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ORIGINAL PAPER

Area 1

On solutions to evolution equations defined by lattice operators

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Abstract We discuss a specific form of evolution equations defined by lattice operators. We give exact solutions for a class of those equations and evaluate the complexity of the solutions. Moreover we discuss the relationship between them and binary cellular automata, and analyze their asymptotic behavior utilizing the explicit expression of the solution.

Keywords Lattice operator · Max-plus algebra · Ultradiscretization · Cellular automaton · Integrable system

Mathematics Subject Classification 39A14 · 06D99 · 37B15 · 68Q80

1 Introduction

Lattices are basic algebraic objects that provide a natural way to formalize and study the ordering of objects. They play an important role in various fields such as mathematics, computer science and engineering [1].

Lattices have binary operations on the elements of (partially) ordered sets called join ($\vee$) and meet ($\wedge$). By these operations, we can investigate the structure of ordered sets and manipulate elements of them algebraically. One of well-known examples is the Boolean lattice, which gives a general expression for logic circuits. Physical

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application of lattice is reported in [2] where it is used to analyze the dynamics of sandpile models.

Meanwhile, various dynamical systems described by differential or difference equations have been studied by many researchers. They are often classified into some different groups by their characteristics; periodic or non-periodic, integrable or non-integrable, and so on. To judge the integrability of the equations, several entropy-based complexity measures have been proposed and employed [3,4], which are relevant to the Lyapunov exponent used in the chaos theory [5].

In this article, we propose evolution equations defined on lattices, which we call 'L-equation' for short. We give exact general solutions for a class of L-equations, and evaluate the complexity of the solutions. Since L-equations are reduced to a certain class of max-plus equations or binary cellular automata (CA) [6–8] when we restrict dependent variables to the ordered set $(\mathbb{R}, \leq)$ or binary numbers, exact solutions of them apply to max-plus equations and CA. Though numerical or statistical studies about the behavior of solutions have been vastly done for CA, direct evaluation of behavior of solution by means of explicit expressions is quite few. Therefore, our approach provides a novel viewpoint to analysis of complex dynamical systems, that is, analytical mathematics on L-equations.

We note our result is related to ultradiscretization procedure in the field of integrable systems that connects discrete max-plus equations, CA, and continuous differential or difference equations [9–12]. For example, the Burgers equation,

$$\frac{\partial u}{\partial t} = 2u\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

has the discrete analogue called 'discrete Burgers equation',

$$u_j^{n+1} = u_j^n \frac{u_{j+1}^n + 1/u_j^n}{u_j^n + 1/u_{j-1}^n}, \tag{2}$$

and the ultradiscrete analogue called 'ultradiscrete Burgers equation',

$$u_j^{n+1} = u_j^n + \min(u_{j-1}^n, 1 - u_j^n) - \min(u_j^n, 1 - u_{j+1}^n). \tag{3}$$

We obtain (1) by the continuum limit of (2) and obtain (3) by the ultradiscrete limit of (2). Moreover, if we restrict the value set of u to $\{0, 1\}$ for (3), it is equivalent to the elementary cellular automaton (ECA) of rule number 184 explained in Sect. 2. The solutions to (1), (2) and (3) are also directly connected one another through these procedure. The evolution equation of ECA and its solutions can be expressed by the lattice operators. In this context, the L-equations proposed with exact solutions will be useful in finding wider perspective between continuous and digital systems.

The contents of this article are as follows. In Sect. 2, we first introduce the general definition of lattice and define a class where the complexity of their solutions are of polynomial order. Secondly we introduce the definition of elementary cellular automata (ECA) and show the relation between L-equations and ECA. In Sect. 3, we show a list of L-equations of the above class together with their exact solutions. In

Sect. 4, we show some examples analyzing the asymptotic behavior of ECA using the exact solutions. In Sect. 5, we give concluding remarks. In Appendix A, we give a list of the L-equations and corresponding ECA which are equivalent to each other under the transformation of variables. In Appendix B, we prove some of the expressions in Sect. 3 are solutions to the evolution equations.

2 Lattice and elementary cellular automaton

2.1 Lattice

A partially ordered set (poset) $(L, \leq)$ is a lattice [1] if the supremum (join) and the infimum (meet) of $\{a, b\}$ always exist for any $a, b \in L$. Let us define $\vee$ and $\wedge$ by

$$a \vee b = \sup\{a, b\}, \qquad a \wedge b = \inf\{a, b\}. \tag{4}$$

We have the following laws for any lattice,

$$a \vee b = b \vee a, \quad a \wedge b = b \wedge a, \qquad \text{(commutative laws)} \tag{5}$$

$$a \vee (b \vee c) = (a \vee b) \vee c, \quad a \wedge (b \wedge c) = (a \wedge b) \wedge c, \qquad \text{(associative laws)} \tag{6}$$

$$a \vee (a \wedge b) = a, \quad a \wedge (a \vee b) = a, \qquad \text{(absorption laws)} \tag{7}$$

$$a \vee a = a, \quad a \wedge a = a. \qquad \text{(idempotent laws)} \tag{8}$$

Note that we can define $\vee$ and $\wedge$ by

$$a \vee b = \max(a, b), \qquad a \wedge b = \min(a, b), \tag{9}$$

for the totally ordered set.

The poset $(L, \leq)$ is a 'distributive lattice' if it satisfies (one of) the following equivalent distributive laws for any a, b and $c \in L$,

$$\begin{aligned} &a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c), \\ &a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \qquad \text{(distributive laws)} \\ &(a \vee b) \wedge (b \vee c) \wedge (c \vee a) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a). \end{aligned} \tag{10}$$

For example, the totally ordered set $(\mathbb{R}, \leq)$ for the real number field $\mathbb{R}$ with the usual order $\leq$ is a distributive lattice since the above distributive laws hold.

Let us consider a 'conjugate' element $\overline{a}$ for each $a \in L$ by the conditions,

$$\overline{a} \in L, \quad \overline{\overline{a}} = a, \quad \overline{a} \geq \overline{b} \quad \text{if } a \leq b. \tag{11}$$

Note that conjugate elements can not be defined for a general lattice and the definition is not unique even if they can be. For example, we can define the conjugate element for $(\mathbb{R}, \leq)$ by

$$\overline{a} = c - a \quad (c\text{: constant}) \tag{12}$$

and also by

$$\overline{a} = \begin{cases} -ca & (a \leq 0) \\ -a/c & (a > 0) \end{cases}, \tag{13}$$

where $c > 0$. For the conjugation, the following laws always hold,

$$\overline{a \vee b} = \overline{a} \wedge \overline{b}, \qquad \overline{a \wedge b} = \overline{a} \vee \overline{b}. \tag{14}$$

2.2 Evolution equations expressed by lattice operators

Let us consider the following form of evolution equations using lattice operators,

$$u_j^{n+1} = f(u_{j-1}^n, u_j^n, u_{j+1}^n), \tag{15}$$

where j denotes an integer space site number, n an integer time and $u \in L$. The function $f(a, b, c)$ is constructed by $\vee$, $\wedge$ and $\overline{\cdot}$. We call the equation in the form of (15) 'L-equation' for short. Let us consider a general solution to (15) from the initial data $\{u_j^0\}$ assuming $n = 0$ is an initial time without loss of generality. If we use the above equation recursively, we obtain the formal solution expressed by u_j^0 as

$$u_j^n = f^{(n)}(u_{j-n}^0, u_{j-n+1}^0, \ldots, u_{j+n-1}^0, u_{j+n}^0), \tag{16}$$

where $f^{(n)}$ is defined recursively by

$$\begin{aligned} &f^{(k+1)}(a_{-k-1}, a_{-k}, \ldots, a_k, a_{k+1}) \\ &\quad = f(f^{(k)}(a_{-k-1}, \ldots, a_{k-1}), f^{(k)}(a_{-k}, \ldots, a_k), f^{(k)}(a_{-k+1}, \ldots, a_{k+1})). \end{aligned} \tag{17}$$

However this solution contains 3^n terms of u_j^0 and it does not give any information on the solution since it is formal. In this article, we show the list of evolution equations with a general solution containing the terms u_j^0 of which the number is of polynomial order. If the solution u_j^n can be expressed by the $O(n^m)$ terms among $\{u_j^0\}$, let us classify as the equation is of class P_m (polynomial class of order m).

For example, let us consider the equation

$$u_j^{n+1} = u_{j-1}^n \wedge u_j^n \wedge u_{j+1}^n. \tag{18}$$

Using the basic laws of lattice shown above, we can obtain a reduced expression of solution,

$$u_j^n = u_{j-n}^0 \wedge u_{j-n+1}^0 \wedge \ldots \wedge u_{j+n-1}^0 \wedge u_{j+n}^0. \tag{19}$$

Equation (18) is of class P_1 since the RHS of (19) contains $2n + 1$ $(= O(n))$ terms which is equal to the infimum of the terms from u_{j-n}^0 to u_{j+n}^0.

There are equivalent equations through the transformation of variables and coordinates. If $f_1(a, b, c) = f_2(c, b, a)$, then the solution to $u_j^{n+1} = f_1(u_{j-1}^n, u_j^n, u_{j+1}^n)$ is obtained from the solution to $u_j^{n+1} = f_2(u_{j-1}^n, u_j^n, u_{j+1}^n)$ through the transformation $j \to -j$. Similarly, the following evolution rules given by f_i's are equivalent each other,

$$f_1(a, b, c) = f_2(c, b, a), \quad \text{(reflection)} \tag{20}$$

$$f_1(a, b, c) = \overline{f_2(\overline{a}, \overline{b}, \overline{c})}, \quad \text{(conjugation)} \tag{21}$$

$$f_1(a, *, *) = f_2(*, b, *) = f_3(*, *, c), \quad \text{(Galilean transformation)} \tag{22}$$

$$f_1(a, b, *) = f_2(*, b, c), \quad \text{(Galilean transformation)} \tag{23}$$

$$f_1(a, *, c) = f_2(a, b, *), \quad \text{(separation of even and odd sites)} \tag{24}$$

where the symbol '$*$' of arguments denotes f_i does not depend on the corresponding argument. Moreover if f_1 and f_2 satisfy the condition

$$f_1(a, b, c) = \overline{f_2(a, \overline{b}, c)} = \overline{f_2(\overline{a}, b, \overline{c})}, \tag{25}$$

they are equivalent through the transformation $u_{2j}^n \to \overline{u_{2j}^n}$ and $u_{2j+1}^n \to u_{2j+1}^n$. Similarly, if f_1 and f_2 satisfy the condition

$$f_1(a, b, c) = f_2(\overline{a}, \overline{b}, \overline{c}) = \overline{f_2(a, b, c)}, \tag{26}$$

they are equivalent through the transformation $u_j^{2n} \to \overline{u_j^{2n}}$ and $u_j^{2n+1} \to u_j^{2n+1}$.

2.3 Elementary cellular automaton

Elementary cellular automaton is a simple evolutional digital system and the evolution rule is given by

$$u_j^{n+1} = f_b(u_{j-1}^{n+1}, u_j^n, u_{j+1}^n), \tag{27}$$

where j denotes an integer space site number, n an integer time and u a binary state value ($= 0$ or 1) [7,8]. The binary-valued function f_b determines the time evolution rule and gives the state value at the next time. Thus f_b can be defined by a rule table in the following form,

$a\,b\,c$	111	110	101	100	011	010	001	000
$f_b(a, b, c)$	b_7	b_6	b_5	b_4	b_3	b_2	b_1	b_0

(28)

where the upper row shows combinations of binary variables a, b, c and the lower the binary values of $f(a, b, c)$. Since f is uniquely defined if all b_k's are given, we can identify the evolution rule by the rule number $(r)_{10} = (b_7 b_6 \ldots b_0)_2$ after Wolfram

[7,8]. Let us call the ECA of the decimal rule number r 'ECAr'. There exist 256 rules from ECA0 to ECA255.

Let us consider the lattice $(\mathbb{R}, \leq)$ and assume that the conjugation operator is given by

$$\overline{a} = 1 - a. \tag{29}$$

If we define the rule $f_b(a, b, c)$ using $\vee$, $\wedge$ and $\overline{\cdot}$, they are equivalent to the bitwise operators AND, OR and NOT, respectively, for the binary values as follows:

$$\begin{aligned} &a \vee b = \max(a, b) = a \,\text{OR}\, b, \qquad a \wedge b = \min(a, b) = a \,\text{AND}\, b, \\ &\overline{a} = 1 - a = \text{NOT}\, a, \end{aligned} \tag{30}$$

where $a, b \in \{0, 1\}$. Therefore every ECA rule can be expressed by the lattice operators.

Moreover, if we consider the lattice $(\mathbb{R}, \leq)$, (18) becomes

$$u_j^n = \min(u_{j-1}^n, u_j^n, u_{j+1}^n), \tag{31}$$

and it gives ECA128 in the binary case. The solution

$$u_j^n = \min(u_{j-n}^0, u_{j-n+1}^0, \dots, u_{j+n-1}^0, u_{j+n}^0), \tag{32}$$

is 1 if $u_{j-n}^0 = u_{j-n+1}^0 = \dots = u_{j+n-1}^0 = u_{j+n}^0 = 1$ and 0 otherwise in the binary case.

Considering the equivalent relations described in Sect. 2.2, there exist 75 independent rules of ECA. We list the rule numbers as follows:

$$\begin{aligned} &0, 1, 2, 3, 4, 6, 8, 9, 11, 12, 13, 14, 18, 19, 21, 22, 24, 25, 26, \\ &27, 28, 29, 30, 32, 33, 35, 36, 37, 38, 40, 41, 42, 43, 44, 45, \\ &46, 50, 54, 56, 57, 58, 60, 62, 72, 73, 74, 76, 78, 94, 104, 106, \\ &108, 110, 122, 126, 128, 130, 132, 134, 136, 138, 140, 146, \\ &150, 152, 154, 156, 162, 164, 168, 170, 172, 184, 200, 232. \end{aligned} \tag{33}$$

3 Solutions to evolution equations expressed by lattice operators

In this section, we list up the L-equations satisfying the following conditions.

(a) The terms a, b and c do not appear or appear once in $f(a, b, c)$.
(b) The solution is of polynomial class.

In addition, we show some equations which satisfy (b) but not (a) as an exceptional case. All solutions are derived exactly by arranging and reducing terms included in the formal solution using lattice formulas shown in Sect. 2.1. Two examples of proofs on such solutions are shown in Appendix B. In the list blow, we show the form of $f(a, b, c)$, the rule number of corresponding ECA and its solution.

3.1 Class P_0

- $f(a,b,c) = e \ (\in L)$ (ECA0)

$$u_j^n = e.$$

- $f(a,b,c) = a$ (ECA240)

$$u_j^n = u_{j-n}^0.$$

- $f(a,b,c) = \overline{a} \wedge b$ (ECA12)

$$u_j^n = \overline{u_{j-1}^0} \wedge u_j^0.$$

- $f(a,b,c) = \overline{a} \wedge \overline{b}$ (ECA3)

$$u_j^1 = \overline{u_{j-1}^0} \wedge \overline{u_j^0}, \quad u_j^2 = u_{j-1}^0 \vee (u_{j-2}^0 \wedge u_j^0),$$
$$u_j^{2m+1} = u_{j-m}^1, \quad u_j^{2m+2} = u_{j-m}^2 \quad (m \geq 1).$$

- $f(a,b,c) = \overline{a} \wedge \overline{b} \wedge c$ (ECA2)

$$u_j^n = \overline{u_{j+n-2}^0} \wedge \overline{u_{j+n-1}^0} \wedge u_{j+n}^0.$$

- $f(a,b,c) = \overline{a} \wedge b \wedge \overline{c}$ (ECA4)

$$u_j^n = \overline{u_{j-1}^0} \wedge u_j^0 \wedge \overline{u_{j+1}^0}.$$

- $f(a,b,c) = \overline{a} \wedge \overline{b} \wedge \overline{c}$ (ECA1)

$$u_j^{2m-1} = \overline{u_{j-1}^0} \wedge \overline{u_j^0} \wedge \overline{u_{j+1}^0},$$
$$u_j^{2m} = (u_{j-2}^0 \vee u_{j-1}^0 \vee u_j^0) \wedge (u_{j-1}^0 \vee u_j^0 \vee u_{j+1}^0) \wedge (u_j^0 \vee u_{j+1}^0 \vee u_{j+2}^0).$$

- $f(a,b,c) = (\overline{a} \vee \overline{b}) \wedge c$ (ECA42)

$$u_j^n = (\overline{u_{j+n-2}^0} \vee \overline{u_{j+n-1}^0}) \wedge u_{j+n}^0.$$

- $f(a,b,c) = (a \vee c) \wedge b$ (ECA200)

$$u_j^n = (u_{j-1}^0 \vee u_{j+1}^0) \wedge u_j^0.$$

- $f(a,b,c) = (\overline{a} \vee \overline{c}) \wedge b$ (ECA76)

$$u_j^n = (\overline{u_{j-1}^0} \vee \overline{u_{j+1}^0}) \wedge u_j^0.$$

- $f(a,b,c) = (\overline{a} \vee \overline{c}) \wedge \overline{b}$ (ECA19)

$$\begin{aligned}
u_j^1 &= (\overline{u_{j-1}^0} \vee \overline{u_{j+1}^0}) \wedge \overline{u_j^0},\\
u_j^{2m} &= (u_{j-1}^0 \vee u_j^0) \wedge (u_j^0 \vee u_{j+1}^0) \wedge (u_{j-2}^0 \vee u_{j-1}^0 \vee u_{j+1}^0 \vee u_{j+2}^0),\\
u_j^{2m+1} &= \overline{u_j^{2m}} \quad (m \geq 1).
\end{aligned}$$

- $f(a,b,c) = (b \vee (a \wedge c)) \wedge (\overline{b} \vee (\overline{a} \wedge \overline{c}))$ (ECA36)

$$\begin{aligned}
u_j^1 &= (u_j^0 \vee (u_{j-1}^0 \wedge u_{j+1}^0)) \wedge (\overline{u_j^0} \vee (\overline{u_{j-1}^0} \wedge \overline{u_{j+1}^0}),\\
u_j^n &= (u_j^0 \vee (u_{j-1}^0 \wedge u_{j+1}^0)) \wedge (\overline{u_j^0} \vee (\overline{u_{j-1}^0} \wedge \overline{u_{j+1}^0})\\
&\quad \wedge (\overline{u_{j-1}^0} \vee u_j^0 \vee \overline{u_{j+1}^0} \vee (u_{j-2}^0 \wedge u_{j+2}^0))\\
&\quad \wedge (u_{j-1}^0 \vee \overline{u_j^0} \vee u_{j+1}^0 \vee (\overline{u_{j-2}^0} \wedge \overline{u_{j+2}^0}))\\
&\quad \wedge (u_{j-1}^0 \vee \overline{u_{j-1}^0} \vee u_{j+1}^0 \vee \overline{u_{j+1}^0})) \quad (n \geq 2).
\end{aligned}$$

3.2 Class P_1

We define new symbols about $\vee$ and $\wedge$ as

$$\begin{aligned}
\bigvee_{k_0 \leq k \leq k_1} x_k &= x_{k_0} \vee x_{k_0+1} \vee \cdots \vee x_{k_1},\\
\bigwedge_{k_0 \leq k \leq k_1} x_k &= x_{k_0} \wedge x_{k_0+1} \wedge \cdots \wedge x_{k_1}.
\end{aligned}$$

- $f(a,b,c) = a \wedge b$ (ECA192)

$$\begin{aligned}
u_j^n &= \bigwedge_{-n \leq k \leq 0} u_{j+k}^0\\
&= u_{j-n}^0 \wedge u_{j-n+1}^0 \wedge \ldots \wedge u_j^0.
\end{aligned}$$

- $f(a,b,c) = a \wedge b \wedge c$ (ECA128)

$$\begin{aligned}
u_j^n &= \bigwedge_{-n \leq k \leq n} u_{j+k}^0\\
&= u_{j-n}^0 \wedge u_{j-n+1}^0 \wedge \ldots \wedge u_{j+n-1}^0 \wedge u_{j+n}^0.
\end{aligned}$$

- $f(a,b,c) = \overline{a} \wedge b \wedge c$ (ECA8)

$$\begin{aligned}
u_j^n &= \left(\bigwedge_{-1 \leq k \leq n-2} \overline{u_{j+k}^0}\right) \wedge \left(\bigwedge_{0 \leq k \leq n} u_{j+k}^0\right)\\
&= \overline{u_{j-1}^0} \wedge \overline{u_j^0} \wedge \cdots \wedge \overline{u_{j+n-2}^0} \wedge u_j^0 \wedge u_{j+1}^0 \wedge \cdots \wedge u_{j+n}^0.
\end{aligned}$$

- $f(a, b, c) = a \wedge \overline{b} \wedge c$ (ECA32)

$$\begin{aligned} u_j^n &= \left(\bigwedge_{0 \le k \le n} u^0_{j-n+2k} \right) \wedge \left(\bigwedge_{0 \le k \le n-1} u^0_{j-n+k+1} \right) \\ &= u^0_{j-n} \wedge \overline{u^0_{j-n+1}} \wedge u^0_{j-n+2} \wedge \overline{u^0_{j-n+3}} \wedge \cdots \wedge \overline{u^0_{j+n-1}} \wedge u^0_{j+n}. \end{aligned}$$

3.3 Class P_2

- $f(a, b, c) = (\overline{a} \vee b) \wedge c$ (ECA138)

$$\begin{aligned} u_j^n &= \bigwedge_{0 \le l \le n} \left(\left(\bigvee_{l-1 \le k \le n-2} \overline{u^0_{j+k}} \right) \vee u^0_{j+l} \right) \\ &= (\overline{u^0_{j-1}} \vee \overline{u^0_j} \vee \cdots \vee \overline{u^0_{j+n-2}} \vee u^0_j) \\ &\quad \wedge (\overline{u^0_j} \vee \overline{u^0_{j+1}} \vee \cdots \vee \overline{u^0_{j+n-2}} \vee u^0_{j+1}) \\ &\quad \ldots \\ &\quad \wedge (\overline{u^0_{j+n-3}} \vee \overline{u^0_{j+n-2}} \vee u^0_{j+n-2}) \\ &\quad \wedge (\overline{u^0_{j+n-2}} \vee u^0_{j+n-1}) \wedge u^0_{j+n}. \end{aligned}$$

- $f(a, b, c) = (a \vee \overline{b}) \wedge c$ (ECA162)

$$\begin{aligned} u_j^n &= \bigwedge_{0 \le l \le n} \left(u_{j-n+2l} \vee \left(\bigvee_{l \le k \le n-1} \overline{u^0_{j-n+2k+1}} \right) \right) \\ &= (u^0_{j-n} \vee \overline{u^0_{j-n+1}} \vee \overline{u^0_{j-n+3}} \vee \cdots \vee \overline{u^0_{j+n-3}} \vee \overline{u^0_{j+n-1}}) \\ &\quad \wedge (u^0_{j-n+2} \vee \overline{u^0_{j-n+3}} \vee \overline{u^0_{j-n+5}} \vee \cdots \vee \overline{u^0_{j+n-3}} \vee \overline{u^0_{j+n-1}}) \\ &\quad \ldots \\ &\quad \wedge (u^0_{j+n-4} \vee \overline{u^0_{j+n-3}} \vee \overline{u^0_{j+n-1}}) \\ &\quad \wedge (u^0_{j+n-2} \vee \overline{u^0_{j+n-1}}) \wedge u^0_{j+n}. \end{aligned}$$

- $f(a, b, c) = (\overline{a} \vee c) \wedge b$ (ECA140)

$$\begin{aligned} u_j^n &= \bigwedge_{0 \le l \le n} \left(\left(\bigvee_{-1 \le k \le n-l-2} u^0_{j+k} \right) \vee u^0_{j+n-l} \right) \\ &= (\overline{u^0_{j-1}} \vee \overline{u^0_j} \vee \cdots \vee \overline{u^0_{j+n-3}} \vee \overline{u^0_{j+n-2}} \vee u^0_{j+n}) \\ &\quad \wedge (\overline{u^0_{j-1}} \vee \overline{u^0_j} \vee \cdots \vee \overline{u^0_{j+n-3}} \vee u^0_{j+n-1}) \\ &\quad \ldots \\ &\quad \wedge (\overline{u^0_{j-1}} \vee \overline{u^0_j} \vee u^0_{j+2}) \\ &\quad \wedge (\overline{u^0_{j-1}} \vee u^0_{j+1}) \wedge u^0_j. \end{aligned}$$

- $f(a, b, c) = (a \vee c) \wedge \overline{b}$ (ECA50)

$$
\begin{aligned}
u_j^{2m-1} = & \left\{ \bigvee_{0\le l\le m-1} \left(u^0_{j-2m+2l+1} \wedge \left(\bigwedge_{-m+l+1\le k\le m-l-1} \overline{u^0_{j+2k}} \right)\right)\right\} \\
& \vee \left\{ \bigvee_{0\le l\le m-1} \left(\left(\bigwedge_{-m+l+1\le k\le m-l-1} \overline{u^0_{j+2k}} \right) \wedge u^0_{j+2m-2l-1} \right)\right\} \\
& \vee \left\{ \bigvee_{0\le l\le m-2} \left(\overline{u^0_{j-2m+2l+2}} \wedge \left(\bigwedge_{-m+l+1\le k\le m-l-2} u^0_{j+2k+1} \right)\right)\right\} \\
& \vee \left\{ \bigvee_{0\le l\le m-2} \left(\left(\bigwedge_{-m+l+1\le k\le m-l-2} u^0_{j+2k+1} \right) \wedge \overline{u^0_{j-2m+2l+1}} \right)\right\} \\
= & (u^0_{j-2m+1} \wedge \overline{u^0_{j-2m+2}} \wedge \overline{u^0_{j-2m+4}} \wedge \dots \wedge \overline{u^0_{j+2m-4}} \wedge \overline{u^0_{j+2m-2}}) \\
& \vee (\overline{u^0_{j-2m+2}} \wedge \overline{u^0_{j-2m+4}} \wedge \dots \wedge \overline{u^0_{j+2m-4}} \wedge \overline{u^0_{j+2m-2}} \wedge u^0_{j+2m-1}) \\
& \vee (\overline{u^0_{j-2m+2}} \wedge u^0_{j-2m+3} \wedge u^0_{j-2m+5} \wedge \dots \wedge u^0_{j+2m-5} \wedge u^0_{j+2m-3}) \\
& \vee (u^0_{j-2m+3} \wedge u^0_{j-2m+5} \wedge \dots \wedge u^0_{j+2m-5} \wedge u^0_{j+2m-3} \wedge \overline{u^0_{j+2m-2}}) \\
& \dots \\
& \vee (\overline{u^0_{j-2}} \wedge u^0_{j-1} \wedge u^0_{j+1}) \vee (u^0_{j-1} \wedge u^0_{j+1} \wedge \overline{u^0_{j+2}}) \\
& \vee (u^0_{j-1} \wedge \overline{u^0_{j}}) \vee (\overline{u^0_{j}} \wedge u^0_{j+1}),
\end{aligned}
$$

$$
\begin{aligned}
u_j^{2m} = & \left\{ \bigvee_{0\le l\le m} \left(u^0_{j-2m+2l} \wedge \left(\bigwedge_{-m+l\le k\le m-l-1} \overline{u^0_{j+2k+1}} \right)\right)\right\} \\
& \vee \left\{ \bigvee_{0\le l\le m} \left(\left(\bigwedge_{-m+l\le k\le m-l-1} \overline{u^0_{j+2k+1}} \right) \wedge u^0_{j+2m-2l} \right)\right\} \\
& \vee \left\{ \bigvee_{0\le l\le m-1} \left(\overline{u^0_{j-2m+2l+1}} \wedge \left(\bigwedge_{-m+l+1\le k\le m-l-1} u^0_{j+2k} \right)\right)\right\} \\
& \vee \left\{ \bigvee_{0\le l\le m-1} \left(\left(\bigwedge_{-m+l+1\le k\le m-l-1} u^0_{j+2k} \right) \wedge \overline{u^0_{j+2m-2l-1}} \right)\right\} \\
= & (u^0_{j-2m} \wedge \overline{u^0_{j-2m+1}} \wedge \overline{u^0_{j-2m+3}} \wedge \dots \wedge \overline{u^0_{j+2m-3}} \wedge \overline{u^0_{j+2m-1}}) \\
& \vee (\overline{u^0_{j-2m+1}} \wedge \overline{u^0_{j-2m+3}} \wedge \dots \wedge \overline{u^0_{j+2m-3}} \wedge \overline{u^0_{j+2m-1}} \wedge u^0_{j+2m}) \\
& \vee (\overline{u^0_{j-2m+1}} \wedge u^0_{j-2m+2} \wedge u^0_{j-2m+4} \wedge \dots \wedge u^0_{j+2m-4} \wedge u^0_{j+2m-2}) \\
& \vee (u^0_{j-2m+2} \wedge u^0_{j-2m+4} \wedge \dots \wedge u^0_{j+2m-4} \wedge u^0_{j+2m-2} \wedge \overline{u^0_{j+2m-1}}) \\
& \dots \\
& \vee (u^0_{j-2} \wedge \overline{u^0_{j-1}} \wedge \overline{u^0_{j+1}}) \vee (\overline{u^0_{j-1}} \wedge \overline{u^0_{j+1}} \wedge u^0_{j+2}) \\
& \vee (\overline{u^0_{j-1}} \wedge u^0_{j}) \vee (u^0_{j} \wedge \overline{u^0_{j+1}}).
\end{aligned}
$$

- $f(a,b,c) = (\overline{a} \vee \overline{b}) \wedge \overline{c}$ (ECA21)

$$\begin{aligned} u_j^{2m-1} = &\left\{\bigvee_{0\le l\le m-2} \overline{u_{j+2l+1}^0}\right\} \\ &\wedge \left\{\bigwedge_{0\le l\le m-2} \left(\overline{u_{j+l-1}^0} \vee \overline{u_{j+l}^0} \vee \left(\bigvee_{0\le k\le m-l-2} \overline{u_{j+2k+l+3}^0}\right)\right)\right\} \\ &\wedge \left\{\bigwedge_{0\le l\le m-2} \left(\left(\bigvee_{0\le k\le m-l-2} \overline{u_{j+2k+l+2}^0}\right) \vee \overline{u_{j+2m-l-1}^0}\right)\right\} \\ &\wedge \left\{\bigwedge_{0\le l\le m-1} \left(\overline{u_{j+l-1}^0} \vee \left(\bigvee_{0\le k\le m-l-1} \overline{u_{j+2k+l}^0}\right)\right)\right\}, \\ u_j^{2m} = &\left\{\bigwedge_{0\le l\le m} u_{j+2l}^0\right\} \\ &\vee \left\{u_{j-2}^0 \wedge u_{j-1}^0 \wedge \left(\bigwedge_{0\le l\le m-1} u_{j+2l+2}^0\right)\right\} \\ &\vee \left\{\bigvee_{0\le l\le m-2} \left(u_{j+l-1}^0 \wedge u_{j+l}^0 \wedge \left(\bigwedge_{0\le k\le m-l-2} u_{j+2k+l+3}^0\right) \wedge u_{j+2m-l}^0\right)\right\} \\ &\vee \left\{\bigvee_{0\le l\le m-2} \left(u_{j+l}^0 \wedge u_{j+l+1}^0 \wedge \left(\bigwedge_{0\le k\le m-l-2} u_{j+2k+l+4}^0\right)\right)\right\} \\ &\vee \left\{\bigvee_{0\le l\le m-1} \left(\left(\bigwedge_{0\le k\le m-l-1} u_{j+2k+l+1}^0\right) \wedge u_{j+2m-l}^0\right)\right\} \\ &\vee \left\{\bigvee_{0\le l\le m-1} \left(u_{j+l}^0 \wedge \left(\bigwedge_{0\le k\le m-l-1} u_{j+2k+l+1}^0\right)\right)\right\}. \end{aligned}$$

- $f(a,b,c) = (\overline{a} \vee b) \wedge \overline{c}$ (ECA69)

$$\begin{aligned} u_j^{2m-1} = &(u_j^0 \wedge \overline{u_{j+1}^0}) \\ &\vee \left\{\bigvee_{1\le l\le m-1} \left(\left(\bigwedge_{0\le k\le l} \overline{u_{j+2k-1}^0}\right) \wedge u_{j+2l}^0\right)\right\} \\ &\vee \left\{\bigvee_{1\le l\le m-1} \left(\left(\bigwedge_{0\le k\le l} u_{j+2k}^0\right) \wedge \overline{u_{j+2l+1}^0})\right)\right\} \\ &\vee \left\{\bigvee_{1\le l\le m-1} \left(u_{j-2}^0 \wedge \overline{u_{j-1}^0} \wedge \left(\bigwedge_{1\le k\le l} u_{j+2k}^0\right) \wedge \overline{u_{j+2l+1}^0}\right)\right\} \\ &\vee \left\{\bigwedge_{0\le l\le m} \overline{u_{j+2l-1}^0}\right\}, \\ u_j^{2m} = \quad &(u_j^0 \wedge \overline{u_{j+1}^0}) \\ &\vee \left\{\bigvee_{1\le l\le m} \left(\left(\bigwedge_{0\le k\le l} \overline{u_{j+2k-1}^0}\right) \wedge u_{j+2l}^0\right)\right\} \\ &\vee \left\{\bigvee_{1\le l\le m-1} \left(\left(\bigwedge_{0\le k\le l} u_{j+2k}^0\right) \wedge \overline{u_{j+2l+1}^0}\right)\right\} \end{aligned}$$

$$\vee \left\{ \bigvee_{1\le l\le m-1} \left(u^0_{j-2} \wedge \overline{u^0_{j-1}} \wedge \left(\bigwedge_{1\le k\le l} u^0_{j+2k} \right) \wedge \overline{u^0_{j+2l+1}} \right) \right\}$$
$$\vee \left\{ u^0_{j-2} \wedge \overline{u^0_{j-1}} \wedge \left(\bigwedge_{1\le l\le m} u^0_{j+2l} \right) \right\}$$
$$\vee \left\{ \bigwedge_{0\le l\le m} u^0_{j+2l} \right\}.$$

- $f(a,b,c) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$ (ECA232)

$$\begin{aligned} u^n_j = & \left\{ \bigwedge_{0\le l\le n-1} \left(u^0_{j-l-1} \vee \left(\bigvee_{0\le k\le l} u^0_{j+2k-l} \right) \right) \right\} \\ & \wedge \left\{ \bigwedge_{0\le l\le n-1} \left(\left(\bigvee_{0\le k\le l} u^0_{j+2k-l} \right) \vee u^0_{j+l+1} \right) \right\} \\ & \wedge \left\{ \bigvee_{0\le l\le n} u^0_{j+2l-n} \right\} \\ = & (u^0_{j-1} \vee u^0_j) \wedge (u^0_j \vee u^0_{j+1}) \\ & \wedge (u^0_{j-2} \vee u^0_{j-1} \vee u^0_{j+1}) \wedge (u^0_{j-1} \vee u^0_{j+1} \vee u^0_{j+2}) \\ & \wedge (u^0_{j-3} \vee u^0_{j-2} \vee u^0_j \vee u^0_{j+2}) \wedge (u^0_{j-2} \vee u^0_j \vee u^0_{j+2} \vee u^0_{j+3}) \\ & \cdots \\ & \wedge (u^0_{j-n} \vee u^0_{j-n+1} \vee u^0_{j-n+3} \vee \cdots \vee u^0_{j+n-1}) \\ & \wedge (u^0_{j-n+1} \vee u^0_{j-n+3} \vee \cdots \vee u^0_{j+n-1} \vee u^0_{j+n}) \\ & \wedge (u^0_{j-n} \vee u^0_{j-n+2} \vee \cdots \vee u^0_{j+n-2} \vee u^0_{j+n}). \end{aligned}$$

- $f(a,b,c) = b \wedge (\overline{a} \vee c) \wedge (a \vee \overline{c})$ (ECA132)

$$\begin{aligned} u^n_j = & u^0_j \\ & \wedge \left\{ \bigwedge_{1\le l\le n} \left(\left(\bigvee_{0\le k\le 2(l-1)} \overline{u^0_{j+k-l}} \right) \vee u^0_{j+l} \right) \right\} \\ & \wedge \left\{ \bigwedge_{1\le l\le n} \left(u^0_{j-l} \vee \left(\bigvee_{0\le k\le 2(l-1)} \overline{u^0_{j-k+l}} \right) \right) \right\} \\ = & u^0_j \\ & \wedge (\overline{u^0_{j-1}} \vee u^0_{j+1}) \\ & \wedge (\overline{u^0_{j-2}} \vee \overline{u^0_{j-1}} \vee \overline{u^0_j} \vee u^0_{j+2}) \\ & \cdots \\ & \wedge (\overline{u^0_{j-n}} \vee \overline{u^0_{j-n+1}} \vee \cdots \vee \overline{u^0_{j+n-2}} \vee u^0_{j+n}) \\ & \wedge (u^0_{j-1} \vee \overline{u^0_{j+1}}) \end{aligned}$$

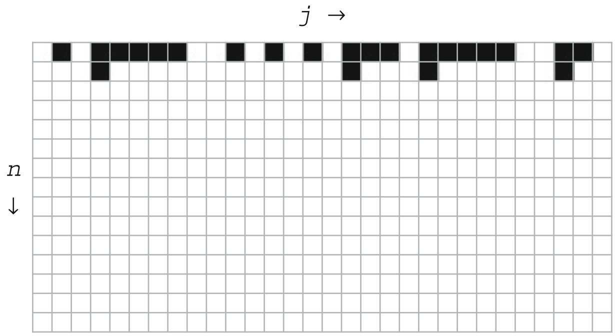

Fig. 1 Evolution pattern of ECA8. *open squares* and *filled squares* denote 0 and 1, respectively

$$\begin{aligned}&\wedge(u^0_{j-2} \vee \overline{u^0_j} \vee \overline{u^0_{j+1}} \vee \overline{u^0_{j+2}})\\ &\dots\\ &\wedge(u^0_{j-n} \vee \overline{u^0_{j-n+2}} \vee \overline{u^0_{j-n+3}} \vee \dots \vee \overline{u^0_{j+n}}).\end{aligned}$$

4 Asymptotic behavior of ECA

For the solutions of polynomial class, we can easily grasp its asymptotic behavior for $n \gg 0$ in the binary case. We show it using two examples in Sect. 3.

The evolution rule in the case of $f(a, b, c) = \overline{a} \wedge b \wedge c$ is equivalent to ECA8 in the binary case. We show again the solution,

$$u^n_j = \overline{u^0_{j-1}} \wedge \overline{u^0_j} \wedge \dots \wedge \overline{u^0_{j+n-2}} \wedge u^0_j \wedge u^0_{j+1} \wedge \dots \wedge u^0_{j+n}.$$

In the binary case, $\overline{u} \wedge u = 0$ since $u \in \{0, 1\}$. Thus the above solution is simplified as

$$\begin{aligned}&u^1_j = \overline{u^0_{j-1}} \wedge u^0_j \wedge u^0_{j+1} = \begin{cases} 1 & (u^0_{j-1} = 0 \text{ and } u^0_j = u^0_{j+1} = 1)\\ 0 & (\text{otherwise})\end{cases},\\ &u^n_j = 0 \quad (n \geq 2).\end{aligned}$$

Figure 1 shows a typical evolution pattern of ECA8.

Another example is $f(a, b, c) = (\overline{a} \vee c) \wedge b$ (ECA140). The solution is

$$\begin{aligned}u^n_j =&(\overline{u^0_{j-1}} \vee \overline{u^0_j} \vee \dots \vee \overline{u^0_{j+n-3}} \vee \overline{u^0_{j+n-2}} \vee u^0_{j+n})\\ &\wedge (\overline{u^0_{j-1}} \vee \overline{u^0_j} \vee \dots \vee \overline{u^0_{j+n-3}} \vee u^0_{j+n-1})\\ &\dots,\\ &\wedge (\overline{u^0_{j-1}} \vee \overline{u^0_j} \vee u^0_{j+2})\\ &\wedge (\overline{u^0_{j-1}} \vee u^0_{j+1}) \wedge u^0_j.\end{aligned} \tag{34}$$

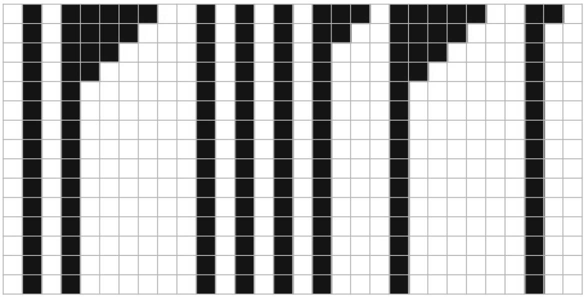

Fig. 2 Evolution pattern of ECA140

In the binary case, it is clear that $u_j^n = 0$ $(n \geq 1)$ if $u_j^0 = 0$ and that $u_j^n = 1$ $(n \geq 1)$ if $u_{j-1}^0 = 0$ and $u_j^0 = 1$. Moreover, $u_j^n = 0$ $(n \gg 0)$ if there exist $r \geq 0$ such that $u_{j-1}^0 = u_j^0 = \cdots = u_{j+r}^0 = 1$ and $u_{j+r+1}^0 = 0$. Therefore, we can see the asymptotic behavior of u_j^n for large enough n as

$$u_j^n = \begin{cases} 1 & (u_{j-1}^0 = 0 \text{ and } u_j^0 = 1) \\ 0 & (\text{otherwise}) \end{cases} \qquad (n \gg 0)$$

Note that $u_j^n \equiv 1$ $(n \geq 1)$ if $u_j^0 \equiv 1$ as a special case. Figure 2 shows a typical evolution pattern of ECA140.

Let us consider the asymptotic behavior of solution from real valued initial data for the same $f(a, b, c) = (\overline{a} \vee c) \wedge b$ of the lattice $(\mathbb{R}, \leq)$. The lattice operations $a \vee b$, $a \wedge b$ and $\overline{a}$ can be replaced by $\max(a, b)$, $\min(a, b)$ and $1 - a$, respectively, for the lattice $(\mathbb{R}, \leq)$. Thus (34) can be rewritten as

$$\begin{aligned} u_j^n = \min(\, & \max(1 - u_{j-1}^0, 1 - u_j^0, \ldots, 1 - u_{j+n-3}^0, 1 - u_{j+n-2}^0, u_{j+n}^0), \\ & \ldots, \\ & \max(1 - u_{j-1}^0, 1 - u_j^0, \ldots, 1 - u_{j+r-2}^0, 1 - u_{j+r-1}^0, u_{j+r+1}^0), \\ & \underline{\max(1 - u_{j-1}^0, 1 - u_j^0, \ldots, 1 - u_{j+r-2}^0, u_{j+r}^0)}, \\ & \ldots, \\ & \max(1 - u_{j-1}^0, 1 - u_j^0, u_{j+2}^0), \\ & \max(1 - u_{j-1}^0, u_{j+1}^0), u_j^0). \end{aligned} \tag{35}$$

If there exist $r \geq 1$ such that $u_{j+r}^0 < 1/2$, the max terms above the underlined terms in (35) are all greater than $\max(1 - u_{j-1}^0, 1 - u_j^0, \ldots, 1 - u_{j+r-2}^0, u_{j+r}^0)$ since they include the same $1 - u_{j-1}^0$, $1 - u_j^0$, ..., $1 - u_{j+r-2}^0$, and $1 - u_{j+r}^0$ which is greater than u_{j+r}^0. They can be neglected since RHS of (35) is a minimum of all max terms.

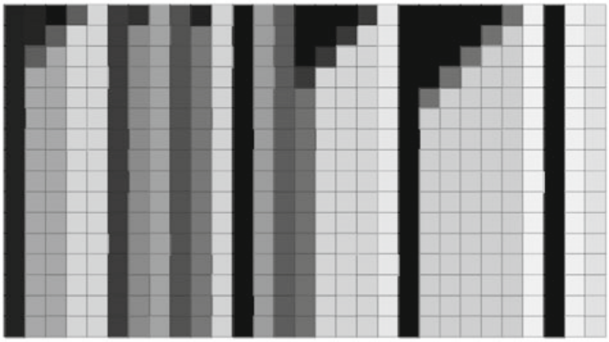

Fig. 3 Evolution pattern of real valued solution

Thus we obtain

$$\begin{aligned} u_j^n = \min(&\max(1-u_{j-1}^0, 1-u_j^0, \dots, 1-u_{j+r-2}^0, 1-u_{j+r-1}^0, u_{j+r+1}^0),\\ &\max(1-u_{j-1}^0, 1-u_j^0, \dots, 1-u_{j+r-2}^0, u_{j+r}^0),\\ &\dots,\\ &\max(1-u_{j-1}^0, 1-u_j^0, u_{j+2}^0),\\ &\max(1-u_{j-1}^0, u_{j+1}^0), u_j^0), \end{aligned}$$

and $u_j^{n+1} = u_j^n$. Figure 3 shows an example of this type of solution. Otherwise $u_k^0 \geq 1/2$ for any $k \geq j-1$. In this case,

$$\max(1-u_{j-1}^0, 1-u_j^0, \dots, 1-u_{j+r-2}^0, u_{j+r}^0) = u_{j+r}^0$$

and we have

$$u_j^n = \min(u_j^0, u_{j+1}^0, \dots, u_{j+n}^0),$$

from (35). Then all u become the same value $\min_{1\leq k\leq K} u_k^0$ for $n \gg 0$ under the periodic boundary condition for the space cites with a finite period K. In any case, the asymptotic behavior of solution is static ($u_j^{n+1} = u_j^n$) for $n \gg 0$.

The behaviors of asymptotic solution shown in Figs. 2 and 3 are similar each other in a sense that they become static. Though the binary solution is a special case of real valued solution, it implies that ECA can become a good test of the corresponding L-equation.

5 Concluding remarks

We proposed the equations constructed from lattice operators $\vee$, $\wedge$ and $\overline{\cdot}$ of which solutions are expressed by the initial data of polynomial order. Since ECA are embedded in these equations as a special case, we can grasp the asymptotic behavior of ECA easily using the solutions.

We proposed 24 equations and their solutions. Rule numbers of corresponding ECA are as follows:

0, 1, 2, 3, 4, 8, 12, 19, 21, 32, 36, 42, 50, 69, 76, 128, 132, 138, 140, 162, 192, 200, 232, 240.

Therefore about one third of ECA listed in (33) are solved. ECA are roughly classified into 4 classes (Class 1 to 4) by Wolfram. Most ECA expressed by the L-equation shown in this article are in Class 1 or 2, and some are in Class 3. This classification was done by the geometrical complexity of solution pattern. On the other hand, we classified the L-equations from a viewpoint of algebraic complexity, that is, the order of initial data included in the solution. It is interesting that some rule numbers in the same Wolfram class are not in the same polynomial class.

The lattice operations in the lattice $(\mathbb{R}, \leq)$ can be embedded in the max-plus algebra where max (min) is the addition, $+$ is the production, and $-$ is the subtraction [6]. The max-plus equation

$$u_j^{n+1} = u_j^n + \min(u_{j-1}^n, 1 - u_j^n) - \min(u_j^n, 1 - u_{j+1}^n),$$

can not be expressed only by the lattice operations and is equivalent to ECA184 in the binary case [11]. However, this equation is transformed into the L-equation of P_1,

$$f_j^{n+1} = \max(f_{j-1}^n, f_{j+1}^n),$$

through the transformation

$$u_j^n = f_j^n - f_{j-1}^n + \frac{1}{2}.$$

There exist some max-plus equations which reduce to the L-equation of polynomial class like the above example. It is a future problem to clarify the more general relation between the max-plus and the lattice expressions from the viewpoint of solvability of equation.

We analyzed the L-equations constructed from a simple combination of lattice operations. Though they are simple, we can not confirm the polynomial class of some equations, for example, those defined by the following $f(a, b, c)$,

$$(\overline{a} \vee c) \wedge \overline{b} \quad \text{(ECA35)}, \quad (a \vee b) \wedge c \quad \text{(ECA168)},$$
$$(a \vee b) \wedge \overline{c} \quad \text{(ECA84)}, \quad (a \vee \overline{b}) \wedge \overline{c} \quad \text{(ECA81)}.$$

They may not be in a polynomial class. It is another future problem to evaluate the complexity of general solution to the above equations. In addition, there exist many equations defined by the more complicated combination of lattice operations. These propose a vast target to be analyzed.

Appendix A: Equivalent L-equations and ECA rule numbers

In this appendix, we show in Table 1 the equivalent L-equations of polynomial class in the form,

$$u_j^{n+1} = f(u_{j-1}^n, u_j^n, u_{j+1}^n), \tag{36}$$

and its corresponding rule number of ECA. The first column of each row shows equivalent $f(a, b, c)$'s and the first one is a representative. The second column shows

Table 1 Equivalent L-equations and their class

$e \in L$ (0, 255)	P_0
a (240), b (204), c (170) , $\overline{a}$ (15), $\overline{b}$ (51), $\overline{c}$ (85)	P_0
$a \wedge b$ (192), $b \wedge c$ (136), $a \wedge c$ (160), $a \vee b$ (152), $b \vee c$ (238), $a \vee c$ (250)	P_1
$\overline{a} \wedge b$ (12), $\overline{b} \wedge c$ (34), $\overline{a} \vee b$ (207), $\overline{b} \vee c$, (187) $a \wedge \overline{b}$ (48), $b \wedge \overline{c}$ (68), $a \vee \overline{b}$ (243), $b \vee \overline{c}$ (221), $\overline{a} \wedge c$ (10), $\overline{a} \vee c$ (175), $a \wedge \overline{c}$ (80), $a \vee \overline{c}$ (245)	P_0
$\overline{a} \wedge \overline{b}$ (3), $\overline{b} \wedge \overline{c}$ (17), $\overline{a} \vee \overline{b}$ (63), $\overline{b} \vee \overline{c}$ (119), $\overline{a} \wedge \overline{c}$ (5), $\overline{a} \vee \overline{c}$ (95)	P_0
$a \wedge b \wedge c$ (128), $a \vee b \vee c$ (254)	P_1
$\overline{a} \wedge b \wedge c$ (8), $a \wedge b \wedge \overline{c}$ (64), $\overline{a} \vee b \vee c$ (239), $a \vee b \vee \overline{c}$ (253)	P_1
$a \wedge \overline{b} \wedge c$ (32), $a \vee \overline{b} \vee c$ (251)	P_1
$\overline{a} \wedge \overline{b} \wedge c$ (2), $a \wedge \overline{b} \wedge \overline{c}$ (16), $\overline{a} \vee \overline{b} \vee c$ (191), $a \vee \overline{b} \vee \overline{c}$ (247)	P_0
$\overline{a} \wedge b \wedge \overline{c}$ (4), $\overline{a} \vee b \vee \overline{c}$ (223)	P_0
$\overline{a} \wedge \overline{b} \wedge \overline{c}$ (1), $\overline{a} \vee \overline{b} \vee \overline{c}$ (127)	P_0
$(a \vee c) \wedge b$ (200), $(a \wedge c) \vee b$ (236)	P_0
$(\overline{a} \vee c) \wedge b$ (140), $(a \vee \overline{c}) \wedge b$ (196), $(\overline{a} \wedge c) \vee b$ (206), $(a \wedge \overline{c}) \vee b$ (220)	P_2
$(\overline{a} \vee \overline{c}) \wedge b$ (76), $(\overline{a} \wedge \overline{c}) \vee b$ (205)	P_0
$(a \vee c) \wedge \overline{b}$ (50), $(a \wedge c) \vee \overline{b}$ (179)	P_2
$(\overline{a} \vee \overline{c}) \wedge \overline{b}$ (19), $(\overline{a} \wedge \overline{c}) \vee \overline{b}$ (55)	P_1
$(\overline{a} \vee b) \wedge c$ (138), $a \wedge (b \vee \overline{c})$ (208), $(\overline{a} \wedge b) \vee c$ (174), $a \vee (b \wedge \overline{c})$ (244)	P_2
$(a \vee \overline{b}) \wedge c$ (162), $a \wedge (\overline{b} \vee c)$ (176), $(a \wedge \overline{b}) \vee c$ (186), $a \vee (\overline{b} \wedge c)$ (242)	P_2
$(\overline{a} \vee \overline{b}) \wedge c$ (42), $a \wedge (\overline{b} \vee \overline{c})$ (112), $(\overline{a} \wedge \overline{b}) \vee c$ (171), $a \vee (\overline{b} \wedge \overline{c})$ (241)	P_0
$(\overline{a} \vee b) \wedge \overline{c}$ (69), $\overline{a} \wedge (b \vee \overline{c})$ (13), $(\overline{a} \wedge b) \vee \overline{c}$ (93), $\overline{a} \vee (b \wedge \overline{c})$ (79)	P_2
$(\overline{a} \vee \overline{b}) \wedge \overline{c}$ (21), $\overline{a} \wedge (\overline{b} \vee \overline{c})$ (7), $(\overline{a} \wedge \overline{b}) \vee \overline{c}$ (87), $\overline{a} \vee (\overline{b} \wedge \overline{c})$ (31)	P_2
$(a \vee b) \wedge (b \vee c) \wedge (c \vee a)$ (232), $(\overline{a} \wedge \overline{b}) \vee (\overline{b} \wedge \overline{c}) \vee (\overline{c} \wedge \overline{a})$ (23)	P_2
$b \wedge (\overline{a} \vee c) \wedge (a \vee \overline{c})$ (132), $\overline{b} \vee (\overline{a} \wedge c) \vee (a \wedge \overline{c})$ (123)	P_2
$(b \vee (a \wedge c)) \wedge (\overline{b} \vee (\overline{a} \wedge \overline{c}))$ (36), $(b \wedge (a \vee c)) \vee (\overline{b} \wedge (\overline{a} \vee \overline{c}))$ (219)	P_0

Table 2 Equivalent L-equations of unconfirmed class

$(\overline{a} \vee c) \wedge \overline{b}$ (35), $(a \vee \overline{c}) \wedge \overline{b}$ (49), $(\overline{a} \wedge c) \vee \overline{b}$ (59), $(a \wedge \overline{c}) \vee \overline{b}$ (115)
$(a \vee b) \wedge \overline{c}$ (84), $\overline{a} \wedge (b \vee c)$ (14), $(a \wedge b) \vee \overline{c}$ (213), $\overline{a} \vee (b \wedge c)$ (143)
$(a \vee b) \wedge c$ (168), $a \wedge (b \vee c)$ (224), $(a \wedge b) \vee c$ (234), $a \vee (b \wedge c)$ (248)
$(a \vee \overline{b}) \wedge \overline{c}$ (81), $\overline{a} \wedge (\overline{b} \vee c)$ (11), $(a \wedge \overline{b}) \vee \overline{c}$ (117), $\overline{a} \vee (\overline{b} \wedge c)$ (47)

the class of its general solution. The rule number of corresponding ECA is noted together with $f(a, b, c)$. Note that classes of some L-equations in the simple form described in Sect. 3 are not yet confirmed. We show them in Table 2.

Appendix B: Proof of solutions

The solutions described in the Sect. 3 are all derived only by using the formulas in the Sect. 2.1. In this appendix, we show two examples of proofs of solutions. The solutions to the other equations can also be proved by the similar procedure.

The first example is about the equation

$$u_j^{n+1} = \overline{u_{j-1}^n} \wedge u_j^n \wedge u_{j+1}^n, \tag{37}$$

which corresponds to ECA8. The solution is

$$w_j^n = \overline{u_{j-1}^0} \wedge \overline{u_j^0} \wedge \cdots \wedge \overline{u_{j+n-2}^0} \wedge u_j^0 \wedge u_{j+1}^0 \wedge \cdots \wedge u_{j+n}^0. \tag{38}$$

It is of class P_1. For $n = 1$, it is easy to check $w_j^1 = \overline{u_{j-1}^0} \wedge u_j^0 \wedge u_{j+1}^0$ from (37) and (38). If we substitute it into the RHS of (37), we obtain

$$\begin{aligned}
&\overline{w_{j-1}^n} \wedge w_j^n \wedge w_{j+1}^n \\
&= \overline{(\overline{u_{j-2}^0} \wedge \cdots \wedge \overline{u_{j+n-3}^0} \wedge u_{j-1}^0 \wedge \cdots \wedge u_{j+n-1}^0)} \\
&\quad \wedge (\overline{u_{j-1}^0} \wedge \cdots \wedge \overline{u_{j+n-2}^0} \wedge u_j^0 \wedge \cdots \wedge u_{j+n}^0) \\
&\quad \wedge (\overline{u_j^0} \wedge \cdots \wedge \overline{u_{j+n-1}^0} \wedge u_{j+1}^0 \wedge \cdots \wedge u_{j+n+1}^0) \\
&= (u_{j-2}^0 \vee \cdots \vee u_{j+n-3}^0 \vee \overline{u_{j-1}^0} \vee \cdots \vee \overline{u_{j+n-1}^0}) \\
&\quad \wedge (\overline{u_{j-1}^0} \wedge \cdots \wedge \overline{u_{j+n-1}^0} \wedge u_j^0 \wedge \cdots \wedge u_{j+n+1}^0) \\
&= \big((u_{j-2}^0 \vee \cdots \vee u_{j+n-3}^0 \vee \overline{u_{j-1}^0} \vee \cdots \vee \overline{u_{j+n-1}^0}) \\
&\quad \wedge \overline{u_{j-1}^0}\big) \wedge (\overline{u_j^0} \cdots \wedge \overline{u_{j+n-1}^0} \wedge u_j^0 \wedge \cdots \wedge u_{j+n+1}^0) \\
&= \overline{u_{j-1}^0} \wedge \overline{u_j^0} \cdots \wedge \overline{u_{j+n-1}^0} \wedge u_j^0 \wedge \cdots \wedge u_{j+n+1}^0 \\
&= w_j^{n+1}.
\end{aligned}$$

Thus induction on the solution holds.

The second example is about the equation

$$u_j^{n+1} = (\overline{u_{j-1}^n} \vee u_{j+1}^n) \wedge u_j^n, \tag{39}$$

which corresponds to ECA140. The solution is

$$\begin{aligned} w_j^n = &(\overline{u_{j-1}^0} \vee \overline{u_j^0} \vee \cdots \vee \overline{u_{j+n-3}^0} \vee \overline{u_{j+n-2}^0} \vee u_{j+n}^0) \\ &\wedge (\overline{u_{j-1}^0} \vee \overline{u_j^0} \vee \cdots \vee \overline{u_{j+n-3}^0} \vee u_{j+n-1}^0) \\ &\cdots \\ &\wedge (\overline{u_{j-1}^0} \vee \overline{u_j^0} \vee u_{j+2}^0) \\ &\wedge (\overline{u_{j-1}^0} \vee u_{j+1}^0) \wedge u_j^0. \end{aligned} \tag{40}$$

It is more complicated than that of the first example and is of class P_2. For $n = 1$, it is easy to check $w_j^1 = (\overline{u_{j-1}^0} \vee u_{j+1}^0) \wedge u_j^0$ from (39) and (40). We can rewrite $\overline{w_{j-1}^n}$, w_j^n and w_{j+1}^n as follows:

$$\begin{aligned} \overline{w_{j-1}^n} &= \overline{(\overline{u_{j-2}^0} \vee \cdots \vee \overline{u_{j+n-3}^0} \vee u_{j+n-1}^0) \wedge \cdots \wedge (\overline{u_{j-2}^0} \vee u_j^0) \wedge u_{j-1}^0} \\ &= \underbrace{(u_{j-2}^0 \wedge \cdots \wedge u_{j+n-3}^0 \wedge \overline{u_{j+n-1}^0}) \vee \cdots \vee (u_{j-2}^0 \wedge \overline{u_j^0})}_{A} \vee \underbrace{\overline{u_{j-1}^0}}_{B}, \\ w_j^n &= (\overline{u_{j-1}^0} \vee \cdots \vee \overline{u_{j+n-2}^0} \vee u_{j+n}^0) \wedge \cdots \wedge (\overline{u_{j-1}^0} \vee u_{j+1}^0) \wedge u_j^0 \\ &= \Big(\underbrace{\overline{u_{j-1}^0}}_{B} \vee \Big(\underbrace{(\overline{u_j^0} \vee \cdots \vee \overline{u_{j+n-2}^0} \vee u_{j+n}^0) \wedge \cdots \wedge (\overline{u_j^0} \vee u_{j+2}^0) \wedge u_{j+1}^0}_{C} \Big)\Big) \\ &\quad \wedge \underbrace{u_j^0}_{D}, \\ w_{j+1}^n &= (\overline{u_j^0} \vee \cdots \vee \overline{u_{j+n-1}^0} \vee u_{j+n+1}^0) \wedge \cdots \wedge (\overline{u_j^0} \vee u_{j+2}^0) \wedge u_{j+1}^0 \\ &= \underbrace{(\overline{u_j^0} \vee \cdots \vee \overline{u_{j+n-1}^0} \vee u_{j+n+1}^0)}_{E} \\ &\qquad \wedge \Big(\underbrace{(\overline{u_j^0} \vee \cdots \vee \overline{u_{j+n-2}^0} \vee u_{j+n}^0) \wedge \cdots \wedge (\overline{u_j^0} \vee u_{j+2}^0) \wedge u_{j+1}^0}_{C} \Big). \end{aligned}$$

If we substitute (40) into the RHS of (39), we obtain

$$\begin{aligned} (\overline{w_{j-1}^n} \vee w_{j+1}^n) \wedge w_j^n &= ((A \vee B) \vee (E \wedge C)) \wedge ((B \vee C) \wedge D) \\ &= (A \vee B \vee E) \wedge (A \vee B \vee C) \wedge (B \vee C) \wedge D \\ &= (A \vee B \vee E) \wedge (B \vee C) \wedge D \end{aligned}$$

Moreover, we have

$$\begin{aligned} A \vee E = &\underbrace{(u^0_{j-2} \wedge \cdots \wedge u^0_{j+n-3} \wedge \overline{u^0_{j+n-1}}) \vee \cdots \vee (u^0_{j-2} \wedge \overline{u^0_j})}_{A} \\ &\vee \underbrace{(\overline{u^0_j} \vee \cdots \vee \overline{u^0_{j+n-1}} \vee u^0_{j+n+1})}_{E} \\ &= ((u^0_{j-2} \wedge \cdots \wedge u^0_{j+n-3} \wedge \overline{u^0_{j+n-1}}) \vee \overline{u^0_{j+n-1}}) \\ &\quad \vee \cdots \vee ((u^0_{j-2} \wedge \overline{u^0_j}) \vee \overline{u^0_j}) \vee u^0_{j+n+1} \\ &= \overline{u^0_{j+n-1}} \vee \cdots \vee \overline{u^0_j} \vee u^0_{j+n+1} \\ &= E. \end{aligned}$$

Therefore,

$$\begin{aligned} &(\overline{w^n_{j-1}} \vee w^n_{j+1}) \wedge w^n_j \\ &\quad = (B \vee E) \wedge (B \vee C) \wedge D \\ &\quad = (\overline{u^0_{j-1}} \vee \overline{u^0_j} \vee \cdots \vee \overline{u^0_{j+n-1}} \vee u^0_{j+n+1}) \\ &\qquad \wedge (\overline{u^0_{j-1}} \vee \cdots \vee \overline{u^0_{j+n-2}} \vee u^0_{j+n}) \wedge \cdots \wedge (\overline{u^0_{j-1}} \vee u^0_{j+1}) \wedge u^0_j \\ &\quad = w^{n+1}_j, \end{aligned}$$

is derived and induction on the solution holds.

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