The q-exponential generating function for permutations by consecutive patterns and inversions

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Abstract

The inverse of Fedou's insertion-shift bijection is used to deduce a general form for the q-exponential generating function for permutations by consecutive patterns (overlaps allowed) and inversion number from a result due to Jackson and Goulden for enumerating words by distinguished factors. Explicit q-exponential generating functions are then derived for permutations by the consecutive patterns $12\dots m$, $12\dots(m \quad 2)m(m \quad 1)$, $1m(m \quad 1)\dots 2$, and by the pair of consecutive patterns $(123, 132)$.

1. Introduction

Denoting the set of permutations of $\{1, 2, \dots, n\}$ by S_n, let $\sigma = \sigma_1\sigma_2\dots\sigma_n \in S_n$ and $p \in S_m$ with $m \leqslant n$. For $i \leqslant n - m + 1$, $\sigma_i\sigma_{i+1}\dots\sigma_{i+m-1}$ is said to be a consecutive pattern p if $\bar{\sigma}_i\bar{\sigma}_{i+1}\dots\bar{\sigma}_{i+m-1} = p$ where, for $0 \leqslant k \leqslant m-1$, $\bar{\sigma}_{i+k}$ denotes the number of elements in the set $\{\sigma_i, \sigma_{i+1}, \dots, \sigma_{i+m-1}\}$ that are less than or equal to σ_{i+k}. For example, $p = 132 \in S_3$ occurs consecutively twice in $\sigma = 14253 \in S_5$: $\bar{\sigma}_1\bar{\sigma}_2\bar{\sigma}_3 = \overline{1}\overline{4}\overline{2} = 132$ and $\bar{\sigma}_3\bar{\sigma}_4\bar{\sigma}_5 = \overline{2}\overline{5}\overline{3} = 132$. The two consecutive occurrences of $p = 132$ in $\sigma = 14253$ overlap in the sense that $\sigma_3 = 2$ is common to both.

A number of results for the enumeration of permutations by consecutive patterns have recently been discovered. Kitaev [7,8] determined a formula that related the exponential generating function for permutations by the maximal number of nonoverlapping consecutive occurrences of any pattern p to the exponential generating function for permutations having no consecutive occurrences of p. Subsequently, Mendes and Remmel [9] used the theory of symmetric functions to reprove and extend Kitaev's result in three ways: They replaced the single pattern p by a set of patterns, expanded the theory to deal with tuples of permutations, and threw the inversion number in so as to obtain q-analogs. Mendes and Remmel further obtained some results on permutations by consecutive patterns (overlaps allowed) subject to a certain constraint. Prior to [9], Elizalde and Noy [2] also obtained several results on permutations by consecutive patterns (overlaps allowed). Their approach was based on solving systems of linear ordinary differential equations.

The purpose here is to present yet another approach to the problem of enumerating permutations by consecutive patterns (overlaps allowed). In brief, the inverse of Fedou's [3] insertion-shift bijection is used to convert Jackson and Goulden's [5] result [6, Theorem 2.8.6] for enumerating words by distinguished factors into an identity for the q-exponential generating function for permutations by consecutive patterns and inversion number. To be more explicit, the number of times a pattern p occurs consecutively in a permutation σ will be denoted by $p(\sigma)$. A nonempty finite set P of patterns of possibly varying lengths, each greater than or equal to 2, is said to be *permissible* if no pattern in P occurs as a consecutive pattern in another pattern in P. The *inversion number* of $\sigma \in S_n$, denoted by $\operatorname{inv}\sigma$, is defined to be the number of integer pairs (i, j) such that $1 \leqslant i < j \leqslant n$ and $\sigma_i > \sigma_j$. The *q-analog* and *q-factorial* of a nonnegative integer n are respectively defined by

$$[n] = 1 + q + q^2 + \cdots + q^{n-1} \quad \text{and} \quad [n]! = [1][2]\cdots[n].$$

After presenting Jackson and Goulden's result in Section 2 and the inverse of Fedou's bijection in Section 3, the following corollary is deduced in Section 4.

Corollary 1. *If P is a permissible set of consecutive patterns and*

$$A_{n,P}(q, \mathbf{t}) = \sum_{\sigma \in S_n} q^{\operatorname{inv}\sigma} \prod_{p \in P} t_p^{p(\sigma)},$$

then

$$\sum_{n \geqslant 0} \frac{A_{n,P}(q, \mathbf{t}) z^n}{[n]!} = \bigl(1 - z - C_P\bigl(q, \mathbf{t} - \mathbf{1}, (1-q)z\bigr)\bigr)^{-1}, \tag{1}$$

where $C_P(q, \mathbf{t}, z)$ is as in (7).

Although not done so here, Corollary 1 may easily be extended to tuples of permutations. Such extensions have been considered by Carlitz, Scoville, and Vaughan [1], by Fedou and Rawlings [4], and by Mendes and Remmel [9].

The difficulty in using (1) to determine the q-exponential generating function for permutations by a particular permissible set P lies in computing a certain q-sum that arises in connection with $C_P(q, \mathbf{t}, z)$. The details when $P = \{p\}$ for $p = 12\ldots(m-2)m(m-1)$, $p = 1m(m-1)\ldots 2$, and $p = 12\ldots m \in S_m$ are worked out in Sections 5–7. The q-exponential generating function for the joint distribution of 123 and 132 is derived in Section 8.

2. Words by distinguished factors

For the purposes at hand, Jackson and Goulden's [6] Theorem 2.8.6 is best as formulated by Stanley [10, pp. 266–267] in the context of formal series over the free monoid. Let X be a nonempty alphabet and X^* be the free monoid of words formed with letters from X. The *length* of a word $w \in X^*$, denoted by $l(w)$, is the number of letters in w. The set of words of length n in X^* is signified by X^n. A word $v \in X^*$ is said to be a *factor* of $w \in X^*$ if there exist words r and s (possibly empty) in X^* such that $w = rvs$. The ith letter of a nonempty word w will be denoted by w_i.

A nonempty set $D \subset X^*$ is said to be *distinguished* if no word in D is a factor of another word in D. A *cluster* of distinguished factors (or D*-cluster*) is an ordered triple (w, δ, β) where

$$\begin{aligned}
&w = w_1 w_2 \dots w_{l(w)} \in X^*,\\
&\delta = (d_1, d_2, \dots, d_k) \quad \text{for some } k \geqslant 1 \text{ with each } d_i \in D, \quad \text{and}\\
&\beta = (b_1, b_2, \dots, b_k) \quad \text{is a } k\text{-tuple of positive integers}
\end{aligned}$$

such that

- $d_i = w_{b_i} w_{b_i+1} \dots w_{b_i+l(d_i)-1}$ for $1 \leqslant i \leqslant k$ (each d_i appears as a factor of w),
- $1 = b_1 < b_2 < \cdots < b_{k-1} < b_k = l(w) - l(d_k) + 1$ (w begins with a copy of d_1, a copy of d_{i+1} begins to the right of the initial letter of a copy of d_i in w, and w ends with d_k), and
- $b_{i+1} \leqslant b_i + l(d_i) - 1$ (the copy of d_i beginning at w_{b_i} and the copy of d_{i+1} beginning at $w_{b_{i+1}}$ overlap in w).

Roughly speaking, a D-cluster is a word w together with a recipe for covering w with overlapping distinguished factors. A word w which appears as the first component in a D-cluster is said to be D*-coverable*.

Let C_D denote the set of D-clusters. The *cluster generating function* is then defined to be the formal series

$$C_D(\mathbf{y}, w) = \sum_{(w,\delta,\beta)\in C_D} \left(\prod_{d\in D} y_d^{d(\delta)} \right) w \in \Re[[y_d\colon d \in D]]\langle\langle X \rangle\rangle, \tag{2}$$

where $d(\delta)$ denotes the number of times d appears as a component in δ. Theorem 2.8.6 in [6] as reformulated by Stanley reads as follows.

Theorem 1 *(Jackson and Goulden). For a nonempty alphabet X and a distinguished set $D \subset X^*$, the generating function for words by distinguished factors is given by*

$$\sum_{w\in X^*} \left(\prod_{d\in D} y_d^{d(w)} \right) w = \left(1 - \sum_{x\in X} x - C_D(\mathbf{y} - \mathbf{1}, w) \right)^{-1}, \tag{3}$$

where $d(w)$ denotes the number of times d appears as a factor of w and $C_D(\mathbf{y} - \mathbf{1}, w)$ is obtained by replacing each y_d in $C_D(\mathbf{y}, w)$ with $y_d - 1$.

Independently of Jackson and Goulden, Zeilberger presented an erroneous version of Theorem 1 in [11]. Although stated and proven in both [6,10] for X finite, Theorem 1 actually holds for any nonempty alphabet. The proof for a finite alphabet works without modification for an infinite alphabet.

Results for enumerating permutations by consecutive patterns may be extracted directly from (3). For instance, consider the problem of determining the number $c_{n,k}$ of permutations in S_n with exactly k occurrences of the consecutive pattern $p = 12\dots m \in S_m$ where $m \geqslant 2$.

In order to apply (3), take $X=\{1,2,\dots,n\}$ and let $D=\{w\in X^m\colon\ 1\leqslant w_1\leqslant w_2\leqslant\cdots\leqslant w_m\}$. Then set $y_d=y$ for all $d\in D$ and map $w\mapsto z_{w_1}z_{w_2}\cdots z_{w_n}$ where $z_1,z_2,\dots,z_n$ are commuting indeterminates. With these choices in (3), $c_{n,k}$ is the coefficient of $y^k z_1z_2\cdots z_n$. The derivation of the exponential generating function for $c_{n,k}$ requires much more work. Corollary 1 provides a more elegant means of obtaining the generating function for $c_{n,k}$.

3. The inverse of Fedou's bijection

Let $N=\{0,1,2,\dots\}$ and $\Lambda_n=\{\lambda\in N^n\colon\ 0\leqslant\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_n\}$. For $w\in N^n$, set $\|w\|=w_1+w_2+\cdots+w_n$. An element $\lambda\in\Lambda_n$ satisfying $\|\lambda\|=k$ may be viewed as a partition of k with at most n parts. Note that

$$\sum_{\lambda\in\Lambda_n} q^{\|\lambda\|}=\frac{1}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

The number of inversions in $\sigma\in S_n$ induced by σ_i is defined to be $\operatorname{inv}_i\sigma=|\{j\colon\ i<j\leqslant n,\ \sigma_i>\sigma_j\}|$.

The inverse of Fedou's [3] insertion-shift bijection f_n as described by Foata (personal communication) maps $S_n\times\Lambda_n$ onto N^n by the rule $f_n(\sigma,\lambda)=w$ where $w_i=\operatorname{inv}_i\sigma+\lambda_{\sigma_i}$. For instance, f_6 sends $(314652, 011112)\in S_6\times\Lambda_6$ to $302421\in N^6$.

Tracking the pattern and inversion counts through the bijection f_n is key. Let $\sigma\in S_n$ and $p\in S_m$. Further, suppose that $f_n(\sigma,\lambda)=w$. For the inversion count, note that the definition of f_n implies that

$$\operatorname{inv}\sigma+\|\lambda\|=\|w\|.$$

For the pattern count, observe for $1\leqslant i<k\leqslant n$ that

$$\begin{aligned}&\sigma_i<\sigma_k\quad\text{if and only if } w_i\leqslant w_k+\big|\{j\colon\ i<j<k,\ \sigma_i>\sigma_j\}\big|\quad\text{and}\\ &\sigma_i>\sigma_k\quad\text{if and only if } w_i> w_k+\big|\{j\colon\ i<j<k,\ \sigma_i>\sigma_j\}\big|.\end{aligned}\tag{4}$$

To see how (4) may be used to track a pattern through f_n, consider the case when $p=14352\in S_5$ occurs at $\sigma_i\sigma_{i+1}\sigma_{i+2}\sigma_{i+3}\sigma_{i+4}$ in σ. Then

$$\sigma_{i+3}>\sigma_{i+1}>\sigma_{i+2}>\sigma_{i+4}>\sigma_i.\tag{5}$$

In view of (4), (5) holds if and only if

$$w_{i+3}+1\geqslant w_{i+1}>w_{i+2}>w_{i+4}\geqslant w_i.$$

So (4) provides a one-to-one correspondence between the occurrences of the consecutive pattern $p=14352$ in σ and the factors in w belonging to the distinguished set

$$D_{14352}=\big\{v=v_1v_2v_3v_4v_5\in N^5\colon\ v_4+1\geqslant v_2>v_3>v_5\geqslant v_1\big\}.$$

In general, for any pattern p, there is a one-to-one correspondence of the set of consecutive occurrences of p in σ with the set of factors in w belonging to a distinguished set D_p completely determined by (4). Furthermore,

$$p(\sigma)=\sum_{d\in D_p} d(w).\tag{6}$$

Tracking a permissible set of patterns is much the same. If P is a permissible set of patterns, then $D_P=\bigcup_{p\in P} D_p$ is a distinguished set of factors and (6) holds for each $p\in P$.

4. Proof of Corollary 1

Let P be a permissible set of patterns. The map ϕ that sends

- $w \in N^*$ to $q^{\|w\|} z^{l(w)}$ where q and z are commuting indeterminates and
- y_d to t_p for each pattern $p \in P$ and each $d \in D_p$

extends to a continuous homomorphism from $\Re[\![y_d\colon\ d \in C_{D_P}]\!]\langle\langle N\rangle\rangle$ to $\Re[\![t_p\colon\ p \in P]\!][\![q, z]\!]$. The image under ϕ of the cluster generating function $C_{D_P}(\mathbf{y}, w)$ is

$$C_P(q, \mathbf{t}, z) = \sum_{(w,\delta,\beta)\in C_{D_P}} q^{\|w\|} z^{l(w)} \prod_{p\in P} \prod_{d\in D_p} t_p^{d(\delta)}. \tag{7}$$

The properties of f_n and the application of ϕ to Theorem 1 justify the following computation:

$$\begin{aligned}
&\sum_{n\geqslant 0} \frac{A_{n,P}(q,\mathbf{t})z^n}{(1-q)(1-q^2)\cdots(1-q^n)} \\
&\quad = \sum_{n\geqslant 0} z^n \sum_{(\sigma,\lambda)\in S_n\times\Lambda_n} q^{\operatorname{inv}\sigma+\|\lambda\|} \prod_{p\in P} t_p^{p(\sigma)} \\
&\quad = \sum_{w\in N^*} q^{\|w\|} z^{l(w)} \prod_{p\in P} \prod_{d\in D_p} t_p^{d(w)} \\
&\quad = \left(1 - \frac{z}{1-q} - \sum_{(w,\delta,\beta)\in C_{D_P}} q^{\|w\|} z^{l(w)} \prod_{p\in P} \prod_{d\in D_p} \left(t_p^{d(\delta)} - 1\right)\right)^{-1}.
\end{aligned}$$

The replacement of z by $z(1-q)$ then yields (1).

The above derivation is just an extension of the proof used by Fedou and Rawlings [4] to obtain the q-exponential generating function for the q-Eulerian polynomials. In contrast to the methods used to enumerate permutations by consecutive patterns in [2,9], the inversion number is indispensable in the approach embodied in Corollary 1.

5. Consecutive $12\ldots(m-2)m(m-1)$ patterns

Beginning with the case $m = 3$, let $p = 132$. The associated set of distinguished factors is $D_{132} = \{w \in N^3\colon\ w_2 > w_3 \geqslant w_1\}$. Note that two distinct distinguished factors may overlap in a word in at most one letter. So for $w \in N^*$ to be D_{132}-coverable, $l(w)$ must be odd and greater than or equal to three. Moreover, a D_{132}-coverable word can be covered with distinguished factors in only one way. For instance, $w = 27586 \in \{0, 1, \ldots, 8\}^5$ may only be covered with $d_1 = 275 \in D_{132}$ beginning at $w_1 = 2$ and $d_2 = 586 \in D_{132}$ beginning at $w_3 = 5$. In other words, $w = 27586$ appears in but one D_{132}-cluster, namely $(27586, (275, 586), (1, 3))$.

In general, $(w, \delta, \beta) \in C_{D_{132}}$ is such that

$$w \in N^{2k+1}, \quad \delta = (d_1, d_2, \ldots, d_k) \quad \text{with each } d_i \in D_{132}, \quad \text{and} \quad \beta = (1, 3, \ldots, 2k-1),$$

where $k = \sum_{d\in D_{132}} d(w)$. Thus, there is a natural one-to-one correspondence between $C_{D_{132}}$ and the set

$$\bigcup_{k\geqslant 1} \left\{w \in N^{2k+1}\colon\ w_{2j} > w_{2j+1} \geqslant w_{2j-1} \text{ for } 1 \leqslant j \leqslant k\right\}. \tag{8}$$

Denoting the kth set appearing in the union in (8) by N_{132}^{2k+1} and putting $t = t_{132}$ in (7), we have

$$C_{132}(q,t,z) = \sum_{k \geqslant 1} t^k z^{2k+1} \sum_{w \in N_{132}^{2k+1}} q^{\|w\|}. \tag{9}$$

As an example of how to compute the "q-sum" on the right-hand side of (9), note when $k = 2$ that

$$\begin{aligned} \sum_{w \in N_{132}^5} q^{\|w\|} &= \sum_{w_1 \geqslant 0} q^{w_1} \sum_{w_3 \geqslant w_1} q^{w_3} \sum_{w_5 \geqslant w_3} q^{w_5} \sum_{w_2 > w_3} q^{w_2} \sum_{w_4 > w_5} q^{w_4} \\ &= \frac{q^2}{(1-q)^2(1-q^2)(1-q^4)(1-q^5)}. \end{aligned}$$

In general,

$$\sum_{w \in N_{132}^{2k+1}} q^{\|w\|} = \frac{q^k}{(1-q)^k(1-q^2)(1-q^4)\cdots(1-q^{2k})(1-q^{2k+1})}.$$

Thus,

$$C_{132}\big(q, t-1, (1-q)z\big) = \sum_{k \geqslant 1} \frac{q^k (t-1)^k z^{2k+1}}{[2][4]\cdots[2k][2k+1]}. \tag{10}$$

By defining the *q-integral* of a formal series in z to be

$$\int \sum_{k \geqslant 0} a_k z^k \, dz_q = \sum_{k \geqslant 0} \frac{a_k z^{k+1}}{[k+1]},$$

(10) may be expressed as

$$C_{132}\big(q, t-1, (1-q)z\big) = \int \sum_{k \geqslant 1} \frac{(q(t-1)z^2)^k}{[2][4]\cdots[2k]} \, dz_q.$$

In view of Corollary 1, it follows that

$$\sum_{n \geqslant 0} \frac{A_{n,132}(q,t) z^n}{[n]!} = \left(1 - \int \sum_{k \geqslant 0} \frac{(q(t-1)z^2)^k}{[2][4]\cdots[2k]} \, dz_q\right)^{-1}. \tag{11}$$

Elizalde and Noy [2] deduced (11) for the case $q = 1$. Mendes and Remmel [9] subsequently indicated how to obtain (11) for general q.

Treatment of the pattern $p = 12\ldots(m-2)m(m-1) \in S_m$ for $m > 3$ is much the same as for $m = 3$. Omitting the details, the extension of (11) to $m \geqslant 3$ is recorded below.

Corollary 2. *The q-exponential generating function for permutations by the consecutive pattern $p = 12\ldots(m-2)m(m-1) \in S_m$ ($m \geqslant 3$ and $t = t_p$) and by the inversion number is*

$$\sum_{n \geqslant 0} \frac{A_{n,p}(q,t) z^n}{[n]!} = \left(1 - \int \sum_{k \geqslant 0} \frac{(q(t-1)z^{m-1})^k \prod_{j=1}^{k-1}[j(m-1)+1]}{[k(m-1)]!} \, dz_q\right)^{-1}.$$

6. Consecutive $1m(m-1)\dots 2$ patterns

In general, consecutive patterns $p \in S_m$ with a single peak ($p_i < p_{i+1} > p_{i+2}$ for but one i), such as the one of Section 5, are easily dealt with. As a second example of such a pattern, consider $p = 1m(m-1)\dots 2 \in S_m$ where $m \geqslant 3$. Arguing in much the same way as in Section 5, there is a natural one-to-one correspondence between C_{D_p} and $\bigcup_{k\geqslant 1} N_p^{(m-1)k+1}$ where $N_p^{(m-1)k+1}$ is the set

$$\big\{w \in N^{(m-1)k+1}\colon\ w_{(m-1)j+2} > w_{(m-1)j+3} > \cdots > w_{(m-1)j+m} \text{ and } w_{(m-1)j+1} \leqslant w_{(m-1)(j+1)+1} \text{ for } 0 \leqslant j \leqslant k-1\big\}.$$

As

$$\sum_{w\in N_p^{(m-1)k+1}} q^{\|w\|} = \frac{q^{k\binom{m-1}{2}}}{(1-q^{(m-1)k+1})\prod_{j=1}^{m-2}(1-q^j)^k \prod_{i=1}^{k}(1-q^{(m-1)i})},$$

the corresponding cluster generating function (with $t = t_p$) is

$$C\big(q,t,(1-q)z\big) = \int \sum_{k\geqslant 1} \frac{(q^{\binom{m-1}{2}} t z^{m-1})^k}{([m-2]!)^k \prod_{i=1}^{k}[(m-1)i]}\, dz_q.$$

Therefore, by Corollary 1, we have

Corollary 3. *The q-exponential generating function for permutations by the consecutive pattern $p = 1m(m-1)\dots 2 \in S_m$ ($m \geqslant 3$ and $t = t_p$) and by the inversion number is*

$$\sum_{n\geqslant 0} \frac{A_{n,p}(q,t)z^n}{[n]!} = \left(1 - \int \sum_{k\geqslant 0} \frac{(q^{\binom{m-1}{2}}(t-1)z^{m-1})^k}{([m-2]!)^k \prod_{i=1}^{k}[(m-1)i]}\, dz_q\right)^{-1}.$$

The case $m = 4$ and $q = 1$ of Corollary 3 is due to Elizalde and Noy [2].

7. Consecutive $12\dots m$ patterns

For $m \geqslant 2$, let $p = 12\dots m \in S_m$. The associated set of distinguished factors is $D_p = \{w \in N^m\colon\ w_1 \leqslant w_2 \leqslant \cdots \leqslant w_m\} = \Lambda_m$. Note that, for $n \geqslant m$, $w \in N^n$ is D_p-coverable if and only if $w \in \Lambda_n$. Generally, $w \in \Lambda_n$ may be covered by distinguished factors in more than one way (since two distinct distinguishable factors may overlap in a word in as many as $m-1$ letters). For instance, the word $w = 24789 \in \{0,1,\dots,9\}^5$ appears as the first component in both of the D_{123}-clusters

$$\big(24789, (247, 789), (1,3)\big) \quad \text{and} \quad \big(24789, (247, 478, 789), (1,2,3)\big).$$

Let $a_{n,k}(m)$ be the number of times a $w \in \Lambda_n$ appears in a D_p-cluster (w,δ,β) with δ having k components. The cluster generating function (with $t = t_p$) is then

$$\begin{aligned} C(q,t,z) &= \sum_{(w,\delta,\beta)} z^{l(w)} q^{\|w\|} \prod_{d\in D_p} t^{d(\delta)} = \sum_{n\geqslant m} z^n \sum_{k=1}^{n-m+1} a_{n,k}(m) t^k \sum_{w\in\Lambda_n} q^{\|w\|} \\ &= \sum_{n\geqslant m} \frac{z^n}{(1-q)(1-q^2)\cdots(1-q^n)} \sum_{k=1}^{n-m+1} a_{n,k}(m) t^k. \end{aligned} \tag{12}$$

To determine $a_{n,k}(m)$, consider a typical D_p-cluster $(w, \delta, (b_1, b_2, \dots, b_k))$. As b_2 may be $2, 3, \dots, m-1$, or m, it follows that

$$a_{n,k}(m) = \sum_{j=1}^{m-1} a_{n-j,k-1}(m), \quad k \geqslant 2. \tag{13}$$

Define $G(x, z) = \sum_{n\geqslant 0} \sum_{k\geqslant 1} a_{n,k}(m) x^k z^n$. Then (13) implies that

$$G(x, z) = \frac{xz^m}{1 - xz(1 + z + \cdots + z^{m-2})} = xz^m \sum_{j\geqslant 0} (xz)^j \left(\frac{1 - z^{m-1}}{1 - z} \right)^j. \tag{14}$$

Extraction of the coefficient of $x^k z^n$ from (14) yields

$$a_{n,k}(m) = \begin{cases} \chi(n = m) & \text{if } k = 1, \\ \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \binom{n-jm-m+j-1}{k-2} & \text{if } k \geqslant 2, \end{cases} \tag{15}$$

where $\chi(n = m)$ is 1 if $n = m$ and is 0 otherwise. The cases $m = 2$ and $m = 3$ have particularly simple forms:

$$a_{n,k}(2) = \begin{cases} 1 & \text{if } n = k + 1 \geqslant 2, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad a_{n,k}(3) = \binom{k-1}{n-k-2}, \tag{16}$$

with the latter equality holding for $1 \leqslant k \leqslant n - 2$.

By noting (12) and invoking Corollary 1, we obtain

Corollary 4. *The q-exponential generating function for permutations by the consecutive pattern $p = 12\dots m \in S_m$ ($m \geqslant 2$ and $t = t_p$) and by the inversion number is*

$$\sum_{n\geqslant 0} \frac{A_{n,p}(q, t) z^n}{[n]!} = \left(1 - z - \sum_{n\geqslant m} \frac{z^n}{[n]!} \sum_{k=1}^{n-m+1} a_{n,k}(m)(t - 1)^k \right)^{-1},$$

where $a_{n,k}(m)$ is as in (15) *or* (16).

A system of differential equations for the $q = 1$ case of the reciprocal of the generating function in Corollary 4 may be found in [2]. For $p = 12$, Corollary 4 gives the generating function for the classical q-Eulerian polynomials associated with the inversion number: If we set $e_q(z) = \sum_{n\geqslant 0} z^n/[n]!$, then

$$\sum_{n\geqslant 0} \frac{A_{n,12}(q, t) z^n}{[n]!} = \frac{1 - t}{-t + e_q((t - 1)z)}.$$

A consecutive occurrence of $p = 12$ in σ is known as a rise.

8. The joint distribution (123, 132, inv)

The set of distinguished factors associated with the permissible set $P = \{123, 132\}$ is

$$D_P = D_{123} \cup D_{132} = \{w \in N^3\colon\ w_1 \leqslant w_2 \leqslant w_3\} \cup \{w \in N^3\colon\ w_2 > w_3 \geqslant w_1\}.$$

For a nonempty word v, let $FL(v)$ and $LL(v)$, respectively, denote the first and last letters of v. Note that $w \in N^n$ is D_P-coverable if and only if there is an integer $k \geqslant 0$ and positive integers $n_k, \ldots, n_0$ with $n_0 \neq 2$ such that

$$w = v_k x_k v_{k-1} x_{k-1} \cdots v_1 x_1 v_0, \tag{17}$$

where $x_1, \ldots, x_k \in N$, $v_i \in \Lambda_{n_i}$, and $x_i > FL(v_{i-1}) \geqslant LL(v_i)$. A word w as in (17) contains exactly k distinguished factors from D_{132}, namely $LL(v_i)x_i FL(v_{i-1})$ for $1 \leqslant i \leqslant k$. Let $N^n_{P,n_k,\ldots,n_0}$ denote the set of words satisfying (17). For instance,

$$N^9_{P,1,3,3} = \{w \in N^9\colon\ w_2 > w_3,\ w_6 > w_7,\ w_1 \leqslant w_3 \leqslant w_4 \leqslant w_5 \leqslant w_7 \leqslant w_8 \leqslant w_9\}.$$

Note that

$$\sum_{w \in N^9_{P,1,3,3}} q^{\|w\|} = \frac{q^2[3][7]}{(1-q)(1-q^2)\cdots(1-q^9)}.$$

In general,

$$\sum_{w \in N^n_{P,n_k,\ldots,n_0}} q^{\|w\|} = \frac{q^k \prod_{i=1}^k [n_0 + \cdots + n_{i-1} + i - 1]}{(1-q)(1-q^2)\cdots(1-q^n)}. \tag{18}$$

Let $a_{n_k,\ldots,n_0,j}$ be the number of times that a word w as in (17) appears in a D_P-cluster (w, δ, β) where k signifies the number of components in δ belonging to D_{132} and j denotes the number of components in δ belonging to D_{123}. Noting that x_i may or may not be the last letter in a distinguished factor from D_{123}, repeated use of (16) leads to the conclusion that $a_{n_k,\ldots,n_0,j}$ is equal to

$$\sum_{\substack{A \subseteq \{1,2,\ldots,k\} \\ |A| \leqslant j}} \sum_{j_0 + \cdots + j_k = j} \prod_{i \in A \setminus \{l\colon\, n_l = 1\}} \binom{j_i - 1}{n_i - j_i - 1} \prod_{i \in \overline{A} \setminus \{l\colon\, n_l = 1\}} \binom{j_i - 1}{n_i - j_i - 2},$$

where $\overline{A}$ is the complement of A relative to $\{0, 1, \ldots, k\}$ (so $0 \in \overline{A}$) and, for convenience, $\binom{\mu}{\nu} = 0$ if either $\nu > \mu$ or $\nu < 0$.

Relative to (7), put $t_{132} = t$ and $t_{123} = s$. The cluster generating function $C(q, t, s, z)$ is then

$$\sum_{n \geqslant 3} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{j=0}^{n-2} t^k s^j z^n \sum_{\substack{n_k + \cdots + n_0 = n-k \\ n_k, \ldots, n_0 \geqslant 1}} a_{n_k,\ldots,n_0,j} \sum_{w \in N^n_{P,n_k,\ldots,n_0}} q^{\|w\|}.$$

So (18) implies that $C(q, t, s, (1-q)z)$ equals

$$\sum_{n \geqslant 3} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{j=0}^{n-2} \frac{q^k t^k s^j z^n}{[n]!} \sum_{n_k, \ldots, n_0} a_{n_k,\ldots,n_0,j} \prod_{i=1}^k [n_0 + \cdots + n_{i-1} + i - 1], \tag{19}$$

where the intermediate sum is over positive integers $n_k, \ldots, n_0$ such that $n_k + \cdots + n_0 = n - k$. Invoking Corollary 1 leads to

Corollary 5. *The q-exponential generating function for permutations by the permissible set of patterns $P = \{132, 123\}$ ($t_{132} = t$, $t_{123} = s$) and by the inversion number is*

$$\sum_{n\geqslant 0} \frac{A_{n,P}(q,t,s)z^n}{[n]!} = \bigl(1 - z - C\bigl(q, t-1, s-1, (1-q)z\bigr)\bigr)^{-1},$$

where $C(q,t,s,(1-q)z)$ *is as in* (19).

A system of differential equations for the $q = 1$ case of the reciprocal of the generating function in Corollary 5 is given in [2].

9. Concluding remarks

As noted in the introduction, the problem that arises in using (1) lies in the computation of a certain q-sum. In the examples of Sections 5–8, the q-sums were relatively straightforward to compute. However, many of the q-sums that arise appear to be intractable.

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