Elementary orbifold differential topology

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A B S T R A C T

Taking an elementary and straightforward approach, we develop the concept of a regular value for a smooth map $f : \mathcal{O} \to \mathcal{P}$ between smooth orbifolds $\mathcal{O}$ and $\mathcal{P}$. We show that Sard's theorem holds and that the inverse image of a regular value is a smooth full suborbifold of $\mathcal{O}$. We also study some constraints that the existence of a smooth orbifold map imposes on local isotropy groups. As an application, we prove a Borsuk no retraction theorem for compact orbifolds with boundary and some obstructions to the existence of real-valued orbifold maps from local model orbifold charts.

1. Introduction

Inspired by the elementary and elegant treatment of differential topology found in J. Milnor's book [7], *Topology from a differentiable viewpoint*, we generalize some of the fundamental material of that book to the category of smooth orbifolds in a manner that is elementary.

2. Smooth orbifolds

Although there are many references for this background material, we will use our previous work [3,4] as our standard reference. While much of what we discuss here works equally well for smooth C^r orbifolds, to simplify the exposition, we restrict ourselves to smooth C^∞ orbifolds. Throughout, the term *smooth* means C^∞. This results in no loss of generality [3, Proposition 3.11], [6]. Note that the classical definition of orbifold given below is modeled on the definition in Thurston [12] and that these orbifolds are referred to as *classical effective orbifolds* in [1].

Definition 2.1. An n-dimensional *smooth orbifold* $\mathcal{O}$, consists of a paracompact, Hausdorff topological space $X_\mathcal{O}$ called the *underlying space*, with the following local structure. For each $x \in X_\mathcal{O}$ and neighborhood U of x, there is a neighborhood $U_x \subset U$, an open set $\tilde{U}_x$ diffeomorphic to $\mathbb{R}^n$, a finite group Γ_x acting smoothly and effectively on $\tilde{U}_x$ which fixes $0 \in \tilde{U}_x$, and a homeomorphism $\phi_x : \tilde{U}_x/\Gamma_x \to U_x$ with $\phi_x(0) = x$. These actions are subject to the condition that for a neighborhood $U_z \subset U_x$ with corresponding $\tilde{U}_z \cong \mathbb{R}^n$, group Γ_z and homeomorphism $\phi_z : \tilde{U}_z/\Gamma_z \to U_z$, there is a smooth embedding

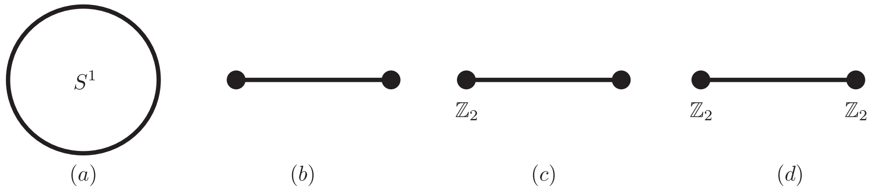
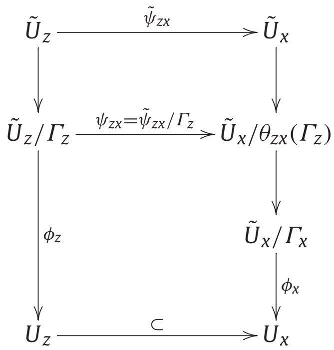


Fig. 1. Compact connected 1-orbifolds.

$\tilde{\psi}_{zx} : \tilde{U}_z \to \tilde{U}_x$ and an injective homomorphism $\theta_{zx} : \Gamma_z \to \Gamma_x$ so that $\tilde{\psi}_{zx}$ is equivariant with respect to θ_{zx} (that is, for $\gamma \in \Gamma_z$, $\tilde{\psi}_{zx}(\gamma \cdot \tilde{y}) = \theta_{zx}(\gamma) \cdot \tilde{\psi}_{zx}(\tilde{y})$ for all $\tilde{y} \in \tilde{U}_z$), such that the following diagram commutes:



We will refer to the neighborhood U_x or $(\tilde{U}_x, \Gamma_x)$ or $(\tilde{U}_x, \Gamma_x, \rho_x, \phi_x)$ as an *orbifold chart*, and write $U_x = \tilde{U}_x/\Gamma_x$. In the 4-tuple notation, we are making explicit the representation $\rho_x : \Gamma_x \to \mathrm{Diff}^\infty(\tilde{U}_x)$. The *isotropy group of x* is the group Γ_x. The definition of orbifold implies that the germ of the action of Γ_x in a neighborhood of the origin of $\mathbb{R}^n$ is unique, so that by shrinking $\tilde{U}_x$ if necessary, Γ_x is well defined up to isomorphism. The *singular set* of $\mathcal{O}$ is the set of points $x \in \mathcal{O}$ with $\Gamma_x \neq \{1\}$. More detail can be found in [3].

Definition 2.2. A *smooth orbifold with boundary* $\mathcal{X}$, is an orbifold as in Definition 2.1 where one replaces the requirement that $\tilde{U}_x$ be diffeomorphic to $\mathbb{R}^n$ with the requirement that $\tilde{U}_x$ be diffeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^n_+$, the closed upper half-space. The boundary $\partial\mathcal{X}$ of $\mathcal{X}$ consists of those points $x \in \mathcal{X}$ where $\tilde{U}_x$ is diffeomorphic to $\mathbb{R}^n_+$. Throughout the rest of the article, we will use $\mathcal{X}$ to denote a smooth orbifold with nonempty boundary.

2.1. Compact 1-dimensional orbifolds

Using the classification of compact 1-dimensional manifolds, it is easy to classify all 1-dimensional compact connected orbifolds with or without boundary. There are four types: (a) the circle S^1, (b) the closed interval $[0, 1]$ with trivial orbifold structure, (c) the closed interval $[0, 1]$ with where $\{0\}$ is a singular point with $\mathbb{Z}_2$ isotropy, and (d) the closed interval $[0, 1]$ where both $\{0, 1\}$ have $\mathbb{Z}_2$ isotropy. Thus, the compact 1-orbifolds must be finite unions of orbifolds of these types. See Fig. 1.

2.2. Smooth suborbifolds

The definition of suborbifold is somewhat subtle and we distinguish two types of suborbifolds.

Definition 2.3. An (embedded) *suborbifold* $\mathcal{P}$ of an orbifold $\mathcal{O}$ consists of the following.

(1) A subspace $X_\mathcal{P} \subset X_\mathcal{O}$ equipped with the subspace topology;
(2) For each $x \in X_\mathcal{P}$ and neighborhood W of x in $X_\mathcal{O}$ there is an orbifold chart $(\tilde{U}_x, \Gamma_x, \rho_x, \phi_x)$ about x in $\mathcal{O}$ with $U_x \subset W$, a subgroup $\Lambda_x \subset \Gamma_x$ of the isotropy group of x in $\mathcal{O}$ and a $\rho_x(\Lambda_x)$ invariant linear submanifold $\tilde{V}_x \subset \tilde{U}_x \cong \mathbb{R}^n$, so that $(\tilde{V}_x, \Lambda_x/\Omega_x, \rho_x|_{\Lambda_x}, \psi_x)$ is an orbifold chart for $\mathcal{P}$ where $\Omega_x = \{\gamma \in \Lambda_x\colon\ \rho_x(\gamma)|_{\tilde{V}_x} = \mathrm{Id}\}$ (in particular, the *intrinsic* isotropy subgroup at $x \in \mathcal{P}$ is Λ_x/Ω_x); and
(3) $V_x = \psi_x(\tilde{V}_x/\rho_x(\Lambda_x)) = U_x \cap X_\mathcal{P}$ is an orbifold chart for x in $\mathcal{P}$.

Remark 2.4. Originally, in [12], the notion of an m-suborbifold $\mathcal{P}$ of an n-orbifold $\mathcal{O}$ required $\mathcal{P}$ to be locally modeled on $\mathbb{R}^m \subset \mathbb{R}^n$ modulo finite groups. That is, the local action on $\mathbb{R}^m$ is induced by the local action on $\mathbb{R}^n$. This is equivalent to adding the condition that $\Lambda_x = \Gamma_x$ at all x in the underlying topological space of $\mathcal{P}$.

Given this remark, we make the following definition:

Definition 2.5. $\mathcal{P} \subset \mathcal{O}$ is a *full suborbifold* of $\mathcal{O}$ if $\mathcal{P}$ is a suborbifold with $\Lambda_x = \Gamma_x$ for all $x \in \mathcal{P}$.

Example 2.6. Let $\mathcal{Q} = \mathbb{R}/\mathbb{Z}_2$ be the smooth orbifold (without boundary) where $\mathbb{Z}_2$ acts on $\mathbb{R}$ via $\gamma \cdot x = -x$. The underlying topological space $X_{\mathcal{Q}}$ of $\mathcal{Q}$ is $[0, \infty)$ and the isotropy subgroups are $\{1\}$ for $x \in (0, \infty)$ and $\mathbb{Z}_2$ for $x = 0$. Let $\mathcal{O} = \mathcal{Q} \times \mathcal{Q}$ be the smooth product orbifold (without boundary). See [3, Definition 2.12]. The underlying space for $\mathcal{O}$ can be identified with the closed first quadrant and the singular points of $\mathcal{O}$ lie in one of three connected singular strata: the positive x axis, the positive y axis (corresponding to those points with $\mathbb{Z}_2$ isotropy), and the origin which has $\mathbb{Z}_2 \times \mathbb{Z}_2$ isotropy. Then both $\mathcal{Q} \times \{0\}$ and $\{0\} \times \mathcal{Q}$ are full suborbifolds of $\mathcal{O}$. On the other hand, the diagonal $\mathrm{diag}(\mathcal{Q}) = \{(x, x)\colon\ x \in \mathcal{Q}\} \subset \mathcal{O}$ is merely a suborbifold. See [3, Example 2.15].

Example 2.7. Let $\mathcal{O}$ be as in Example 2.6. Consider the circle $\mathcal{S} \subset \mathcal{O}$ of radius 1 centered at $(1, 1)$. Then $\mathcal{S}$ is a suborbifold of $\mathcal{O}$ that is not a full suborbifold. To see this, just note that at the point $x = (1, 0) \in \mathcal{O}$ any lift of $\mathcal{S}$ to $\tilde{U}_x \cong \mathbb{R}^2$ in a neighborhood of x, cannot be an invariant linear submanifold unless we choose $\Lambda_x = \{1\}$. In this case, we see that the intrinsic isotropy group of $\mathcal{S}$ at x is trivial which it must be since $\mathcal{S}$ is actually a compact 1-dimensional *manifold*. That is, a compact 1-dimensional orbifold with trivial orbifold structure.

Remark 2.8. Let $\mathcal{P} \subset \mathcal{O}$ be a suborbifold. Note that even though a point $p \in X_{\mathcal{P}}$ may be in the singular set of $\mathcal{O}$, it need not be in the singular set of $\mathcal{P}$.

3. Smooth mappings between orbifolds

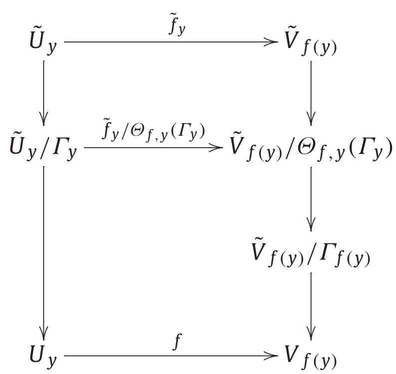
In the literature, there are four related definitions of maps between orbifolds which are based on the classical Satake–Thurston approach to orbifolds via atlases of orbifold charts. In this paper, we only need to use the notion of complete orbifold map. It is distinguished from the other notions of orbifold map in that we are going to keep track of all defining data. All other notions of orbifold map descend from the complete orbifold maps by forgetting information. It turns out that the results of this paper also follow using any of the four notions of orbifold map. This requires only an understanding of how these notions of orbifold map are related to one another. We point this out explicitly in our exposition below. More detail can be found in [4] and in what follows we use the notation of [3, Section 2].

The original motivation for defining the notion of complete orbifold map was to make meaningful and well defined certain geometric constructions involving orbifolds and their maps. The need to be careful in defining an adequate notion of orbifold map was already noted in the work of Moerdijk and Pronk [8] and Chen and Ruan [5] and was missing from Satake's original work on V-manifolds [10,11].

3.1. Complete orbifold maps

Definition 3.1. A C^∞ *complete orbifold map* $(f, \{\tilde{f}_x\}, \{\Theta_{f,x}\})$ between smooth orbifolds $\mathcal{O}$ and $\mathcal{P}$ consists of the following:

(1) A continuous map $f : X_{\mathcal{O}} \to X_{\mathcal{P}}$ of the underlying topological spaces.
(2) For each $y \in \mathcal{O}$, a group homomorphism $\Theta_{f,y} : \Gamma_y \to \Gamma_{f(y)}$.
(3) A smooth $\Theta_{f,y}$-equivariant lift $\tilde{f}_y : \tilde{U}_y \to \tilde{V}_{f(y)}$ where $(\tilde{U}_y, \Gamma_y)$ is an orbifold chart at y and $(\tilde{V}_{f(y)}, \Gamma_{f(y)})$ is an orbifold chart at $f(y)$. That is, the following diagram commutes:



(⋆4) (Equivalence) Two complete orbifold maps $(f, \{\tilde{f}_x\}, \{\Theta_{f,x}\})$ and $(g, \{\tilde{g}_x\}, \{\Theta_{g,x}\})$ are considered equivalent if for each $x \in \mathcal{O}_1$, $\tilde{f}_x = \tilde{g}_x$ as germs and $\Theta_{f,x} = \Theta_{g,x}$. That is, there exists an orbifold chart $(\tilde{U}_x, \Gamma_x)$ at x such that $\tilde{f}_x|_{\tilde{U}_x} = \tilde{g}_x|_{\tilde{U}_x}$ and $\Theta_{f,x} = \Theta_{g,x}$. Note that this implies that $f = g$.

The set of smooth complete orbifold maps from $\mathcal{O}$ to $\mathcal{P}$ will be denoted by $C^\infty_{\star\mathrm{Orb}}(\mathcal{O},\mathcal{P})$. For $\mathcal{O}$ compact (without boundary), $C^\infty_{\star\mathrm{Orb}}(\mathcal{O},\mathcal{P})$ carries the structure of a smooth Fréchet manifold [4].

3.2. Regular and critical values

Definition 3.2. Let ${}_\star f = (f, \{\tilde{f}_x\}, \{\Theta_{f,x}\}) : \mathcal{O} \to \mathcal{P}$ be a smooth complete orbifold map between smooth orbifolds. A point $p \in \mathcal{P}$ is a *regular value* for ${}_\star f$ if $d\tilde{f}_x(\tilde{x}) : T_{\tilde{x}}\tilde{U}_x \to T_{\tilde{p}}\tilde{V}_p$ is surjective for all $x \in f^{-1}(p)$. Otherwise, p is a *critical value* for ${}_\star f$. By convention, if $p \notin f(\mathcal{O})$, then p is a regular value.

Remark 3.3. Because all local lifts of an orbifold map $f : \mathcal{O} \to \mathcal{P}$ at x differ from one another by the action of an element of $\Gamma_{f(x)}$ (which acts by diffeomorphisms on $\tilde{V}_{f(x)}$), it is clear that the notion of regular value is well defined for any of the four notions of orbifold map.

4. Sard's theorem and preimage theorem

The local structure of a smooth orbifold is that of a quotient by a finite action by diffeomorphisms which is measure non-increasing. Hence the usual Sard's theorem for manifolds [7] yields a Sard's theorem for smooth orbifolds.

Theorem 4.1 *(Sard's theorem for orbifolds). Let $f : \mathcal{O} \to \mathcal{P}$ be a (complete) smooth orbifold map. Then the set of critical values for f has measure 0 in $\mathcal{P}$ and thus the set of regular values is everywhere dense in $\mathcal{P}$.*

We are ready to state our first main result which is the analogue of the so-called preimage theorem:

Theorem 4.2 *(Preimage theorem for orbifolds). Let $\mathcal{O}, \mathcal{P}$ be smooth orbifolds (without boundary) with $\dim \mathcal{O} \geqslant \dim \mathcal{P}$. Let $f : \mathcal{O} \to \mathcal{P}$ be a (complete) smooth orbifold map and $p \in \mathcal{P}$ a regular value for f. Then $f^{-1}(p) = \mathcal{S}$ has the structure of a full, smooth suborbifold of dimension $\dim(\mathcal{S}) = \dim(\mathcal{O}) - \dim(\mathcal{P})$. Moreover, the local isotropy groups $\Gamma_{x,\mathcal{S}} = \Gamma_{x,\mathcal{O}}/G_{x,\mathcal{O}}$ where $G_{x,\mathcal{O}} = \{\gamma \in \Gamma_{x,\mathcal{O}}\colon d\gamma|_{\ker(d\tilde{f}_x(\tilde{x}))} = \mathrm{Id}\}$.*

Proof. It suffices to work in a chart. For $x \in \mathcal{S}$, $\tilde{f}_x^{-1}(\tilde{p})$ is a submanifold $\tilde{S}_x$ of $\tilde{U}_x$ of dimension $\dim(\mathcal{O}) - \dim(\mathcal{P})$ and $T_{\tilde{x}}\tilde{S}_x = \ker(d\tilde{f}_x(\tilde{x}))$, by the preimage theorem for manifolds [7]. The submanifold $\tilde{S}_x$ is $\Gamma_{x,\mathcal{O}}$-invariant. To see this, let $\tilde{y} \in \tilde{S}_x$ and $\gamma \in \Gamma_{x,\mathcal{O}}$. Then

$$\tilde{f}_x(\gamma \cdot \tilde{y}) = \Theta_{f,x}(\gamma) \cdot \tilde{f}_x(\tilde{y}) = \Theta_{f,x}(\gamma) \cdot \tilde{p} = \tilde{p}$$

since $\Theta_{f,x}(\gamma) \in \Gamma_{p,\mathcal{P}}$. Thus, $\gamma \cdot \tilde{y} \in \tilde{S}_x$ and we have shown that $\tilde{S}_x$ is $\Gamma_{x,\mathcal{O}}$-invariant. Thus, a neighborhood of $x \in \mathcal{S}$ can be realized as the quotient $\tilde{S}_x/\Gamma_{x,\mathcal{O}} \cong \tilde{S}_x/(\Gamma_{x,\mathcal{O}}/\Omega_{x,\mathcal{O}})$, where $\Omega_{x,\mathcal{O}} = \{\gamma \in \Gamma_{x,\mathcal{O}}\colon \gamma|_{\tilde{S}_x} = \mathrm{Id}\}$. Since $\Gamma_{x,\mathcal{O}}/\Omega_{x,\mathcal{O}}$ acts effectively, we have shown that $\mathcal{S}$ has the structure of a full suborbifold of $\mathcal{O}$ with local isotropy groups $\Gamma_{x,\mathcal{S}} = \Gamma_{x,\mathcal{O}}/\Omega_{x,\mathcal{O}}$. The Bochner–Cartan theorem [3,9] implies that the smooth action of $\Gamma_{x,\mathcal{O}}$ is smoothly conjugate to the linear action on $\tilde{U}_x$ given by the differential of the action. Since $\Gamma_{x,\mathcal{S}} = \Gamma_{x,\mathcal{O}}/\Omega_{x,\mathcal{O}}$, the representation of $\Gamma_{x,\mathcal{S}}$ given in the last statement of the theorem follows. □

More generally, as in the case for manifolds we get a preimage theorem for orbifolds with boundary. We omit the proof.

Theorem 4.3 *(Preimage theorem for orbifolds with boundary). Let $\mathcal{X}$ be a smooth orbifold with boundary and $\mathcal{P}$ a smooth orbifold with $\dim \mathcal{X} > \dim \mathcal{P}$. Let $f : \mathcal{X} \to \mathcal{P}$ be a (complete) smooth orbifold map and $p \in \mathcal{P}$ a regular value for f and for the restriction $f|_{\partial\mathcal{X}}$. Then $f^{-1}(p) = \mathcal{S}$ has the structure of a full, smooth suborbifold with boundary of dimension $\dim(\mathcal{S}) = \dim(\mathcal{X}) - \dim(\mathcal{P})$. Moreover, the boundary $\partial(f^{-1}(p))$ is the intersection $f^{-1}(p) \cap \partial\mathcal{X}$.*

Remark 4.4. By Remark 3.3, it follows that each of the results of this section also holds for any of the four notions of orbifold map. Specifically, because all local lifts of an orbifold map $f : \mathcal{O} \to \mathcal{P}$ at x differ from one another by the action of an element of $\Gamma_{f(x)}$ (which acts by diffeomorphisms on $\tilde{V}_{f(x)}$), we have that $T_{\tilde{x}}\tilde{S}_x = \ker(d\tilde{f}_x(\tilde{x})) = \ker(d(\eta_x \cdot \tilde{f}_x)(\tilde{x}))$ for any $\eta_x \in \Gamma_{f(x)}$.

5. Implications of the existence of smooth map between orbifolds

Unsurprisingly, there are obstructions (which are manifested in the local orbifold chart structure) to the existence of a smooth map between orbifolds. In this section, we give the main tool we will use later. To avoid cumbersome notation, the induced action of $\gamma \in \Gamma_x$ on tangent vectors $\tilde{v} \in T_{\tilde{x}}\tilde{U}_x$ will be denoted by left multiplication as well: $\gamma \cdot \tilde{v} = d\gamma_{\tilde{x}}(\tilde{v})$ when convenient.

Let ${}_\star f = (f, \{\tilde{f}_x\}, \{\Theta_{f,x}\}) : \mathcal{O} \to \mathcal{P}$ be a smooth (complete) orbifold map. Let $K_x = \ker(d\tilde{f}_x(\tilde{x}))$ and $N_x = \ker \Theta_{f,x} \subset \Gamma_{x,\mathcal{O}}$, a normal subgroup. For all $\tilde{v} \in T_{\tilde{x}}\tilde{U}_x$ and $\gamma \in N_x$ we have

$$d\tilde{f}_x(\tilde{x})(\gamma \cdot \tilde{v}) = \Theta_{f,x}(\gamma) \cdot d\tilde{f}_x(\tilde{x})(\tilde{v}) = d\tilde{f}_x(\tilde{x})(\tilde{v}).$$

Thus, $\gamma \cdot \tilde{v} - \tilde{v} \in K_x$. In other words, for each $\gamma \in N_x$ we have a linear map $A_\gamma = (\gamma - I) \in \mathrm{Hom}(T_{\tilde{x}}\tilde{U}_x, K_x)$. Here, I denotes the identity map.

We have $\gamma \cdot \tilde{v} = (I + A_\gamma)\tilde{v}$ and thus, $(\gamma\delta) \cdot \tilde{v} = (I + A_{\gamma\delta})\tilde{v}$. On the other hand, we have

$$\begin{aligned}(I + A_{\gamma\delta})\tilde{v} &= (\gamma\delta) \cdot \tilde{v} = \gamma \cdot (\delta \cdot \tilde{v}) = \gamma \cdot (I + A_\delta)\tilde{v} = \gamma \cdot \tilde{v} + \gamma \cdot A_\delta \tilde{v} \\ &= (I + A_\gamma)\tilde{v} + \gamma \cdot A_\delta \tilde{v} = (I + A_\gamma + \gamma \cdot A_\delta)\tilde{v}.\end{aligned}$$

Also,

$$\begin{aligned}(I + A_{\gamma\delta})\tilde{v} &= (\gamma\delta) \cdot \tilde{v} = \gamma \cdot (\delta \cdot \tilde{v}) = (I + A_\gamma)(\delta \cdot \tilde{v}) = \delta \cdot \tilde{v} + A_\gamma(\delta \cdot \tilde{v}) \\ &= (I + A_\delta)\tilde{v} + A_\gamma(\delta \cdot \tilde{v}) = (I + A_\delta + A_\gamma\delta)\tilde{v}.\end{aligned}$$

Similarly,

$$\begin{aligned}(I + A_{\gamma\delta})\tilde{v} &= (\gamma\delta) \cdot \tilde{v} = \gamma \cdot (\delta \cdot \tilde{v}) = (I + A_\gamma)(\delta \cdot \tilde{v}) = \delta \cdot \tilde{v} + A_\gamma(\delta \cdot \tilde{v}) \\ &= \delta \cdot \tilde{v} + A_\gamma(I + A_\delta)\tilde{v} = \delta \cdot \tilde{v} + A_\gamma\tilde{v} + A_\gamma A_\delta \tilde{v} \\ &= (I + A_\delta)\tilde{v} + A_\gamma\tilde{v} + A_\gamma A_\delta\tilde{v} = (I + A_\delta + A_\gamma + A_\gamma A_\delta)\tilde{v}.\end{aligned}$$

We thus have three expressions for $A_{\gamma\delta}$:

$$\begin{aligned}A_{\gamma\delta} &= A_\gamma + \gamma \cdot A_\delta \\ &= A_\delta + A_\gamma\delta\cdot \\ &= A_\delta + A_\gamma + A_\gamma A_\delta.\end{aligned}$$

Proposition 5.1. *With ${}_\star f$ and notation as above, for each $x \in \mathcal{O}$, there exists an N_x-invariant linear projection $A_x \in \mathrm{Hom}_{N_x}(T_{\tilde{x}}\tilde{U}_x, K_x)$.*

Proof. Define the linear map $A \in \mathrm{Hom}(T_{\tilde{x}}\tilde{U}_x, K_x)$, by $A = \frac{1}{|N_x|}\sum_{\delta \in N_x} A_\delta$. Then $\gamma \cdot A = \frac{1}{|N_x|}\sum_{\delta \in N_x} \gamma \cdot A_\delta = \frac{1}{|N_x|}\sum_{\delta \in N_x}(A_{\gamma\delta} - A_\gamma) = A - A_\gamma$. Therefore, $A_\gamma = A - \gamma \cdot A$. Similarly, $A\delta\cdot = \frac{1}{|N_x|}\sum_{\gamma \in N_x} A_\gamma\delta\cdot = \frac{1}{|N_x|}\sum_{\gamma \in N_x}(A_{\gamma\delta} - A_\delta) = A - A_\delta$. Therefore, $A_\delta = A - A\delta\cdot$. Putting this together, we conclude that $\gamma \cdot A = A\gamma\cdot$ and thus, A is N_x-invariant. To show that A is a projection, we compute

$$\begin{aligned}A^2 &= \frac{1}{|N_x|^2}\sum_{\gamma \in N_x}\sum_{\delta \in N_x} A_\gamma A_\delta = \frac{1}{|N_x|^2}\sum_{\gamma \in N_x}\sum_{\delta \in N_x}(A_{\gamma\delta} - A_\gamma - A_\delta) \\ &= \frac{1}{|N_x|^2}\sum_{\gamma \in N_x} |N_x|(A - A_\gamma - A) = -A.\end{aligned}$$

Thus, $A_x = -A$ is the required N_x-invariant linear projection. $\square$

Lemma 5.2. *For all $\tilde{v} \in \ker A_x$ and $\gamma \in N_x$, $\gamma \cdot \tilde{v} = \tilde{v}$. That is, $\gamma|_{\ker A_x} = \mathrm{Id}$.*

Proof. Using Proposition 5.1, since A_x is a projection, the tangent space decomposes $T_{\tilde{x}}\tilde{U}_x = \ker A_x \oplus \mathrm{im}\, A_x$ and furthermore, since A_x is N_x-invariant, so is this decomposition. For $\tilde{v} \in T_{\tilde{x}}\tilde{U}_x$ and $\gamma \in N_x$, we have $\gamma \cdot \tilde{v} - \tilde{v} = A_\gamma\tilde{v} = (\gamma \cdot A_x - A_x)\tilde{v}$. Thus, $\gamma \cdot \tilde{v} - \tilde{v} \in \mathrm{im}\, A_x$ since $\mathrm{im}\, A_x$ is N_x-invariant. If we further suppose that $\tilde{v} \in \ker A_x$, then since $\ker A_x$ is N_x-invariant, we must have $\gamma \cdot \tilde{v} - \tilde{v} \in \ker A_x \cap \mathrm{im}\, A_x = \{0\}$. This implies that $\gamma \cdot \tilde{v} = \tilde{v}$ for all $\tilde{v} \in \ker A_x$ and $\gamma \in N_x$. $\square$

Proposition 5.3. *Let $\mathcal{O}$, $\mathcal{P}$ be smooth orbifolds with $\dim \mathcal{O} \geqslant \dim \mathcal{P}$. Let ${}_\star f = (f, \{\tilde{f}_x\}, \{\Theta_{f,x}\}) : \mathcal{O} \to \mathcal{P}$ be a smooth (complete) orbifold map and $p \in \mathcal{P}$ a regular value for f. Then there is a faithful representation of $N_x = \ker \Theta_{f,x}$ in $\Gamma_{x,\mathcal{S}}$ where $\mathcal{S} = f^{-1}(p)$ is the full, smooth suborbifold given by the preimage Theorem 4.2.*

Proof. Let $K_x = \ker(d\tilde{f}_x(\tilde{x}))$. By Theorem 4.2, $\Gamma_{x,\mathcal{S}} = \Gamma_{x,\mathcal{O}}/G_{x,\mathcal{O}}$ where $G_{x,\mathcal{O}} = \{\gamma \in \Gamma_{x,\mathcal{O}}\colon\ \gamma|_{K_x} = \mathrm{Id}\}$. Then $G_{x,\mathcal{O}} \cap N_x = \{\mathrm{Id}\}$. For, if $\gamma \in G_{x,\mathcal{O}} \cap N_x$, then by Lemma 5.2, $\gamma|_{\ker A_x} = \mathrm{Id}$. Also, since $\gamma|_{K_x} = \mathrm{Id}$ and $\mathrm{im}\, A_x \subset K_x$, then $\gamma|_{\mathrm{im}\, A_x} = \mathrm{Id}$. Since

$T_{\tilde{x}}\tilde{U}_x = \ker A_x \oplus \operatorname{im} A_x$, we conclude that $\gamma = \mathrm{Id}$. Consider the quotient homomorphism $\Gamma_{x,\mathcal{O}} \to \Gamma_{x,\mathcal{O}}/G_{x,\mathcal{O}} \cong \Gamma_{x,\mathcal{S}}$ and restrict to the normal subgroup N_x:

$$N_x \to N_x G_{x,\mathcal{O}}/G_{x,\mathcal{O}} \cong N_x/(N_x \cap G_{x,\mathcal{O}}) \cong N_x.$$

From this we see that N_x is faithfully represented in $\Gamma_{x,\mathcal{S}}$. □

Remark 5.4. It follows that each of the results of this section also holds for any of the four notions of orbifold map by our previous Remarks 3.3 and 4.4, and the observation that $N_x = \ker \Theta_{f,x} = \ker \eta_x \Theta_{f,x} \eta_x^{-1}$, for all $\eta_x \in \Gamma_{f(x)}$.

6. Applications

In this section we give some applications of our results.

Example 6.1. Let Γ be a finite group. Suppose that $\mathcal{O} = \mathbb{R}^n/\Gamma$, with Γ acting linearly on $\mathbb{R}^n$ and $\mathcal{P} = \mathbb{R}^n$ (with the trivial orbifold structure). Let ${}_\star f = (f, \{\tilde{f}_x\}, \{\Theta_{f,x}\}) : \mathcal{O} \to \mathcal{P}$ be a smooth (complete) orbifold map. Assume $f(0) = p$. Then $p \in \mathcal{P}$ is never a regular value. For otherwise, Γ would be forced to act effectively on 0-dimensional singleton by Proposition 5.3, which is impossible.

Example 6.2. Let Γ be a finite group. Suppose that $\mathcal{O} = \mathbb{R}^n/\Gamma$, with Γ acting linearly on $\mathbb{R}^n$ via an irreducible representation. Let $\mathcal{P} = \mathbb{R}^k/\Gamma$ where $k < n$ and Γ any effective action on $\mathbb{R}^k$. Let ${}_\star f = (f, \{\tilde{f}_x\}, \{\Theta_{f,x}\}) : \mathcal{O} \to \mathcal{P}$ be a smooth (complete) orbifold map. Assume $f(0) = p$. Then $p \in \mathcal{P}$ is never a regular value. For, otherwise, Γ would be forced leave an $(n-k)$-dimensional subspace of $\mathbb{R}^n$ invariant by Theorem 4.2, which cannot happen by our assumption of irreducibility of the action of Γ on $\mathbb{R}^n$.

Example 6.3. Let ${}_\star f = (f, \{\tilde{f}_x\}, \{\Theta_{f,x}\}) : \mathcal{O} \to \mathbb{R}$ be a smooth (complete) orbifold map where $\mathcal{O}$ is a smooth n-dimensional orbifold (without boundary) and $\mathbb{R}$ has been given the trivial orbifold structure. Suppose p is a regular value of ${}_\star f$. Then $f^{-1}(p) = \mathcal{S}$ is a full suborbifold of dimension $(n-1)$. For $x \in \mathcal{S}$, we have $N_x = \Gamma_{x,\mathcal{O}}$. Since $G_{x,\mathcal{O}} \cap N_x = \{\mathrm{Id}\}$ (see the proof of Proposition 5.3), we have that $\Gamma_{x,\mathcal{S}} = \Gamma_{x,\mathcal{O}}$ and thus $\Gamma_{x,\mathcal{O}}$ acts effectively on $K_x = \ker(d\tilde{f}_x(\tilde{x})) = T_{\tilde{x}}\tilde{\mathcal{S}}_x \cong \mathbb{R}^{n-1}$. Since $\tilde{v} \in \ker A_x$ implies $\Gamma_{x,\mathcal{O}} \cdot \tilde{v} = \tilde{v}$ by Lemma 5.2 and $\Gamma_{x,\mathcal{O}}$ acts effectively on K_x, we see that $\ker A_x \cap K_x = \{0\}$. This implies that $K_x \subset \operatorname{im} A_x$ and since $\operatorname{im} A_x \subset K_x$ by definition, $K_x = \operatorname{im} A_x$ and hence $\ker A_x \cong \mathbb{R}$. Thus we have a $\Gamma_{x,\mathcal{O}}$-invariant decomposition of the tangent space $T_{\tilde{x}}\tilde{U}_x = \ker A_x \oplus \operatorname{im} A_x = \mathbb{R} \oplus K_x$. In particular, again by Lemma 5.2, the $\mathbb{R}$ factor of this decomposition is fixed by the action of $\Gamma_{x,\mathcal{O}}$ and thus we conclude that $\Sigma_{\mathcal{O}}(x)$, the connected component of the singular set of $\mathcal{O}$ that contains x, must be empty or have dimension $\dim(\Sigma_{\mathcal{O}}(x)) \geqslant 1$.

Example 6.4. As an application of Example 6.3 we conclude that if ${}_\star f : \mathcal{O} \to \mathbb{R}$ is a smooth (complete) orbifold map and $p \in \mathbb{R}$ is a regular value, then $f^{-1}(p)$ cannot contain any isolated points in the singular set of $\mathcal{O}$.

A generalization of K. Borsuk's so-called *no retraction theorem* [2,7] states that there is no smooth map from a compact manifold with boundary to its boundary that leaves the boundary fixed. We prove an analogue of this result for orbifolds. The following example shows that some extra assumptions are necessary in the orbifold case.

Example 6.5. Let $\mathcal{X}$ be the compact 1-orbifold with boundary of type (c) given in Section 2.1. Then a smooth (complete) orbifold map ${}_\star f : \mathcal{X} \to \partial\mathcal{X}$ with ${}_\star f|_{\partial\mathcal{X}} = \mathrm{Id}$ is given by the constant map $x \mapsto 1 = \partial\mathcal{X}$.

Theorem 6.6. *Let $\mathcal{X}$ be a smooth n-dimensional compact orbifold with boundary $\partial\mathcal{X}$ and assume that the interior,* $\operatorname{int}\mathcal{X}$, *does not have any codimension* 1 *singular strata. Then there is no smooth* (*complete*) *orbifold map ${}_\star f : \mathcal{X} \to \partial\mathcal{X}$ with ${}_\star f|_{\partial\mathcal{X}} = \mathrm{Id}$.*

Proof. Suppose such ${}_\star f$ exists. By Sard's theorem there exists a regular value $p \in \partial\mathcal{X}$. Furthermore, since the singular set of an orbifold is nowhere dense, we may further assume that p is not in the singular set of $\partial\mathcal{X}$. Therefore, by Theorem 4.3, $f^{-1}(p) = \mathcal{S}$ is a full, smooth 1-orbifold with boundary and $\partial\mathcal{S} = \mathcal{S} \cap \partial\mathcal{X} = \{p\}$ since ${}_\star f|_{\partial\mathcal{X}} = \mathrm{Id}$. Because $\partial\mathcal{S}$ consists of a single point, there must be a connected component $\mathcal{S}_c$ of $\mathcal{S}$ isomorphic to a compact 1-orbifold of type (c). Consider the unique point $z \in \mathcal{S}_c \cap \operatorname{int}\mathcal{X}$ where $\Gamma_{z,\mathcal{S}_c} = \mathbb{Z}_2$. Arguing as in Example 6.3, we can conclude that $\Gamma_{z,\mathcal{X}} = \Gamma_{z,\mathcal{S}_c} = \mathbb{Z}_2$ and that we have a $\mathbb{Z}_2$-invariant decomposition of the tangent space $T_{\tilde{z}}\tilde{U}_z = \mathbb{R}^{n-1} \oplus \mathbb{R}$ which leaves the $\mathbb{R}^{n-1}$ factor fixed. This implies that the dimension of the singular stratum containing z has codimension 1. By assumption, no such points $z \in \mathcal{X}$ exist and we have our desired contradiction. □

Corollary 6.7 *(No retraction theorem for orbifolds). Let $\mathcal{X}$ be a smooth compact orbifold with boundary $\partial\mathcal{X}$. Assume the singular set of $\mathcal{X}$ has codimension greater than* 1. *Then $\partial\mathcal{X}$ is not a smooth orbifold retract of $\mathcal{X}$.*

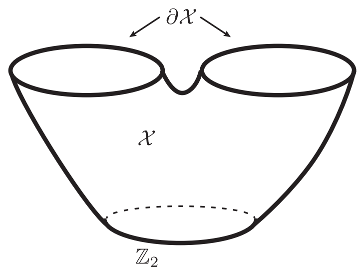


Fig. 2. An orbifold $\mathcal{X}$ with only codimension 1 strata that does not retract to $\partial\mathcal{X}$.

Remark 6.8. Orbifolds can be regarded as rational homology manifolds and Corollary 6.7 provides a nice subclass of such rational homology manifolds for which a Borsuk no retraction result holds.

In light of Example 6.5, one might suspect that the existence of codimension 1 strata is enough to guarantee a retraction to the boundary. The following two examples show that this is not the case.

Example 6.9 *(A pair of pants with mirror).* See Fig. 2.

Example 6.10 *(A knot complement).* Consider the closed 3-ball D^3 and let K denote any embedded tubular neighborhood of a knot in the interior of D^3. The boundary of $D^3 - K$ is the disjoint union $S^2 \amalg T^2$ of a 2-sphere and 2-torus. Consider the 3-orbifold with boundary $\mathcal{X}$ whose underlying topological space is $D^3 - K$ where we consider the S^2 (topological) boundary component as a $\mathbb{Z}_2$ mirror. Thus, as an orbifold $\partial\mathcal{X} = T^2$. Because $H_1(\partial\mathcal{X}, \mathbb{Z}) \cong \mathbb{Z}^2$ and $H_1(\mathcal{X}, \mathbb{Z}) \cong \mathbb{Z}$ since $\mathcal{X}$ is a knot complement, there can be no retraction $r : \mathcal{X} \to \partial\mathcal{X}$ by elementary homology considerations.

Elaborating on the ideas in the proof of Theorem 6.6, we can give hypotheses that guarantee that the preimage of a regular value is, in fact, a 1-*manifold* (an orbifold with trivial orbifold structure).

Theorem 6.11. *Let $\mathcal{X}$ be a smooth n-dimensional orbifold with boundary and $\mathcal{P}$ a smooth orbifold with* $\dim\mathcal{P} = n - 1$. *Suppose that $p \in \mathcal{P}$ is a regular value for a smooth (complete) orbifold map ${}_\star f : \mathcal{X} \to \mathcal{P}$. This will happen, for example, if ${}_\star f$ is surjective. Let $\mathcal{S} = f^{-1}(p)$. Suppose further that for $x \in \mathcal{S}$, $\Gamma_{x,\mathcal{X}}$ has no index 2 subgroups acting on $\mathbb{R}^n$ as $\mathbb{R}^{n-1} \oplus \mathbb{R}$ with trivial action on the $\mathbb{R}$ factor. Then $\mathcal{S}$ is a compact 1-manifold with an even number of boundary points.*

Proof. As before, by Theorem 4.3, $\mathcal{S}$ is a compact 1-orbifold and thus is a disjoint union of 1-orbifolds of types (a)–(d). The goal is to show that cases (c) and (d) do not occur. To this end, suppose a component $\mathcal{C}$ of $\mathcal{S}$ is of type (c) or (d) and choose one of the points $z \in \mathcal{C}$ where $\Gamma_{z,\mathcal{C}} = \mathbb{Z}_2$. At this point, the kernel $G_{z,\mathcal{X}}$ of the quotient homomorphism $\Gamma_{z,\mathcal{X}} \to \Gamma_{z,\mathcal{C}}$ has index 2 and acts on $\mathbb{R}^n = \mathbb{R}^{n-1} \oplus \mathbb{R}$ trivially on the $\mathbb{R}$ factor. By assumption, no such points $z \in \mathcal{X}$ exist and we have our desired contradiction. We conclude, therefore, that $\mathcal{S}$ is a compact 1-manifold with an even number of boundary points. □

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