Sadulla Z. Jafarov

Approximation of functions by de la Vallée-Poussin sums in weighted Orlicz spaces

Abstract We investigate problems of estimating the deviation of functions from their de la Vallée-Poussin sums in weighted Orlicz spaces $L_M(T, \omega)$ in terms of the best approximation $E_n(f)_{M, \omega}$.

Mathematics Subject Classification 30E10 · 41A10 · 41A25 · 46E30

الملخص

نبحث في مسائل تقريب انحراف الدوال عن مجاميع فالي – بوسن الخاصة بها في فضاءات أورلتز الموزونة $L_M(T, \omega)$ بدلالة أفضل تقريب $E_n(f)_{M,\omega}$.

1 Introduction, some auxiliary results and main results

Let $M(u)$ be a continuous increasing convex function on $[0, \infty)$ such that $M(u)/u \to 0$ if $u \to 0$, and $M(u)/u \to \infty$ if $u \to \infty$. We denote by N *the complementary of M in Young's sense*, i.e., $N(u) = \max \{uv - M(v) : v \geq 0\}$ if $u \geq 0$. We will say that M satisfies the Δ_2−condition if $M(2u) \leq cM(u)$ for any $u \geq u_0 \geq 0$ with some constant c, independent of u.

Let $\mathbb{T}$ denote the interval $[-\pi, \pi]$, $\mathbb{C}$ the complex plane, and $L_p(\mathbb{T})$, $1 \leq p \leq \infty$, the Lebesgue space of measurable complex-valued functions on $\mathbb{T}$.

For a given Young function M, let $\widetilde{L}_M(\mathbb{T})$ denote the set of all Lebesgue measurable functions $f : \mathbb{T} \to \mathbb{C}$ for which
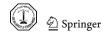
$$\int_{\mathbb{T}} M(|f(x)|)\, dx < \infty.$$

Let N be the complementary Young function of M. It is well-known [28, p.69], [47, pp.52–68] that the linear span of $\widetilde{L}_M(\mathbb{T})$ equipped with the *Orlicz norm*

$$\|f\|_{L_M(\mathbb{T})} := \sup \left\{ \int_{\mathbb{T}} |f(x)g(x)|\, dx : g \in \widetilde{L}_N(\mathbb{T}), \int_{\mathbb{T}} N(|g(x)|)\, dx \leq 1 \right\},$$

S. Z. Jafarov
Department of Mathematics, Faculty of Art and Science, Pamukkale University, 20070 Denizli, Turkey

S. Z. Jafarov (✉)
Mathematics and Mechanics Institute, Azerbaijan National Academy of Sciences, 9, B. Vahabzade St., Az-1141 Baku, Azerbaijan
E-mail: sjafarov@pau.edu.tr

Springer

or with the *Luxemburg norm*

$$\|f\|_{L_M(\mathbb{T})}^* := \inf \left\{ k > 0 : \int_{\mathbb{T}} M\left(\frac{|f(x)|}{k}\right) dx \le 1 \right\}$$

becomes a Banach space. This space is denoted by $L_M(\mathbb{T})$ and is called an *Orlicz space* [28, p.26]. The Orlicz spaces are known as the generalizations of the Lebesgue spaces $L_p(\mathbb{T})$, $1 < p < \infty$. If $M(x) = M(x, p) := x^p$, $1 < p < \infty$, then Orlicz spaces $L_M(\mathbb{T})$ coincide with the usual Lebesgue spaces $L_p(\mathbb{T})$, $1 < p < \infty$. Note that the Orlicz spaces play an important role in many areas such as applied mathematics, mechanics, regularity theory, fluid dynamics and statistical physics. Therefore, the investigation into the approximation of the functions by means of Fourier trigonometric series in Orlicz spaces is also important in these areas of research.

The Luxemburg norm is equivalent to the Orlicz norm as

$$\|f\|_{L_M(\mathbb{T})}^* \le \|f\|_{L_M(\mathbb{T})} \le 2 \|f\|_{L_M(\mathbb{T})}^*, \quad f \in L_M(\mathbb{T})$$

holds true [28, p.80].

If we choose $M(u) = u^p/p$, $1 < p < \infty$, then the complementary function is $N(u) = u^q/q$ with $1/p + 1/q = 1$ and we have the relation

$$p^{-1/p} \|u\|_{L_p(\mathbb{T})} = \|u\|_{L_M(\mathbb{T})}^* \le \|u\|_{L_M(\mathbb{T})} \le q^{1/q} \|u\|_{L_p(\mathbb{T})},$$

where $\|u\|_{L_p(\mathbb{T})} = \left(\int_{\mathbb{T}} |u(x)|^p dx\right)^{1/p}$ stands for the usual norm of the $L_p(\mathbb{T})$ space.

If N is complementary to M in Young's sense and $f \in L_M(\mathbb{T})$, $g \in L_N(\mathbb{T})$, then the so-called strong Hölder inequalities [28, p.80]

$$\int_{\mathbb{T}} |f(x)g(x)| dx \le \|f\|_{L_M(\mathbb{T})} \|g\|_{L_N(\mathbb{T})}^*,$$

$$\int_{\mathbb{T}} |f(x)g(x)| dx \le \|f\|_{L_M(\mathbb{T})}^* \|g\|_{L_N(\mathbb{T})}$$

are satisfied.

The Orlicz space $L_M(\mathbb{T})$ is *reflexive* if and only if the $N-$function M and its complementary function N both satisfy the Δ_2-condition [47, p.113].

Let $M^{-1} : [0, \infty) \to [0, \infty)$ be the inverse function of the $N-$function M. The *lower* and *upper indices* [4, p.350]

$$\alpha_M := \lim_{t \to +\infty} -\frac{\log h(t)}{\log t}, \quad \beta_M := \lim_{t \to o^+} -\frac{\log h(t)}{\log t}$$

of the function

$$h : (0, \infty) \to (0, \infty], \quad h(t) := \lim_{y \to \infty} \sup \frac{M^{-1}(y)}{M^{-1}(ty)}, \quad t > 0$$

first considered by Matuszewska and Orlicz [38], are called the *Boyd indices* of the Orlicz spaces $L_M(T)$.

It is known that the indices α_M and β_M satisfy $0 \le \alpha_M \le \beta_M \le 1$, $\alpha_N + \beta_M = 1$, $\alpha_M + \beta_N = 1$ and the space $L_M(\mathbb{T})$ is reflexive if and only if $0 < \alpha_M \le \beta_M < 1$. The detailed information about the Boyd indices can be found in [3,5–7,39].

A measurable function $\omega : \mathbb{T} \to [0, \infty]$ is called a *weight function* if the set $\omega^{-1}(\{0, \infty\})$ has Lebesgue measure zero. With any given weight ω we associate the ω-*weighted Orlicz space* $L_M(\mathbb{T}, \omega)$ consisting of all measurable functions f on $\mathbb{T}$ such that

$$\|f\|_{L_M(\mathbb{T}, \omega)} := \|f\omega\|_{L_M(\mathbb{T})}.$$

Let $1 < p < \infty$, $1/p + 1/p' = 1$ and let ω be a weight function on $\mathbb{T}$. ω is said to satisfy *Muckenhoupt's A_p-condition* on $\mathbb{T}$ if

$$\sup_J \left(\frac{1}{|J|} \int_J \omega^p (t)\, \mathrm{d}t \right)^{1/p} \left(\frac{1}{|J|} \int_J \omega^{-p'} (t)\, \mathrm{d}t \right)^{1/p'} < \infty,$$

where J is any subinterval of $\mathbb{T}$ and $|J|$ denotes its length.

Let us indicate by $A_p(\mathbb{T})$ the set of all weight functions satisfying Muckenhoupt's A_p-condition on $\mathbb{T}$.

According to [33, 34, Lemma 3.3], and [34, Section 2.3] if $L_M(\mathbb{T})$ is reflexive and ω weight function satisfying the condition $\omega \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}(\mathbb{T})$, then the space $L_M(\mathbb{T}, \omega)$ is also reflexive.

Let $L_M(\mathbb{T}, \omega)$ be a weighted Orlicz space, let $0 < \alpha_M \le \beta_M < 1$ and let $\omega \in A_{\frac{1}{\alpha_M}}(\mathbb{T}) \cap A_{\frac{1}{\beta_M}}(\mathbb{T})$. For $f \in L_M(\mathbb{T}, \omega)$ we set

$$(\nu_h f)(x) := \frac{1}{2h} \int_{-h}^{h} f(x + t)\, \mathrm{d}t, \ 0 < h < \pi, \ x \in T.$$

By reference [18], Lemma 1.4, the shift operator ν_h is a bounded linear operator on $L_M(\mathbb{T}, \omega)$:

$$\|\nu_h(f)\|_{L_M(\mathbb{T}, \omega)} \le c \|f\|_{L_M(\mathbb{T}, \omega)}.$$

The function

$$\Omega_{M,\omega}^l (\delta, \ f) := \sup_{\substack{0 < h_i \le \delta \\ 1 \le i \le l}} \left\| \prod_{i=1}^{l} (I - \nu_{h_i}) f \right\|_{L_M(\mathbb{T}, \omega)}, \quad \delta > 0, \quad l = 1, 2, \ldots$$

is called *k-th modulus of smoothness* of $f \in L_M(\mathbb{T}, \omega)$, where I is the identity operator.

It can easily be shown that $\Omega_{M,\omega}^k(\cdot, \ f)$ is a continuous, nonnegative and nondecreasing function satisfying the conditions

$$\lim_{\delta \to 0} \Omega_{M,\omega}^k (\delta, \ f) = 0, \quad \Omega_{M,\omega}^k (\delta, \ f + g) \le \Omega_{M,\omega}^k (\delta, \ f) + \Omega_{M,\omega}^k (\delta, g)$$

for $f, \ g \in L_M(\mathbb{T}, \omega)$.

The *function conjugate* to a $2\pi-$periodic summable function on $[-\pi, \pi]$ given by

$$\widetilde{f}(x) = \lim_{\varepsilon \to +0} \left\{ -\frac{1}{\pi} \int_\varepsilon^\pi \frac{f(x + t) - f(x - t)}{2 \tan \frac{t}{2}} \mathrm{d}t \right\} = -\frac{1}{\pi} \int_0^\pi \frac{f(x + t) - f(x - t)}{2 \tan \frac{t}{2}} \mathrm{d}t$$

exists almost everywhere.

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} A_k(x, \ f) \tag{1.1}$$

be the Fourier series of the function $f \in L_1(\mathbb{T})$, where $A_k(x, \ f) := (a_k(f) \cos kx + b_k(f) \sin kx)$, $k = 1, 2, \ldots$, $a_k(f)$ and $b_k(f)$ are Fourier coefficients of the function $f \in L_1(\mathbb{T})$. For given $f \in L_1(\mathbb{T})$, let

$$\widetilde{f} \sim \sum_{k=1}^{\infty} (a_k(f) \sin kx - b_k(f) \cos kx) = \sum_{k=-\infty}^{\infty} \{-i \, sign k\} c_k(f) e^{ikx}$$

be the *conjugate Fourier series* of f with $c_k(f) = (1/2)(a_k(f) - i b_k(f))$. It is known that the conjugate series to Fourier series $f \in L_{[0, 2\pi]}$ will not always be the Fourier series (see, e.g., [53, p.155]).

The *n-th partial sums*, and *de la Vallé e-Poussin sums* [57] of series (1.1) are defined, respectively, as

$$S_n(x,\ f) = \frac{a_0}{2} + \sum_{k=1}^{n} A_k(x,\ f),$$

$$V_{n,m}(x,\ f) = \frac{1}{m+1} \sum_{v=n-m}^{n} S_v(x,\ f), \quad (0 \leq m \leq n,\ m, n \in Z_+ := \{1, 2, 3, \ldots\}).$$

The best approximation to $f \in L_M(\mathbb{T}, \omega)$ in the class $\prod_n$ of trigonometric polynomials of degree not exceeding n is defined by

$$E_n(f)_{M,\omega} := \inf \left\{ \|f - T_n\|_{L_M(\mathbb{T},\ \omega)} : T_n \in \prod_n \right\}.$$

Note that the existence of $T_n^* \in \Pi_n$ such that

$$E_n(f)_{M,\omega} = \left\| f - T_n^* \right\|_{L_M(\mathbb{T},\omega)}$$

follows, for example, from Theorem 1.1 in [10, p.59].

Let $W_M^r(\mathbb{T}, \omega)$, $(r = 1, 2, \ldots)$ be the class of functions such that $f^{(r-1)}$ is absolutely continuous and $f^{(r)} \in L_M(\mathbb{T}, \omega)$ becomes a Banach space under the consideration of the norm

$$\| f \|_{W_M(\mathbb{T},\omega)} := \| f \|_{L_M(\mathbb{T},\omega)} + \| f^{(r)} \|_{L_M(\mathbb{T},\omega)}.$$

Let G be a finite domain in the complex plane $\mathbb{C}$, bounded by a rectifiable Jordan curve Γ, and let $G^- := \text{ext}\Gamma$. Further let

$$T := \{w \in \mathbb{C} : |w| = 1\}, \quad D := \text{int } T \text{ and } D^- := \text{ext } T.$$

Let $w = \varphi(z)$ be the conformal mapping of G^- onto D^- normalized by

$$\varphi(\infty) = \infty, \quad \lim_{z \to \infty} \frac{\varphi(z)}{z} > 0,$$

and ψ stands for the inverse of φ.

Let $w = \varphi_1(z)$ indicate a function that maps the domain G conformally onto the disk $|w| < 1$. The inverse mapping of φ_1 will be shown by ψ_1. Let Γ_r be the image of the circle $|\varphi_1(z)| = r$, $0 < r < 1$ under the mapping $z = \psi_1(w)$.

Let us denote by E_p, where $p > 0$, the class of all functions $f(z) \neq 0$ that are analytic in G and have the property that the integral

$$\int_{\Gamma_r} |f(z)|^p \, |dz|$$

is uniformly bounded for $0 < r < 1$. We shall call the E_p-class the *Smirnov class*. If the function $f(z)$ belongs to E_p, then $f(z)$ has definite limiting values $f(z')$ almost every where on Γ, over all nontangential paths; $\left| f(z') \right|$ is summable on Γ; and

$$\lim_{r \to 1} \int_{\Gamma_r} |f(z)|^p \, |dz| = \int_{\Gamma} \left| f(z') \right|^p \, |dz'|.$$

It is known that $\varphi' = E_1(G^-)$ and $\psi' \in E_1(D^-)$. Note that the general information about Smirnov classes can be found in the books [12, pp.438–453], and [8, pp.168–185].

Let $L_M(\Gamma, \omega)$ be a weighted Orlicz space defined on Γ. We also define the *ω-weighted Smirnov–Orlicz class* $E_M(G, \omega)$ as

$$E_M(G, \omega) := \{f \in E_1(G) : f \in L_M(\Gamma, \omega)\}.$$

With every weight function ω on Γ, we associate another weight ω_0 on T defined by

$$\omega_0(t) := \omega(\psi(t)), \quad t \in T.$$

For $f \in L_M(\Gamma, \omega)$ we define the function

$$f_0(t) := f(\psi(t)), \quad t \in T.$$

Let h be a continuous function on $[0, 2\pi]$. Its modulus of continuity is defined by

$$\omega(t, h) := \sup\{|h(t_1) - h(t_2)| : t_1, t_2 \in [0, 2\pi], |t_1 - t_2| \leq t\}, \quad t \geq 0.$$

The curve Γ is called *Dini-smooth* if it has a parameterization

$$\Gamma : \varphi_0(s), \quad 0 \leq s \leq 2\pi$$

such that $\varphi_0'(s)$ is Dini-continuous, i.e.,

$$\int_0^\pi \frac{\omega(t, \varphi_0')}{t} \, dt < \infty$$

and $\varphi_0'(s) \neq 0$ [44, p.48].

If Γ is a Dini-smooth curve, then there exist [58] the constants c_1 and c_2 such that

$$0 \leq c_1 \leq |\psi'(t)| \leq c_2 < \infty, \quad |t| > 1. \tag{1.2}$$

Note that if Γ is a Dini-smooth curve, then by (1.2) we have $f_0 \in L_M(\mathbb{T}, \omega_0)$ if $f \in L_M(\Gamma, \omega)$.

Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and let ω be a weight function on Γ. ω is said to satisfy Muckenhoupt's A_p-condition on Γ if

$$\sup_{z \in \Gamma} \sup_{r > 0} \left(\frac{1}{r} \int_{\Gamma \cap D(z,r)} |\omega(\tau)|^p \, |d\tau| \right)^{1/p} \left(\frac{1}{r} \int_{\Gamma \cap D(z,r)} [\omega(\tau)]^{-p'} \, |d\tau| \right)^{1/p'} < \infty,$$

where $D(z, r)$ is an open disk with radius r and centered z.

Let us denote by $A_p(\Gamma)$ the set of all weight functions satisfying Muckenhoupt's A_p-condition on Γ. For a detailed discussion of Muckenhoupt weights on curves (see, e.g., [4]).

Let Γ be a rectifiable Jordan curve and $f \in L_1(\Gamma)$. Then, the function f^+ defined by

$$f^+(z) := \frac{1}{2\pi i} \int_\Gamma \frac{f(s) ds}{s - z}, \quad z \in G$$

is analytic in G. Note that if $0 < \alpha_M \leq \beta_M < 1$, $\omega \in A_{1/\alpha_M}(\Gamma) \cap A_{1/\beta_M}(\Gamma)$ and $f \in L_M(\Gamma, \omega)$, then by Lemma 1.4 in [20] $f^+ \in E_M(G, \omega)$.

Let $\varphi_k(z)$, $k = 0, 1, 2, \ldots$ be the Faber polynomials for G. The Faber polynomials $\varphi_k(z)$, associated with $G \cup \Gamma$, are defined through the expansion

$$\frac{\psi'(w)}{\psi(w) - z} = \sum_{k=0}^\infty \frac{\varphi_k(z)}{t^{k+1}}, \quad z \in G, \quad t \in D^- \tag{1.3}$$
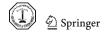
and the equalities

$$\varphi_k(z) = \frac{1}{2\pi i} \int_T \frac{w^k \psi'(w)}{\psi(w) - z} \, dw, \quad z \in G, \tag{1.4}$$

$$\varphi_k(z) = \varphi^k(z) + \frac{1}{2\pi i} \int_\Gamma \frac{\varphi^k(s)}{s - z} \, ds, \quad z \in G^-, \quad k = 0, 1, 2, \ldots \tag{1.5}$$

hold [51, p.33–38].

Let $f \in E_M(G, \omega)$. Since $f \in E_1(G)$, we have

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(s) ds}{s - z} = \frac{1}{2\pi i} \int_T \frac{f(\psi(w)) \psi'(w)}{\psi(w) - z} \, dw,$$

for every $z \in G$. Considering this formula and expansion (1.3), we can associate with f the *Faber* series

$$f(z) \sim \sum_{k=0}^{\infty} a_k(f)\varphi_k(z), \quad z \in G , \tag{1.6}$$

where

$$a_k(f) := \frac{1}{2\pi i} \int_T \frac{f(\psi(w))}{w^{k+1}} \mathrm{d}w, \quad k = 0, 1, 2, \ldots$$

This series is called the *Faber series* expansion *of* f, and the coefficients $a_k(f)$, $k = 0, 1, 2, \ldots$ are said to be the *Faber coefficients* of f.

The *n-th partial sums and de la Vallée-Poussin sums of the series* (1.6) are defined, respectively, as

$$S_n(z, f) = \sum_{k=o}^{n} a_k(f)\varphi_k(z) ,$$

$$V_{n,m}(z, f) = \frac{1}{m+1} \sum_{v=n-m}^{n} S_v(z, f), \quad (0 \le m \le n, \ m, n \in Z_+) .$$

Let Γ be a Dini-smooth curve. Using the nontangential boundary values of f_0^+ on T we define the r-th modulus of smoothness of $f \in L_M(\Gamma, \omega)$ as

$$\Omega_{\Gamma, M, \omega}^l(\delta, f) := \Omega_{M, \omega_0}^l(\delta, f_0^+), \quad \delta > 0,$$

for $l = 1, 2, 3, \ldots$

Let $\mathcal{P} := \{$all polynomials (with no restriction on the degree)$\}$, and let $\mathcal{P}(D)$ be the set of traces of members of $\mathcal{P}$ on D. We define the operator T as follows:

$$T := \mathcal{P}(D) \longrightarrow E_M(G, \omega) ,$$

$$T(P)(z) := \frac{1}{2\pi i} \int_T \frac{P(w)\psi'(w)}{\psi(w) - z} \mathrm{d}t, \quad z \in G.$$

Then, taking into account (1.4) and (1.5) we have

$$T\left(\sum_{k=0}^{n} b_k w^k\right) = \sum_{k=0}^{n} b_k \varphi_k(z), \quad z \in G.$$

We use the constants $c, c_1, c_2, \ldots$ (in general, different in different relations) which depend only on the quantities that are not important for the questions of interest

We need the following results.

Theorem 1.1 [18] *Let* $L_M(T, \omega)$ *be a weighted Orlicz space with Boyd indices* $0 < \alpha_M \le \beta_M < 1$, *and let* $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$. *Then for every*
$f \in W_M^r(T, \omega)$ $(r = 0, 1, 2, \ldots)$ *the inequality*

$$E_n(f)_{M,\omega} \le \frac{c_3}{(n+1)^r} E_n(f^{(r)})_{M,\omega}$$

holds with a constant $c_3 > 0$ *independent of* n.

Theorem 1.2 [18] *Let* $L_M(T, \omega)$ *be a weighted Orlicz space with Boyd indices* $0 < \alpha_M \le \beta_M < 1$, *and let* $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$. *Then for every*
$f \in L_M(T, \omega)$ *the inequality*

$$E_n(f)_{M,\omega} \le c_4 \Omega_{M,\omega}^l \left(\frac{1}{n+1}, f\right)$$

holds with a constant $c_4 > 0$ *independent of* n.

Theorem 1.3 [26] *Let $L_M(T, \omega)$ be a weighted Orlicz space with Boyd indices $0 < \alpha_M \leq \beta_M < 1$, and let $f \in L_M(T, \omega)$, $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$. If the condition*

$$\sum_{n=1}^{\infty} n^{r-1} E_n(f)_{M,\omega.} < \infty$$

is satisfied for $r \in Z_+$, then $\widetilde{f}^{(r)} \in L_M(T, \omega)$ and

$$E_n(\widetilde{f}^{(r)})_{M,\omega} \leq c_5 \left\{ (n+1)^r E_n(f)_{M,\omega.} + \left(\sum_{\mu=n+1}^{\infty} \mu^{r-1} E_\mu(f)_{M,\omega.} \right) \right\},$$

where the constant c_5 is independent of n.

Lemma 1.4 *Let $L_M(T, \omega)$ be a weighted Orlicz space with Boyd indices $0 < \alpha_M \leq \beta_M < 1$, and $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$.*

1. *Then, for $r \in N$ and $f^{(r)} \in L_M(T, \omega)$ the estimate*

$$\| f^{(r)} \|_{L_M(\mathbb{T},\omega)} \leq c_6 \left\{ n^r \| f \|_{L_M(\mathbb{T},\omega)} + E_n(f^{(r)})_{M,\omega.} \right\} \tag{1.7}$$

   holds with a constant $c_6 > 0$ independent of n.
2. *If $\widetilde{f}^{(r)} \in L_M(T, \omega)$, then the estimate*

$$\| \widetilde{f}^{(r)} \|_{L_M(\mathbb{T},\omega)} \leq c_7 \left\{ n^r \| f \|_{L_M(\mathbb{T},\omega)} + E_n(\widetilde{f}^{(r)})_{M,\omega.} \right\} \tag{1.8}$$

   holds with a constant $c_7 > 0$ independent of n.

Proof The function $f^{(r)}$ can be written in following form:

$$f^{(r)}(x) = \left(f^{(r)}(x) - V_{2n,n}(x, f^{(r)}) \right) + V_{2n,n}(x, f^{(r)}). \tag{1.9}$$

Then, by (1.9) we obtain

$$\| f^{(r)} \|_{L_M(\mathbb{T},\omega)} \leq \| f^{(r)} - V_{2n,n}(., f^{(r)}) \|_{L_M(\mathbb{T},\omega)} + \| V_{2n,n}(., f^{(r)}) \|_{L_M(\mathbb{T},\omega)}. \tag{1.10}$$

By [18] the following inequality holds:

$$\| f - S_n(., f) \|_{L_M(\mathbb{T},\omega)} \leq c_8 E_n(f)_{M,\omega}. \tag{1.11}$$

Then, from the inequality (1.11) we conclude that

$$\| f^{(r)} - V_{2n,n}(., f^{(r)}) \|_{L_M(\mathbb{T},\omega)} \leq E_n(f^{(r)})_{M,\omega}. \tag{1.12}$$

Using the Bernstein inequality for weighted Orlicz spaces [18], we have

$$\| V_{2n,n}(., f^{(r)}) \|_{L_M(\mathbb{T},\omega)} = \| \frac{d^r}{dx^r} V_{2n,n}(x, f) \|_{L_M(\mathbb{T},\omega)}$$
$$\leq c_9 (2n)^r \| V_{2n,n}(., f) \|_{L_M(\mathbb{T},\omega)} \leq c_{10} n^r \| f \|_{L_M(\mathbb{T},\omega)}.$$

Now combining (1.10), (1.12) and last relation, we obtain the inequality (1.7) of Lemma 2.1. The inequality (1.8) is proved to be similar.

The proof of Lemma 1.4 is completed. □

Theorem 1.5 [20] *Let Γ be a Dini-smooth curve and $L_M(\Gamma)$ be a reflexive Orlicz space. If $\omega \in A_{1/\alpha_M}(\Gamma) \cap A_{1/\beta_M}(\Gamma)$, then the linear operator $T := P(D) \longrightarrow E_M(G, \omega)$ is bounded.*

Theorem 1.6 [20] *If Γ is a Dini-smooth curve, $0 < \alpha_M \leq \beta_M < 1$, and $\omega \in A_{1/\alpha_M}(\Gamma) \cap A_{1/\beta_M}(\Gamma)$, then the operator*

$$T : E_M(D, \omega_0) \longrightarrow E_M(G, \omega)$$

is one-to-one and onto.

The problems of approximation theory in weighted and nonweighted Lebesgue spaces, weighted and nonweighted Orlicz spaces have been investigated by several authors (see, e.g., [13–27, 30–32, 36, 37, 45, 46]).

Note that the approximation problems by trigonometric polynomials in weighted Lebesgue spaces with weights belonging to the Muckenhoupt class $A_p(\mathbb{T})$ were studied in [13, 36, 37]. Detailed information on the weighted polynomial approximation can be found in the books [9, 41].

In the present paper, we investigate the problems of estimating the deviation of functions from their de la Vallée-Poussin sums in weighted Orlicz spaces $L_M(T, \omega)$.

This result is applied to estimate of approximation of de la Vallé e-Poussin sums of Faber series in weighted Smirnov–Orlicz classes defined on simply connected domains of the complex plane in terms of the modulus of smoothness. We also study the approximation of conjugate function by de la Vallée-Poussin sums of the Fourier series of the conjugate function in weighted Orlicz spaces $L_M(\mathbb{T}, \omega)$. Note that the estimates obtained in this work depend on sequence of the best approximation $E_n(f)_{M, \omega}$. Similar problems in different spaces have been investigated by several researchers (see, e.g., [1, 2, 11, 27, 42, 43, 48–50, 52–57, 59, 60]).

Our main results are as follows.

Theorem 1.7 *Let $L_M(T)$ be a reflexive Orlicz space and $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$. Then for $f \in L_M(T, \omega)$, $0 \leq m \leq n$, $m, n \in Z_+$ the inequality*

$$\| f - V_{n,m}(., f) \|_{L_M(\mathbb{T},\omega)} \leq \frac{c_{11}}{m+1} \sum_{k=n-m}^{n} E_k(f)_{M,\omega} \tag{1.13}$$

holds with a constant $c_{11} > 0$ independent of n.

Note that this result in the spaces of continuous functions and Lebesgue space L_p $(1 < p < \infty)$ have been investigated in [49, 50, 60].

Corollary 1.8 *Let $L_M(T, \omega)$ be a weighted Orlicz space with Boyd indices $0 < \alpha_M \leq \beta_M < 1$, and let $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$, $m, n \in Z_+$, $0 \leq m \leq n$. Then for every $f \in L_M(T, \omega)$, the estimate*

$$\| f - V_{n,m}(., f) \|_{L_M(\mathbb{T},\omega)} \leq \frac{c_{12}}{m+1} \sum_{k=n-m}^{n} \Omega^l_{M,\omega}\left(\frac{1}{k+1}, f\right) \tag{1.14}$$

holds with a constant $c_{12} > 0$ independent of n.

Similar result for the other modulus of smoothness in the spaces of continuous functions has been obtained in [49]. Also, similar results for the Cesaro means, Zygmund means of order 2 and Abel–Poisson means in weighted Orlicz spaces can be found in [15].

Theorem 1.9 *Let $L_M(T, \omega)$ be a weighted Orlicz space with Boyd indices $0 < \alpha_M \leq \beta_M < 1$, and let $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$, $m, n \in Z_+$, $0 \leq m \leq n$. If the inequality*

$$\sum_{\nu=1}^{\infty} \frac{E_\nu(f)_{M,\omega}}{\nu} < \infty$$

is satisfied for $f \in L_M(T, \omega)$, then the estimate

$$\| \widetilde{f} - V_{n,m}(., \widetilde{f}) \|_{L_M(\mathbb{T},\omega)} \leq c_{13} \left\{ \frac{1}{m+1} \sum_{\nu=0}^{n} E_{n-m+\nu}(f)_{M,\omega.} + \sum_{\nu=n+1}^{\infty} \frac{E_\nu(f)_{M,\omega.}}{\nu} \right\}$$

holds with a constant $c_{13} > 0$ independent of n.

This result for the spaces of continuous functions has been obtained in [1].

Theorem 1.10 *Let Γ be a Dini-smooth curve. Also, let $L_M(\Gamma, \omega)$ be a Orlicz space with Boyd indices $0 < \alpha_M \leq \beta_M < 1$, and $\omega \in A_{1/\alpha_M}(\Gamma) \cap A_{1/\beta_M}(\Gamma)$. Then for $f \in E_M(G, \omega)$ the inequality*

$$\| f - V_{n,m}(., f) \|_{L_M(T,\omega)} \leq \frac{c_{14}}{m+1} \sum_{k=n-m}^{n} \Omega^l_{\Gamma,M,\omega}\left(\frac{1}{k+1}, f\right)$$

holds with a constant $c_{14} > 0$ independent of n.

Similar results for the other means of Fourier trigonometric series in the Smirnov classes $E_p(G)$ ($1 < p < \infty$) and weighted Orlicz spaces $E_M(G, \omega)$ can be found in [15,29].

2 Proofs of the main results

Proof of Theorem 1.7 We take the integer j such that the inequality $2^j \leq m + 1 < 2^{j+1}$ is satisfied. The following identity holds:

$$f(x) - V_{n,m}(x, f) = \frac{1}{m+1}\left[f(x) - S_{n-m}(x, f)\right]$$
$$+ \frac{1}{m+1}\left\{\sum_{k=1}^{j} \sum_{i=n-m+2^{k-1}}^{n-m+2^k-1} [f(x) - S_i(x, f)]\right\}$$
$$+ \frac{1}{m+1}\left\{\sum_{k=n-m+2^j}^{n} [f(x) - S_k(x, f)]\right\}. \tag{2.1}$$

Taking into account of (2.1), we have

$$\| f - V_{n,m}(., f) \|_{L_M(\mathbb{T},\omega)} \leq \frac{1}{m+1} \| f - S_{n-m}(., f) \|_{L_M(\mathbb{T},\omega)}$$
$$+ \frac{1}{m+1}\left\{\sum_{k=1}^{j} \sum_{i=n-m+2^{k-1}}^{n-m+2^k-1} \| f - S_i(., f) \|_{L_M(\mathbb{T},\omega)}\right\}$$
$$+ \frac{1}{m+1}\left\{\sum_{k=n-m+2^j}^{n} \| f - S_k(., f) \|_{L_M(\mathbb{T},\omega)}\right\}. \tag{2.2}$$

Consideration of (1.11) and (2.1) gives us

$$\| f - V_{n,m}(., f) \|_{L_M(\mathbb{T},\omega)} \leq c_{15} \frac{1}{m+1}\left\{E_{n-m}(f)_{M,\omega.} + \sum_{k=1}^{j} 2^{k-1} E_{n-m+2^{k-1}}(f)_{M,\omega}\right\}$$
$$+ c_{15} \frac{1}{m+1}\left(m - 2^j + 1\right) E_{n-m+2^j}(f)_{M,\omega}. \tag{2.3}$$

On the other hand, the following inequality holds:

$$\sum_{k=1}^{j} 2^{k-1} E_{n-m+2^{k-1}}(f)_{M,\omega} \leq E_{n-m+1}(f)_{M,\omega}$$
$$+ 2 \sum_{k=2}^{j} \sum_{i=n-m+2^{k-2}}^{n-m+2^{k-1}-1} E_i(f)_{M,\omega} \leq c_{16} \sum_{k=n-m}^{n-m+2^{j-1}} E_k(f)_{M,\omega}. \tag{2.4}$$

From inequality $2^j \leq m + 1 < 2^{j+1}$, we have $2^j > m - 2^j + 1$. Then,

$$\left(m - 2^j + 1\right) E_{n-m+2^j}(f)_{M,\omega} \leq \sum_{k=n-m}^{n-m+2^j-1} E_k(f)_{M,\omega} \tag{2.5}$$

Using (2.3), (2.4) and (2.5), we finally conclude that

$$\| f - V_{n,m}(., \, f) \|_{L_M(\mathbb{T},\omega)} \leq \frac{c_{17}}{m+1} \left\{ E_{n-m}(f)_{M,\omega} + \sum_{k=n-m}^{n-m+2^j-1} E_k(f)_{M,\omega} + \sum_{k=n-m}^{n-m+2^j-1} E_k(f)_{M,\omega} \right\}$$

$$\leq \frac{c_{18}}{m+1} \sum_{k=n-m}^{n} E_k(f)_{M,\omega}.$$

Thus, the proof of Theorem 1.7 is completed.

Proof of Corollary 1.8 According to Theorem 1.2 and (1.13), we obtain the inequality (1.14) of Corollary 1.8.

Proof of Theorem 1.9 We consider two cases: 1. Take $0 \leq 2m \leq n$. Suppose that $R(x)$ is an antiderivative of the function $f(x) - V_{n,m}(x, \, f)$ and $U(x)$ is an antiderivative of the function $f(x) - a_0/2$. Then, according to Theorem 1.1 we obtain

$$E_\nu(U)_{M,\omega} \leq \frac{c_{20}}{\nu + 1} E_\nu(f)_{M,\omega}, \quad \nu \in Z_+. \tag{2.6}$$

Since

$$\sum_{\nu=1}^{\infty} \frac{E_\nu(f)_{M,\omega}}{\nu} < \infty,$$

the inequality (2.6) yields

$$\sum_{\nu=1}^{\infty} E_\nu(U)_{M,\omega.} < \infty.$$

By virtue of Lemma 1.4 we get

$$\| \widetilde{R}' \|_{L_M(\mathbb{T},\omega)} \leq c_{21} \left\{ n \, \|R\|_{L_M(\mathbb{T},\omega)} + E_n(\widetilde{R}')_{M,\omega} \right\},$$

where $R = U - V_{n,m}(., \, U) + c_{22}$ and c_{22} is a constant. Then, taking Theorem 1.3 into account, we conclude that

$$E_n(\widetilde{R}')_{M,\omega} \leq c_{23} \left\{ (n+1) \, E_n(U)_{M,\omega.} + \sum_{\nu=n+1}^{\infty} E_\nu(U)_{M,\omega} \right\}.$$

Then, the last inequality yields

$$\| \widetilde{f} - V_{n,m}(., \, \widetilde{f}) \|_{L_M(\mathbb{T},\omega)} \leq c_{24} \left\{ (n+1) \, \left\| f - V_{n,m}(., \, U) \right\|_{L_M(\mathbb{T},\omega)} + \sum_{\nu=n+1}^{\infty} E_n(U)_{M,\omega} \right\}.$$

From Theorem 1.7 we have

$$\left\| f - V_{n,m}(., \, U) \right\|_{L_M(\mathbb{T},\omega)} \leq \frac{c_{25}}{m+1} \sum_{\nu=n-m}^{n} E_\nu(U)_{M,\omega}$$

$$\leq \frac{c_{26}}{m+1} \sum_{\nu=n-m}^{n} \frac{E_\nu(f)_{M,\omega}}{\nu+1} = \frac{c_{26}}{m+1} \sum_{\nu=0}^{n} \frac{E_{n-m+\nu}(f)_{M,\omega}}{n-m+\nu+1}.$$

Since $0 \leq 2m \leq n$, this implies $(n+1)/(n-m+1) \leq 3$. Then, using the last inequality we reach

$$\| \widetilde{f} - V_{n,m}(., \ \widetilde{f}) \|_{L_M(\mathbb{T},\omega)} \leq c_{27} \left\{ \frac{1}{m+1} \sum_{\nu=0}^{n} \frac{E_{n-m+\nu}(f)_{M,\omega}}{n-m+\nu+1}(n+1) + \sum_{\nu=n+1}^{\infty} E_\nu(U)_{M,\omega} \right\}$$

$$\leq c_{28} \left\{ \frac{1}{m+1} \sum_{\nu=0}^{n} E_{n-m+\nu}(f)_{M,\omega} + \sum_{\nu=n+1}^{\infty} E_\nu(f)_{M,\omega} \right\}.$$

2. Suppose that the inequality $n < 2m \leq 2n$ is satisfied. From Theorem 1.3 we have

$$E_\nu(\widetilde{f})_{M,\omega} \leq c_{29} \left\{ E_\nu(f)_{M,\omega} + \sum_{\mu=\nu+1}^{\infty} \frac{E_\mu(f)_{M,\omega}}{\mu} \right\}.$$

Using the last inequality and Theorem 1.7, we find that

$$\| \widetilde{f} - V_{n,m}(., \ \widetilde{f}) \|_{L_M(\mathbb{T},\omega)} \leq \frac{c_{30}}{m+1} \sum_{\nu=n-m}^{n} E_\nu(\widetilde{f})_{M,\omega}$$

$$\leq \frac{c_{31}}{m+1} \sum_{\nu=n-m}^{n} \left\{ E_\nu(f)_{M,\omega} + \sum_{\mu=\nu+1}^{\infty} \frac{E_\mu(f)_{M,\omega}}{\mu} \right\}$$

$$\leq c_{32} \left\{ \frac{1}{m+1} \sum_{\nu=n-m}^{n} E_\nu(f)_{M,\omega} + \sum_{\nu=n+1}^{\infty} \frac{E_\nu(f)_{M,\omega}}{\nu} \right\}.$$

Hence, the proof of Theorem 1.9 is completed.

Proof of Theorem 1.10. Suppose that $f \in E_M(G,\omega)$. By virtue of Theorem 1.6 the operator $T : E_M(D,\omega_0) \longrightarrow E_M(G,\omega)$ is bounded, one-to-one and onto and $T(f_0^+) = f$. For the function f, the following Faber series holds:

$$f(z) \backsim \sum_{k=0}^{\infty} a_k(f)\varphi_k(z).$$

Since $\omega_0 \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$, considering Lemma 1.4 given in Ref. [20] we get $f_0^+ \in E_M(D,\omega_0)$. Then, function f_0^+ has the following Taylor expansion

$$f_0^+(w) = \sum_{k=0}^{\infty} a_k(f)w^k.$$

Note that $f_0^+ \in E_1(D)$ and boundary function $f_0^+ \in L_M(T,\omega_0)$. Then, using Theorem 3.4 [8, p.38] for the function $f_0^+(w)$ we get Fourier expansion

$$f_0^+(t) \backsim \sum_{k=0}^{\infty} a_k(f)e^{itk}.$$

Using the boundedness of the operator T, Theorems 1.7 and 1.2 we reach

$$\| f - V_{n,m}(., \ f) \|_{L_M(\Gamma,\omega)} = \| T(f_0^+) - T(V_{n,m}(., \ f_0^+)) \|_{L_M(\Gamma,\omega)}$$

$$\leq c_{33} \| f_0^+ - V_{n,m}(., \ f_0^+) \|_{L_M(T,\omega_0)} \leq \frac{c_{34}}{m+1} \sum_{k=n-m}^{n} E_k(f_0^+)_{M,\omega}$$

$$\leq \frac{c_{35}}{m+1} \sum_{k=n-m}^{n} \Omega_{M,\omega_0}^l \left(\frac{1}{k+1}, \ f_0^+ \right) = \frac{c_{35}}{m+1} \sum_{k=n-m}^{n} \Omega_{\Gamma,M,\omega}^l \left(\frac{1}{k+1}, \ f \right).$$

Thus, the theorem is proved.

References

1. Baiborodov, S.P.: Approximation of functions by Vallé e-Poussin sums. Matem. Zametki, **24**(1), 33–47 (1980) (**in Russian**)
2. Baiborodov, S.P.: Approximation of conjugate functions by Fourier series sums in $L_{2\pi}^p$. Anal. Math. **11**, 3–12 (1985)
3. Bennett, C., Sharpley, R.: Interpolation of Operators. Academic Press, New York (1988)
4. Böttcher, A.; Karlovich, Y.I.: Carleson Curves, Muckenhoupt Weights and Teoplitz Operators. Birkhauser, Verlag (1997)
5. Boyd, D.W.: Spaces between a pair of reflexive Lebesgue spaces. Proc. Am. Math. Soc. **18**, 215–219 (1967)
6. Boyd, D.W.: Indices of function spaces and their relationship to interpolation. Can. J. Math. **21**, 1245–1254 (1969)
7. Boyd, D.W.: Indices for the orlicz spaces. Pacific J. Math. **38**, 315–325 (1971)
8. Duren, P.L.: Theory of H^p Spaces. Academic Press, New York (1970)
9. Ditzian, Z., Totik, V.: Moduli of Smoothness. Springer Ser, Copmut. Math. 9, Springer, New York (1987)
10. Devore, R.A., Lorentz, G.G.: Constructive approximation. Springer, New York (1993)
11. Gavriljuk, V.T.: Linear summability methods for Fourier series and best approximation. Ukrain. Math. Zh. **15**(4), 412—418 (1963) (**in Russian**)
12. Goluzin, G.M.: Geometric Theory of Functions of a Complex Variable. Translation of Mathematical Monographs, vol. 26, Providence, RI: AMS, 1968
13. Gadjieva, E.A.: Investigation of the properties of functions with quasimonotone Fourier coefficients in generalized Nikolskii–Besov spaces. Author's summary of dissertation, Tbilisi, 1986 (**in Russian**)
14. Guven, A.: Trigonometric approximation of functions in weighted L^p spaces. Sarajevo J. Math. **5**(17), 99–108 (2009)
15. Guven, A., Israfilov, D.M.: Approximation by Means of Fourier trigonometric series in weighted Orlicz spaces. Adv. Stud. Contemp. Math. (Kyundshang), **19**(2), 283–295 (2009)
16. Ibragimov, I.I., Mamedkhanov, D.I.: A constructive characterization of a certain class of functions. Dokl. Akad. Nauk. SSSR **223**(1), 35–37 (1975) (**in Russian**)
17. Israfilov, D.M., Approximation by p-Faber polynomials in the weighted Smirnov class $E^p(G, \omega)$ and the Bieberbach polynomials. Constr. Approx. **17**(3), 335–351 (2001)
18. Israfilov, D.M., Guven, A.: Approximation by trigonometric polynomials in weighted Orlicz spaces. Stud. Math. **174**(2), 147–168 (2006)
19. Israfilov, D.M., Oktay, B., Akgün, R.: Approximation in Smirnov–Orlicz classes. Glas. Mat. Ser. III **40**(60), 87–102 (2005)
20. Israfilov, D.M., Akgün, R.: Approximation in weighted Smirnov–Orlicz classes. J. Math. Kyoto Univ. **46**(4), 755–770 (2006)
21. Israfilov, D.M., Kokilashvili, V.M., Samko, S.G.: Approximation in weighted Lebesgue and Smirnov spaces with variable exponent. Proc. A. Razmadze Math. Inst. **143**, 25–35 (2007)
22. Jafarov, S.Z.: Approximation by rational functions in Smirnov–Orlicz classes. J. Math. Anal. Appl. **379**, 870–877 (2011)
23. Jafarov, S.Z.: Approximation of functions by rational functions on closed curves of the complex plane. Arab. J. Sci. Eng. **36**(8), 1529–1531 (2011)
24. Jafarov, S.Z.: On approximation in weighted Smirnov–Orlicz classes. Complex Var. Elliptic Equ. **57**(5), 567–577 (2012)
25. Jafarov, S.Z.: The inverse theorem of approximation of the function in Smirnov–Orlicz classes. Math. Ineq. Appl. **12**(4), 835–844 (2012)
26. Jafarov, S.Z.: Approximation of conjugate functions by trigonometric polynomials in weighted Orlicz spaces. J. Math. Ineq. **7**(2), 271–281 (2013)
27. Jafarov, S.Z.: Approximation by Fejér sums of Fourier trigonometric series in weighted Orlicz spaces. Hacet. J. Math. Stat. **42**(3), 259–268 (2013)
28. Krasnoselskii, M.A., Rutickii, Y.B.: Convex Functions and Orlicz Spaces. P. Norrdhoff Ltd., Groningen (1961)
29. Kokilasvili, V.M.: On approximation of analytic functions from E_p classes. Trudy Tbiliss. Mat. Inst. im Razmadze Akad. Nauk Gruzin. SSR **34**, 82–102 (1968)(in Russian)
30. Kokilashvili, V.M.: On analytic functions of Smirnov–Orlicz classes. Stud. Math. **31**, 43–59 (1968)
31. Kokilashvili, V.M., Samko, S.G.: Operators of harmonic analysis in weighted spaces with non-standard growth. J. Math. Anal. Appl. **352**, 15–34 (2009)
32. Kokilashvili, V.M., Samko, S.G.: A refined inverse inequality of approximation in weighted variable exponent Lebesgue spaces. Proc. A. Razmadze Matgh. Inst. **151**, 132–138 (2009)
33. Karlovich, A.Y.: Algebras of Singular integral operators with piecewise continuous coefficients on reflexive Orlicz spaces. Math. Nachr. **179**, 187–222 (1996)
34. Karlovich, A.Y.: Singular integral operators with PC coefficients in reflexive rearrangement invariant spaces. Integr. Eq. Oper. Theory **32**, 436–481 (1998)
35. Karlovich, A.Y.: Fredholmness of singular integral operators with piecewise continuous coefficients on weighted Banach function spaces. J. Integr. Eq. Appl. **15**, 263–320 (2003)
36. Ky, N.X.: On approximation by trigonometric polynomials in L_u^p-spaces. Stud. Sci. Math. Hungar. **28**, 183–188 (1993)
37. Ky, N.X.: Moduli of mean smoothness and approximation with A_p weights. Ann. Univ. Sci. Budapest **40**, 37–48 (1997)

38. Matuszewska, W., Orlicz, W.: On certain properties of ϕ -functions. Bull. Acad. Polon. Sci. Ser. Math. Aster. Phys. **8**(7), 439–443 (1960)
39. Maligranda, L.: Indices and interpolation. Dissertationes Math. vol. 234 (1985)
40. Mamedkhanov, D.I.: Approximation in complex plane and singular operators with a Cauchy kernel. Dissertation Doct. Phys-math.nauk. The University of Tblisi, 1984 (in Russian)
41. Mhaskar, H.N.: Introduction to the theory of weighted polynomial approximation. Series in Approximation and Decompositions 7, World Sci., River Edge, NJ (1996)
42. Ovsi, Y., Serdyuk, A.S.: Approximation of continuous periodic functions by de la Vallée Poussin sums. Zb. Pr. Inst. Mat. NAN Ukr. **8**(1), 151–161 (2011)
43. Prestin, J.: On the approximation by de la Vallée Poussin sums and interpolatory polynomials in Lipschitz norms. Anal. Math. **14**, 251–259 (1987)
44. Pommerenke, C.: Boundary Behavior of Conformal Maps. Springer, Berlin (1992)
45. Ponomarenko, V.G.: Approximation of periodic functions in a Orlicz space. Sibirsk. Math. J. **7**, 1337–1346 (1966) (**in Russian**)
46. Ramazanov, A.R.K.: On approximation by polynomials and rational functions in Orlicz spaces. Anal. Math. **10**, 117–132 (1984)
47. Rao, M.M., Ren, Z.D.: Theory of Orlicz Spaces. Marcel Dekker, New York (1991)
48. Stechkin, S.B.: On best approximation of conjugate functions by trigonometric polynomials. Izv. Akad. Nauk SSSR Ser. Mat. **20**(2), 197–206 (1956) (**in Russian**)
49. Stechkin, S.B.: The approximation of periodic functions by Fejér sums. Trudy Math Inst. Steklov **62**, 48–60 (1961)(**in Russian**)
50. Stechkin, S.B.: On the approximation of periodic functions by de la ValléePoussin sums. Anal. Math. **4**, 61–74 (1978)
51. Suetin, P.K.: Series of Faber Polynomials, vol. 1. Gordob and Breack, Reading (1998)
52. Serdyuk, A.S., Ovsi, Y., Musienko, A.P.: Approximation of classes of analytic functions by de la ValléePoussin sums in uniform metric. Rend. Mat. Appl. **32**, 1–15 (2012)
53. Timan, A.F.: Theory of Approximation of Functions of a Real Variable. Pergamon Press, MacMillan (1963) Russian original published by Fizmatgiz, Moscow, 1960.
54. Timan, M.F.: Best approximation of a function and linear methods of summing Fourier series. Izv. Akad. Nauk SSSR Ser: Math. **29**, 587–604 (1965)(**in Russian**)
55. Timan, M.F.: The approximation of continuous periodic functions by linear operators which are constructed on the basis of their Fourier. Ser. Dokl. Akad. Nauk SSSR **181**, 1339–1342 (1968)(**in Russian**)
56. Timan, M.F.: Some linear summation processes for Fourier series and best approximation. Dokl. Akad. Nauk SSSR **145**, 741–743 (1962)
57. Vallée Poussin, C.L.: Sur la meilleure approximation des fonctions d'une variable reelle par des expressions d'ordre donné. Comptes dendus de l'Académie des Sciences **166**(4), 799–802 (1918)
58. Warschawskii, S.: Über das Randverhalten der Ableitung der Abbildungsfunktionen bei Konformer Abbildung. Math. Z. **35**, 321–456 (1932)
59. Zhizhiashvili, L.V.: Trigonometric Fourier series and their conjugates, Tbilis. Gos. Univ. Tbilisi, 1993 (in Russian); English transl.: Kluwer Acad. Publ., Dordrecht, 1996.
60. Zakharov, A.A.: Bound on deviations of continuous periodic functions from their De La Vallée-Poussin sums. Matem. Zametki, **3**(1), 77–84 (1968)(**in Russian**)