut relation we may then conclude that the entire geodesic γ lies in a region of M that corresponds to points where the profile curve is $\geq c_\gamma$. Since γ is not a parallel, (otherwise, γ would have to coincide with ρ_0 which is not geodesic), $h(u(t)) > c_\gamma$ for some $t \in \mathbb{R}$. Thus, γ is a subset of a connected component M_γ of M^{c_γ}. Since ρ_0 is non-geodesic, u_0 is not a critical point of h and thus h is monotone in a neighborhood of $u = u_0$. The Clairaut relation then implies that γ lies on one side of ρ_0. That is, for all t, $u(t) \in (u_N, u_0]$ or $u(t) \in [u_0, u_S)$. This shows that $\rho_0 \subset \partial M_\gamma$ and that intersections of γ with ρ_0 are tangential.

Without loss of generality, we may assume that $u(t) \in [u_0, u_S)$. Let $u_1 \in (u_0, u_S)$ be the smallest number such that $h(u_1) = c_\gamma$. Let $\rho_1 : t \to \mathbf{x}(u_1, t)$ be the corresponding parallel. Define $b = \sup_{t \in \mathbb{R}} u(t)$.

If there is a t_b such that $u(t_b) = b$, then $u'(t_b) = 0$. Thus, γ is parallel to $\mathbf{x}_v$ at t_b and hence $c_\gamma = h(b)$. By the choice of u_1, we must have $b = u_1$ and thus, $\gamma \cap \rho_1 \neq \emptyset$. As before, we may conclude that ρ_1 is non-geodesic, all intersections are tangential, and γ lies on one side of ρ_1. In particular, the set $\{u(t) \mid t \in \mathbb{R}\} = [u_0, u_1]$ and γ oscillates back and forth between the two parallels ρ_0 and ρ_1 which form the boundary ∂M_γ.

On the other hand, if no such t_b exists, then $\lim_{t \to \infty} u(t) = b$ and γ is asymptotic to the parallel at $u = b$. Since γ is geodesic, this implies that the parallel ρ_b at $u = b$ is geodesic with slant $c_{\rho_b} = h(b)$. By taking a limits we conclude that $c_\gamma = \lim_{t \to \infty} \left[h(u(t)) \sin \angle(\gamma'(t), \mathbf{x}_u) \right] = h(b) \sin \angle(\rho_b', \mathbf{x}_u) = h(b)$. By the choice of u_1 we conclude that $b = u_1$ and that $\rho_b = \rho_1$. In particular, in this case, γ spirals asymptotically to a geodesic parallel ρ_1 and $\partial M_\gamma = \rho_0 \cup \rho_1$. $\qquad \square$

Geodesics which exhibit the oscillating behavior of proposition 3.1 will be called *oscillating geodesics* and those with asymptotic behavior will be called *asymptotic geodesics*. There is actually one last type of geodesic, called a *bi-asymptotic geodesic*. This is a geodesic that spirals into a geodesic parallel as $t \to -\infty$ and another geodesic parallel as $t \to \infty$. The existence of bi-asymptotic geodesics will be considered in proposition 3.2 where we consider conditions that imply the existence of (bi-)asymptotic geodesics.

Proposition 3.2. *Let $\alpha = (g, h) : [u_N, u_S] \to \mathbb{R}^2$ be the profile curve of M. Then Γ_M contains an asymptotic geodesic if and only if h has a critical point in the interval (u_N, u_S) that is not a local maximum.*

Proof. Suppose h has a critical point at $u = u_0$ that is not a local maximum. Without loss of generality, we may assume there is $u_1 \in h^{-1}(h(u_0))$ such that the open interval $(u_0, u_1) \subset \{u \in (u_N, u_S) \mid h(u) > h(u_0)\}$.

If $h'(u_1) = 0$ then the corresponding parallel at u_1 is non-geodesic and by proposition 3.1 there is a geodesic γ through $\mathbf{x}(u_1, 0)$ and parallel to $\mathbf{x}_v(u_1, 0)$ so that γ is asymptotic to the parallel at u_0.

On the other hand, if $h'(u_1) = 0$, then pick a point $\hat{u} \in (u_0, u_1)$. Since $h(u_0) < h(\hat{u})$ we can find $\hat{\varphi} \in (0, \pi/2)$ with $\hat{\varphi} = \arcsin\left(\dfrac{h(u_0)}{h(\hat{u})}\right)$. Now, let γ be the geodesic with $\gamma(0) = \mathbf{x}(\hat{u}, 0)$ and with $\angle(\gamma'(0), \mathbf{x}_u) = \hat{\varphi}$. Then the slant of γ, $c_\gamma = h(\hat{u}) \sin \hat{\varphi} = h(u_0)$. By proposition 3.1, we may conclude that γ is asymptotic to the geodesic parallels at u_0 and u_1. In this case, γ is bi-asymptotic. $\square$

We have shown that a geodesic on spherical surface of revolution is either a meridian, a geodesic parallel, an oscillating geodesic, an asymptotic geodesic or a bi-asymptotic geodesic. We now define the boundary values and boundary function on the set of geodesics on M.

Definition 3.3. *Let $\gamma(t) = \mathbf{x}(u(t), v(t))$ be a non-meridional geodesic on a spherical surface of revolution M. Define $b_0(\gamma) = \inf_{t \in \mathbb{R}}(u(t))$ and $b_1(\gamma) = \sup_{t \in \mathbb{R}}(u(t))$ to be the* left *and* right *boundary values of γ, respectively. If γ is a meridian we define $b_0(\gamma) = u_N$ and $b_1(\gamma) = u_S$. The boundary function b of γ is defined by $b(\gamma) = (b_0(\gamma), b_1(\gamma)) \in [u_N, u_S] \times [u_N, u_S]$.*

In the case of a non-meridional geodesic, by proposition 3.1 we have that $b_0(\gamma)$ and $b_1(\gamma)$ are the corresponding u values for the parallels ρ_0 and ρ_1. When the geodesic under consideration is clear, we will often drop the reference to γ and refer to the boundary values of γ as b_0 and b_1.

Definition 3.4. *Two geodesics γ_1 and γ_2 on M are* equivalent *if γ_1 and γ_2 are in the same orbit of the natural S^1 action on M. We denote the set of all equivalence classes $[\gamma]$ by Γ_M.*

Since the S^1 action on M preserves parallels, we conclude from proposition 3.1 that the boundary function $b : \Gamma_M \to [u_N, u_S] \times [u_N, u_S]$ is well-defined and injective. We adopt the common abuse of notation by simply referring to a geodesic $\gamma \in \Gamma_M$.

We can classify all geodesics on a spherical surface of revolution by boundary function:

Definition 3.5. *Let γ be a geodesic in Γ_M.*

(1) *If $b = (u_N, u_S)$ then γ is a meridian. Otherwise, if γ is not a meridian,*
(2) *If $b_0 = b_1$, γ is a geodesic parallel,*
(3) *If $b_0 = b_1$ with $h'(b_0)$ and $h'(b_1)$ both non-zero, then γ is an oscillating geodesic.*
(4) *If $b_0 = b_1$ with both $h'(b_0)$ and $h'(b_1)$ equal to zero, then γ is a bi-asymptotic geodesic, otherwise*
(5) *If $b_0 = b_1$ with either $h'(b_0)$ or $h'(b_1)$ equal to zero, then γ is an asymptotic geodesic.*

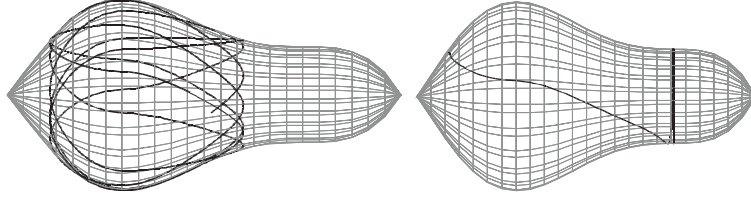Examples of an oscillating and asymptotic geodesics are given in figure 1.

FIGURE 1. An oscillating and asymptotic geodesic on a spherical surface of revolution

Let γ be an oscillating geodesic. If $u(t_0) = b_0 = u(t_1)$ for $t_0 = t_1$ and there is a unique $t \in (t_0, t_1)$ such that $u(t) = b_1$, we call the segment of γ corresponding to the interval $[t_0, t_1]$ an *oscillation*. Since it is not important for what follows, we will refer to bi-asymptotic geodesics as simply asymptotic geodesics also.

4. A TOPOLOGY ON THE SET OF OSCILLATING GEODESICS

Definition 4.1. *A geodesic γ is* closed *if there exist real numbers $t_0 = t_1$ such that*

$$\gamma(t_0) = \gamma(t_1) \quad \text{and} \quad \gamma'(t_0) = \gamma'(t_1).$$

Equality of the derivatives distinguish closed geodesics from the more general notion of geodesic loop. Every geodesic parallel is closed, and no asymptotic geodesic or meridian (using our definition) is closed. Oscillating geodesics, however, may or may not be closed. Since we are interested in closed geodesics the set

$$\Gamma_M^O = \{[\gamma] : \gamma \text{ is an oscillating geodesic}\}$$

will be the most interesting to us.

Note that if γ is oscillating, then γ is the unique geodesic with left boundary $b_0(\gamma)$. This is because $h'(b_0(\gamma)) = 0$, so the parallel at $u = b_0(\gamma)$ is not geodesic and there can be no geodesic asymptotic to it. Thus, by our classification, any geodesic which shares a left boundary with γ must be oscillating itself. But, any oscillating geodesic intersects its left boundary tangentially, so by the definition of our equivalence relation and the uniqueness of geodesics we conclude that γ is the unique geodesic in its equivalence class with left boundary $b_0(\gamma)$. Thus, the map $b_0 : \Gamma_M^O \to (u_N, u_S)$ is injective. In particular, for oscillating geodesics, the right boundary value is determined by the left boundary value.

Proposition 4.2. *Let $b_1(u_1) = \min\{u > u_1 : h(u) = h(u_1)\}$ and let $\mathcal{U} = \{u_1 \in (u_N, u_S) : h'(u_1) > 0$ and $h'(b_1(u_1)) < 0\}$. Then $\mathcal{U}$ is an open subset of the interval (u_N, u_S) and $b_0 : \Gamma_M^O \to \mathcal{U}$ is a bijection.*

Proof. We first show that b_0 is a bijection. Indeed, $b_0(\gamma) \in \mathcal{U}$ for any $\gamma \in \Gamma_M^O$ by proposition 3.1. For any $u_1 \in \mathcal{U}$, there is a geodesic γ with the initial conditions $u(0) = u_1$, $u'(0) = 0$. Then $h'(u_1) > 0$ implies γ is not a geodesic parallel and $b_0(\gamma) = u_1$. Thus, $b_1(\gamma) = b_1(u_1)$, and $h'(b_1(u_1)) < 0$ implies γ is not asymptotic, so $\gamma \in \Gamma_M^O$.

Smoothness of h implies that if $u_1 \in \mathcal{U}$ there exists an $\epsilon > 0$ such that $(u_1 - \epsilon, u_1 + \epsilon) \subset \mathcal{U}$. Thus, $\mathcal{U}$ is an open subset of the real interval (u_N, u_S). $\square$

We can now topologize on Γ_M^O by pull-back: We declare a subset $U \subset \Gamma_M^O$ to be open if and only if $b_0(U)$ is open in $\mathcal{U}$. Hence Γ_M^O is homeomorphic to a disjoint union of open intervals of (u_N, u_S). This allows us, for example, to speak of a sequence of geodesics in the space Γ_M^O as a sequence of (left) boundary values from $\mathcal{U}$. Consequently, we can also easily define convergence of oscillating geodesics, and, more importantly, continuous functions defined on Γ_M^O.

5. The Period Function

We now present our main analytic tool for detecting closed geodesics on spherical surfaces of revolution.

In the case of oscillating or asymptotic geodesics, the geodesic equations (2.1) and (2.2) can be reduced to a first-order system and solved explicitly. Equation (2.2), after dividing by v' (which is never zero for oscillating or asymptotic geodesics) and integrating, becomes

$$\int \frac{v''}{v'} \, dt = - \int \frac{G_u}{G} u' \, dt = - \int \frac{G'}{G} \, dt,$$

since $G_v = 0$. Then $v' = c/G$ for some constant $c \in \mathbb{R}$ and $c = Gv' = c_\gamma$, again showing the slant c_γ to be constant. Using $v' = c_\gamma/G$ in the unit speed relation $Eu'^2 + Gv'^2 = 1$ gives

$$u' = \pm \sqrt{\frac{G - c_\gamma^2}{EG}}.$$

Hence,

$$\frac{dv}{du} = \frac{v'}{u'} = \pm \frac{c_\gamma \sqrt{E}}{\sqrt{G}\sqrt{G - c_\gamma^2}},$$

and

$$v = \pm \int \frac{c_\gamma \sqrt{E}}{\sqrt{G}\sqrt{G - c_\gamma^2}} \, du.$$

As we will soon see, by measuring the total change in v an oscillating geodesic makes between its boundaries one can determine if it is closed. This motivates the following definition.

Definition 5.1. *The* period function $\Phi_M : \Gamma_M^O \to (0, \infty)$ *is defined by*

$$\Phi_M(\gamma) = 2 \int_{b_0(\gamma)}^{b_1(\gamma)} \frac{c_\gamma \sqrt{E}}{\sqrt{G}\sqrt{G - c_\gamma^2}} \, du = 2 \int_{b_0(\gamma)}^{b_1(\gamma)} \frac{c_\gamma \sqrt{E}}{h(u)\sqrt{h^2(u) - c_\gamma^2}} \, du.$$

We denote the integrand by $f_\gamma(u)$.

Geometrically, the period function gives the change in v as γ undergoes one oscillation. Since $h^2(b_0) = h^2(b_1) = c_\gamma^2$, the integral is improper for every geodesic γ, however, because it represents the change in v between b_0 and b_1 it must converge for every $\gamma \in \Gamma_M^O$. We can use this geometric interpretation to see that the period function is invariant under reparametrization and scaling of M and to extend the domain of the period function to include the asymptotic geodesics, by setting $\Phi_M(\gamma_0) = \infty$ for any asymptotic geodesic γ_0.

The next theorem shows how the period function can be used to detect closed geodesics.

Theorem 5.2. *An oscillating geodesic γ on a spherical surface of revolution M is closed if and only if $\Phi_M(\gamma) = 2q\pi$ for some rational number $q \in \mathbb{Q}$.*

Proof. We can assume without loss of generality that γ satisfies the initial conditions

$$\gamma(0) = \mathbf{x}(b_0, 0) \quad \text{and} \quad \gamma'(0) = \frac{\mathbf{x}_v(b_0, 0)}{\|\mathbf{x}_v(b_0, 0)\|}.$$

If $\gamma(t) = \mathbf{x}(u(t), v(t))$ is closed, there exists a $t_0 \in \mathbb{R}^+$ such that $\gamma(t_0) = \gamma(0)$ and $\gamma'(t_0) = \gamma'(0)$. In particular, $\gamma(t_0) = \mathbf{x}(b_0, 2r\pi)$ for some positive integer r. Note that the period function does not depend on the value $v(0)$, so by rotational symmetry, v changes the same amount during every oscillation of γ. Clearly, between $t = 0$ and $t = t_0$, γ has completed, say, s oscillations. That is, there have been s times subsequent to $t = 0$ that $u(t)$ has re-attained the boundary value b_0. Therefore, $\Phi_M(\gamma) = 2(r/s)\pi$.

Conversely, suppose $\Phi_M(\gamma) = 2(r/s)\pi$ for some $r, s \in \mathbb{Z}^+$, where γ is taken to have the same initial conditions. Then there exists a $t_0 \in \mathbb{R}^+$ such that

$$\gamma(t_0) = \mathbf{x}(b_0, 2r\pi) = \mathbf{x}(b_0, 0) = \gamma(0).$$

Since $u(t_0) = b_0$, by proposition 3.1, we must have $\gamma'(t_0)$ tangent to $\mathbf{x}_v(b_0, 0)$, and thus, $\gamma'(t_0) = \gamma'(0)$. Hence γ is closed. $\square$

The next theorem shows that the period function is continuous. A sketch of the proof of this result first appeared in the unpublished manuscript [6]. For clarity of the exposition, we relegate to section 9 the rather technical proof of this result.

Theorem 5.3. *If $\gamma_0 \in \Gamma_M^O$, then Φ_M is continuous at γ_0.*

As we will see, the continuity of Φ_M at every oscillating geodesic implies the existence of infinitely many geodesics on many spherical surfaces of revolution.

Definition 5.4. *A non-empty open subset $U \subseteq \Gamma_M^O$ on which Φ_M is a constant, irrational multiple of π is said to be* void.

The definition is motivated by theorem 5.2, which implies that all oscillating geodesics γ with $b_0(\gamma) \in U$ fail to close smoothly, so U is *void* of closed geodesics.

Corollary 5.5. *Suppose a spherical surface of revolution M has a non-empty open subset U of Γ_M^O that is not void. Then M has infinitely many closed geodesics.*

Proof. Let $\Phi_M(U) = \{\Phi_M(\gamma) : \gamma \in U\}$. If $\Phi_M(U)$ is a constant rational multiple of π we are done by theorem 5.2, so suppose Φ_M is not constant on U. By continuity of Φ_M, there exists a nonempty open interval $I \subset \Phi_M(O)$. $\mathbb{Q}\pi$ is dense in any such I yielding an infinite number of closed geodesics in U by theorem 5.2. $\square$

The following corollary shows that the existence of an asymptotic geodesic on M implies the existence of such a non-void subset of Γ_M^O, and hence the existence of infinitely many closed geodesics.

Corollary 5.6. *Let M be a spherical surface of revolution with an asymptotic geodesic γ_0 asymptotic to the geodesic parallel at $b_0(\gamma_0)$. Then if $\gamma_n \to \gamma_0$ is a sequence of oscillating geodesics,*

$$\lim_{n \to \infty} \Phi_M(\gamma_n) = \Phi_M(\gamma_0) = \infty.$$

Thus, by corollary 5.5, M has infinitely many closed geodesics.

Proof. Let $A > 0$. $\Phi_M(\gamma_0) = \int_{b_0(\gamma_0)}^{b_1(\gamma_0)} f_{\gamma_0} = \infty$, so there exists $\delta, \mu > 0$ so that

$A < \int_{b_0(\gamma_0)+\delta}^{b_1(\gamma_0)-\mu} f_{\gamma_0}$. Choose $N > 0$ large enough so that $b_0(\gamma_n) < b_0(\gamma_0) + \delta$ and $b_1(\gamma_n) > b_1(\gamma_0) - \mu$ for $n > N$. Thus,

$$\Phi_M(\gamma_n) = \int_{b_0(\gamma_n)}^{b_1(\gamma_n)} f_{\gamma_n} > \int_{b_0(\gamma_0)+\delta}^{b_1(\gamma_0)-\mu} f_{\gamma_n}$$

On the interval $(b_0(\gamma_0)+\delta, b_1(\gamma_0)-\mu)$, $f_{\gamma_n} \to f_{\gamma_0}$, and both f_{γ_n} and f_{γ_0} are bounded hence integrable. Thus, by dominated convergence, for $\varepsilon > 0$, there is $N' > N$ so that

$$\Phi_M(\gamma_n) > \int_{b_0(\gamma_0)+\delta}^{b_1(\gamma_0)-\mu} f_{\gamma_n} > \left[\int_{b_0(\gamma_0)+\delta}^{b_1(\gamma_0)-\mu} f_{\gamma_0} \right] - \varepsilon > A - \varepsilon$$

for $n > N'$. This implies $\Phi_M(\gamma_n) \to \infty$. $\square$

Corollary 5.7. *A spherical surface of revolution whose profile curve has more than one critical point necessarily has an infinite number of closed geodesics.*

Proof. This follows from proposition 3.2 and corollary 5.6. $\square$

Corollary 5.8. *A spherical surface of revolution whose profile curve has a single critical point (which must be a maximum), has exactly one closed geodesic or infinitely many.*

Proof. The parallel at the critical point is necessarily geodesic. If Γ_M^O is not void, then there are an infinite number of closed geodesics by corollary 5.5. The only other possibility is that Φ_M is a constant irrational multiple of π over its entire domain Γ_M^O. Then by theorem 5.2, no oscillating geodesic is closed, and M has exactly one closed geodesic. $\square$

Definition 5.9. *A spherical surface of revolution with exactly one closed geodesic will be called a* void surface.

An explicit example of a void surface will be given in section 8.

6. Surfaces of Revolution with Constant Period Function

Since, ultimately, we wish to show that no void spherical 2-orbifolds of revolution exist, corollary 5.8 implies we should look for general conditions that imply the period function is constant. We do exactly that in this section.

If a spherical surface of revolution $\mathbf{x}(u,v) = (g(u), h(u)\cos v, h(u)\sin v)$ obtained from the profile curve $\alpha(u) = (g(u), h(u))$ is to have a constant period function, we can, without loss of generality, assume that $h(u)$ is a smooth function from $[0, L]$ to $[0, 1]$ satisfying:

(1) $h(0) = h(L) = 0$
(2) h has a unique critical point, say u_0, on $[0, L]$
(3) $h(u_0) = 1$

where

$$g(u) = \int_0^u \sqrt{1 - [h'(t)]^2}\, dt.$$

As a result, we may assume the metric on M is of the form

$$ds^2 = du^2 + h^2(u)dv^2.$$

If the period function Φ_M is to be constant, proposition 3.2 and corollary 5.6 imply that condition (2) is necessary. (1) and (3) may be satisfied by an appropriate reparametrization and scaling of the profile curve, which does not affect the period function. The following proposition is adapted from [3].

Proposition 6.1. *Let M be a spherical surface of revolution satisfying conditions (1),(2) and (3). We can define new coordinates (r, v) on M so that the metric in these coordinates has the form*

$$ds^2 = E(r)dr^2 + \sin^2 r\, dv^2,$$

where $\hat{E}(\cos r) = E(r)$ is a function from $[0, \pi]$ to $\mathbb{R}^+$.

Proof. Define $f : [0, L] \to [0, \pi]$ and $c : [-1, 1] \to [0, L]$ by

$$f(u) = \begin{cases} \arcsin h(u) & \text{if } u \in [0, u_0] \\ \pi - \arcsin h(u) & \text{if } u \in [u_0, L] \end{cases}$$

$$c(\cos r) = \begin{cases} (h|_{[0,u_0]})^{-1}(\sin r) & \text{if } r \in [0, \frac{\pi}{2}] \\ (h|_{[u_0,L]})^{-1}(\sin r) & \text{if } r \in [\frac{\pi}{2}, \pi]. \end{cases}$$

Then, setting $r = f(u)$, we have $c(\cos r) = f^{-1}(r) = f^{-1}(f(u)) = u$ and

$$h(u) = h[c(\cos r)] = \sin r.$$

Hence,

$$du^2 = \left(\frac{1}{f'(u)}\right)^2 dr^2 = \left(\frac{\cos r}{h'[c(\cos r)]}\right)^2 dr^2,$$

and we can now write the metric on M as

$$ds^2 = E(r)dr^2 + \sin^2 r\, dv^2,$$

where $\hat{E}(\cos r) = E(r)$ is a function from $[0, \pi]$ to $\mathbb{R}^+$ defined by

$$E(r) = \begin{cases} \hat{E}(\cos r) = \dfrac{\cos^2 r}{\left(h'[c(\cos r)]\right)^2} & \text{if } r = \frac{\pi}{2} \\[2ex] \hat{E}(0) = \dfrac{1}{[f'(u_0)]^2} = \dfrac{-1}{h''(u_0)} & \text{if } r = \frac{\pi}{2}. \end{cases}$$

Note that condition (2) implies $h''(u_0) < 0$ and that by differentiating the relation $h(u) = \sin f(u)$ twice and evaluating at $u = u_0$ shows that $f'(u_0) = -h''(u_0)$. Thus E is continuous on $[0, \pi]$. $\qquad\square$

We can now take as a starting point in our search for surfaces with constant constant period function those surfaces of revolution with metric of the form

$$ds^2 = E(u)\, du^2 + \sin^2 u\, dv^2,$$

where $E(u)$ is a function from $[0, \pi]$ to $\mathbb{R}^+$. This corresponds to the spherical surface of revolution M with profile curve $\alpha(u) = (g(u), \sin u)$, where

$$g(u) = \int_0^u \sqrt{E(t) - \cos^2 t}\, dt.$$

If γ_x is the geodesic with left boundary value $b_0 = x$, then the right boundary value $b_1 = \pi - x$ and the period function may then be written as a function of $x \in (0, \frac{\pi}{2})$:

$$\Phi_M(\gamma_x) = \Phi_M(x) = \int_x^{\pi-x} \frac{\sin x \cdot \sqrt{E(u)}}{\sin u \quad \sin^2 u - \sin^2 x} \, du,$$

which is continuous on $(0, \frac{\pi}{2})$ by theorem 5.3. The following technical lemma adapted from [3] will be essential in our characterization of surfaces of revolution with constant period function.

Lemma 6.2. *Consider the function*

$$F(x) = \int_x^{\pi-x} \frac{\sin x \cdot f(u)}{\sin u \quad \sin^2 u - \sin^2 x} \, du$$

Define a function $\hat{f}$ by the formula $f(u) = \hat{f}(\cos u)$. Then $F(x)$ is identically zero on $(0, \frac{\pi}{2})$ if and only if $\hat{f}$ is an odd function over $[-1, 1]$.

Proof. Let $\hat{f}^e(\cos u) = (\hat{f}(\cos u) + \hat{f}(-\cos u))/2$ be the even part of $\hat{f}$. Then $\hat{f}$ is odd if and only if $\hat{f}^e$ is identically zero. We have

$$F(x) = \int_x^{\frac{\pi}{2}} \frac{\sin x \cdot \hat{f}(\cos u)}{\sin u \quad \sin^2 u - \sin^2 x} \, du + \int_{\frac{\pi}{2}}^{\pi-x} \frac{\sin x \cdot \hat{f}(\cos u)}{\sin u \quad \sin^2 u - \sin^2 x} \, du$$

$$= \int_x^{\frac{\pi}{2}} \frac{\sin x \cdot \hat{f}(\cos u)}{\sin u \quad \sin^2 u - \sin^2 x} \, du$$

$$+ \int_{\frac{\pi}{2}}^{\pi-x} \frac{\sin x \cdot 2\hat{f}^e(\cos u)}{\sin u \quad \sin^2 u - \sin^2 x} \, du - \int_{\frac{\pi}{2}}^{\pi-x} \frac{\sin x \cdot \hat{f}(-\cos u)}{\sin u \quad \sin^2 u - \sin^2 x} \, du$$

$$= \int_x^{\frac{\pi}{2}} \frac{\sin x \cdot \hat{f}(\cos u)}{\sin u \quad \sin^2 u - \sin^2 x} \, du - \int_x^{\frac{\pi}{2}} \frac{\sin x \cdot \hat{f}(\cos u)}{\sin u \quad \sin^2 u - \sin^2 x} \, du$$

$$+ \int_x^{\frac{\pi}{2}} \frac{\sin x \cdot 2\hat{f}^e(\cos u)}{\sin u \quad \sin^2 u - \sin^2 x} \, du$$

$$= 2 \int_x^{\frac{\pi}{2}} \frac{\sin x \cdot \hat{f}^e(\cos u)}{\sin u \quad \sin^2 u - \sin^2 x} \, du.$$

So $\hat{f}^e(\cos u) \equiv 0$ gives $F(x) = 0$ for all $x \in (0, \frac{\pi}{2})$.

For the converse, we follow a proof given in [3]. Assume $F(x)$ is zero for all $x \in (0, \frac{\pi}{2})$, then

$$F(x) = \int_x^{\pi-x} \frac{\sin x \cdot f(u)}{\sin u \quad \sin^2 u - \sin^2 x} \, du = \int_x^{\frac{\pi}{2}} \frac{\sin x \cdot 2\hat{f}^e(\cos u)}{\sin u \quad \sin^2 u - \sin^2 x} \, du \equiv 0.$$

So the function

$$I(a) = \int_a^{\frac{\pi}{2}} \frac{\cos x \cdot F(x)}{\sin^2 x - \sin^2 a} \, dx$$

is zero for $a \in (0, \frac{\pi}{2}]$. Also, for such a, the function

$$\frac{1}{\left(\quad \sin^2 u - \sin^2 x \right) \left(\quad \sin^2 x - \sin^2 a \right)}$$

is (Lebesgue) integrable on the set $\{(u, x) \in [a, \frac{\pi}{2}] \times [a, \frac{\pi}{2}] \mid x \leq u\}$. Applying Fubini's Theorem we have

$$I(a) = \int_a^{\frac{\pi}{2}} \frac{2\hat{f}^e(\cos u)}{\sin u} \left(\int_a^u \frac{\sin x \cos x}{\sqrt{\sin^2 u - \sin^2 x} \quad \sin^2 x - \sin^2 a} \, dx \right) du.$$

The substitution $t = \sqrt{\sin^2 x - \sin^2 a} / \sqrt{\sin^2 u - \sin^2 x}$ gives

$$I(a) = 2 \int_a^{\frac{\pi}{2}} \frac{\hat{f}^e(\cos u)}{\sin u} \left(\int_0^\infty \frac{dt}{1 + t^2} \right) du = \pi \int_a^{\frac{\pi}{2}} \frac{\hat{f}^e(\cos u)}{\sin u} \, du.$$

As $\sin u$ is strictly positive on $(0, \frac{\pi}{2}]$, $I(a) = 0$ for all $a \in [0, \frac{\pi}{2}]$ implies $\hat{f}^e(\cos u) = 0$ for all $\cos u \in [-1, 1]$. That is, $\hat{f}$ is odd on $[-1, 1]$. $\qquad \square$

Proposition 6.3. *For a spherical surface of revolution M with metric*

$$ds^2 = E(u) \, du^2 + \sin^2(u) \, dv^2,$$

define $a_c(u) = \sqrt{E(u)} - c$ for any $c \in \mathbb{R}^+$. Then $\Phi_M(x) \equiv 2c\pi$ on $(0, \frac{\pi}{2})$ if and only if the function $\hat{a}_c$ defined by $\hat{a}_c(\cos u) = a_c(u)$ is an odd function from $[-1, 1]$ to $[-c, c]$.

Proof. Let S^2 be the standard 2-sphere of constant curvature 1 in $\mathbb{R}^3$ generated as a surface of revolution by the profile curve $\alpha(u) = (\cos u, \sin u)$. Thus, S^2 is parametrized by

$$\mathbf{x}(u, v) = (\cos u, \sin u \cos v, \sin u \sin v).$$

The geodesics on S^2 are great circles, so

$$\Phi_{S^2}(x) = 2 \int_x^{\pi - x} \frac{\sin x}{\sin u \quad \sqrt{\sin^2 u - \sin^2 x}} \, du \equiv 2\pi.$$

Then, for all $x \in (0, \frac{\pi}{2})$,

$$\Phi_M(x) = 2 \int_x^{\pi - x} \frac{\sin x \cdot \sqrt{E(u)}}{\sin u \quad \sqrt{\sin^2 u - \sin^2 x}} \, du = 2 \int_x^{\pi - x} \frac{\sin x \cdot (c + \hat{a}_c(\cos u))}{\sin u \quad \sqrt{\sin^2 u - \sin^2 x}} \, du$$

$$= c\Phi_{S^2}(x) + 2 \int_x^{\pi - x} \frac{\sin x \cdot \hat{a}_c(\cos u)}{\sin u \quad \sqrt{\sin^2 u - \sin^2 x}} \, du$$

$$= 2c\pi + 2 \int_x^{\pi - x} \frac{\sin x \cdot \hat{a}_c(\cos u)}{\sin u \quad \sqrt{\sin^2 u - \sin^2 x}} \, du.$$

The proof of the proposition now follows from lemma 6.2, which implies that $\hat{a}_c$ must be odd. For $u \in (0, \pi)$, $c + \hat{a}_c(\cos u) = \sqrt{E(u)} > 0$ so $\hat{a}_c(\cos u) > -c$ for $u \in (0, \pi)$. This implies that $\hat{a}_c(-\cos u) > -c$, so since $\hat{a}_c$ is odd, we have $\hat{a}_c(\cos u) = a_c(u) \in [-c, c]$ for $u \in (0, \pi)$. $\qquad \square$

At this point we are able to recover Bangert's result for spherical surfaces of revolution which have (smooth) Riemannian metrics, such as ellipsoids of revolution. We first need a computation.

Let ϕ_N, resp. ϕ_S, be the angle between the profile curve $\alpha(u) = (g(u), h(u)) = (g(u), \sin u)$ and the axis of rotation at $g(0)$, resp. $g(\pi)$. Then

$$(6.1a) \qquad \sin \phi_N = \frac{h'(0)}{\sqrt{[g'(0)]^2 + [h'(0)]^2}} = \frac{\cos(0)}{\sqrt{E(0)}} = \frac{1}{c + \hat{a}_c(1)}$$

and

$$(6.1\text{b}) \qquad \sin\phi_S = \frac{-h'(\pi)}{\sqrt{[g'(\pi)]^2 + [h'(\pi)]^2}} = \frac{-\cos(\pi)}{E(\pi)} = \frac{1}{c + \hat{a}_c(-1)} = \frac{1}{c - \hat{a}_c(1)},$$

with the last equality following since $\hat{a}_c$ is odd on $[-1, 1]$.

We now easily deduce Bangert's result for Riemannian spherical surfaces of revolution.

Corollary 6.4. *Every smooth Riemannian S^2 arising as a surface of revolution has infinitely many closed geodesics.*

Proof. The result follows if the surface has non-constant period function by corollary 5.5. Thus, we assume the surface has constant period function. Since the surface is a smooth manifold, the profile curve meets the x-axis at right angles, so that $\sin\phi_N = \sin\phi_S = 1$. Equations (6.1a) and (6.1b) imply that $c + \hat{a}_c(1) = c - \hat{a}_c(1) = 1$ so $0 = \hat{a}_c(1) = \hat{a}_c(-1)$ and $c = 1$. Hence $\Phi_M \equiv 2\pi$ and all oscillating geodesics close up after one oscillation. $\square$

7. Orbifolds of Revolution

Our work up to this point is valid for spherical surfaces of revolution in general. Since our main theorem 1.1 concerns orbifolds, we now specialize to that case. Spherical orbifolds of revolution are easily identifiable by their tangent cones at the poles. Namely, the tangent cone at a pole must be isometric to the metric quotient of the flat plane $\mathbb{R}^2$ by a finite cyclic group of rotations fixing the origin. Note that the tangent cone at a pole is generated by rotating the tangent line to the profile curve at the pole about the axis of rotation. If the cyclic groups at the poles are of different orders, the orbifold is commonly referred to as *bad* since it will not arise as a quotient of a Riemannian S^2 by a finite cyclic group of isometries [16].

In general, a flat right circular cone with vertex angle ϕ is obtained by identifying the edges of a plane circular sector of angle θ. The relation between θ and ϕ is easily computed: $\theta = 2\pi\sin\phi$. See figure 2. Thus, if the tangent cone at a pole of spherical orbifold of revolution is isometric to $\mathbb{R}^2/\mathbb{Z}_m$, then $\theta = 2\pi/m$ for a positive integer m. So, for an orbifold of revolution, if ϕ_N and ϕ_S are as in equations (6.1), we must have $\sin\phi_N = 1/m$ and $\sin\phi_S = 1/k$ for some positive integers m and k.

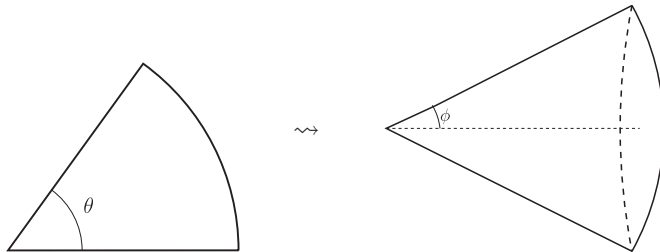FIGURE 2. Cone as quotient of a planar sector

We have the following restriction for spherical orbifolds of revolution of constant period function.

Theorem 7.1. *Let M be a spherical orbifold of revolution with metric $ds^2 = E(u)\,du^2 + \sin^2(u)\,dv^2$. Then $\Phi_M(x) \equiv 2c\pi$ on $(0, \frac{\pi}{2})$ implies c is rational.*

Proof. Equations (6.1) give

$$\sin\phi_N = \frac{1}{c + \hat{a}_c(1)} \quad \text{and} \quad \sin\phi_S = \frac{1}{c + \hat{a}_c(-1)} = \frac{1}{c - \hat{a}_c(1)},$$

since $\hat{a}_c$ is odd on $[-1, 1]$. As noted above, if M is an orbifold, then $c + \hat{a}_c(1)$ and $c - \hat{a}_c(1)$ must be integers. This easily implies that c be rational. In fact, $c = n/2$ for some positive integer n and $\Phi_M(x) = n\pi$ on $(0, \frac{\pi}{2})$. $\qquad\square$

We are now in a position to prove the main result of this paper.

Theorem 7.2. *There are no void spherical orbifolds of revolution. Hence, every orbifold of revolution has infinitely many closed geodesics.*

Proof. Suppose one such example existed. By corollary 5.7, we may assume that the profile curve has a single critical point and hence by proposition 6.1 that the metric on M is of the form required in theorem 7.1. By theorem 5.2 and corollary 5.5, Φ_M must be a constant, irrational multiple of π. However, by Theorem 7.1, an orbifold of revolution with constant Φ_M must have $\Phi_M \equiv 2c\pi$ with $c \in \mathbb{Q}$. Hence no such void spherical orbifold exists and all spherical orbifolds of revolution must have infinitely many geodesics. $\qquad\square$

8. Two examples

In summary, we can characterize all spherical surfaces of revolution with constant period function as having a metric of the form $ds^2 = (c + f(\cos u))^2\,du^2 + \sin^2(u)\,dv^2$ where

(1) c is a real constant,
(2) $f(\cos u)$ is an odd function from $[-1, 1]$ to $[-c, c]$.

The void spherical surfaces of revolution satisfy these conditions but have $c \notin \mathbb{Q}$, and hence are not orbifolds. The orbifolds of revolution with constant period function must satisfy (1), (2) and

(3) $c + f(1)$ and $c - f(1)$ are positive integers.

Example 8.1 (Tannery's pear). Take $c = 2$ and $a_c(u) = \cos u$ (so $\hat{a}_c(\cos u)$ is the identity map on $[-1, 1]$, and hence odd). This surface, known as Tannery's pear, has a period function that is constant 4π, so all non-meridional geodesics are closed. It also is an orbifold.

Taking as a profile curve $\alpha(u) = (g(u), h(u))$, where $h(u) = \sin u$ and

$$g(u) = \int_0^u E(t) - (h'(t))^2\,dt = \int_0^u (2 + \cos t)^2 - \cos^2 t\,dt$$
$$= \int_0^u \sqrt{4 + 4\cos t}\,dt = 4\sqrt{2}\sin(u/2)$$

gives a parametrization for Tannery's pear in $\mathbb{R}^3$:

$$\mathbf{x}(u, v) = (4\sqrt{2}\sin(u/2), \sin u \cos v, \sin u \sin v),$$

where $u \in [0, \pi]$ and $v \in [0, 2\pi]$.

We also have that

$$\sin \phi_N = \frac{\cos(0)}{\sqrt{E(0)}} = \frac{1}{3} \quad \text{and} \quad \sin \phi_S = \frac{-\cos(\pi)}{\sqrt{E(\pi)}} = 1,$$

so Tannery's pear is an orbifold. In orbifold terminology, Tannery's pear is a $\mathbb{Z}_3$-teardrop, as the metric is actually smooth at $u = \pi$ and the single cone point at $u = 0$ is of order 3. See figure 3.
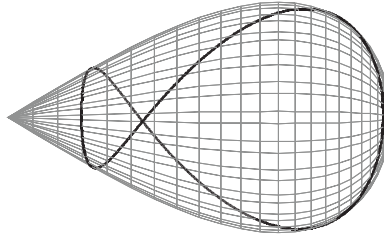

FIGURE 3.   A typical closed geodesic on a Tannery pear

Example 8.2 (A void surface). In the previous example, take $c = \sqrt{5}$. A surface with this metric can be isometrically embedded in $\mathbb{R}^3$ by the parametrization

$$\mathbf{x}(u, v) = \left(\int_0^u \sqrt{5 + 2\sqrt{5} \cos t} \, dt, \sin u \cos v, \sin u \sin v \right).$$

It has constant period function $2\pi\sqrt{5}$ and hence, its only closed geodesic is the parallel at $u = \pi/2$. However,

$$\sin \phi_N = \frac{1}{\sqrt{5} + 1} \quad \text{and} \quad \sin \phi_S = \frac{1}{\sqrt{5} - 1},$$

so, like all void spherical surfaces of revolution, this one is not an orbifold. See figure 4.
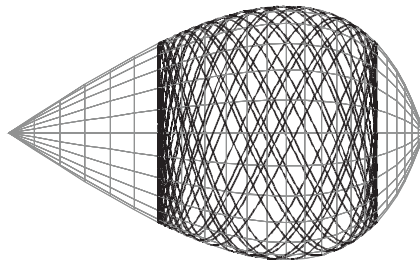

FIGURE 4.   An oscillating geodesic that is not closed

9. Proof of the Continuity of the Period Function

In this section, we prove theorem 5.3, which asserts the continuity of the period function. The notation used will be that from section 5.

Proof. Without loss of generality, we will assume the profile curve $\alpha = (g, h)$ of M is parametrized by arclength. Thus, $E = [g'(u)]^2 + [h'(u)]^2 \equiv 1$. Let γ_0 be an oscillating geodesic. By proposition 4.2 there is $0 < \varepsilon_0 < 1$ so that $h'(u) > 0$ on $B(b_0(\gamma_0), \varepsilon_0)$ and $h'(u) < 0$ on $B(b_1(\gamma_0), \varepsilon_0)$. Here $B(p, r)$ is the open interval of radius r centered at p. By shrinking ε_0 if necessary, we may assume $|h'(u)| \geq \eta > 0$ on $B = B(b_0(\gamma_0), \varepsilon_0) \cup B(b_1(\gamma_0), \varepsilon_0)$.

Let γ_n be a sequence of oscillating geodesics with $b_0(\gamma_n) \to b_0(\gamma_0)$ and $b_0(\gamma_n) \in B(b_0(\gamma_0), \frac{1}{4}\varepsilon_0^2)$. We may also assume $b_1(\gamma_n) \in B(b_1(\gamma_0), \frac{1}{4}\varepsilon_0^2)$, by choosing n large enough.

Consider the integrand $f_{\gamma_n}(u)$ of the period function Φ_M:

$$f_{\gamma_n}(u) = \frac{c_{\gamma_n}}{h(u)\sqrt{h^2(u) - c_{\gamma_n}^2}} = \frac{c_{\gamma_n}}{h(u)} \cdot \frac{1}{h(u) + c_{\gamma_n}} \cdot \frac{1}{h(u) - c_{\gamma_n}}$$

for $u \in (b_0(\gamma_n), b_1(\gamma_n))$. Since $c_{\gamma_n} = h(b_0(\gamma_n))$, applying the mean value theorem to the second factor in the last equality yields:

$$f_{\gamma_n}(u) = \frac{h(b_0(\gamma_n))}{h(u)} \cdot \frac{1}{h(u) + h(b_0(\gamma_n))} \cdot \frac{1}{h'(\xi_u)} \cdot \frac{1}{u - b_0(\gamma_n)}$$

for some $\xi_u \in (b_0(\gamma_n), u)$. Let $m = \inf_B h$ and define $\lambda = (2m\eta)^{-1/2}$. Then since $h(b_0(\gamma_n)) < h(u)$ for $u \in (b_0(\gamma_n), b_1(\gamma_n))$, we have

$$f_{\gamma_n}(u) \leq \frac{\lambda}{u - b_0(\gamma_n)} \qquad \text{on } (b_0(\gamma_n), b_0(\gamma_n) + \varepsilon_0),$$

and similarly,

$$f_{\gamma_n}(u) \leq \frac{\lambda}{b_1(\gamma_n) - u} \qquad \text{on } (b_1(\gamma_n) - \varepsilon_0, b_1(\gamma_n))$$

To show continuity at γ_0, we now prove that $|\Phi_M(\gamma_n) - \Phi_M(\gamma_0)| \to 0$ as $n \to \infty$. Unfortunately, to do this, we must consider separate cases.

Consider first the case where $b_0(\gamma_n) \nearrow b_0(\gamma_0)$. Define the positive numbers $\delta_n = b_0(\gamma_0) - b_0(\gamma_n) + \frac{1}{4}\varepsilon_0^2$ and $\mu_n = b_1(\gamma_n) - b_1(\gamma_0) + \frac{1}{4}\varepsilon_0^2$. Then

$$\frac{1}{2}|\Phi_M(\gamma_n) - \Phi_M(\gamma_0)| \leq$$

(9.1a)
$$\left| \int_{b_0(\gamma_n)}^{b_0(\gamma_n)+\delta_n} f_{\gamma_n} - \int_{b_0(\gamma_0)}^{b_0(\gamma_0)+\delta_n} f_{\gamma_0} \right| +$$

(9.1b)
$$\left| \int_{b_1(\gamma_n)-\mu_n}^{b_1(\gamma_n)} f_{\gamma_n} - \int_{b_1(\gamma_0)-\mu_n}^{b_1(\gamma_0)} f_{\gamma_0} \right| +$$

(9.1c)
$$\left| \int_{b_0(\gamma_n)+\delta_n}^{b_1(\gamma_n)-\mu_n} f_{\gamma_n} - \int_{b_0(\gamma_0)+\delta_n}^{b_1(\gamma_0)-\mu_n} f_{\gamma_0} \right|$$

We now turn our attention to each of the three terms in equations (9.1). For equation (9.1a), we have

$$\int_{b_0(\gamma_n)}^{b_0(\gamma_n)+\delta_n} f_{\gamma_n} - \int_{b_0(\gamma_0)}^{b_0(\gamma_0)+\delta_n} f_{\gamma_0}$$

(9.2) $$\leq \int_{b_0(\gamma_n)}^{b_0(\gamma_0)} f_{\gamma_n} + \int_{b_0(\gamma_0)}^{b_0(\gamma_n)+\delta_n} (f_{\gamma_n} - f_{\gamma_0}) + \int_{b_0(\gamma_n)+\delta_n}^{b_0(\gamma_0)+\delta_n} f_{\gamma_0}$$

We show that each of the terms in (9.2) can be made arbitrarily small. For the first term of (9.2):

$$\int_{b_0(\gamma_n)}^{b_0(\gamma_0)} f_{\gamma_n} \leq \int_{b_0(\gamma_n)}^{b_0(\gamma_0)} \lambda \left[u - b_0(\gamma_n)\right]^{-1/2} = 2\lambda(b_0(\gamma_0) - b_0(\gamma_n))^{1/2} < \lambda\varepsilon_0$$

For the third term of (9.2):

$$\int_{b_0(\gamma_n)+\delta_n}^{b_0(\gamma_0)+\delta_n} f_{\gamma_0} \leq \int_{b_0(\gamma_0)+\frac{1}{4}\varepsilon_0^2}^{b_0(\gamma_0)+\delta_n} \lambda \left[u - b_0(\gamma_0)\right]^{-1/2} = 2\lambda[(\delta_n)^{1/2} - \varepsilon_0/2]$$

which goes to 0 as $n \to \infty$. We handle the second term of (9.2) by applying the dominated convergence theorem: Note that $b_0(\gamma_n) + \delta_n = b_0(\gamma_0) + \frac{1}{4}\varepsilon_0^2$ and on the interval $(b_0(\gamma_0), b_0(\gamma_0) + \frac{1}{4}\varepsilon_0^2)$, $f_{\gamma_n} \to f_{\gamma_0}$ pointwise. Furthermore, on this interval

$$f_{\gamma_n} \leq \lambda \left[u - b_0(\gamma_n)\right]^{-1/2} < \lambda \left[u - b_0(\gamma_0)\right]^{-1/2} = g, \text{ and } \int_{b_0(\gamma_0)}^{b_0(\gamma_0)+\frac{1}{4}\varepsilon_0^2} g = \lambda\varepsilon_0. \text{ Thus,}$$

by dominated convergence

$$\int_{b_0(\gamma_0)}^{b_0(\gamma_n)+\delta_n} (f_{\gamma_n} - f_{\gamma_0}) \to 0 \text{ as } n \to \infty$$

For (9.1b) of equations (9.1) we write:

$$\int_{b_1(\gamma_n)-\mu_n}^{b_1(\gamma_n)} f_{\gamma_n} - \int_{b_1(\gamma_0)-\mu_n}^{b_1(\gamma_0)} f_{\gamma_0}$$

(9.3) $$\leq \int_{b_1(\gamma_0)}^{b_1(\gamma_n)} f_{\gamma_n} + \int_{b_1(\gamma_n)-\mu_n}^{b_1(\gamma_0)} (f_{\gamma_n} - f_{\gamma_0}) + \int_{b_1(\gamma_0)-\mu_n}^{b_1(\gamma_n)-\mu_n} f_{\gamma_0}$$

Arguing similarly, we conclude that each term of (9.3) can be made arbitrarily small. We omit the details.

Finally, for (9.1c) we have:

$$\int_{b_0(\gamma_n)+\delta_n}^{b_1(\gamma_n)-\mu_n} f_{\gamma_n} - \int_{b_0(\gamma_0)+\delta_n}^{b_1(\gamma_0)-\mu_n} f_{\gamma_0}$$

(9.4) $$\leq \int_{b_0(\gamma_n)+\delta_n}^{b_0(\gamma_0)+\delta_n} f_{\gamma_n} + \int_{b_0(\gamma_0)+\delta_n}^{b_1(\gamma_0)-\mu_n} (f_{\gamma_n} - f_{\gamma_0}) + \int_{b_1(\gamma_0)-\mu_n}^{b_1(\gamma_n)-\mu_n} f_{\gamma_n}$$

We now show that each of the terms in (9.4) can be made arbitrarily small. Note that $b_0(\gamma_0)+\delta_n = b_0(\gamma_0)+[b_0(\gamma_0)-b_0(\gamma_n)]+\frac{1}{4}\varepsilon_0^2 < b_0(\gamma_0)+\frac{1}{2}\varepsilon_0^2 = b_0(\gamma_n)+[b_0(\gamma_0)-$

$b_0(\gamma_n)] + \frac{1}{2}\varepsilon_0^2 < b_0(\gamma_n) + \frac{3}{4}\varepsilon_0^2 < b_0(\gamma_n) + \varepsilon_0$. Thus, for the first term of (9.4):

$$\int_{b_0(\gamma_n)+\delta_n}^{b_0(\gamma_0)+\delta_n} f_{\gamma_n} \le \int_{b_0(\gamma_n)+\delta_n}^{b_0(\gamma_0)+\delta_n} \lambda \left[u - b_0(\gamma_n)\right]^{-1/2}$$

$$= 2\lambda \left[(2\delta_n - \varepsilon_0^2/4)^{1/2} - (\delta_n)^{1/2})\right] \to 0 \text{ as } n \to \infty$$

For the third term of (9.4) we note that, similar to before, $b_1(\gamma_0) - \mu_n > b_1(\gamma_n) - \varepsilon_0$ thus:

$$\int_{b_1(\gamma_0)-\mu_n}^{b_1(\gamma_n)-\mu_n} f_{\gamma_n} \le \int_{b_1(\gamma_0)-\mu_n}^{b_1(\gamma_n)-\mu_n} \lambda \left[b_1(\gamma_n) - u\right]^{-1/2}$$

$$= -2\lambda \left[(\mu_n)^{1/2} - (2\mu_n - \varepsilon_0^2/4)^{1/2})\right] \to 0 \text{ as } n \to \infty$$

For the middle term of (9.4), just note that f_{γ_n} and f_{γ_0} are both bounded on the interval $(b_0(\gamma_0) + \delta_n, b_1(\gamma_0) - \mu_n)$ and that $f_{\gamma_n} \to f_{\gamma_0}$ pointwise. Dominated convergence then implies that this term approaches zero as $n \to \infty$.

This is enough to verify continuity of the period function in the case when $b_0(\gamma_n) \nearrow b_0(\gamma_0)$. We now complete the continuity proof by treating the case where $b_0(\gamma_n) \searrow b_0(\gamma_0)$. The proof here is essentially obtained by interchanging the roles of γ_n and γ_0 in what has gone before. However, there are some minor technical differences, which we point out.

To this end, define the positive numbers $\delta_n = b_0(\gamma_n) - b_0(\gamma_0) + \frac{1}{4}\varepsilon_0^2$ and $\mu_n = b_1(\gamma_0) - b_1(\gamma_n) + \frac{1}{4}\varepsilon_0^2$. Then

$$\frac{1}{2}|\Phi_M(\gamma_0) - \Phi_M(\gamma_n)| \le$$

(9.5a)
$$\int_{b_0(\gamma_0)}^{b_0(\gamma_0)+\delta_n} f_{\gamma_0} - \int_{b_0(\gamma_n)}^{b_0(\gamma_n)+\delta_n} f_{\gamma_n} +$$

(9.5b)
$$\int_{b_1(\gamma_0)-\mu_n}^{b_1(\gamma_0)} f_{\gamma_0} - \int_{b_1(\gamma_n)-\mu_n}^{b_1(\gamma_n)} f_{\gamma_n} +$$

(9.5c)
$$\int_{b_0(\gamma_0)+\delta_n}^{b_1(\gamma_0)-\mu_n} f_{\gamma_0} - \int_{b_0(\gamma_n)+\delta_n}^{b_1(\gamma_n)-\mu_n} f_{\gamma_n}$$

For equation (9.5a), we have

$$\int_{b_0(\gamma_0)}^{b_0(\gamma_0)+\delta_n} f_{\gamma_0} - \int_{b_0(\gamma_n)}^{b_0(\gamma_n)+\delta_n} f_{\gamma_n}$$

(9.6)
$$\le \int_{b_0(\gamma_0)}^{b_0(\gamma_n)} f_{\gamma_0} + \int_{b_0(\gamma_n)}^{b_0(\gamma_0)+\delta_n} (f_{\gamma_0} - f_{\gamma_n}) + \int_{b_0(\gamma_0)+\delta_n}^{b_0(\gamma_n)+\delta_n} f_{\gamma_n}$$

As before, we show that each of the terms in (9.6) can be made arbitrarily small. For the first term of (9.6):

$$\int_{b_0(\gamma_0)}^{b_0(\gamma_n)} f_{\gamma_0} \le \int_{b_0(\gamma_0)}^{b_0(\gamma_n)} \lambda \left[u - b_0(\gamma_0)\right]^{-1/2} = 2\lambda(b_0(\gamma_n) - b_0(\gamma_0))^{1/2} < \lambda\varepsilon_0$$

For the third term of (9.6):

$$\int_{b_0(\gamma_0)+\delta_n}^{b_0(\gamma_n)+\delta_n} f_{\gamma_n} \leq \int_{b_0(\gamma_n)+\frac{1}{4}\varepsilon_0^2}^{b_0(\gamma_n)+\delta_n} \lambda\left[u - b_0(\gamma_n)\right]^{-1/2} = 2\lambda[(\delta_n)^{1/2} - \varepsilon_0/2]$$

which goes to 0 as $n \to \infty$. We now handle the second term of (9.6). Since the functions f_{γ_n} are not defined on the entire domain of f_{γ_0}, there is a minor technical difference between this situation and the analogous one for (9.2). Define

$$\hat{f}_{\gamma_n} = \begin{array}{ll} f_{\gamma_n} & \text{on } \left(b_0(\gamma_n), b_0(\gamma_n) + \frac{1}{4}\varepsilon_0^2\right), \\ f_{\gamma_0} & \text{on } \left(b_0(\gamma_0), b_0(\gamma_n)\right) \cup \left(b_0(\gamma_n) + \frac{1}{4}\varepsilon_0^2, b_0(\gamma_0) + \frac{1}{4}\varepsilon_0^2\right) \end{array}$$

Then $\displaystyle\int_{b_0(\gamma_n)}^{b_0(\gamma_0)+\delta_n} (f_{\gamma_0} - f_{\gamma_n}) = \int_{b_0(\gamma_0)}^{b_0(\gamma_0)+\frac{1}{4}\varepsilon_0^2} (f_{\gamma_0} - \hat{f}_{\gamma_n}).$

Then $\hat{f}_{\gamma_n} \to f_{\gamma_0}$ a.e. on $(b_0(\gamma_0), b_0(\gamma_0) + \frac{1}{4}\varepsilon_0^2)$ and $\hat{f}_{\gamma_n} \leq g_n$ where

$$g_n = \begin{array}{ll} \lambda[u - b_0(\gamma_n)]^{-1/2} & \text{on } \left(b_0(\gamma_n), b_0(\gamma_n) + \frac{1}{4}\varepsilon_0^2\right), \\ \lambda[u - b_0(\gamma_0)]^{-1/2} & \text{on } \left(b_0(\gamma_0), b_0(\gamma_n)\right) \cup \left(b_0(\gamma_n) + \frac{1}{4}\varepsilon_0^2, b_0(\gamma_0) + \frac{1}{4}\varepsilon_0^2\right) \end{array}$$

Furthermore, on $(b_0(\gamma_0), b_0(\gamma_0) + \frac{1}{4}\varepsilon_0^2)$, $g_n \to g$ a.e. where $g = \lambda[u - b_0(\gamma_0)]^{-1/2}$ and $\displaystyle\lim_{n\to\infty} \int_{b_0(\gamma)}^{b_0(\gamma_0)+\frac{1}{4}\varepsilon_0^2} g_n = \int_{b_0(\gamma)}^{b_0(\gamma_0)+\frac{1}{4}\varepsilon_0^2} g$. By a modified dominated convergence theorem which may be found, for example, in [15], we may conclude that

$$\int_{b_0(\gamma_n)}^{b_0(\gamma_0)+\delta_n} (f_{\gamma_0} - f_{\gamma_n}) = \int_{b_0(\gamma_0)}^{b_0(\gamma_0)+\frac{1}{4}\varepsilon_0^2} (f_{\gamma_0} - \hat{f}_{\gamma_n}) \to 0 \text{ as } n \to \infty$$

Arguing similarly, we conclude that (9.5b) can be made arbitrarily small. We omit the details. Lastly, for (9.5c) we have:

$$\int_{b_0(\gamma_0)+\delta_n}^{b_1(\gamma_0)-\mu_n} f_{\gamma_0} - \int_{b_0(\gamma_n)+\delta_n}^{b_1(\gamma_n)-\mu_n} f_{\gamma_n}$$

$$(9.7) \qquad \leq \int_{b_0(\gamma_0)+\delta_n}^{b_0(\gamma_n)+\delta_n} f_{\gamma_0} + \int_{b_0(\gamma_n)+\delta_n}^{b_1(\gamma_n)-\mu_n} (f_{\gamma_0} - f_{\gamma_n}) + \int_{b_1(\gamma_n)-\mu_n}^{b_1(\gamma_0)-\mu_n} f_{\gamma_0}$$

Arguing as we did for expression (9.4), we may conclude that the first and third terms of (9.7) approach 0 as $n \to \infty$. For the middle term of (9.7), proceed in the same way as we did to handle the second term of (9.6) by defining

$$\hat{f}_{\gamma_n} = \begin{array}{ll} f_{\gamma_n} & \text{on } \left(b_0(\gamma_n) + \delta_n, b_1(\gamma_n) - \mu_n\right), \\ f_{\gamma_0} & \text{on } \left(b_0(\gamma_0) + \delta_n, b_0(\gamma_n) + \delta_n\right) \cup \left(b_1(\gamma_n) - \mu_n, b_1(\gamma_0) - \mu_n\right) \end{array}$$

and applying dominated convergence. This completes the proof of the continuity of the period function. $\qquad\square$

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