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A note on truncations in fractional Sobolev spaces

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Abstract We study the Nemytskii operators $u \mapsto |u|$ and $u \mapsto u^{\pm}$ in fractional Sobolev spaces $H^s(\mathbb{R}^n)$, $s > 1$.

Keywords Fractional Laplacian · Sobolev spaces · Truncation operators

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1 Introduction and main result

In this paper we discuss the relation between the map $u \mapsto |u|$ and the *Dirichlet Laplacian*. Recall that the Dirichlet Laplacian $(-\Delta_{\mathbb{R}^n})^s u$ of order $s > 0$ of a function $u \in L^2(\mathbb{R}^n)$, $n \geq 1$, is the distribution

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$$\langle(-\Delta_{\mathbb{R}^n})^s u, \varphi\rangle \equiv \int\limits_{\mathbb{R}^n} u \ (-\Delta_{\mathbb{R}^n})^s \varphi \, dx := \int\limits_{\mathbb{R}^n} |\xi|^{2s} \mathcal{F}[\varphi]\overline{\mathcal{F}[u]} \, d\xi \ , \quad \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n),$$

where

$$\mathcal{F}[u](\xi) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x) \, dx$$

is the Fourier transform in $\mathbb{R}^n$. The Sobolev–Slobodetskii space

$$H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) \mid (-\Delta_{\mathbb{R}^n})^{\frac{s}{2}} u \in L^2(\mathbb{R}^n) \}$$

naturally inherits an Hilbertian structure from the scalar product

$$(u, v) = \langle(-\Delta_{\mathbb{R}^n})^s u, v\rangle + \int\limits_{\mathbb{R}^n} uv \, dx \, .$$

The standard reference for the operator $(-\Delta_{\mathbb{R}^n})^s$ and functions in $H^s(\mathbb{R}^n)$ is the monograph [8] by Triebel.

For any positive order $s \notin \mathbb{N}$ we introduce the constant

$$C_{n,s} = \frac{2^{2s}s}{\pi^{\frac{n}{2}}} \frac{\Gamma\left(\frac{n}{2}+s\right)}{\Gamma\left(1-s\right)}. \tag{1}$$

Notice that

$$C_{n,s} > 0 \quad \text{if } \lfloor s \rfloor \text{ is even;} \qquad C_{n,s} < 0 \quad \text{if } \lfloor s \rfloor \text{ is odd,} \tag{2}$$

where $\lfloor s \rfloor$ stands for the integer part of s. It is well known that for $s \in (0, 1)$ and $u, v \in H^s(\mathbb{R}^n)$ one has

$$\langle(-\Delta_{\mathbb{R}^n})^s u, v\rangle = \frac{C_{n,s}}{2} \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} \, dxdy \, . \tag{3}$$

Let us recall some known facts about the Nemytskii operator $|\cdot| : u \mapsto |u|$.

1. $|\cdot|$ is a Lipschitz transform of $H^0(\mathbb{R}^n) \equiv L^2(\mathbb{R}^n)$ into itself.
2. Let $0 < s \le 1$. Then $|\cdot|$ is a continuous transform of $H^s(\mathbb{R}^n)$ into itself, by general results about Nemytskii operators in Sobolev/Besov spaces, see [7, Theorem 5.5.2/3]. Also it is obvious that for $u \in H^1(\mathbb{R}^n)$

$$\langle -\Delta|u|, |u|\rangle = \langle -\Delta u, u\rangle = \int\limits_{\mathbb{R}^n} |\nabla u|^2 dx\ ,$$

$$\langle -\Delta u^+, u^-\rangle = \int\limits_{\mathbb{R}^n} \nabla u^+ \cdot \nabla u^- dx = 0.$$

Here and elsewhere $u^\pm = \max\{\pm u, 0\} = \frac{1}{2}(|u| \pm u)$, so that $u = u^+ - u^-$, $|u| = u^+ + u^-$. On the other hand, for $s \in (0,1)$ and $u \in H^s(\mathbb{R}^n)$ formula (3) gives

$$\langle (-\Delta_{\mathbb{R}^n})^s u^+, u^-\rangle = -C_{n,s} \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{u^+(x)u^-(y)}{|x-y|^{n+2s}}\, dxdy. \tag{4}$$

From (4) we infer by the polarization identity

$$4\langle (-\Delta_{\mathbb{R}^n})^s u^+, u^-\rangle = \langle (-\Delta_{\mathbb{R}^n})^s |u|, |u|\rangle - \langle (-\Delta_{\mathbb{R}^n})^s u, u\rangle$$

that if u changes sign then

$$\langle (-\Delta_{\mathbb{R}^n})^s |u|, |u|\rangle < \langle (-\Delta_{\mathbb{R}^n})^s u, u\rangle, \qquad s \in (0,1). \tag{5}$$

We mention also [4, Theorem 6] for a different proof and explanation of (5), that includes the case when $(-\Delta_{\mathbb{R}^n})^s$ is replaced by the *Navier* (or *spectral Dirichlet*) *Laplacian* on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$.

3. Let $1 < s < \frac{3}{2}$. The results in [2] and [6] (see also Section 4 of the exhaustive survey [3]) imply that $|\cdot|$ is a bounded transform of $H^s(\mathbb{R}^n)$ into itself. That is, there exists a constant $c(n,s)$ such that

$$\langle (-\Delta_{\mathbb{R}^n})^s |u|, |u|\rangle \le c(n,s)\langle (-\Delta_{\mathbb{R}^n})^s u, u\rangle, \qquad u \in H^s(\mathbb{R}^n).$$

In particular, $|\cdot|$ is continuous at $0 \in H^s(\mathbb{R}^n)$.

It is easy to show that the assumption $s < \frac{3}{2}$ can not be improved, see Example 1 below and [2, Proposition p. 357], where a more general setting involving Besov spaces $B^{s,q}_p(\mathbb{R}^n)$, $s \ge 1 + \frac{1}{p}$, is considered.

At our knowledge, the continuity of $|\cdot| : H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$, $s \in (1, \frac{3}{2})$, is an open problem. We can only point out the next simple result.

Proposition 1 *Let* $0 < \tau < s < \frac{3}{2}$. *Then* $|\cdot| : H^s(\mathbb{R}^n) \to H^\tau(\mathbb{R}^n)$ *is continuous.*

Proof Recall that $H^s(\mathbb{R}^n) \hookrightarrow H^\tau(\mathbb{R}^n)$ for $0 < \tau < s$. Actually, the Hölder inequality readily gives the well known interpolation inequality

$$\begin{aligned}\langle (-\Delta_{\mathbb{R}^n})^\tau v, v\rangle &= \int\limits_{\mathbb{R}^n} |\xi|^{2\tau} |\mathcal{F}[v]|^2 d\xi \\ &\le \Big(\langle (-\Delta_{\mathbb{R}^n})^s v, v\rangle\Big)^{\frac{\tau}{s}} \Big(\int\limits_{\mathbb{R}^n} |v|^2 dx\Big)^{\frac{s-\tau}{s}}, \quad v \in H^s(\mathbb{R}^n).\end{aligned}$$

Since $|\cdot|$ is continuous $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and bounded $H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$, the statement follows immediately. □

Now we formulate our main result. It provides the complete proof of [5, Theorem 1] for s below the threshold $\frac{3}{2}$ and gives a positive answer to a question raised in [1, Remark 4.2] by Nicola Abatangelo, Sven Jahros and Albero Saldaña.

Theorem 1 *Let $s \in (1, \frac{3}{2})$ and $u \in H^s(\mathbb{R}^n)$. Then formula (4) holds. In particular, if u changes sign then*

$$\langle (-\Delta_{\mathbb{R}^n})^s |u|, |u| \rangle > \langle (-\Delta_{\mathbb{R}^n})^s u, u \rangle .$$

Our proof is deeply based on the continuity result in Proposition 1. The knowledge of continuity of $|\cdot| : H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ could considerably simplify it.

We denote by c any positive constant whose value is not important for our purposes. Its value may change line to line. The dependance of c on certain parameters is shown in parentheses.

2 Preliminary results and proof of Theorem 1

We begin with a simple but crucial identity that has been independently pointed out in [5, Lemma 1] and [1, Lemma 3.11] (without exact value of the constant). Notice that it holds for general fractional orders $s > 0$.

Theorem 2 *Let $s > 0$, $s \notin \mathbb{N}$. Assume that $v, w \in H^s(\mathbb{R}^n)$ have compact and disjoint supports. Then*

$$\langle (-\Delta_{\mathbb{R}^n})^s v, w \rangle = -C_{n,s} \iint\limits_{\mathbb{R}^n \times \mathbb{R}^n} \frac{v(x)w(y)}{|x-y|^{n+2s}}\, dxdy. \tag{6}$$

Proof Let ρ_h be a sequence of mollifiers, and put $w_h := w * \rho_h$. Formula (3) gives

$$\begin{aligned}\langle (-\Delta_{\mathbb{R}^n})^s v, w_h \rangle &= \langle (-\Delta_{\mathbb{R}^n})^{s-\lfloor s \rfloor} v, (-\Delta)^{\lfloor s \rfloor} w_h \rangle \\ &= \frac{C_{n,s-\lfloor s \rfloor}}{2} \iint\limits_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\big(v(x)-v(y)\big)\big((-\Delta)^{\lfloor s \rfloor} w_h(x) - (-\Delta)^{\lfloor s \rfloor} w_h(y)\big)}{|x-y|^{n+2(s-\lfloor s \rfloor)}}\, dxdy.\end{aligned}$$

Since for large h the supports of v and w_h are separated, we have

$$\langle (-\Delta_{\mathbb{R}^n})^s v, w_h \rangle = -C_{n,s-\lfloor s \rfloor} \iint\limits_{\mathbb{R}^n \times \mathbb{R}^n} \frac{v(x)\ (-\Delta)^{\lfloor s \rfloor} w_h(y)}{|x-y|^{n+2(s-\lfloor s \rfloor)}}\, dydx.$$

Here we can integrate by parts. Using (1) one computes for $a > 0$

$$\Delta \frac{C_{n,a}}{|x-y|^{n+2a}} = \frac{C_{n,a}(n+2a)(2a+2)}{|x-y|^{n+2a+2}} = -\frac{C_{n,a+1}}{|x-y|^{n+2(a+1)}}$$

and obtains (6) with w_h instead of w.

Since the supports of v and w are separated, it is easy to pass to the limit as $h \to \infty$ and to conclude the proof. $\square$

Remark 1 Motivated by (6) and (2), A.I. Nazarov conjectured in [5] that

$$\begin{aligned} &\langle (-\Delta_{\mathbb{R}^n})^s |u|, |u|\rangle - \langle (-\Delta_{\mathbb{R}^n})^s u, u\rangle < 0 \ \text{ if } \lfloor s \rfloor \ \text{ is even;} \\ &\langle (-\Delta_{\mathbb{R}^n})^s |u|, |u|\rangle - \langle (-\Delta_{\mathbb{R}^n})^s u, u\rangle > 0 \ \text{ if } \lfloor s \rfloor \ \text{ is odd} \end{aligned}$$

for any not integer exponent $s > 0$ and for any changing sign function $u \in H^s(\mathbb{R}^n)$ such that $u^\pm \in H^s(\mathbb{R}^n)$.

Lemma 1 *Let $s \in (1, \frac{3}{2})$ and $\varepsilon > 0$. If a function $u \in H^s(\mathbb{R}^n)$ has compact support then $(u-\varepsilon)^+ \in H^s(\mathbb{R}^n)$, and*

$$\langle (-\Delta_{\mathbb{R}^n})^s (u-\varepsilon)^+, (u-\varepsilon)^+\rangle \le c(n,s)\langle (-\Delta_{\mathbb{R}^n})^s u, u\rangle + c(n,s,\mathrm{supp}(u))\varepsilon^2 .$$

Proof Take a nonnegative function $\eta \in \mathcal{C}^\infty_0(\mathbb{R}^n)$ such that $\eta \equiv 1$ on $\mathrm{supp}(u)$. Clearly $u - \varepsilon\eta \in H^s(\mathbb{R}^n)$. Hence, by Item 3 in the Introduction we have that $(u-\varepsilon\eta)^+ = (u-\varepsilon)^+ \in H^s(\mathbb{R}^n)$ and

$$\begin{aligned} \langle (-\Delta_{\mathbb{R}^n})^s (u-\varepsilon)^+, (u-\varepsilon)^+\rangle &\le c(n,s)\,\langle (-\Delta_{\mathbb{R}^n})^s (u-\varepsilon\eta), u-\varepsilon\eta\rangle \\ &\le c(n,s)\big(\langle (-\Delta_{\mathbb{R}^n})^s u, u\rangle + \varepsilon^2 \langle (-\Delta_{\mathbb{R}^n})^s \eta, \eta\rangle\big). \end{aligned}$$

The proof is complete. $\square$

In order to simplify notation, for $u : \mathbb{R}^n \to \mathbb{R}$ and $s > 0$ we put

$$\Phi^s_u(x,y) = \frac{u^+(x)u^-(y)}{|x-y|^{n+2s}} .$$

Lemma 2 *Let $s \in (1, \frac{3}{2})$ and $u \in H^s(\mathbb{R}^n) \cap \mathcal{C}^0_0(\mathbb{R}^n)$. Then (4) holds, and in particular $\Phi^s_u \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$.*

Proof Thanks to Lemma 1 we have that $(u^- - \varepsilon)^+ \in H^s(\mathbb{R}^n) \cap \mathcal{C}^0_0(\mathbb{R}^n)$ for any $\varepsilon > 0$. Next, the supports of the functions u^+ and $(u^- - \varepsilon)^+$ are compact and disjoint. Thus we can apply Theorem 2 to get

$$\langle (-\Delta_{\mathbb{R}^n})^s u^+, (u^- - \varepsilon)^+\rangle = -\,C_{n,s} \iint\limits_{\mathbb{R}^n \times \mathbb{R}^n} \frac{u^+(x)(u(y)^- - \varepsilon)^+}{|x-y|^{n+2s}}\, dxdy. \tag{7}$$

Take a decreasing sequence $\varepsilon \searrow 0$. From Lemma 1 we infer that $(u^- - \varepsilon)^+ \to u^-$ weakly in $H^s(\mathbb{R}^n)$, as $(u^- - \varepsilon)^+ \to u^-$ in $L^2(\mathbb{R}^n)$. Hence the duality product in (7) converges to the the duality product in (4). Next, the integrand in the right-hand side of (7) increases to Φ^s_u a.e. on $\mathbb{R}^n \times \mathbb{R}^n$. By the monotone convergence theorem we get the convergence of the integrals, and the conclusion follows immediately. $\square$

Lemma 3 *Let $s \in (1, \frac{3}{2})$ and $u \in H^s(\mathbb{R}^n)$. Then $\Phi_u^s \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$.*

Proof Take a sequence of functions $u_h \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ such that $u_h \to u$ in $H^s(\mathbb{R}^n)$ and almost everywhere. Since $\Phi_{u_h}^s \to \Phi_u^s$ a.e. on $\mathbb{R}^n \times \mathbb{R}^n$, Fatou's Lemma, Lemma 2 for u_h and the boundeness of $v \mapsto v^\pm$ in $H^s(\mathbb{R}^n)$ give

$$\begin{aligned}\iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \Phi_u^s(x,y)\,dxdy &\le \liminf_{h\to\infty} \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \Phi_{u_h}^s(x,y)\,dxdy \\ &= c(n,s)\liminf_{h\to\infty}\langle(-\Delta_{\mathbb{R}^n})^s u_h^+, u_h^-\rangle \\ &\le c(n,s)\lim_{h\to\infty}\langle(-\Delta_{\mathbb{R}^n})^s u_h, u_h\rangle = c(n,s)\langle(-\Delta_{\mathbb{R}^n})^s u, u\rangle,\end{aligned}$$

that concludes the proof. □

Proof of Theorem 1 Take a sequence $u_h \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ such that $u_h \to u$ in $H^s(\mathbb{R}^n)$ and almost everywhere. Consider the nonnegative functions

$$v_h := u_h^+ \wedge u^+ = u^+ - (u^+ - u_h^+)^+ , \quad w_h := u_h^- \wedge u^- = u^- - (u^- - u_h^-)^+.$$

Then $v_h, w_h \in H^s(\mathbb{R}^n)$. Next, take any exponent $\tau \in (1,s)$. By Proposition 1 we have that $u^\pm - u_h^\pm \to 0$ in $H^\tau(\mathbb{R}^n)$; hence $(u^\pm - u_h^\pm)^+ \to 0$ in $H^\tau(\mathbb{R}^n)$ by Item 3 in the Introduction. Thus,

$$v_h \to u^+ , \quad w_h \to u^- \quad \text{in } H^\tau(\mathbb{R}^n) \text{ and almost everywhere,} \quad \text{as } h \to \infty. \tag{8}$$

Now we take a small $\varepsilon > 0$. Recall that $(v_h - \varepsilon)^+ \in H^\tau(\mathbb{R}^n)$ by Lemma 1. Moreover, from $0 \le v_h \le u_h^+$, $0 \le w_h \le u_h^-$ it follows that

$$\operatorname{supp}((v_h - \varepsilon)^+) \subseteq \{u_h \ge \varepsilon\}; \qquad \operatorname{supp}(w_h) \subseteq \operatorname{supp}(u_h^-).$$

In particular, the functions $(v_h - \varepsilon)^+$, w_h have compact and disjoint supports. Thus we can apply Theorem 2 to infer

$$\langle(-\Delta_{\mathbb{R}^n})^\tau (v_h - \varepsilon)^+, w_h\rangle = -C_{n,\tau} \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(v_h(x) - \varepsilon)^+ w_h(y)}{|x-y|^{n+2\tau}}\,dxdy.$$

We first take the limit as $\varepsilon \searrow 0$. The argument in the proof of Lemma 2 gives

$$\langle(-\Delta_{\mathbb{R}^n})^\tau v_h, w_h\rangle = -C_{n,\tau} \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{v_h(x) w_h(y)}{|x-y|^{n+2\tau}}\,dxdy. \tag{9}$$

Next we push $h \to \infty$. By (8) we get

$$\lim_{h\to\infty}\langle(-\Delta_{\mathbb{R}^n})^\tau v_h, w_h\rangle = \langle(-\Delta_{\mathbb{R}^n})^\tau u^+, u^-\rangle.$$

Further, since the integrand in the right-hand side of (9) does not exceed $\Phi_u^\tau(x,y)$, Lemma 3, (8) and Lebesgue's theorem give

$$\lim_{h\to\infty} \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{v_h(x)w_h(y)}{|x-y|^{n+2\tau}}\,dxdy = \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \Phi_u^\tau(x,y)\,dxdy.$$

Thus, we proved (4) with s replaced by τ. It remains to pass to the limit as $\tau \nearrow s$. By Lebesgue's theorem, we have

$$\begin{aligned}\lim_{\tau\nearrow s} \langle(-\Delta_{\mathbb{R}^n})^\tau u^+, u^-\rangle &= \lim_{\tau\nearrow s} \int\limits_{\mathbb{R}^n} |\xi|^{2\tau}\mathcal{F}[u^+]\overline{\mathcal{F}[u^-]}d\xi \\ &= \int\limits_{\mathbb{R}^n} |\xi|^{2s}\mathcal{F}[u^+]\overline{\mathcal{F}[u^-]}d\xi = \langle(-\Delta_{\mathbb{R}^n})^s u^+, u^-\rangle.\end{aligned}$$

Now we fix $\tau_0 \in (1,s)$ and notice that $0 \le \Phi_u^\tau \le \max\{\Phi_u^{\tau_0}, \Phi_u^s\}$ for any $\tau \in (\tau_0, s)$. Therefore, Lemma 3 and Lebesgue's theorem give

$$\lim_{\tau\nearrow s} \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \Phi_u^\tau(x,y)\,dxdy = \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \Phi_u^s(x,y)\,dxdy.$$

The proof of (4) is complete. The last statement follows immediately from (4), polarization identity and (2). □

Example 1 It is easy to construct a function $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ such that $u^+ \in H^s(\mathbb{R}^n)$ if and only if $s < \frac{3}{2}$.

Take $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ satisfying $\varphi(0) = 0$, $\varphi'(0) > 0$ and $x\varphi(x) \ge 0$ on $\mathbb{R}$. By direct computation one checks that $\varphi^+ = \chi_{(0,\infty)}\varphi \in H^s(\mathbb{R})$ if and only if $s < \frac{3}{2}$. If $n = 1$ we are done. If $n \ge 2$ we take $u(x_1, x_2, \dots, x_n) = \varphi(x_1)\varphi(x_2)\dots\varphi(x_n)$.

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