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Lieb–Thirring inequalities for generalized magnetic fields

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Abstract Following an approach by Exner et al. (Commun Math Phys 26:531–541, 2014), we establish Lieb–Thirring inequalities for general self-adjoint and second-degree differential operators with matrix valued potentials acting in one space-dimension. These include and generalize the magnetic Schrödinger operator. Three different settings are considered, with functions defined on the whole real line, a semi-axis and an interval, respectively, leading to different types of bounds. An interpretation of the result in terms of Schrödinger operators acting on star graphs and graphs with two vertices is also given.

Keywords Schrödinger operators · Lieb–Thirring inequalities · Commutation method

Mathematics Subject Classification 35P15 · 81Q10

1 Introduction

Let V be a real-valued function and consider the self-adjoint Schrödinger operator

$$H = -\Delta + V(x) \tag{1.1}$$

acting on $L^2(\mathbb{R}^d)$. In addition to describing the spectral properties of H as a detailed function of the potential V, it is also of interest to obtain general bounds on the moments

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of the negative eigenvalues of H, which physically correspond to states bound in the potential well. If we denote these eigenvalues by $\{-\lambda_i(V)\}$, bounds can be achieved through so-called Lieb–Thirring inequalities of the form

$$\sum_{i=1}^{\infty} \lambda_i^{\gamma}(V) \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_-^{\gamma+\frac{1}{2}} \mathrm{d}x, \tag{1.2}$$

for $L_{\gamma,d}$ constants depending only on γ and d. Here, $V_{\pm} = \frac{|V| \pm V}{2}$ denote the positive and negative parts of $V(x)$, respectively. For the case $\gamma \geq 3/2$, it was shown in [13] that

$$L_{\gamma,d} = L_{\gamma,d}^{\mathrm{cl}} := (4\pi)^{-\frac{d}{2}} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\frac{d}{2}+1)}, \tag{1.3}$$

and it holds for all γ that $L_{\gamma,d}^{\mathrm{cl}} \leq L_{\gamma,d}$.

The Lieb–Thirring inequalities have been proven to hold if and only if $\gamma \geq 1/2$ for $d = 1$ [9,20,25], $\gamma > 0$ for $d = 2$ [20], and $\gamma \geq 0$ for $d \geq 3$ [4,11,16,20,23]. Achieving estimates for the constants in Eq. (1.2) is also of importance in applications and is discussed in e.g. [7,8,25]. For the case where $V(x)$ is matrix-valued, see also the paper in [13].

Originally shown by Lieb and Thirring [20] in order to give an alternative proof of the stability of matter, these inequalities have also found additional applications. See [3,14,15,17,22] for examples in physics, including the Navier–Stokes equations, and other areas of mathematics; Laptev [12], Seiringer [24] also give detailed surveys of the area and a comprehensive treatment can be found in [19].

This article aims at generalizing the setting of Lieb–Thirring inequalities to more general operators than the magnetic or non-magnetic Schrödinger operators. More precisely, take $P(x)$, $\widetilde{Q}(x)$ as complex-valued $n \times n$-matrix functions and let Ω be either [0, 1], $\mathbb{R}_+$ or $\mathbb{R}$. We consider all self-adjoint operators of the type

$$\mathcal{L} := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes \mathbb{I} + P(x) \frac{\mathrm{d}}{\mathrm{d}x} \otimes \mathbb{I} + \widetilde{Q}(x) \tag{1.4}$$

acting on $L^2(\Omega; \mathbb{C}^n)$, and obtain bounds on the moments of their negative eigenvalues, $\{-\lambda_n\}$. Note that this form of the operator includes, but is not restricted to, both the magnetic and non-magnetic Schrödinger operators. For some symmetric and real-valued $n \times n$-matrix function $A(x)$, these can be written as

$$\left(i \frac{\mathrm{d}}{\mathrm{d}x} \otimes \mathbb{I} + A(x)\right)^2 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes \mathbb{I} + 2iA(x) \frac{\mathrm{d}}{\mathrm{d}x} \otimes \mathbb{I} + iA'(x) + A^2(x), \tag{1.5}$$

which is of the form in Eq. (1.4). The operator in Eq. (1.4) can therefore be interpreted as a generalized magnetic Schrödinger operator.

Remark 1 The operator in Eq. (1.4) is similar to the magnetic Schrödinger operator in Eq. (1.5). When the potential functions are scalar-valued, the magnetic Schrödinger operator takes the form $\left(i \frac{\mathrm{d}}{\mathrm{d}x} + a(x)\right)^2$, for some real-valued $a(x)$. This can be reduced to the non-magnetic Schrödinger operator through a gauge transformation. The main contribution of this paper lies in generalizing this operator in three different ways: to also include matrix-valued potential functions as in Eq. (1.5), to the more general operator in Eq. (1.4), and lastly to settings where the operator acts on functions defined on subsets of the real line. This last part follows and slightly extends the work in [5].

The approach in this article uses the ideas from [2,5] and the commutation method. We will show below that the proper setting for this approach is to have P element-wise weakly differentiable together with some integrability assumptions on P and $\widetilde{Q}$. More precisely, while studying functions defined on some $\Omega \subseteq \mathbb{R}^d$, we take

$$P(x)^2,\, P'(x),\, \widetilde{Q}(x) \in \begin{cases} L^1(\Omega;\mathbb{C}^{n\times n}) + L^\infty(\Omega;\mathbb{C}^{n\times n}), & d=1,\\ L^{1+\varepsilon}(\Omega;\mathbb{C}^{n\times n}) + L^\infty(\Omega;\mathbb{C}^{n\times n}), & d=2,\\ L^{d/2}(\Omega;\mathbb{C}^{n\times n}) + L^\infty(\Omega;\mathbb{C}^{n\times n}), & d\geq 3,\end{cases} \tag{1.6}$$

where $\varepsilon \in \mathbb{R}_+ := [0,\infty)$ is arbitrary and the integrability is to be interpreted element-wise. Arguing by approximation, we will in the following assume that P is indeed element-wise weakly differentiable. We will show below that any $\mathcal{L}$ in Eq. (1.4) is then self-adjoint if and only if P is anti-Hermitian with $\widetilde{Q}^* = \widetilde{Q} - P'$. We will therefore rewrite Eq. (1.4) on the form

$$\mathcal{L} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\otimes\mathbb{I} + P(x)\frac{\mathrm{d}}{\mathrm{d}x}\otimes\mathbb{I} + \frac{P'(x)}{2} - \frac{P(x)^2}{4} + Q(x), \tag{1.7}$$

where P is anti-Hermitian, $Q = \widetilde{Q} - \frac{P'}{2} + \frac{P^2}{4}$ is Hermitian and $\mathcal{L}$ acts on some subset of $L^2(\mathbb{R};\mathbb{C}^n)$. Note that this is slightly more general than the operator in Eq. (1.5), since P need not be of the form iA, for A a real $n\times n$-matrix.

We will study the operator in Eq. (1.7) for the three different cases of $\mathcal{L}$ acting on functions defined on the whole real line, a semi-axis and an interval, respectively. The last two cases will be endowed with Robin boundary conditions at the endpoints. More precisely, we let $\mathcal{L}$ act on

$$\mathfrak{D}(\mathcal{L}) := L^2(\mathbb{R};\mathbb{C}^n), \tag{1.8}$$

$$\mathfrak{D}_+(\mathcal{L}) := \{u \in L^2(\mathbb{R}_+;\mathbb{C}^n) : u'(0) - \mathfrak{S}u(0) = 0\},\quad P(0)=0, \tag{1.9}$$

$$\mathfrak{D}_{[0,1]}(\mathcal{L}) := \{u \in L^2([0,1];\mathbb{C}^n) : \tag{1.10}$$

$$u'(0) - \mathfrak{S}_0 u(0) = 0 = u'(1) - \mathfrak{S}_1 u(1)\},\quad P(0)=P(1)=0, \tag{1.11}$$

respectively. Here, $\mathfrak{S}, \mathfrak{S}_0, \mathfrak{S}_1$ are $n\times n$ Hermitian matrices encoding Robin boundary conditions. Note that the conditions that $P(0)=0$ in Eq. (1.9) and that $P(0)=P(1)=0$ in Eq. (1.11) are necessary and sufficient to make $\mathcal{L}$ self-adjoint when acting on these spaces, as will be shown below. Note also that this is a generalization of the settings of [2,5], and we extend their results by establishing Lieb–Thirring inequalities on the moments of the negative eigenvalues of $\mathcal{L}$ on these spaces. Setting $P=0$ in Theorems 1, 2 and Corollaries 1, 2 below, we recover the results in [2,5]. The main results are as follows.

Theorem 1 *Let $\mathcal{L}$ act on $\mathfrak{D}(\mathcal{L})$ in Eq. (1.8). Let also $P(x)$ be anti-Hermitian, termwise weakly differentiable and $Q(x)$ Hermitian. If $\mathcal{L}$ is bounded from below and* $\mathrm{Tr}(Q_-^2) \in L^1(\mathbb{R})$, *then*

$$\sum_{i=1}^{\infty} \kappa_i \lambda_i^{3/2} \leq \frac{3}{16} \int_{\mathbb{R}} \mathrm{Tr}(Q_-^2) \mathrm{d}x, \tag{1.12}$$

where κ_i denotes the multiplicity of the negative eigenvalue $-\lambda_i$. This bound is also sharp, in the sense that there are P, Q such that the resulting inequality is false if the constant is replaced by a smaller number.

Theorem 2 *Let $P(x)$ be anti-Hermitian, termwise weakly differentiable with $P(0) = 0$ and $Q(x)$ Hermitian. If $\mathcal{L}$ acts on $\mathfrak{D}_+(\mathcal{L})$ in Eq. (1.9), is bounded from below and if* $\mathrm{Tr}(Q_-^2) \in L^1(\mathbb{R}_+)$, *then*

$$\frac{3}{4}\lambda_1 \mathrm{Tr}\mathfrak{S} + \frac{1}{2}(2\kappa_1 - n)\lambda_1^{3/2} + \sum_{i=2}^{\infty} \kappa_i \lambda_i^{3/2} \leq \frac{3}{16} \int_{\mathbb{R}_+} \mathrm{Tr}(Q_-^2) \mathrm{d}x + \frac{1}{4}\mathrm{Tr}\mathfrak{S}^3. \tag{1.13}$$

Theorem 3 *Let $P(x)$ be anti-Hermitian, termwise weakly differentiable with $P(0) = P(1) = 0$ and $Q(x)$ Hermitian. If $\mathcal{L}$ acts on $\mathfrak{D}_{[0,1]}(\mathcal{L})$ in Eq. (1.11), is bounded from below and if* $\mathrm{Tr}(Q_-^2) \in L^1([0,1])$, *then*

$$\frac{3}{4}\lambda_1 \mathrm{Tr}\left(\mathfrak{S}_0 - \mathfrak{S}_1\right) + \sum_{i=2}^{\infty} \kappa_i \lambda_i^{3/2} \leq \frac{3}{16} \int_0^1 \mathrm{Tr}(Q_-^2) \mathrm{d}x + \frac{1}{4}\mathrm{Tr}\left(\mathfrak{S}_0^3 - \mathfrak{S}_1^3\right). \tag{1.14}$$

Remark 2 The same techniques used to prove Theorems 1–3 can be used to obtain Lieb–Thirring bounds also when $\mathcal{L}$ acts on functions defined on unions of intervals and semi-axes. Each interval will then give a contribution of the form in Eq. (1.14) and each semi-axis contributes like in Eq. (1.13).

These bounds can be extended to higher moments using an Aizenman–Lieb argument from [1]. We can then conclude the following corollary.

Corollary 1 *Let $\mathcal{L}$ act on $\mathfrak{D}(\mathcal{L})$ in Eq. (1.8) and take $\gamma \geq 3/2$. Let $P(x)$ be anti-Hermitian, termwise weakly differentiable and $Q(x)$ Hermitian. If $\mathcal{L}$ is bounded from below and* $\mathrm{Tr}(Q_-^{\gamma+\frac{1}{2}}) \in L^1(\mathbb{R})$, *then*

$$\sum_{i=1}^{\infty} \kappa_i \lambda_i^{\gamma} \leq L_{\gamma,1} \int_{\mathbb{R}} \mathrm{Tr}(Q_-^{\gamma+\frac{1}{2}}) \mathrm{d}x. \tag{1.15}$$

These bounds are also sharp.

In the case of the semi-axis, we can use the same techniques to obtain bounds on higher moments, in case $\mathrm{Tr}\mathfrak{S}^3 \leq 0$. The bound in Theorem 2 then becomes

$$\frac{3}{4}\lambda_1 \mathrm{Tr}\mathfrak{S} + \frac{1}{2}(2\kappa_1 - n)\lambda_1^{3/2} + \sum_{i=2}^{\infty} \kappa_i \lambda_i^{3/2} \leq \frac{3}{16} \int_{\mathbb{R}_+} \mathrm{Tr}(Q_-^2) \mathrm{d}x, \tag{1.16}$$

which allows for the Aizenman–Lieb treatment. We obtain the following corollary.

Corollary 2 *Take* $\gamma \geq 3/2$ *and assume that* $\mathrm{Tr}\mathfrak{S}^3 \leq 0$. *Let* $P(x)$ *be anti-Hermitian, termwise weakly differentiable with* $P(0) = 0$ *and* $Q(x)$ *Hermitian. If* $\mathcal{L}$ *acts on* $\mathfrak{D}_+(\mathcal{L})$ *in Eq.* (1.9), *is bounded from below and if* $\mathrm{Tr}(Q_-^{\gamma+\frac{1}{2}}) \in L^1(\mathbb{R}_+)$, *then*

$$\frac{3}{4}\frac{B(\gamma-3/2,2)}{B(\gamma-3/2,5/2)}\lambda_1^{\gamma-1/2}\mathrm{Tr}\mathfrak{S} + \frac{1}{2}\left(2\kappa_1 - n\right)\lambda_1^\gamma + \sum_{i=2}^{\infty}\kappa_i\lambda_i^\gamma \tag{1.17}$$

$$\leq L_{\gamma,1}\int_0^\infty \mathrm{Tr}\left(Q_-^{\gamma+1/2}\right)\mathrm{d}x, \tag{1.18}$$

where $B(p,q)$ *denotes the Beta function*

$$B(p,q) = \int_0^1 (1-t)^{q-1}t^{p-1}\mathrm{d}t. \tag{1.19}$$

Analogously, in the setting of the interval, we assume $\mathrm{Tr}\left(\mathfrak{S}_0^3 - \mathfrak{S}_1^3\right) \leq 0$ in order to prove the following.

Corollary 3 *Take* $\gamma \geq 3/2$ *and assume that* $\mathrm{Tr}\left(\mathfrak{S}_0^3 - \mathfrak{S}_1^3\right) \leq 0$. *Let* $P(x)$ *be anti-Hermitian, termwise weakly differentiable with* $P(0) = P(1) = 0$ *and* $Q(x)$ *Hermitian. If* $\mathcal{L}$ *acts on* $\mathfrak{D}_{[0,1]}(\mathcal{L})$ *in Eq.* (1.11), *is bounded from below and if* $\mathrm{Tr}\left(Q_-^{\gamma+\frac{1}{2}}\right) \in L^1(\mathbb{R}_+)$, *then*

$$\frac{3}{4}\frac{B(\gamma-3/2,2)}{B(\gamma-3/2,5/2)}\lambda_1^{\gamma-1/2}\mathrm{Tr}\left(\mathfrak{S}_0 - \mathfrak{S}_1\right) + \sum_{i=2}^{\infty}\kappa_i\lambda_i^\gamma \leq L_{\gamma,1}\int_0^1 \mathrm{Tr}\left(Q_-^{\gamma+1/2}\right)\mathrm{d}x. \tag{1.20}$$

We next turn to an application of the results. We have the following.

Theorem 4 *If* P *and* Q *are diagonal matrices, then* $\mathcal{L}$ *acting on* $\mathfrak{D}_+(\mathcal{L})$ *can be interpreted as a Schrödinger operator acting on a star graph with* n *semi-infinite edges and a matching condition at the common vertex. Its negative spectrum satisfies Eqs.* (1.13) *and* (1.17). $\mathcal{L}$ *acting on* $\mathfrak{D}_{[0,1]}(\mathcal{L})$ *can similarly be seen as a Schrödinger operator on a graph with two vertices,* n *edges between these and separate matching conditions at both vertices. The negative eigenvalues of* $\mathcal{L}$ *satisfy Eqs.* (1.14) *and* (1.20).

2 Auxiliary results

In this section, we will establish auxiliary results needed for the proofs of the main propositions. We will follow the commutation method approach used in [2,5], suitably generalized to our setting. First, we record the conditions on P and $\widetilde{Q}$ needed to make $\mathcal{L}$ in Eq. (1.4) self-adjoint.

Lemma 1 *Let P in Eq. (1.4) be termwise weakly differentiable. $\mathcal{L}$ acting on $\mathfrak{D}(\mathcal{L})$ is then symmetric if and only if $P^* = -P$ and $\widetilde{Q}^* = \widetilde{Q} - P'$ almost everywhere. Moreover, $\mathcal{L}$ acting on $\mathfrak{D}_+(\mathcal{L})$ is symmetric if and only if additionally $P(0) = 0$ and $\mathcal{L}$ acting on $\mathfrak{D}_{[0,1]}(\mathcal{L})$ is symmetric if and only if also $P(0) = P(1) = 0$.*

Proof Starting with the case of $\mathfrak{D}(\mathcal{L})$, we take $u, v \in \mathfrak{D}(\mathcal{L})$, integrate by parts and obtain

$$\langle \mathcal{L}u, v\rangle_{L^2(\mathbb{R};\mathbb{C}^n)} = \int_{\mathbb{R}} -v^* u'' + v^* P u' + v^* \widetilde{Q} u \ \mathrm{d}x \tag{2.1}$$

$$= \int_{\mathbb{R}} -v''^* u - v'^* P u + v^*(\widetilde{Q} - P')u \ \mathrm{d}x, \tag{2.2}$$

$$\langle u, \mathcal{L}v\rangle_{L^2(\mathbb{R};\mathbb{C}^n)} = \int_{\mathbb{R}} -v''^* u + v'^* P^* u + v^* \widetilde{Q}^* u \ \mathrm{d}x, \tag{2.3}$$

so $\mathcal{L}$ is symmetric if and only if

$$0 = \langle \mathcal{L}u, v\rangle_{L^2(\mathbb{R};\mathbb{C}^n)} - \langle u, \mathcal{L}v\rangle_{L^2(\mathbb{R};\mathbb{C}^n)} = \int_{\mathbb{R}} v'^*(P + P^*)u + v^*(\widetilde{Q} - P' - \widetilde{Q}^*)u \ \mathrm{d}x, \tag{2.4}$$

for all $u, v \in \mathfrak{D}(\mathcal{L})$. Let now e_i be the ith Euclidean basis vector, and fix any $[a, b] \subseteq \mathbb{R}$, $\varepsilon \in \mathbb{R}$. By setting $v = e_i$ on $[a - 2\varepsilon, b + 2\varepsilon]$, $v = 0$ outside $[a - 3\varepsilon, b + 3\varepsilon]$, we can let u_n be a smooth approximation to $\mathbf{1}_{[a,b]} e_j$, with support contained in $[a - \varepsilon, b + \varepsilon]$. This means that $v'^*(P + P^*)u = 0$ in all of $\mathbb{R}$ and that

$$0 = \int_{\mathbb{R}} v^*(\widetilde{Q} - P' - \widetilde{Q}^*)u_n \ \mathrm{d}x \to \int_a^b \left(\widetilde{Q} - P' - \widetilde{Q}^*\right)_{ij} \ \mathrm{d}x = 0, \tag{2.5}$$

as $n \to \infty$, by Lebesgue's dominated convergence theorem. Since b was arbitrary, it follows that $\left(\widetilde{Q} - P' - \widetilde{Q}^*\right)_{ij} = 0$ a.e. in $[a, b]$ ([6], Theorem 2.14.2), so $\widetilde{Q}^* = \widetilde{Q} - P'$ a.e., since a, i, j were arbitrary. As for the other term in Eq. (2.4), taking $v' = e_i$ and u_n as above implies that also $P^* = -P$.

For $\mathcal{L}$ acting on $\mathfrak{D}_+(\mathcal{L})$ and $u, v \in \mathfrak{D}_+(\mathcal{L})$, the partial integration also produces the two boundary terms

$$\left(v'(0) - \mathfrak{S}v(0)\right)^* u(0) + v^*(0)P(0)u(0) = v^*(0)P(0)u(0). \tag{2.6}$$

If we take u(0) = 0 momentarily, the same argument as in the previous case shows that $\widetilde{Q}^* = \widetilde{Q} - P'$ and $P^* = -P$ in $(0, \infty)$. It follows that

$$v^*(0)P(0)u(0) = 0, \tag{2.7}$$

precisely when $\mathcal{L}$ is symmetric. Since $u(0)$, $v(0)$ are a priori arbitrary, this is the case if and only if $P(0) = 0$.

An analogous argument applied to $\mathcal{L}$ acting on $\mathfrak{D}_{[0,1]}(\mathcal{L})$ concludes the proof. □

Note that, after possibly identifying $\mathcal{L}$ with its Friedrichs extension, we could replace the word "symmetric" in Lemma 1 by "self-adjoint".

For the remainder of this section, we will assume that P is smooth and that both P and Q are compactly supported. By approximation, we can then pass to more general P, Q in appropriate limits [10]. The first step in the argument is to consider a generalized gauge transformation. We let Ψ be the matrix fundamental solution to

$$\Psi' = \frac{1}{2} P\Psi, \quad \Psi(0) = \mathbb{I}. \tag{2.8}$$

Note that Ψ exists globally and is uniquely defined, since P was assumed to be continuous.

Lemma 2 $\Psi(x)$ *has the following properties.*

(i) $\Psi(x)$ *is invertible for all x in its domain of definition.*
(ii) $\Psi(x)$ *is unitary for all x in its domain of definition.*
(iii) $\Psi^{-1}(x)Q(x)\Psi(x)$ *is Hermitian for all x in the domain of definition of* Ψ.
(iv) $\Psi(x)$ *and* $\Psi^{-1}(x)$ *are bounded in norm.*

Proof Each column vector c_i of $\Psi(x)$ satisfies

$$c_i' = \frac{1}{2} P c_i, \quad c_i = e_i, \tag{2.9}$$

where e_i denotes the ith Euclidean basis vector. If $\Psi(x_0)$ were not invertible, then $\{c_i(x_0)\}_{i=1}^n$ would be linearly dependent. By the uniqueness of solutions to Eq. (2.9), it follows that c_i would be linearly dependent for all x, so setting $x = 0$ implies that $\{e_i\}_{i=1}^n$ would be. This is a contradiction, establishing the invertibility of Ψ.

For the second property, we will use the fact that $P^* = -P$. Taking the adjoint of Eq. (2.8), we then obtain

$$\Psi^{*'} = -\frac{1}{2}\Psi^* P, \quad \Psi^*(0) = \mathbb{I}. \tag{2.10}$$

Moreover, differentiation yields

$$\frac{\mathrm{d}}{\mathrm{d}x}\Psi^{-1} = -\Psi^{-1}\Psi'\Psi^{-1} = -\frac{1}{2}\Psi^{-1}P, \quad \Psi^{-1}(0) = \mathbb{I}, \tag{2.11}$$

so we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\Psi^{-1} - \Psi^*\right) = -\frac{1}{2}\left(\Psi^{-1} - \Psi^*\right)P, \quad \left(\Psi^{-1}(0) - \Psi^*(0)\right) = 0. \tag{2.12}$$

By the uniqueness of solutions to linear differential equations, we must then have $\Psi^{-1}(x) = \Psi^*(x)$ for all x where Ψ is defined. Note next that $\left(\Psi^{-1}Q\Psi\right)^* = \Psi^* Q^* \Psi^{*-1} = \Psi^{-1}Q\Psi$, since Q is Hermitian and Ψ unitary.

Lastly, the compact support of P together with Eq. (2.8) imply that Ψ is constant outside some compact interval $[a, b]$, on which the norm of Ψ assumes a maximum. Since also

$$\|\Psi^{-1}(x)\| = \|\Psi^*(x)\| = \|\Psi(x)\|, \tag{2.13}$$

both Ψ and Ψ^{-1} have bounded norm. This establishes the last property and concludes the proof. $\square$

The reason for considering Ψ is that it simplifies the original problem, as seen from the following result. We remark that it is similar to the Liouville normal form of operators in Sturm–Liouville theory [21].

Lemma 3 *$\mathcal{L}$ acting on the spaces in Eqs.* (1.8), (1.9), (1.11) *has the same eigenvalues as the operator*

$$\mathcal{H} := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes \mathbb{I} + \Psi^{-1}(x) Q(x) \Psi(x), \tag{2.14}$$

acting on the spaces

$$\mathfrak{D}(\mathcal{H}) := L^2(\mathbb{R}; \mathbb{C}^n), \tag{2.15}$$

$$\mathfrak{D}_+(\mathcal{H}) := \{\widetilde{u} \in L^2(\mathbb{R}_+; \mathbb{C}^n) : \widetilde{u}'(0) - \mathfrak{S}\widetilde{u}(0) = 0\}, \tag{2.16}$$

$$\mathfrak{D}_{[0,1]}(\mathcal{H}) := \{\widetilde{u} \in L^2([0, 1]; \mathbb{C}^n) : \tag{2.17}$$

$$\widetilde{u}'(0) - \mathfrak{S}_0\widetilde{u}(0) = 0 = \widetilde{u}'(1) - \Psi(1)^{-1}\mathfrak{S}_1\Psi(1)\widetilde{u}(1)\}, \tag{2.18}$$

respectively.

Proof We can write $u(x) = \Psi(x)\widetilde{u}(x)$ and then compute

$$\mathcal{L}u = \Psi\left(-\widetilde{u}'' + \Psi^{-1}\left[-2\Psi' + P\Psi\right]\widetilde{u}' + \Psi^{-1}\left[-\Psi'' + P\Psi' + \widetilde{Q}\Psi\right]\widetilde{u}\right) \tag{2.19}$$

$$= \Psi\left(-\widetilde{u}'' + \Psi^{-1}Q\Psi\widetilde{u}\right), \tag{2.20}$$

meaning that $\mathcal{L}u = \lambda u$ is equivalent to $\mathcal{H}\widetilde{u} = \lambda\widetilde{u}$. Next, the boundedness of Ψ and Ψ^{-1} imply that $u \in L^2$ is equivalent to $\widetilde{u} \in L^2$, for all three settings under consideration. This establishes Eq. (2.15) so it only remains to verify the boundary conditions for $\mathcal{H}$ in Eqs. (2.16, 2.18). In the case of the semi-axis, we have

$$u(0) = \Psi(0)\widetilde{u}(0) = \mathbb{I} \cdot \widetilde{u}(0) = \widetilde{u}(0), \tag{2.21}$$

$$u'(0) = \Psi'(0)\widetilde{u}(0) + \Psi(0)\widetilde{u}'(0) = \frac{1}{2}P(0)\Psi(0)\widetilde{u}(0) + \mathbb{I} \cdot \widetilde{u}'(0) = \widetilde{u}'(0), \tag{2.22}$$

since we insisted on $P(0) = 0$ in this setting. This establishes Eq. (2.16).

Lastly, we use $P(0) = P(1) = 0$ in the setting of the interval to obtain

$$u'(0) = \widetilde{u}'(0), \quad u(0) = \widetilde{u}(0), \tag{2.23}$$

$$u'(1) = \Psi(1)\widetilde{u}'(1), \quad u(1) = \Psi(1)\widetilde{u}(1). \tag{2.24}$$

This is precisely the Robin boundary condition in Eq. (2.18), which concludes the proof. ◻

This result will be sufficient to prove Theorems 1 and 2, so we will in the following only study the operator $\mathcal{H}$ acting on $\mathfrak{D}_{[0,1]}(\mathcal{H})$ in Eq. (2.18). Using the boundedness of Ψ, and the assumptions in Eq. (1.6), we can show that $\mathcal{H}$ is bounded from below precisely as in [18, Sec. 11.3]. The min–max principle then applies, and $\mathcal{H}$ has a ground state ϕ_1 with finite energy $-\lambda_1$. Estimating only the negative eigenvalues, we will assume that $-\lambda_1 < 0$. We next consider the corresponding matrix fundamental solution

$$-M_1''(x) + \Psi(x)^{-1}Q(x)\Psi(x)M_1(x) = -\lambda_1 M_1(x), \tag{2.25}$$
$$M_1'(0) - \mathfrak{S}_0 M_1(0) = 0, \tag{2.26}$$
$$M_1'(1) - \Psi(1)^{-1}\mathfrak{S}_1\Psi(1)M_1(1) = 0. \tag{2.27}$$

The following result is standard [2,5] and will be needed in the remainder of the argument.

Lemma 4 *Take $\phi_1(x)$ as the ground state of $\mathcal{H}$, i.e.*

$$-\phi_1''(x) + \Psi(x)^{-1}Q(x)\Psi(x)\phi_1(x) = -\lambda_1\phi_1(x), \tag{2.28}$$
$$\phi_1'(0) - \mathfrak{S}_0\phi_1(0) = 0, \tag{2.29}$$
$$\phi_1'(1) - \Psi(1)^{-1}\mathfrak{S}_1\Psi(1)\phi_1(1) = 0. \tag{2.30}$$

If $(\phi_1(0), \phi_1'(0))$ is non-trivial, then $\phi_1(x)$ is non-zero for all $x \in [0,1]$ and the ground state has multiplicity at most n.

Proof Assume first that there is an $x_0 < 1$ with $\phi_1(x_0) = 0$. We can then form

$$\widetilde{\phi}(x) = \begin{cases} 0, & x \geq x_0 \\ \phi_1(x), & x < x_0, \end{cases} \tag{2.31}$$

which is weakly differentiable everywhere, so $\widetilde{\phi} \in H^1([0,1];\mathbb{C}^n)$ since ϕ_1 is. We have

$$\begin{aligned}\langle \mathcal{H}\widetilde{\phi}, \widetilde{\phi}\rangle_{L^2([0,1];\mathbb{C}^n)} &= \int_0^1 |\widetilde{\phi}'|^2 \mathrm{d}x + \langle \Psi^{-1}Q\Psi\widetilde{\phi}, \widetilde{\phi}\rangle_{L^2([0,1];\mathbb{C}^n)} \\ &= \int_0^{x_0} |\phi_1'|^2 + \phi_1^*\Psi^{-1}Q\Psi\phi_1 \mathrm{d}x = -\lambda_1 \int_0^{x_0} |\phi_1|^2 \mathrm{d}x \\ &= -\lambda\langle \widetilde{\phi}, \widetilde{\phi}\rangle_{L^2([0,1];\mathbb{C}^n)},\end{aligned}$$

so the min–max principle implies that $\widetilde{\phi}(x)$ is a solution of Eq. (2.28). Since this is a second order ordinary differential equation and $\widetilde{\phi}(x) = 0 = \widetilde{\phi}'(x)$ for $x > x_0$, we must then have $\widetilde{\phi} = 0$ on the whole interval $[0,1]$, contradicting the assumption that

$(\phi_1(0), \phi_1'(0)) \neq 0$. Next, the case $x_0 = 1$ implies that $\phi_1'(x_0) = \phi_1(x_0) = 0$, after using the Robin boundary conditions. This derives a contradiction just as in the case $x_0 < 0$, which concludes the proof of the first part of the lemma.

For the second part, note that if there were $n + 1$ ground state solutions, then these would be linearly dependent when evaluated at a point. By the preceding argument, they would then have to be linearly dependent for all x in $[0, 1]$, meaning that the ground state is of multiplicity at most n. □

By Lemma 4, $M_1(x)$ will then be invertible for any $x \in [0, 1]$, and we can therefore form $F = M_1' M_1^{-1}$, which will be used below.

Lemma 5 *$F(x)$ has the following properties.*

(i) *$F(x)$ satisfies the Ricatti equation $F^2 + F' = \Psi^{-1}(x)Q(x)\Psi(x) + \lambda_1$.*
(ii) *$F(0) = \mathfrak{S}_0$, and $F(1) = \Psi^{-1}(1)\mathfrak{S}_1\Psi(1)$.*
(iii) *F is Hermitian for all x in $[0, 1]$.*

Proof We can calculate

$$\begin{aligned} F' &= M_1'' M_1^{-1} - M_1' M^{-1} M_1' M_1^{-1} = \left(\Psi^{-1} Q \Psi M_1 + \lambda_1 M_1\right) M_1^{-1} - F^2 \\ &= \Psi^{-1} Q \Psi + \lambda_1 - F^2, \end{aligned}$$

showing the Ricatti equation. The second part is shown by using the Robin boundary condition. We have

$$M_1'(0) - \mathfrak{S}_0 M_1(0) = 0, \tag{2.32}$$

so the invertibility of $M(0)$ results in

$$F(0) = M_1'(0) M_1^{-1}(0) = \mathfrak{S}_0. \tag{2.33}$$

$F(1)$ is evaluated analogously.

The third property is obtained by differentiating a Wronskian expression and using the fact that $\Psi^{-1} Q \Psi$ is Hermitian, as shown in Lemma 2. We have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(M_1^* M_1' - M_1^{*'} M_1\right) = M_1^* \left(\Psi^{-1} Q \Psi - \left[\Psi^{-1} Q \Psi\right]^*\right) M_1 = 0,$$

so $M_1^* M_1' - M_1^{*'} M_1 = M_1^* F M_1 - M_1^* F^* M_1 = C$, for $C \in \mathbb{C}^{n\times n}$ constant. Next, for $x = 0$, we have $F(0) = \mathfrak{S}_0$, which is Hermitian. This means that $C = 0$, which results in

$$0 = M_1^* M_1' - M_1^{*'} M_1 = M_1^* F M_1 - M_1^* F^* M_1, \tag{2.34}$$

for all x in $[0, 1]$, i.e. $F^* = F$. This finishes the proof. □

3 Proofs of the main results

Proof of Theorem 1 By the min–max principle, $\{-\lambda_i\}$ will all increase in magnitude if we put the positive part of Q to 0. Since the integral in Lemma 1 is invariant under this change, it suffices to prove the theorem for $Q \leq 0$, which will be assumed in the following. By Lemmas 2 and 3, the eigenvalues of $\mathcal{L}$ are precisely those of the operator $\mathcal{H}$ in Eqs. (2.14)–(2.15), which has a Hermitian potential part. We can therefore apply the Lieb–Thirring inequalities from [13] to conclude that

$$\sum_{i=1}^{\infty} \kappa_i \lambda_i^{3/2} \leq \frac{3}{16} \int_{\mathbb{R}} \operatorname{Tr}\left[\left(\Psi^{-1} Q \Psi\right)_-^2\right] \mathrm{d}x \leq \frac{3}{16} \int_{\mathbb{R}} \operatorname{Tr}\left[\left(\Psi^{-1} Q \Psi\right)^2\right] \mathrm{d}x \tag{3.1}$$

$$= \frac{3}{16} \int_{\mathbb{R}} \operatorname{Tr}\left(Q^2\right) \mathrm{d}x = \frac{3}{16} \int_{\mathbb{R}} \operatorname{Tr}\left(Q_-^2\right) \mathrm{d}x. \tag{3.2}$$

Here, the second to last equality used $\operatorname{Tr}\left[\left(\Psi^{-1} Q \Psi\right)^2\right] = \operatorname{Tr}\left(\Psi^{-1} Q^2 \Psi\right) = \operatorname{Tr}\left(Q^2 \Psi \Psi^{-1}\right) = \operatorname{Tr}\left(Q^2\right)$ and the last equality used $Q^2 = (Q_-)^2$, since $Q \leq 0$.

It only remains to prove the sharpness of the bound. This follows from the sharpness of the original Lieb–Thirring inequalities after setting $P = 0$. The theorem is then proved. $\square$

Proof of Theorem 2 Analogously to the previous proof, we can without loss of generality assume that $Q \leq 0$ and use the results from [5] to conclude that

$$\frac{3}{4}\lambda_1 \operatorname{Tr}\mathfrak{S} + \frac{1}{2}\left(2\kappa_1 - n\right)\lambda_1^{3/2} + \sum_{i=2}^{\infty} \kappa_i \lambda_i^{3/2} \leq \frac{3}{16} \int_{\mathbb{R}} \operatorname{Tr}\left[\left(\Psi^{-1} Q \Psi\right)_-^2\right] \mathrm{d}x + \frac{1}{4}\operatorname{Tr}\mathfrak{S}^3 \tag{3.3}$$

$$\leq \frac{3}{16} \int_{\mathbb{R}} \operatorname{Tr}\left[\left(\Psi^{-1} Q \Psi\right)^2\right] \mathrm{d}x + \frac{1}{4}\operatorname{Tr}\mathfrak{S}^3 = \frac{3}{16} \int_{\mathbb{R}} \operatorname{Tr}\left(Q^2\right) \mathrm{d}x + \frac{1}{4}\operatorname{Tr}\mathfrak{S}^3 \tag{3.4}$$

$$= \frac{3}{16} \int_{\mathbb{R}} \operatorname{Tr}\left(Q_-^2\right) \mathrm{d}x + \frac{1}{4}\operatorname{Tr}\mathfrak{S}^3, \tag{3.5}$$

which is the desired result. $\square$

Proof of Theorem 3 This proof uses the commutation method and shares structure with the proofs in [2,5]. We recall from the preceding proofs that we can without loss of generality assume $Q \leq 0$ and that we aim to bound the negative eigenvalues $\{-\lambda_i\}$ of $\mathcal{H}$. Take now $T := \frac{\mathrm{d}}{\mathrm{d}x} \otimes \mathbb{I} - F$. We can calculate

$$T^*Tu = -u'' + (F - F^*)u' + \left(F^{*\prime} + FF^*\right)u \tag{3.6}$$

$$= -u'' + \left(\Psi^{-1} Q \Psi + \lambda_1\right)u = \mathcal{H}u + \lambda_1 u. \tag{3.7}$$

$$TT^*u = -u'' + (F - F^*)u' + \left(F^*F - F'\right)u = \mathcal{H}u + \lambda_1 u - 2F'u, \tag{3.8}$$

where we used the Ricatti equation and the Hermiticity of F from Lemma 5. Note now that any eigenfunction ψ of TT^* can be written as $T\phi$, for ϕ an eigenfunction of T^*T. Since ϕ satisfies the Robin boundary conditions in Eq. (2.18), ψ will satisfy

$$\psi(0) = T\phi(0) = \phi'(0) - F(0)\phi(0) = \phi'(0) - \mathfrak{S}_0\phi(0) = 0, \tag{3.9}$$

$$\psi(1) = T\phi(1) = \phi'(1) - F(1)\phi(1) = \phi'(1) - \Psi^{-1}(1)\mathfrak{S}_1\Psi(1)\phi(1) = 0, \tag{3.10}$$

where we used Lemma 5. The eigenvectors of TT^* therefore satisfy Dirichlet boundary conditions at $x = 0$. Note also that TT^* and T^*T have the same spectrum, with the zero eigenvalue excluded from the spectrum of TT^*. To see this, assume that $TT^*\psi = 0$. This means that

$$\langle TT^*\psi, \psi\rangle_{L^2([0,1];\mathbb{C}^n)} = \langle T^*\psi, T^*\psi\rangle_{L^2([0,1];\mathbb{C}^n)} = 0, \tag{3.11}$$

so $T^*\psi = 0$ and $\psi'(x) = -F(x)\psi(x)$. This in particular implies that $\psi'(0) = -F(0)\psi(0) = 0$, since ψ satisfies Dirichlet boundary conditions. We therefore obtain $\psi'(0) = \psi(0) = 0$ so $\psi(x) = 0$ for all x, since ψ is an eigenfunction of the second order differential operator in Eq. (3.8).

This procedure therefore reduces the problem of describing the eigenvalues of Eq. (1.4) to bounding the spectrum of

$$TT^*u = \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes \mathbb{I} + \Psi^{-1}Q\Psi - 2F'\right)u, \quad u(0) = u(1) = 0, \tag{3.12}$$

for u in $L^2([0, 1]; \mathbb{C}^n)$. To approach this, we now extend Q and F' by zero to the whole real axis and therefore have $F(x) = F(1)$, for all $x \geq 1$ and $F(y) = F(0)$, for all $y \leq 0$. Note that this results in a self-adjoint operator on the whole real line. We can then use the result in Theorem 1 to obtain

$$\sum_{i=2}^{\infty} \kappa_i \lambda_i^{3/2} \leq \frac{3}{16}\int_0^1 \mathrm{Tr}\left[\left(\Psi^{-1}Q\Psi - 2F'\right)_-^2\right]\mathrm{d}x \tag{3.13}$$

$$\leq \frac{3}{16}\int_0^1 \mathrm{Tr}\left[\left(\Psi^{-1}Q\Psi - 2F'\right)^2\right]\mathrm{d}x. \tag{3.14}$$

This last term can be calculated using the Ricatti equation in Lemma 5. We have

$$\int_0^1 \mathrm{Tr}\left[\left(\Psi^{-1}Q\Psi - 2F'\right)^2\right]\mathrm{d}x = \int_0^1 \mathrm{Tr}\left(\Psi^{-1}Q^2\Psi + 4\left[F'(F' - \Psi^{-1}Q\Psi)\right]\right)\mathrm{d}x \tag{3.15}$$

$$= \int_0^1 \mathrm{Tr}\left(Q^2 + 4\left[F'(\lambda_1 - F^2)\right]\right)\mathrm{d}x = \int_0^1 \mathrm{Tr}\left(Q^2\right)\mathrm{d}x + 4\mathrm{Tr}\left(\left[\lambda_1 F - \frac{1}{3}F^3\right]_0^1\right) \tag{3.16}$$

$$= \int_0^1 \mathrm{Tr}\left(Q^2\right)\mathrm{d}x + 4\lambda_1\mathrm{Tr}\left(\Psi^{-1}(1)\mathfrak{S}_1\Psi(1) - \mathfrak{S}_0\right) - \frac{4}{3}\mathrm{Tr}\left(\Psi^{-1}(1)\mathfrak{S}_1^3\Psi(1) - \mathfrak{S}_0^3\right). \tag{3.17}$$

Inserting this into Eq. (3.13) finally results in

$$\sum_{i=2}^{\infty} \kappa_i \lambda_i^{3/2} \leq \frac{3}{16} \int_0^1 \operatorname{Tr}\left(Q_-^2\right) \mathrm{d}x + \frac{3}{4}\lambda_1 \operatorname{Tr}\left(\mathfrak{S}_1 - \mathfrak{S}_0\right) - \frac{1}{4}\operatorname{Tr}\left(\mathfrak{S}_1^3 - \mathfrak{S}_0^3\right), \quad (3.18)$$

which concludes the proof. □

Proof of Corollaries 1–3 The corollaries can all be proven using standard Aizenman–Lieb arguments from [1]. We will only carry through the details for Corollary 3. Let

$$B(p,q) = \int_0^1 (1-t)^{q-1}\, t^{p-1} \mathrm{d}t \quad (3.19)$$

denote the Beta function and take $\gamma \geq 3/2$. If the negative eigenvalues of $\mathcal{L}$ with potential Q are written as $\{-\lambda_n(Q)\}$, then

$$\frac{3}{4} B(\gamma - 3/2, 2)\lambda_1^{\gamma-1/2}(Q) \operatorname{Tr}\left(\mathfrak{S}_0 - \mathfrak{S}_1\right) + B(\gamma - 3/2, 5/2) \sum_{i=2}^{\infty} \kappa_i \lambda_i^{\gamma}(Q) \quad (3.20)$$

$$= \int_0^{\infty} \left[\frac{3}{4} \operatorname{Tr}\left(\mathfrak{S}_0 - \mathfrak{S}_1\right) \left(\lambda_1(Q) - t\right)_+ + \sum_{i=2}^{\infty} \kappa_i \left(\lambda_i(Q) - t\right)_+^{3/2} \right] t^{\gamma-5/2} \mathrm{d}t \quad (3.21)$$

$$= \int_0^{\infty} \left[\frac{3}{4} \operatorname{Tr}\left(\mathfrak{S}_0 - \mathfrak{S}_1\right) \lambda_1(Q+t) + \sum_{i=2}^{\infty} \kappa_i \lambda_i (Q+t)^{3/2} \right] t^{\gamma-5/2} \mathrm{d}t \quad (3.22)$$

$$\leq L_{3/2,1} \int_0^{\infty} \int_0^1 \operatorname{Tr}\left[(Q+t)_-^2 \right] t^{\gamma-5/2} \mathrm{d}x \mathrm{d}t \quad (3.23)$$

$$= L_{3/2,1} B(\gamma - 3/2, 3) \int_0^{\infty} \int_0^1 \operatorname{Tr}\left(Q_-^{\gamma+1/2}\right) \mathrm{d}x, \quad (3.24)$$

which finishes the proof after noting that $L_{\gamma,1} = L_{3/2,1} \frac{B(\gamma-3/2,3)}{B(\gamma-3/2,5/2)}$ [12]. □

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