CLOSED GEODESICS ON ORBIFOLDS OF REVOLUTION

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Communicated by Charles Hagopian

ABSTRACT. Using the theory of geodesics on surfaces of revolution, we show that any two-dimensional orbifold of revolution homeomorphic to S^2 must contain an infinite number of geometrically distinct closed geodesics. Since any such orbifold of revolution can be regarded as a topological two-sphere with metric singularities, we will have extended Bangert's theorem on the existence of infinitely many closed geodesics on any smooth Riemannian two-sphere. In addition, we give an example of a two-sphere cone-manifold of revolution which possesses a single closed geodesic, thus showing that Bangert's result does not hold in the wider class of closed surfaces with cone manifold structures.

1. INTRODUCTION

In this note, we study closed geodesics on surfaces of revolution with certain types of metric singularities. In particular, we are interested in closed (compact, without boundary) surfaces of revolution that are Riemannian 2-orbifolds. Loosely speaking, a 2-orbifold is modeled locally by convex Riemannian surfaces modulo finite groups of isometries acting with possible fixed points. This means that a neighborhood of each point p of such an orbifold is isometric to a Riemannian quotient U_p/Γ_p where U_p is a convex Riemannian surface diffeomorphic to $\mathbb{R}^2$, and Γ_p is a finite group of isometries acting effectively on U_p. Every Riemannian surface is trivially an orbifold, with each Γ_p being the trivial group. The reader

interested in more background on orbifolds should consult [4], Thurston's classic [15], or the more recent textbook [14]. For the purposes of this note, however, we will only need to apply a simple explicit criterion to determine whether a closed surface of revolution is a 2-orbifold (see section 6).

The existence of closed geodesics on Riemannian manifolds has a long and storied past dating back to Poincaré [2]. It seems that not much has been done on the existence of closed geodesics in singular spaces. The existence of at least one closed geodesic on a compact 2-orbifold was shown in [7] and closed geodesics in orbifolds of higher dimensions have recently been studied in [10]. The paper [11] studies the issue of closed geodesics in spaces with incomplete metrics. The relevance here is that a complete Riemannian orbifold with singular set removed is a Riemannian manifold with incomplete metric and it is known [5] that closed geodesics in a complete Riemannian orbifold may not pass through the singular set, unless they are entirely contained within it.

Here we are interested in the question of the existence of infinitely many closed geodesics. In [1], Bangert used the work of Franks [9] to show that every smooth Riemannian S^2 has infinitely many closed geodesics. For orbifolds with S^2 as the underlying topological space, the existence of an infinity of closed geodesics is an open question. In the general category of closed surfaces of revolution with singular points (which have underlying topological space S^2), one may construct examples with exactly one closed geodesic (see example 7.2), showing that analogue of Bangert's result is false in this category. We call such a surface *void*. A spherical 2-orbifold of revolution is a closed two-dimensional surface of revolution homeomorphic to S^2 that satisfies a certain special orbifold condition at its north and south poles. It is natural to ask whether void orbifolds of revolution exist. In resolving this question we extend Bangert's result by proving that

Theorem 1.1. *Every spherical 2-orbifold of revolution has infinitely many closed geodesics.*

Since we are dealing only with surfaces of revolution, we will use standard facts about the geodesic flow on such surfaces. The basic theory that we need about surfaces of revolution and their geodesics can be found in the textbooks [8], [12], or [13]. For the more refined results that appear in section 5, we often make use of the monograph [3]. For a single self-contained and elementary exposition of all of the results we need, the reader may wish to consult [6].

2. Basic Theory

In what follows the term *smooth function* will refer to a function of class C^∞. In fact, C^2 is sufficient for our needs.

Definition 1. Let $\alpha : [u_N, u_S] \to \mathbb{R}^2$ be a simple (no self intersections) smooth plane curve $\alpha(u) = (g(u), h(u))$ where g and h are smooth functions on the interval $[u_N, u_S]$, with $h \geq 0$, and $h(u) = 0$ if and only if $u = u_N$ or $u = u_S$. A *spherical surface of revolution* M is a surface embedded isometrically in $\mathbb{R}^3$ that admits a parametrization $\mathbf{x} : [u_N, u_S] \times \mathbb{R} \to M$ of the form

$$\mathbf{x}(u, v) = (g(u), h(u) \cos v, h(u) \sin v),$$

That is, M is the surface of revolution obtained by rotating α about the x-axis. The curve α will be called the *profile curve*.

Note that a spherical surface of revolution M is necessarily homeomorphic to S^2 and that by definition the sets $N = \mathbf{x}(u_N, v)$ and $S = \mathbf{x}(u_S, v)$ for $v \in \mathbb{R}$ reduce to single points which will be referred to as the *north* and *south poles* of M. Metric singularities may only occur at these two points. M is smooth everywhere else. Rotation about the x-axis in $\mathbb{R}^3$ descends to a natural S^1-action $S^1 \times M \to M$ on M by isometries. This action is free except at the north and south poles which remain fixed.

For a surface of revolution M, the metric (away from any singular point) is given by $ds^2 = E\,du^2 + G\,dv^2$ where $E = \mathbf{x}_u \cdot \mathbf{x}_u = [g'(u)]^2 + [h'(u)]^2$, and $G = \mathbf{x}_v \cdot \mathbf{x}_v = h^2(u)$. The geodesic equations reduce to

(2.1)
$$u'' + \frac{E_u}{2E} u'^2 - \frac{G_u}{2E} v'^2 = 0$$

(2.2)
$$v'' + \frac{G_u}{G} u'v' = 0.$$

A curve $\gamma(t) = \mathbf{x}(u(t), v(t))$ on M is a geodesic if and only if the above equations are satisfied by the coordinate functions u and v of γ. A geodesic satisfying these equations must be parametrized proportional to arc length, and so we always assume that γ has unit speed. The existence and uniqueness theorem for solutions of ordinary differential equations implies that, given a point in p in M and a vector ξ in $T_p M$, the tangent plane to M at p, there is a unique geodesic γ satisfying $\gamma(0) = p$ and $\gamma'(0) = \xi$. A unit speed curve $\gamma(t) = \mathbf{x}(u(t), v(t))$ with $v(t) \equiv v_0$, a constant, is a *meridional arc*. Such curves are always geodesics. We will use the term *meridian* for those meridional arcs that join N to S. On the other hand, a unit speed curve $\gamma(t) = \mathbf{x}(u(t), v(t))$ with $u(t) \equiv u_0 \in (u_N, u_S)$, a constant, is a

parallel arc. Parallel arcs are geodesic precisely when $h'(u_0) = 0$. We will use the term *parallel* for those parallel arcs which are entire circles.

The main classical tool used to get qualitative information about geodesics on surfaces of revolution is the *Clairaut relation*, which may be regarded as a conservation law for the geodesic flow. It states that the quantity:

$$c_\gamma(t) = h(u(t)) \sin \varphi_\gamma(t)$$

is constant where $\varphi_\gamma(t) = \angle(\gamma', \mathbf{x}_u)$ is the angle between γ' and $\mathbf{x}_u$ at time t. The constant c_γ is called the *slant* of γ. Since $0 \le \sin \varphi_\gamma(t) \le 1$ for all t we must have that $h(u(t)) \ge c_\gamma$ for all t. That is, γ is must lie entirely in the region of the surface M where $h(u) \ge c_\gamma$. It follows that a geodesic with an endpoint at either pole must be a meridional arc, and that non-meridional geodesics γ have unique extensions to unit speed geodesics $\hat{\gamma} : \mathbb{R} \to M$. An analysis [6] based on the Clairaut relation, shows that, besides meridians and geodesic parallels, there are three other types of geodesics: First, we have the *oscillating geodesics*. These are geodesics which oscillate between two non-geodesic parallels $u = u_0$ and $u = u_1$ with $h(u_0) = h(u_1)$. Geometrically, these geodesics bounce off each parallel $u = u_i$ tangentially, rebounding back and forth between them. These are are the geodesics of most interest to us. The other two types are the *asymptotic (homoclinic) geodesics* and *bi-asymptotic (heteroclinic) geodesics*. The former are asymptotic to a single geodesic parallel as $t \to \pm\infty$, while the latter is asymptotic to a geodesic parallel as $t \to -\infty$ and different geodesic parallel as $t \to \infty$. Since it is not important for what follows, we will refer to bi-asymptotic geodesics as simply asymptotic geodesics also. Asymptotic geodesics exist precisely when h has a critical point that is not a local maximum. Illustrations of oscillating and asymptotic geodesics are given in figure 1.
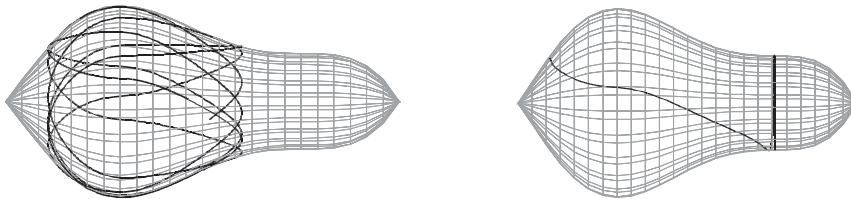

FIGURE 1.   An oscillating and asymptotic geodesic on a spherical surface of revolution

If $\gamma(t) = \mathbf{x}(u(t), v(t))$ is an oscillating or asymptotic geodesic, define $b_0(\gamma) = \inf_{t \in \mathbb{R}}(u(t))$ and $b_1(\gamma) = \sup_{t \in \mathbb{R}}(u(t))$ to be the *left* and *right boundary values* of γ, respectively. That is, $b_i(\gamma) = u_i$ where $u = u_i$ are the bounding or asymptotic parallels described above. For an oscillating geodesic, if $u(t_0) = b_0 = u(t_1)$ for $t_0 \neq t_1$ and there is a unique $t \in (t_0, t_1)$ such that $u(t) = b_1$, we call the segment of γ corresponding to the interval $[t_0, t_1]$ an *oscillation*.

3. A Topology on the Set of Oscillating Geodesics

Because we are interested in geometrically distinct geodesics we regard two geodesics γ_1 and γ_2 to be equivalent if γ_1 and γ_2 are in the same orbit of the natural S^1 action on M. We denote the set of all equivalence classes $[\gamma]$ by Γ_M. We adopt the common abuse of notation by simply referring to a geodesic $\gamma \in \Gamma_M$.

Recall that a geodesic γ is *closed* if there exist real numbers $t_0 \neq t_1$ such that $\gamma(t_0) = \gamma(t_1)$ and $\gamma'(t_0) = \gamma'(t_1)$. Equality of the derivatives distinguish closed geodesics from the more general notion of geodesic loop. Every geodesic parallel is closed, and no asymptotic geodesic or meridian (using our definition) is closed. Oscillating geodesics, however, may or may not be closed and will be our primary focus. We denote the set of oscillating geodesics by Γ_M^O.

If γ is oscillating, then γ is the unique geodesic with left boundary $b_0(\gamma)$. This is because $h'(b_0(\gamma)) \neq 0$, so the parallel at $u = b_0(\gamma)$ is not geodesic and there can be no geodesic asymptotic to it. Thus, by our classification, any geodesic which shares a left boundary with γ must be oscillating itself. But, any oscillating geodesic intersects its left boundary tangentially, so by the definition of our equivalence relation, we conclude that γ is the unique geodesic in its equivalence class with left boundary $b_0(\gamma)$. Thus, the map $b_0 : \Gamma_M^O \to (u_N, u_S)$ is injective.

Proposition 3.1. *Let $b_1(u_1) = \min\{u > u_1 : h(u) = h(u_1)\}$ and let $\mathcal{U} = \{u_1 \in (u_N, u_S) : h'(u_1) > 0 \text{ and } h'(b_1(u_1)) < 0\}$. Then $\mathcal{U}$ is an open subset of the interval (u_N, u_S) and $b_0 : \Gamma_M^O \to \mathcal{U}$ is a bijection.*

PROOF. We first show that b_0 is a bijection. Indeed, $b_0(\gamma) \in \mathcal{U}$ for any $\gamma \in \Gamma_M^O$. For any $u_1 \in \mathcal{U}$, there is a geodesic γ with the initial conditions $u(0) = u_1$, $u'(0) = 0$. Then $h'(u_1) > 0$ implies γ is not a geodesic parallel and $b_0(\gamma) = u_1$. Thus, $b_1(\gamma) = b_1(u_1)$, and $h'(b_1(u_1)) < 0$ implies γ is not asymptotic, so $\gamma \in \Gamma_M^O$. Smoothness of h implies that $\mathcal{U}$ is open. $\square$

We topologize Γ_M^O by declaring a subset $U \subset \Gamma_M^O$ to be open if and only if $b_0(U)$ is open in $\mathcal{U}$. This allows us, for example, to speak of a sequence of geodesics in the space Γ_M^O as a sequence of (left) boundary values from $\mathcal{U}$.

4. The Period Function

We now present our main analytic tool for detecting closed geodesics on spherical surfaces of revolution. In the case of oscillating or asymptotic geodesics, the geodesic equations (2.1) and (2.2) can be reduced to a first-order system and solved explicitly. In particular,

$$v = \pm \int \frac{c_\gamma \sqrt{E}}{\sqrt{G}\sqrt{G - c_\gamma^2}}\, du.$$

This motivates the following definition.

Definition 2. The *period function* $\Phi_M : \Gamma_M^O \to (0, \infty)$ is defined by

$$\Phi_M(\gamma) = 2 \int_{b_0(\gamma)}^{b_1(\gamma)} \frac{c_\gamma \sqrt{E}}{\sqrt{G}\sqrt{G - c_\gamma^2}}\, du = 2 \int_{b_0(\gamma)}^{b_1(\gamma)} \frac{c_\gamma \sqrt{E}}{h(u)\sqrt{h^2(u) - c_\gamma^2}}\, du.$$

We denote the integrand by $f_\gamma(u)$.

Geometrically, the period function gives the change in v as γ undergoes one oscillation. Since $h^2(b_0) = h^2(b_1) = c_\gamma^2$, the integral is improper for every geodesic γ, however, because it represents the change in v between b_0 and b_1 it must converge for every $\gamma \in \Gamma_M^O$. We can use this geometric interpretation to see that the period function is invariant under reparametrization and scaling of M and to extend the domain of the period function to include the asymptotic geodesics, by setting $\Phi_M(\gamma_0) = \infty$ for any asymptotic geodesic γ_0.

The next series of results show how the period function can be used to detect closed geodesics.

Theorem 4.1. *An oscillating geodesic γ on a spherical surface of revolution M is closed if and only if $\Phi_M(\gamma) = 2q\pi$ for some rational number $q \in \mathbb{Q}$.*

PROOF. We can assume without loss of generality that γ satisfies the initial conditions

$$\gamma(0) = \mathbf{x}(b_0, 0) \quad \text{and} \quad \gamma'(0) = \frac{\mathbf{x}_v(b_0, 0)}{\|\mathbf{x}_v(b_0, 0)\|}.$$

If $\gamma(t) = \mathbf{x}(u(t), v(t))$ is closed, there exists $t_0 > 0$ such that $\gamma(t_0) = \gamma(0)$ and $\gamma'(t_0) = \gamma'(0)$. In particular, $\gamma(t_0) = \mathbf{x}(b_0, 2r\pi)$ for some positive integer r. Note that the period function does not depend on the value $v(0)$, so by rotational symmetry, v changes the same amount during every oscillation of γ. Clearly, between $t = 0$ and $t = t_0$, γ has completed, say, s oscillations. That is, there have

been s times subsequent to $t = 0$ that $u(t)$ has re-attained the boundary value b_0. Therefore, $\Phi_M(\gamma) = 2(r/s)\pi$.

Conversely, suppose $\Phi_M(\gamma) = 2(r/s)\pi$ for integers $r, s > 0$, where γ is taken to have the same initial conditions. Then there exists a $t_0 > 0$ such that

$$\gamma(t_0) = \mathbf{x}(b_0, 2r\pi) = \mathbf{x}(b_0, 0) = \gamma(0).$$

Since $u(t_0) = b_0$, we must have $\gamma'(t_0)$ tangent to $\mathbf{x}_v(b_0, 0)$, and thus, $\gamma'(t_0) = \gamma'(0)$. Hence γ is closed. $\qquad\square$

Theorem 4.2. *If $\gamma_0 \in \Gamma_M^O$, then Φ_M is continuous at γ_0.*

A complete proof using elementary techniques appears in [6]. It also follows from the dependency of solutions of ordinary differential equations on initial conditions.

Corollary 4.3. *Suppose a spherical surface of revolution M has a non-empty open subset U of Γ_M^O on which Φ_M is not a constant, irrational multiple of π. Then M has infinitely many closed geodesics.*

PROOF. Let $\Phi_M(U) = \{\Phi_M(\gamma) : \gamma \in U\}$. If $\Phi_M(U)$ is a constant rational multiple of π we are done by theorem 4.1, so suppose Φ_M is not constant on U. By continuity of Φ_M, there exists a nonempty open interval $I \subset \Phi_M(U)$. $\mathbb{Q}\pi$ is dense in any such I yielding an infinite number of closed geodesics in U by theorem 4.1. $\qquad\square$

The next corollary shows that the existence of an asymptotic geodesic on M implies the existence of infinitely many closed geodesics.

Corollary 4.4. *Let M be a spherical surface of revolution with an asymptotic geodesic γ_0 asymptotic to the geodesic parallel at $b_0(\gamma_0)$. Then if $\gamma_n \to \gamma_0$ is a sequence of oscillating geodesics,*

$$\lim_{n \to \infty} \Phi_M(\gamma_n) = \Phi_M(\gamma_0) = \infty.$$

Thus, by corollary 4.3, M has infinitely many closed geodesics.

PROOF. Let $A > 0$. $\Phi_M(\gamma_0) = \int_{b_0(\gamma_0)}^{b_1(\gamma_0)} f_{\gamma_0} = \infty$, so there exists $\delta, \mu > 0$ so that $A < \int_{b_0(\gamma_0)+\delta}^{b_1(\gamma_0)-\mu} f_{\gamma_0}$. Choose $N > 0$ large enough so that $b_0(\gamma_n) < b_0(\gamma_0) + \delta$ and $b_1(\gamma_n) > b_1(\gamma_0) - \mu$ for $n > N$. Thus,

$$\Phi_M(\gamma_n) = \int_{b_0(\gamma_n)}^{b_1(\gamma_n)} f_{\gamma_n} > \int_{b_0(\gamma_0)+\delta}^{b_1(\gamma_0)-\mu} f_{\gamma_n}$$

On the interval $(b_0(\gamma_0) + \delta, b_1(\gamma_0) - \mu)$, $f_{\gamma_n} \to f_{\gamma_0}$, and both f_{γ_n} and f_{γ_0} are bounded hence integrable. Thus, by dominated convergence, for $\varepsilon > 0$, there is $N' > N$ so that

$$\Phi_M(\gamma_n) > \int_{b_0(\gamma_0)+\delta}^{b_1(\gamma_0)-\mu} f_{\gamma_n} > \left[\int_{b_0(\gamma_0)+\delta}^{b_1(\gamma_0)-\mu} f_{\gamma_0} \right] - \varepsilon > A - \varepsilon$$

for $n > N'$. This implies $\Phi_M(\gamma_n) \to \infty$. $\qquad\square$

Corollary 4.5. *A spherical surface of revolution whose profile curve has more than one critical point necessarily has an infinite number of closed geodesics.*

PROOF. This follows from corollary 4.4 since such a surface must contain an asymptotic geodesic. $\qquad\square$

Corollary 4.6. *A spherical surface of revolution whose profile curve has a single critical point has exactly one closed geodesic or infinitely many.*

PROOF. The parallel at the critical point is necessarily geodesic. If Φ_M is a constant irrational multiple of π over the entire domain Γ_M^O, then by theorem 4.1, no oscillating geodesic is closed, and M has exactly one closed geodesic. Otherwise, there are an infinite number of closed geodesics by corollary 4.3. $\qquad\square$

5. SURFACES OF REVOLUTION WITH CONSTANT PERIOD FUNCTION

In light of corollary 4.6, we call a spherical surface of revolution with exactly one closed geodesic a *void surface*. An explicit example of a void surface will be given in section 7. Since, ultimately, we wish to show that no void spherical 2-orbifolds of revolution exist, corollary 4.6 implies we should look for general conditions that imply the period function is constant. We do exactly that in this section.

If a spherical surface of revolution $\mathbf{x}(u, v) = (g(u), h(u) \cos v, h(u) \sin v)$ obtained from the profile curve $\alpha(u) = (g(u), h(u))$ is to have a constant period function, we can, without loss of generality, assume that $h(u)$ is a smooth function from $[0, L]$ to $[0, 1]$ satisfying:

(1) $h(0) = h(L) = 0$
(2) h has a unique critical point, say u_0, on $[0, L]$
(3) $h(u_0) = 1$

with $g(u) = \int_0^u \sqrt{1 - [h'(t)]^2}\, dt$. Thus, the metric on M is of the form $ds^2 = du^2 + h^2(u)dv^2$. If the period function Φ_M is to be constant, corollary 4.4 implies that condition (2) is necessary. (1) and (3) may be satisfied by an appropriate

reparametrization and scaling of the profile curve, which does not affect the period function. The following proposition from [3] shows that the metric on M can be put into a special form.

Proposition 5.1. *Let M be a spherical surface of revolution satisfying conditions (1),(2) and (3). We can define new coordinates (u, v) on M so that the metric in these coordinates has the form $ds^2 = E(u)du^2 + \sin^2 u \, dv^2$, where $\hat{E}(\cos u) = E(u)$ is a function from $[0, \pi]$ to $\mathbb{R}^+$.*

Hence we take as a starting point in our search for surfaces with constant period function those surfaces of revolution with metric of the form

$$ds^2 = E(u) \, du^2 + \sin^2 u \, dv^2,$$

where $E(u)$ is a function from $[0, \pi]$ to $\mathbb{R}^+$. This corresponds to the spherical surface of revolution M with profile curve $\alpha(u) = (g(u), \sin u)$, where

$$g(u) = \int_0^u \sqrt{E(t) - \cos^2 t} \, dt.$$

If γ_x is the geodesic with left boundary value $b_0 = x$, then the right boundary value $b_1 = \pi - x$ and the period function may then be written as a function of $x \in (0, \frac{\pi}{2})$:

$$\Phi_M(\gamma_x) = \Phi_M(x) = \int_x^{\pi-x} \frac{\sin x \cdot \sqrt{E(u)}}{\sin u \sqrt{\sin^2 u - \sin^2 x}} \, du,$$

which is continuous on $(0, \frac{\pi}{2})$ by theorem 4.2. The following technical lemma from [3] is essential in our characterization of surfaces of revolution with constant period function.

Lemma 5.2. *Consider the function*

$$F(x) = \int_x^{\pi-x} \frac{\sin x \cdot f(u)}{\sin u \sqrt{\sin^2 u - \sin^2 x}} \, du$$

Define a function $\hat{f}$ by the formula $f(u) = \hat{f}(\cos u)$. Then $F(x)$ is identically zero on $(0, \frac{\pi}{2})$ if and only if $\hat{f}$ is an odd function over $[-1, 1]$.

We can now characterize those surfaces of revolution with constant period function.

Proposition 5.3. *For a spherical surface of revolution M with metric*

$$ds^2 = E(u) \, du^2 + \sin^2(u) \, dv^2,$$

define $a_c(u) = \sqrt{E}(u) - c$ for any $c \in \mathbb{R}^+$. Then $\Phi_M(x) \equiv 2c\pi$ on $(0, \frac{\pi}{2})$ if and only if the function $\hat{a}_c$ defined by $\hat{a}_c(\cos u) = a_c(u)$ is an odd function from $[-1, 1]$ to $[-c, c]$.

PROOF. Let S^2 be the standard 2-sphere of constant curvature 1 in $\mathbb{R}^3$. The geodesics on S^2 are great circles, so $\Phi_{S^2}(x) \equiv 2\pi$. Then, for all $x \in (0, \frac{\pi}{2})$,

$$\Phi_M(x) = 2 \int_x^{\pi-x} \frac{\sin x \cdot \sqrt{E}(u)}{\sin u \sqrt{\sin^2 u - \sin^2 x}} \, du = 2 \int_x^{\pi-x} \frac{\sin x \cdot (c + \hat{a}_c(\cos u))}{\sin u \sqrt{\sin^2 u - \sin^2 x}} \, du$$

$$= c\Phi_{S^2}(x) + 2 \int_x^{\pi-x} \frac{\sin x \cdot \hat{a}_c(\cos u)}{\sin u \sqrt{\sin^2 u - \sin^2 x}} \, du$$

$$= 2c\pi + 2 \int_x^{\pi-x} \frac{\sin x \cdot \hat{a}_c(\cos u)}{\sin u \sqrt{\sin^2 u - \sin^2 x}} \, du.$$

The proof of the proposition now follows from lemma 5.2, which implies that $\hat{a}_c$ must be odd. For $u \in (0, \pi)$, $c + \hat{a}_c(\cos u) = \sqrt{E}(u) > 0$ so $\hat{a}_c(\cos u) > -c$ for $u \in (0, \pi)$. This implies that $\hat{a}_c(-\cos u) > -c$, so since $\hat{a}_c$ is odd, we have $\hat{a}_c(\cos u) = a_c(u) \in [-c, c]$ for $u \in (0, \pi)$. $\qquad\square$

At this point we are able to recover Bangert's result for spherical surfaces of revolution which have (smooth) Riemannian metrics, such as ellipsoids of revolution. Let ϕ_N, resp. ϕ_S, be the angle between the profile curve $\alpha(u) = (g(u), h(u)) = (g(u), \sin u)$ and the axis of rotation at $g(0)$, resp. $g(\pi)$. Then

(5.1a) $$\sin \phi_N = \frac{h'(0)}{\sqrt{[g'(0)]^2 + [h'(0)]^2}} = \frac{\cos(0)}{\sqrt{E(0)}} = \frac{1}{c + \hat{a}_c(1)}$$

and

(5.1b) $$\sin \phi_S = \frac{-h'(\pi)}{\sqrt{[g'(\pi)]^2 + [h'(\pi)]^2}} = \frac{-\cos(\pi)}{\sqrt{E(\pi)}} = \frac{1}{c + \hat{a}_c(-1)} = \frac{1}{c - \hat{a}_c(1)},$$

with the last equality following since $\hat{a}_c$ is odd on $[-1, 1]$.

Corollary 5.4. *Every smooth Riemannian S^2 arising as a surface of revolution has infinitely many closed geodesics.*

PROOF. The result follows if the surface has non-constant period function by corollary 4.3. Thus, we assume the surface has constant period function. Since the surface is a smooth manifold, the profile curve meets the x-axis at right angles, so that $\sin \phi_N = \sin \phi_S = 1$. Equations (5.1a) and (5.1b) imply that

$c + \hat{a}_c(1) = c - \hat{a}_c(-1) = 1$ so $0 = \hat{a}_c(1) = \hat{a}_c(-1)$ and $c = 1$. Hence $\Phi_M \equiv 2\pi$ and all oscillating geodesics close up after one oscillation. $\qquad\square$

6. Orbifolds of Revolution

Our work up to this point is valid for spherical surfaces of revolution in general. Since our main theorem 1.1 concerns orbifolds, we now specialize to that case. Spherical orbifolds of revolution are easily identifiable by their tangent cones at the poles. Namely, the tangent cone at a pole must be isometric to the metric quotient of the flat plane $\mathbb{R}^2$ by a finite cyclic group of rotations fixing the origin. Note that the tangent cone at a pole is generated by rotating the tangent line to the profile curve at the pole about the axis of rotation. If the cyclic groups at the poles are of different orders, the orbifold is commonly referred to as *bad* since it will not arise as a quotient of a Riemannian S^2 by a finite cyclic group of isometries [15].

In general, a flat right circular cone with vertex angle ϕ is obtained by identifying the edges of a plane circular sector of angle θ. The relation between θ and ϕ is easily computed: $\theta = 2\pi \sin \phi$. See figure 2. Thus, if the tangent cone at a pole of spherical orbifold of revolution is isometric to $\mathbb{R}^2/\mathbb{Z}_m$, then $\theta = 2\pi/m$ for a positive integer m. So, for an orbifold of revolution, if ϕ_N and ϕ_S are as in equations (5.1), we must have $\sin \phi_N = 1/m$ and $\sin \phi_S = 1/k$ for some positive integers m and k.
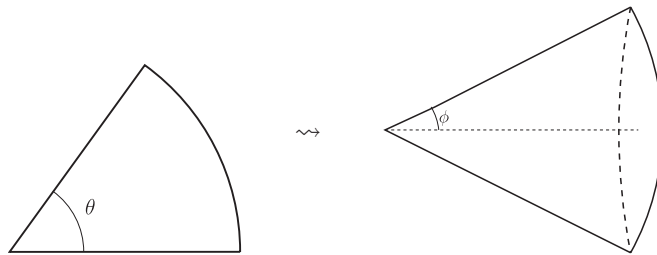

FIGURE 2.   Cone as quotient of a planar sector

We have the following restriction for spherical orbifolds of revolution of constant period function.

Theorem 6.1. *Let M be a spherical orbifold of revolution with metric $ds^2 = E(u)\, du^2 + \sin^2(u)\, dv^2$. Then $\Phi_M(x) \equiv 2c\pi$ on $(0, \frac{\pi}{2})$ implies c is rational.*

PROOF. Equations (5.1) give

$$\sin \phi_N = \frac{1}{c + \hat{a}_c(1)} \quad \text{and} \quad \sin \phi_S = \frac{1}{c + \hat{a}_c(-1)} = \frac{1}{c - \hat{a}_c(1)},$$

since $\hat{a}_c$ is odd on $[-1, 1]$. As noted above, if M is an orbifold, then $c + \hat{a}_c(1)$ and $c - \hat{a}_c(1)$ must be integers. Thus, $c = n/2$ for some positive integer n and $\Phi_M(x) = n\pi$ on $(0, \frac{\pi}{2})$. $\qquad\square$

We are now in a position to prove theorem 1.1.

PROOF. Suppose a counterexample existed. By corollary 4.5, we may assume that the profile curve has a single critical point and hence by proposition 5.1 that the metric on M is of the form required in theorem 6.1. By theorem 4.1 and corollary 4.3, Φ_M must be a constant, irrational multiple of π. However, by Theorem 6.1, an orbifold of revolution with constant Φ_M must have $\Phi_M \equiv 2c\pi$ with $c \in \mathbb{Q}$. Hence no such void spherical orbifold exists and all spherical orbifolds of revolution must have infinitely many closed geodesics. $\qquad\square$

7. TWO EXAMPLES

In summary, we can characterize all spherical surfaces of revolution with constant period function as having a metric of the form $ds^2 = (c + f(\cos u))^2 \, du^2 + \sin^2(u) \, dv^2$ where

(1) c is a real constant,
(2) $f(\cos u)$ is an odd function from $[-1, 1]$ to $[-c, c]$.

The void spherical surfaces of revolution satisfy these conditions but have $c \notin \mathbb{Q}$, and hence are not orbifolds. The orbifolds of revolution with constant period function must satisfy (1), (2) and

(3) $c + f(1)$ and $c - f(1)$ are positive integers.

Example 7.1 (Tannery's pear). Take $c = 2$ and $a_c(u) = \cos u$ (so $\hat{a}_c(\cos u)$ is the identity map on $[-1, 1]$, and hence odd). This surface, known as Tannery's pear, has a period function that is constant 4π, so all non-meridional geodesics are closed. It also is an orbifold. Taking as a profile curve $\alpha(u) = (g(u), h(u))$, where $h(u) = \sin u$ and

$$g(u) = \int_0^u \sqrt{E(t) - (h'(t))^2} \, dt = \int_0^u \sqrt{(2 + \cos t)^2 - \cos^2 t} \, dt = 4\sqrt{2} \sin(u/2)$$

gives a parametrization $\mathbf{x}(u,v) = (4\sqrt{2}\sin(u/2), \sin u \cos v, \sin u \sin v)$ for Tannery's pear in $\mathbb{R}^3$. We also have that

$$\sin\phi_N = \frac{\cos(0)}{\sqrt{E(0)}} = \frac{1}{3} \quad \text{and} \quad \sin\phi_S = \frac{-\cos(\pi)}{\sqrt{E(\pi)}} = 1.$$

Thus, Tannery's pear is a so-called $\mathbb{Z}_3$-teardrop orbifold, as the metric is actually smooth at $u = \pi$ and the single cone point at $u = 0$ is of order 3. See figure 3.
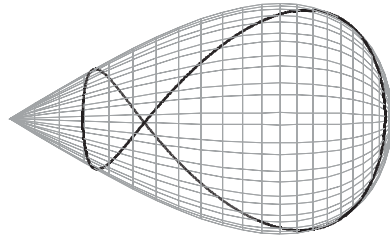

FIGURE 3.   A typical closed geodesic on a Tannery pear

Example 7.2 (A void surface)**.** By taking $c = \sqrt{5}$ in the previous example, we obtain a surface isometrically embedded in $\mathbb{R}^3$ with constant period function $2\pi\sqrt{5}$. Its only closed geodesic is the parallel at $u = \pi/2$. Since $\sin\phi_N = 1/(\sqrt{5}+1)$ and $\sin\phi_S = 1/(\sqrt{5}-1)$, like all void spherical surfaces of revolution, this one is not an orbifold. See figure 4.
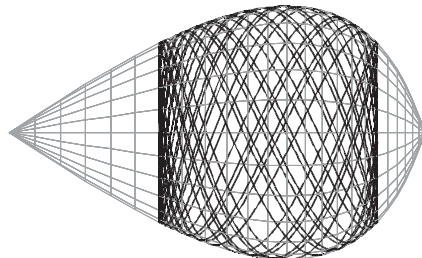

FIGURE 4.   A void surface with a typical oscillating geodesic

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