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On recursions for coefficients of mock theta functions

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Abstract
We use a generalized Lambert series identity due to the first author to present q-series proofs of recent results of Imamoğlu, Raum and Richter concerning recursive formulas for the coefficients of two 3rd order mock theta functions. Additionally, we discuss an application of this identity to other mock theta functions.

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1 Introduction

In [8], the first author proved the following generalized Lambert series identity:

$$\frac{[a_1,\ldots,a_r]_\infty\,(q)_\infty^2}{[b_1,\ldots,b_s]_\infty} = \frac{[a_1/b_1,\ldots,a_r/b_1]_\infty}{[b_2/b_1,\ldots,b_s/b_1]_\infty}\sum_{k=-\infty}^{\infty}\frac{(-1)^{(s-r)k}q^{(s-r)k(k+1)/2}}{1-b_1q^k}\left(\frac{a_1\cdots a_r b_1^{s-r-1}}{b_2\cdots b_s}\right)^k + \mathrm{idem}(b_1;b_2,\ldots,b_s), \tag{1.1}$$

valid for nonnegative integers $r < s$. Here and throughout, we use the following standard q-hypergeometric notation

$$(a)_n := (a;q)_n := \prod_{k=0}^{n}\left(1-aq^{k-1}\right)$$

$$(a_1,\ldots,a_m)_n := (a_1,\ldots,a_m;q)_n := (a_1)_n\cdots(a_m)_n$$

$$[a_1,\ldots,a_m]_n := \left[a_1,\ldots,a_m;q\right]_n = (a_1,q/a_1,\ldots,a_m,q/a_m)_n$$

valid for $n \in \mathbb{N}\cup\{\infty\}$ and $F(a_1,a_2,\ldots,a_m) + \mathrm{idem}(a_1;a_2,\ldots,a_n)$ to denote the sum

$$\sum_{i=1}^{n} F(a_i,a_2,\ldots,a_{i-1},a_1,a_{i+1},\ldots,a_m)$$

where the ith term of the sum is obtained from the first by interchanging a_1 and a_i.

Identity (1.1) is of interest for several reasons. For example, it generalizes a key identity used by Atkin and Swinnerton-Dyer [4] in their proof of Dyson's conjectures on the rank of a partition. Also, (1.1) played a crucial role in the construction of quasimock theta functions [7] and rank-crank PDE's [10], in proving congruences for the mock theta function $\varphi(q)$ [9], Appell-Lerch sums [12], spt-type functions [17] and partition pairs [18] and in obtaining identities for generating functions of other types of partition pairs [11] and various rank differences [19–22].

Recently, Imamoğlu, Raum and Richter [16] proved some intriguing results concerning recursive formulas for the coefficients of the 3rd order mock theta functions

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n^2} = \sum_{n=0}^{\infty} c\left(f;n\right) q^n$$

and

$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2} = \sum_{n=0}^{\infty} c(\omega;n) q^n.$$

Namely, if $\sigma(n) := \sum_{0<d|n} d$, $\text{sgn}^+(n) := \text{sgn}(n)$ for $n \neq 0$ and $\text{sgn}^+(0) := 1$ and

$$d(N,\tilde{N},t,\tilde{t}) := \text{sgn}^+(N)\,\text{sgn}^+(\tilde{N})(|N+t| - |\tilde{N}+\tilde{t}|),$$

then we have the following (see Theorems 1 and 9 in [16], slightly rewritten).

Theorem 1.1. *For a fixed $n \in \mathbb{Z}^+$ and for any $a, b \in \mathbb{Z}$, set $N := \frac{1}{6}(-3a+b-1)$ and $\tilde{N} := \frac{1}{6}(3a+b-1)$. Then*

$$\sum_{\substack{m\in\mathbb{Z}\\ 3m^2+m\le 2n}} \left(m+\frac{1}{6}\right) c\left(f; n-\frac{3}{2}m^2-\frac{1}{2}m\right) = \frac{4}{3}\sigma(n) - \frac{16}{3}\sigma\left(\frac{n}{2}\right) - 2\sum_{\substack{a,b\in\mathbb{Z}\\ ab=2n\\ 6|3a+b-1}} d\left(N,\tilde{N},\frac{1}{6},\frac{1}{6}\right). \tag{1.2}$$

For a fixed $n \in \mathbb{Z}^+$ and for any $a, b \in \mathbb{Z}$, set $N := \frac{1}{12}(3a-b-2)$ and $\tilde{N} := \frac{1}{12}(3a+b-4)$. Then

$$\sum_{\substack{m\in\mathbb{Z}\\ 3m^2+2m\le n}} \left(m+\frac{1}{3}\right) c\left(f; \frac{n}{2}-\frac{3}{2}m^2-m\right) = -2\sum_{\substack{a,b\in\mathbb{Z}\\ ab=4n+1\\ 12|3a-b-2}} d\left(N,\tilde{N},\frac{1}{6},\frac{1}{3}\right). \tag{1.3}$$

Theorem 1.2. *For a fixed $n \in \mathbb{Z}^+$ and for any $a, b \in \mathbb{Z}$, set $N := \frac{1}{12}(3a-b-4)$ and $\tilde{N} := \frac{1}{12}(3a+b-2)$. Then*

$$\sum_{\substack{m\in\mathbb{Z}\\ 3m^2+2m\le n}} \left(m+\frac{1}{6}\right) c\left(\omega; n-3m^2-m\right) = (-1)^{n+1}\sum_{\substack{a,b\in\mathbb{Z}\\ ab=4n+3\\ 12|3a-b-4}} d\left(N,\tilde{N},\frac{1}{3},\frac{1}{6}\right). \tag{1.4}$$

For a fixed $n \in \mathbb{Z}^+$ and for any $a, b \in \mathbb{Z}$, set $N := \frac{1}{6}(a-3b-2)$ and $\tilde{N} := \frac{1}{6}(a+3b-2)$, and let

$$R_n := \begin{cases} \frac{2}{3}\left(\sigma\left(\frac{n}{4}\right) - \sigma\left(\frac{n}{2}\right)\right), & \text{if } n \text{ is even;} \\ \frac{1}{3}\sigma(n), & \text{if } n \text{ is odd.} \end{cases}$$

Then

$$\sum_{\substack{m\in\mathbb{Z}\\ 3m^2+2m+1\le n}} \left(m+\frac{1}{3}\right) c\left(\frac{\omega(q)+\omega(-q)}{2}; n-3m^2-2m-1\right) = R_n - \sum_{\substack{a,b\in\mathbb{Z}\\ ab=n\\ 12|a-3b-8}} d\left(N,\tilde{N},\frac{1}{3},\frac{1}{3}\right), \tag{1.5}$$

and

$$\sum_{\substack{m\in\mathbb{Z}\\ 3m^2+2m+1\leq n}} \left(m+\frac{1}{3}\right) c\left(\frac{\omega(q)-\omega(-q)}{2}; n-3m^2-2m-1\right) = -R_n + \sum_{\substack{a,b\in\mathbb{Z}\\ ab=n\\ 12|a-3b-2}} d\left(N,\tilde{N},\frac{1}{3},\frac{1}{3}\right). \tag{1.6}$$

The identities (1.2)–(1.6) were proven in [16] by applying holomorphic projection to the tensor product of a vector-valued harmonic weak Maass form of weight 1/2 and vector-valued modular form of weight 3/2 [13, 23]. In Remark 1, ii) of [16], it was stated that these identities "can sometimes also be furnished by Appell sums because these are typically expressible in terms of divisors. However, it is not clear whether Theorem 1 and 9 could be obtained using this idea".

Motivated by this remark, the main purpose of this paper is explain how (1.1) can be used to give a q-series proof of these identities. The idea is to compare the coefficients of q^n in identities which express a modular form times either $f(q)$ or $\omega(q)$ (Appell-Lerch sums) in terms of Lambert series (divisor sums). Specifically, (1.2) and (1.3) will basically follow from (2.8) and (3.11), (1.4) from (3.17), (1.5) and (1.6) from (3.24) and (2.9), respectively.

The paper is organized as follows. In Section 2, we discuss some preliminary q-series identities. In Section 3, we prove Theorems 1.1 and 1.2. In Section 4, we discuss another application of (1.1) to other mock theta functions.

2 Preliminaries

To prove (1.2), (1.5) and (1.6), we need the following two results.

Lemma 2.1. *We have*

$$\frac{(q)_\infty^2}{(-q)_\infty^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{1-xq^n} = 4x\frac{(q)_\infty^2\left(q^2;q^2\right)_\infty^2}{[-x]_\infty\left[x^2;q^2\right]_\infty} + 2\sum_{n=-\infty}^{\infty}\frac{q^{n(3n+1)/2}}{(1+xq^n)^2} + \sum_{n=-\infty}^{\infty}\frac{(6n-1)q^{n(3n+1)/2}}{1+xq^n}, \tag{2.1}$$

$$q\frac{\left(q^2;q^2\right)_\infty^2}{\left(-q;q^2\right)_\infty^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n(n+1)}}{1-xq^{2n}} = \frac{q}{x}\frac{\left(q^2;q^2\right)_\infty^4\left(-q;q^2\right)_\infty^2}{\left[x,-xq,-xq;q^2\right]_\infty} + \sum_{n=-\infty}^{\infty}\frac{q^{3n^2}}{\left(1+xq^{2n-1}\right)^2} + \sum_{n=-\infty}^{\infty}\frac{(3n-1)q^{3n^2}}{1+xq^{2n-1}}. \tag{2.2}$$

Proof. Beginning with the case $r=0$, $s=3$ in (1.1), setting $(b_1,b_2,b_3)=(x,-xq/a,-ax)$ and multiplying by $a\left[-a,1/a^2\right]_\infty$, we find

$$a\frac{(q)_\infty^2\left[-a,1/a^2\right]_\infty}{\left[x,-xq/a,-ax\right]_\infty} = \frac{\left[1/a^2\right]_\infty}{[-1/a]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{1-xq^n} + \sum_{n=-\infty}^{\infty}\frac{q^{n(3n+1)/2}a^{-3n-1}}{1+xq^n/a} - \sum_{n=-\infty}^{\infty}\frac{q^{n(3n+1)/2}a^{3n}}{1+axq^n}. \tag{2.3}$$

Differentiating (2.3) with respect to a and letting $a \to 1$, we find that

$$4\frac{(q)_\infty^4(-q)_\infty^2}{[x,-xq,-x]_\infty} = \frac{(q)_\infty^2}{(-q)_\infty^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{1-xq^n} - \sum_{n=-\infty}^{\infty}\frac{q^{n(3n+1)/2}\left[1+3n(1+xq^n)\right]}{(1+xq^n)^2}$$
$$- \sum_{n=-\infty}^{\infty}\frac{q^{n(3n+1)/2}\left[1+(3n-1)(1+xq^n)\right]}{(1+xq^n)^2}. \tag{2.4}$$

Rearranging and simplifying, we find that (2.4) is equivalent to (2.1).

Again, for the case $r = 0$, $s = 3$ in (1.1), setting $(b_1, b_2, b_3) = (x, -x/a, -ax)$, and multiplying by $\left[-a, 1/a^2\right]_\infty$, we find

$$\frac{(q)_\infty^2\left[-a,1/a^2\right]_\infty}{[x,-x/a,-ax]_\infty} = \frac{\left[1/a^2\right]_\infty}{[-1/a]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n(n+1)/2}}{1-xq^n} - \sum_{n=-\infty}^{\infty}\frac{q^{3n(n+1)/2}a^{-3n-2}}{1+xq^n/a}$$
$$+ \sum_{n=-\infty}^{\infty}\frac{q^{3n(n+1)/2}a^{3n+1}}{1+axq^n}. \tag{2.5}$$

Replacing q by q^2 and setting $a = 1/bq$ in (2.5), we find that

$$\frac{(q^2;q^2)_\infty^2\left[-1/(bq),b^2q^2;q^2\right]_\infty}{\left[x,-xbq,-x/(bq);q^2\right]_\infty} = \frac{\left[b^2q^2;q^2\right]_\infty}{\left[-bq;q^2\right]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n(n+1)}}{1-xq^{2n}} - \sum_{n=-\infty}^{\infty}\frac{q^{3n^2-1}b^{3n-1}}{1+xbq^{2n-1}}$$
$$+ \sum_{n=-\infty}^{\infty}\frac{q^{3n^2-1}b^{-3n-1}}{1+xq^{2n-1}/b}. \tag{2.6}$$

Differentiating (2.6) with respect to b and letting $b \to 1$, we arrive at

$$2\frac{(q^2;q^2)_\infty^4\left[-1/q;q^2\right]_\infty}{\left[x,-xq,-x/q;q^2\right]_\infty} = 2\frac{(q^2;q^2)_\infty^2}{\left[-q;q^2\right]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n(n+1)}}{1-xq^{2n}}$$
$$- \sum_{n=-\infty}^{\infty}\frac{q^{3n^2-1}\left[1+(3n-2)(1+xq^{2n-1})\right]}{(1+xq^{2n-1})^2}$$
$$- \sum_{n=-\infty}^{\infty}\frac{q^{3n^2-1}\left[1+3n\left(1+xq^{2n-1}\right)\right]}{\left(1+xq^{2n-1}\right)^2}. \tag{2.7}$$

Rearranging and simplifying, we find that (2.7) is equivalent to (2.2). □

Corollary 2.2. *We have*

$$\frac{(q)_\infty^3}{(-q)_\infty^2}f(q) = -4\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2} - 16\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2} + 1 + 4\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{q^{n(3n+1)/2}}{(1-q^n)^2}$$
$$+2\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(6n-1)q^{n(3n+1)/2}}{1-q^n}, \tag{2.8}$$

$$q\frac{(q^2;q^2)_\infty^3}{(-q;q^2)_\infty^2}\omega(-q) = \sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1+q^{2n-1})^2} - 2\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{q^{3n^2}}{(1-q^{2n})^2}$$
$$+ \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(3n-1)q^{3n^2}}{1-q^{2n}}. \tag{2.9}$$

Proof. For (2.8), multiply both sides of (2.1) by $(1+x)(1+1/x)$, differentiate twice with respect to x, set $x=-1$ and use

$$f(q)=\frac{2}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{1+q^n}. \tag{2.10}$$

Here, we have set $x=1$ in the identity (see (12.2.5) in [2])

$$\sum_{n=0}^{\infty}\frac{q^{2n^2+2n}}{(xq;q^2)_{n+1}(q/x;q^2)_{n+1}}=\frac{1}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n^2+3n}}{1-xq^{2n+1}}.$$

For (2.9), multiply both sides of (2.2) by $(1+q/x)(1+x/q)$, differentiate twice with respect to x, set $x=-q$ and use

$$\omega(q)=\frac{1}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n^2+3n}}{1-q^{2n+1}}. \tag{2.11}$$

The latter follows from taking $x=-1$ in the identity (see (12.2.3) in [2])

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(xq;q)_n(q/x;q)_n}=\frac{(1-x)}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{1-xq^n}.$$

□

To prove (1.3), (1.5) and (1.6), we also need the following.

Theorem 2.3. *For integers l and j, we have*

$$\begin{aligned}\frac{\left[-q^{l+j},q^{2l};q^{6l}\right]_\infty\left(q^{6l};q^{6l}\right)_\infty^2}{\left[-q^{j},q^{l},-q^{2l+j};q^{6l}\right]_\infty}&\left\{j+2\sum_{m=0}^{\infty}\left(\frac{q^{6lm+l}}{1-q^{6lm+l}}-\frac{q^{6lm+5l}}{1-q^{6lm+5l}}+\frac{q^{6lm+l+j}}{1+q^{6lm+l+j}}\right.\right.\\&\left.\left.-\frac{q^{6lm+5l-j}}{1+q^{6lm+5l-j}}-2\frac{q^{6lm+3l-j}}{1+q^{6lm+3l-j}}+2\frac{q^{6lm+3l+j}}{1+q^{6lm+3l+j}}\right)\right\}\\&=\sum_{n=-\infty}^{\infty}\frac{(6n+2+j)(-1)^n q^{3ln(n+1)+ln+l}}{1+q^{6ln+j+2l}}\\&\quad+\sum_{n=-\infty}^{\infty}\frac{(6n+j)(-1)^n q^{3ln(n+1)-ln}}{1+q^{6ln+j}}\\&\quad+\frac{4\left(q^{2l};q^{2l}\right)_\infty^2}{\left(q^{l};q^{2l}\right)_\infty^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3ln(n+1)+2ln+2l}}{1+q^{6ln+j+3l}}.\end{aligned} \tag{2.12}$$

Proof. First, we need to prove the following:

$$\begin{aligned}\frac{\left[-q^{l+j},q^{2l};q^{6l}\right]_\infty\left(q^{6l};q^{6l}\right)_\infty^2}{\left[-q^{j},q^{l},-q^{2l+j};q^{6l}\right]_\infty}&\sum_{m=0}^{\infty}\left(\frac{q^{6lm+l}}{1-q^{6lm+l}}-\frac{q^{6lm+5l}}{1-q^{6lm+5l}}+\frac{q^{6lm+l+j}}{1+q^{6lm+l+j}}-\frac{q^{6lm+5l-j}}{1+q^{6lm+5l-j}}\right)\\&=\left(2\sum_{m=0}^{\infty}\left(\frac{q^{6lm+l}}{1-q^{6lm+l}}-\frac{q^{6lm+5l}}{1-q^{6lm+5l}}\right)\times\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3ln(n+1)+ln+l}}{1+q^{6ln+j+2l}}\right)\\&\quad+\sum_{n=-\infty}^{\infty}\frac{(n+1)(-1)^n q^{3ln(n+1)+ln+l}}{1+q^{6ln+j+2l}}+\sum_{n=-\infty}^{\infty}\frac{n(-1)^n q^{3ln(n+1)-ln}}{1+q^{6ln+j}}\end{aligned} \tag{2.13}$$

and

$$\frac{\left[-q^{l+j},q^{2l};q^{6l}\right]_\infty \left(q^{6l};q^{6l}\right)_\infty^2}{\left[-q^{j},q^{l},-q^{2l+j};q^{6l}\right]_\infty}\sum_{m=0}^{\infty}\left(\frac{q^{6lm+3l-j}}{1+q^{6lm+3l-j}}-\frac{q^{6lm+3l+j}}{1+q^{6lm+3l+j}}\right)$$
$$=\left(\sum_{m=0}^{\infty}\left(\frac{q^{6lm+l}}{1-q^{6lm+l}}-\frac{q^{6lm+5l}}{1-q^{6lm+5l}}\right)\times\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3ln(n+1)+ln+l}}{1+q^{6ln+j+2l}}\right)$$
$$-\sum_{n=-\infty}^{\infty}\frac{n(-1)^n q^{3ln(n+1)+ln+l}}{1+q^{6ln+j+2l}}-\sum_{n=-\infty}^{\infty}\frac{n(-1)^n q^{3ln(n+1)-ln}}{1+q^{6ln+j}}$$
$$-\frac{\left[q^{-2l},q^{2l};q^{6l}\right]_\infty \left(q^{6l};q^{6l}\right)_\infty^2}{\left[q^{3l},q^{l},q^{-l};q^{6l}\right]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3ln(n+1)+2ln+3l}}{1+q^{6ln+j+3l}}. \tag{2.14}$$

As the proofs of (2.13) and (2.14) are similar, we only give details for (2.14). Replacing q, a_1, a_2, b_1, b_2 and b_3 by $q^{6l}, -q^{j+3l}, -q^{j+l}, -bq^{j+3l}, -q^{j+2l}$ and $-q^j$, respectively, in (1.1) with $r=2, s=3$ and after rearranging, we obtain

$$\frac{\left[-q^{3l+j},-q^{j+l},q^{2l},bq^{3l};q^{6l}\right]_\infty \left(q^{6l};q^{6l}\right)_\infty^2}{\left[-q^{j+2l},-q^{j},q^{3}l,q^{l},-bq^{j+3l};q^{6l}\right]_\infty}=\frac{b\left[1/b,q^{-2l}/b,q^{2l};q^{6l}\right]_\infty}{\left[q^{3}l,q^{l},q^{-l}/b;q^{6l}\right]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}q^{3ln(n+1)+2ln+3l}}{1+bq^{6ln+j+3l}}$$
$$+\frac{\left[q^{l},bq^{3l};q^{6l}\right]_\infty}{\left[q^{3l},bq^{l};q^{6l}\right]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1/b)^n q^{3ln(n+1)+ln+l}}{1+q^{6ln+j+2l}}$$
$$+\sum_{n=-\infty}^{\infty}\frac{(-1/b)^n q^{3ln(n+1)-ln}}{1+q^{6ln+j}}. \tag{2.15}$$

Applying the operator $\left.\frac{d}{db}\right|_{b=1}$ on both sides of (2.15) and simplifying, we obtain (2.14). If we multiply four times (2.14), then subtract from twice (2.13), we find

$$\frac{\left[-q^{l+j},q^{2l};q^{6l}\right]_\infty \left(q^{6l};q^{6l}\right)_\infty^2}{\left[-q^{j},q^{l},-q^{2l+j};q^{6l}\right]_\infty}\sum_{m=0}^{\infty}\left\{2\left(\frac{q^{6lm+l}}{1-q^{6lm+l}}-\frac{q^{6lm+5l}}{1-q^{6lm+5l}}+\frac{q^{6lm+l+j}}{1+q^{6lm+l+j}}-\frac{q^{6lm+5l-j}}{1+q^{6lm+5l-j}}\right)\right.$$
$$\left.-4\left(\frac{q^{6lm+3l-j}}{1+q^{6lm+3l-j}}-\frac{q^{6lm+3l+j}}{1+q^{6lm+3l+j}}\right)\right\}$$
$$=\sum_{n=-\infty}^{\infty}\frac{(6n+2)(-1)^n q^{3ln(n+1)+ln+l}}{1+q^{6ln+j+2l}}+\sum_{n=-\infty}^{\infty}\frac{6n(-1)^n q^{3ln(n+1)-ln}}{1+q^{6ln+j}}$$
$$+\frac{4\left(q^{2l};q^{2l}\right)_\infty^2}{\left(q^{l};q^{2l}\right)_\infty^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3ln(n+1)+2ln+2l}}{1+q^{6ln+j+3l}}. \tag{2.16}$$

Replacing q, a_1, b_1, b_2 by $q^{6l}, -q^{l+j}, -q^j, -q^{j+2l}$ in (1.1) with $r=1, s=2$, we obtain

$$\frac{\left[-q^{l+j},q^{2l};q^{6l}\right]_\infty \left(q^{6l};q^{6l}\right)_\infty^2}{\left[-q^{j},q^{l},-q^{2l+j};q^{6l}\right]_\infty}=\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3ln(n+1)+ln+l}}{1+q^{6ln+j+2l}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3ln(n+1)-ln}}{1+q^{6ln+j}} \tag{2.17}$$

Finally, (2.16) plus j times (2.17) yields (2.12). □

To prove (1.4), we need the following result whose proof is analogous to that of Theorem 2.3 and thus is omitted.

Theorem 2.4. *We have*

$$\frac{\left[q^{2l+j},q^{4l};q^{6l}\right]_\infty \left(q^{6l};q^{6l}\right)_\infty^2}{\left[-q^{j},-q^{2l},-q^{4l+j};q^{6l}\right]_\infty}\left\{j+\sum_{m=0}^{\infty}2\left(\frac{q^{6lm+2l}}{1+q^{6lm+2l}}-\frac{q^{6lm+4l}}{1+q^{6lm+4l}}-\frac{q^{6lm+2l+j}}{1-q^{6lm+2l+j}}\right.\right.$$
$$\left.\left.+\frac{q^{6lm+4l-j}}{1-q^{6lm+4l-j}}\right)-4\left(\frac{q^{6lm+4l+j}}{1-q^{6lm+4l+j}}-\frac{q^{6lm+2l-j}}{1-q^{6lm+2l-j}}\right)\right\}$$
$$=-\sum_{n=-\infty}^{\infty}\frac{(6n+4+j)q^{3ln(n+1)+2ln+2l}}{1+q^{6ln+j+4l}}$$
$$+\sum_{n=-\infty}^{\infty}\frac{(6n+j)q^{3ln(n+1)-2ln}}{1+q^{6ln+j}}$$
$$+\frac{2\left(q^{2l};q^{2l}\right)_\infty^2}{\left(-q^{2l};q^{2l}\right)_\infty^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3ln(n+1)+2ln+2l}}{1-q^{6ln+j+4l}}.$$

3 Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. We begin with the proof of (1.2). First, note that

$$\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}+4\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}mq^{mn}+4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}mq^{2mn},$$
$$\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{q^{n(3n+1)/2}}{(1-q^n)^2}=\sum_{n=1}^{\infty}\frac{q^{n(3n+1)/2}}{(1-q^n)^2}+\sum_{n=1}^{\infty}\frac{q^{n(3n-1)/2+2n}}{(1-q^n)^2}$$
$$=\sum_{n=1}^{\infty}q^{n(3n+1)/2}\sum_{m=1}^{\infty}mq^{nm-n}+\sum_{n=1}^{\infty}q^{n(3n+1)/2}\sum_{m=2}^{\infty}(m-1)q^{nm-n}$$
$$=\sum_{m=1}^{\infty}(2m-1)\sum_{n=1}^{\infty}q^{n(3n+2m-1)/2},$$

and

$$\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(6n-1)q^{n(3n+1)/2}}{1-q^n}=\sum_{n=1}^{\infty}\frac{(6n-1)q^{n(3n+1)/2}}{1-q^n}+\sum_{n=1}^{\infty}\frac{(6n+1)q^{n(3n+1)/2}}{1-q^n}$$
$$=12\sum_{n=1}^{\infty}nq^{n(3n+1)/2}\sum_{m=0}^{\infty}q^{mn}$$
$$=12\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}nq^{n(3n+2m-1)/2}.$$

Therefore, (2.8) is equivalent to

$$\frac{(q)_\infty^3}{(-q)_\infty^2}f(q)=1-4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}nq^{mn}-16\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}nq^{2mn}+4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(6n+2m-1)q^{n(3n-1+2m)/2}. \tag{3.1}$$

By applying (see [5], page 114, Entry 8 (ix))

$$\sum_{n=-\infty}^{\infty}(6n+1)q^{n(3n+1)/2}=\frac{(q)_\infty^3}{(-q)_\infty^2}, \tag{3.2}$$

and extracting the coefficient of q^n on both sides of (3.1), we obtain

$$\sum_{\substack{m\in\mathbb{Z}\\ 3m^2+m\le 2n}} (6m+1)\, c\left(f; n-\frac{3}{2}m^2-\frac{1}{2}m\right) = -4\sigma(n) - 16\sigma\left(\frac{n}{2}\right) + 4 \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=2n\\ b>3a\\ a\not\equiv b \pmod 2}} (3a+b). \tag{3.3}$$

We now examine the right-hand side of (1.2). Observe that both a and b have the same sign, and when $a=3b$, N is not an integer. Hence, by splitting the sum according to the values of a and b in the range $1\le b<3a$, $1<3a<b$, $3a<b\le -1$, and $b<3a<-1$, we find that

$$\begin{aligned}\sum_{\substack{a,b\in\mathbb{Z}\\ ab=2n\\ 6|3a+b-1}} d\left(N,\tilde{N},\frac{1}{6},\frac{1}{6}\right) &= \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=2n\\ 1\le b<3a\\ 6|3a+b-1}} \frac{b}{3} - \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=2n\\ b>3a\\ 6|3a+b-1}} a - \sum_{\substack{a,b\in\mathbb{Z}^-\\ ab=2n\\ 3a<b\le -1\\ 6|3a+b-1}} \frac{b}{3} + \sum_{\substack{a,b\in\mathbb{Z}^-\\ ab=2n\\ b<3a<-1\\ 6|3a+b-1}} a\\ &= \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=2n\\ 1\le b<3a\\ 6|3a+b-1 \text{ or } 6|3a+b+1}} \frac{b}{3} - \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=2n\\ b>3a\\ 6|3a+b-1 \text{ or } 6|3a+b+1}} a\\ &= \frac{1}{3}\sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=2n\\ 1\le b<3a\\ b\not\equiv a \pmod 2}} b - \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=2n\\ b>3a\\ b\not\equiv a \pmod 2}} a,\end{aligned} \tag{3.4}$$

where in the third equality, we note that

$$\sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=2n\\ 1\le b<3a\\ 6|b+3a+3}} b = \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=2n\\ 1\le b<3a\\ 3|b\\ a\not\equiv b \pmod 2}} b = \sum_{\substack{a,b\in\mathbb{Z}^+\\ 3ab=2n\\ 1\le 3b<3a\\ a\not\equiv b \pmod 2}} 3b = \sum_{\substack{a,b\in\mathbb{Z}^+\\ 3ba=2n\\ 1\le 3a<3b\\ b\not\equiv a \pmod 2}} 3a = \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=2n\\ b>3a\\ 3|b\\ a\not\equiv b \pmod 2}} 3a = \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=2n\\ b>3a\\ 6|b+3a+3}} 3a.$$

Thus, by (3.3) and (3.4), it suffices to show that

$$\sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=2n\\ b>3a\\ a\not\equiv b \pmod 2}} b = 3\sigma(n) - 4\sigma\left(\frac{n}{2}\right) - \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=2n\\ 1\le b<3a\\ a\not\equiv b \pmod 2}} b. \tag{3.5}$$

By elementary manipulations, we see that

$$\begin{aligned}3\sigma(n) - 4\sigma\left(\frac{n}{2}\right) &= \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=n}} 3b - \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=2n\\ a\equiv b\equiv 0 \pmod 2}} 2b\\ &= \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=n\\ b\equiv 1 \pmod 2}} b + \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=n\\ a\equiv 1 \pmod 2}} 2b\\ &= \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=2n\\ a\equiv 0 \pmod 2,\ b\equiv 1 \pmod 2}} b + \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=2n\\ a\equiv 1 \pmod 2,\ b\equiv 0 \pmod 2}} b\\ &= \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=2n\\ a\not\equiv b \pmod 2}} b\end{aligned}$$

and thus (3.5) is true. This proves (1.2).

Splitting the sum on the right-hand side of (1.3) in a way similar to (3.4), we find that

$$-6\sum_{\substack{a,b\in\mathbb{Z}\\ ab=4n+1\\ 12|3a-b-2}} d\left(N,\tilde{N},\frac{1}{6},\frac{1}{3}\right) = \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=4n+1\\ 1\le b\le 3a-2\\ 12|3a-b-2 \text{ or } 12|3a-b+2}} b - \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=4n+1\\ b\ge 3a+2\\ 12|3a-b-2 \text{ or } 12|3a-b+2}} 3a \tag{3.6}$$

$$=: S_1(n) - 3S_2(n).$$

The generating function for $S_1(n)$ is given by

$$\sum_{n=0}^{\infty} S_1(n)q^n = \sum_{a=1}^{\infty}\sum_{\substack{b=1\\ 12|3a-b-2\\ 4|ab-1}}^{3a-2} bq^{(ab-1)/4} + \sum_{a=1}^{\infty}\sum_{\substack{b=1\\ 12|3a-b+2\\ 4|ab-1}}^{3a-2} bq^{(ab-1)/4}. \tag{3.7}$$

Note to have $12|3a-b-2$ and $4|ab-1$, we must have a odd and $b \equiv 1 \pmod 6$. Hence we replace $a = 2k+1$ and $b = 6l+1$ in the first sum on the right-hand side of (3.7). Similarly, we replace $a = 2k+1$ and $b = 6l+5$ in the second sum. This leads us to

$$\begin{aligned}
\sum_{n=0}^{\infty} S_1(n)q^n &= \sum_{k=0}^{\infty}\sum_{\substack{l=0\\ 2|k-l}}^{k} (6l+1)q^{3kl+(k+3l)/2} + \sum_{k=0}^{\infty}\sum_{\substack{l=0\\ 2|k-l}}^{k} (6l+5)q^{3kl+(5k+3l)/2+1}\\
&= \sum_{l=0}^{\infty}\sum_{\substack{k=l\\ 2|k-l}}^{\infty} (6l+1)q^{3kl+(k+3l)/2} + \sum_{l=0}^{\infty}\sum_{\substack{k=l\\ 2|k-l}}^{\infty} (6l+5)q^{3kl+(5k+3l)/2+1}\\
&= \sum_{l=0}^{\infty}(6l+1)q^{3l^2+2l}\sum_{\substack{k=0\\ 2|k}}^{\infty} q^{k(6l+1)/2} + \sum_{l=0}^{\infty}(6l+5)q^{3l^2+4l+1}\sum_{\substack{k=0\\ 2|k}}^{\infty} q^{k(6l+5)/2}\\
&= \sum_{l=0}^{\infty}(6l+1)\frac{q^{3l^2+2l}}{1-q^{6l+1}} + \sum_{l=0}^{\infty}(6l+5)\frac{q^{3l^2+4l+1}}{1-q^{6l+5}}\\
&= \sum_{l=-\infty}^{\infty}\frac{(6l+1)q^{3l^2+2l}}{1-q^{6l+1}},
\end{aligned} \tag{3.8}$$

where in the last equality, we replaced l by $-l-1$ in the second sum. Similarly, the generating function for $S_2(n)$ is

$$\sum_{n=0}^{\infty} S_2(n)q^n = \sum_{n=-\infty}^{\infty}\frac{(2n-1)q^{3n^2+2n-2}}{1-q^{6n-3}}. \tag{3.9}$$

From (2.10), applying (see [5], page 115, Entry 8 (x))

$$\sum_{n=-\infty}^{\infty}(3n+1)q^{n(3n+2)} = \frac{\left(q^2;q^2\right)_\infty^3}{\left(-q;q^2\right)_\infty^2}, \tag{3.10}$$

(3.6), (3.8) and (3.9), we see that (1.3) follows upon extracting the coefficient of q^n from both sides of

$$2\frac{\left(q^2;q^2\right)_\infty^2}{\left(-q;q^2\right)_\infty^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)}}{1+q^{2n}} = \sum_{n=-\infty}^{\infty}\frac{(6n+1)q^{3n^2+2n}}{1-q^{6n+1}} - 3\sum_{n=-\infty}^{\infty}\frac{(2n-1)q^{3n^2+2n-2}}{1-q^{6n-3}}. \tag{3.11}$$

Now, to prove (3.11), we first set $l = 1$ and $j = -3$ in Theorem 2.3 to obtain

$$\frac{\left[-q^{-2}, q^2; q^6\right]_\infty (q^6;q^6)_\infty^2}{\left[-q^{-3}, q, -1/q; q^6\right]_\infty} \left\{-1 + 2\sum_{m=0}^{\infty}\left(\frac{q^{6m+1}}{1-q^{6m+1}} - \frac{q^{6(m+1)-1}}{1-q^{6(m+1)-1}} + \frac{q^{6m-2}}{1+q^{6m-2}} - \frac{q^{6(m+1)+2}}{1+q^{6(m+1)+2}}\right)\right\}$$
$$= \sum_{n=-\infty}^{\infty} \frac{(6n-1)(-1)^n q^{3n^2+4n+1}}{1+q^{6n-1}} + \sum_{n=-\infty}^{\infty} \frac{(6n-3)(-1)^n q^{3n^2+2n}}{1+q^{6n-3}} + \frac{4\left[q^{-2}, q^2; q^6\right]_\infty (q^6;q^6)_\infty^2}{\left[q^3, q, 1/q; q^6\right]_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)+2n+3}}{1+q^{6n}}. \tag{3.12}$$

Replacing q, a, b and c by q^6, q^5, q^5 and $-1/q^2$, respectively, in ([8], Corollary 3.2), we obtain

$$-1 + 2\sum_{m=0}^{\infty}\left(\frac{q^{6m+1}}{1-q^{6m+1}} - \frac{q^{6(m+1)-1}}{1-q^{6(m+1)-1}} + \frac{q^{6m-2}}{1+q^{6m-2}} - \frac{q^{6(m+1)+2}}{1+q^{6(m+1)+2}}\right)$$
$$= -\frac{\left[q^{10}, -q^3, -q^3; q^6\right]_\infty (q^6;q^6)_\infty^2}{\left[q, q, -1/q^2, -q^8; q^6\right]_\infty},$$

which together with (3.12) gives

$$-\sum_{n=-\infty}^{\infty} \frac{(6n-1)(-1)^n q^{3n^2+4n+1}}{1+q^{6n-1}} - \sum_{n=-\infty}^{\infty} \frac{(6n-3)(-1)^n q^{3n^2+2n}}{1+q^{6n-3}}$$
$$= -\frac{\left[q, q^2, q^2, -q^2, q^3; q^6\right]_\infty (q^6;q^6)_\infty^4}{\left[-q; q^6\right]_\infty} + 4\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+5)+2}}{1+q^{6n}}. \tag{3.13}$$

By ([6], Eq. (2.1)), we have

$$-\frac{q^2\left[q, q^2, q^2, -q^2, q^3; q^6\right]_\infty (q^6;q^6)_\infty^4}{\left[-q; q^6\right]_\infty} = -2\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+3)+2}}{1+q^{6n}}. \tag{3.14}$$

Substituting (3.14) into (3.13), we obtain

$$-\sum_{n=-\infty}^{\infty} \frac{(6n-1)(-1)^n q^{3n^2+4n+1}}{1+q^{6n-1}} - \sum_{n=-\infty}^{\infty} \frac{(6n-3)(-1)^n q^{3n^2+2n}}{1+q^{6n-3}}$$
$$= 4\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+5)+2}}{1+q^{6n}} - 2\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+3)+2}}{1+q^{6n}}$$
$$= 2\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)+2}}{1+q^{2n}}, \tag{3.15}$$

where the last step follows from the fact that

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)}}{1+q^{6n}} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+5)}}{1+q^{6n}}$$

and

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)}}{1+q^{2n}} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)}(1-q^{2n}+q^{4n})}{1+q^{6n}}.$$

As (3.15) is equivalent to (3.11), this proves (1.3). □

Proof of Theorem 1.2. For (1.4), we first split the sum on the right-hand side in a way similar to (3.4) to obtain

$$6(-1)^{n+1} \sum_{\substack{a,b\in\mathbb{Z} \\ ab=4n+3 \\ 12|3a-b-4}} d\left(N, \tilde{N}, \frac{1}{3}, \frac{1}{6}\right) = \sum_{\substack{a,b\in\mathbb{Z}^+ \\ ab=4n+3 \\ b\geq 3a+2 \\ 12|3a-b-4 \text{ or } 12|3a-b+4}} 3a - \sum_{\substack{a,b\in\mathbb{Z}^+ \\ ab=4n+3 \\ 1\leq b\leq 3a-1 \\ 12|3a-b-4 \text{ or } 12|3a-b+4}} b. \tag{3.16}$$

By (2.11), (3.2) and a calculation similar to (3.8) for the generating function of the right-hand side of (3.16), we see that (1.4) follows extracting the coefficient of q^n from both sides of

$$\frac{(q^2;q^2)_\infty^2}{(-q^2;q^2)_\infty^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+3n}}{1-q^{2n+1}} = -3\sum_{n=-\infty}^{\infty} \frac{(2n-1)q^{n(3n+1)-2}}{1+q^{6n-3}} - \sum_{n=-\infty}^{\infty} \frac{(6n-1)q^{n(3n+1)-1}}{1+q^{6n-1}}. \tag{3.17}$$

To prove (3.17), we set $j=-1, l=1$ in Theorem 2.4 to get

$$\begin{aligned} &\frac{[q,q^4;q^6]_\infty (q^6;q^6)_\infty^2}{[-1/q,-q^2,-q^3;q^6]_\infty} \left\{-1+\sum_{m=0}^{\infty} 2\left(\frac{q^{6m+2}}{1+q^{6m+2}} - \frac{q^{6m+4}}{1+q^{6m+4}} - \frac{q^{6m+1}}{1-q^{6lm+1}} + \frac{q^{6m+5}}{1-q^{6m+5}}\right)\right\} \\ &\qquad = -\sum_{n=-\infty}^{\infty} \frac{(6n+3)q^{3n(n+1)+2n+2}}{1+q^{6n+3}} + \sum_{n=-\infty}^{\infty} \frac{(6n-1)q^{3n(n+1)-2n}}{1+q^{6n-1}} \\ &\qquad\quad + \frac{2(q^2;q^2)_\infty^2}{(-q^2;q^2)_\infty^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)+2n+2}}{1-q^{6n+3}}. \end{aligned} \tag{3.18}$$

Replacing q, a, b and c by q^6, q, q and $-q^2$, respectively, in ([8], Corollary 3.2), we obtain

$$\begin{aligned} &-1+\sum_{m=0}^{\infty} 2\left(\frac{q^{6m+2}}{1+q^{6m+2}} - \frac{q^{6m+4}}{1+q^{6m+4}} - \frac{q^{6m+1}}{1-q^{6lm+1}} + \frac{q^{6m+5}}{1-q^{6m+5}}\right) \\ &\quad = -\frac{[q^2,-q^3,-q^3;q^6]_\infty (q^6;q^6)_\infty^2}{[q,q,-q^2,-q^4;q^6]_\infty}, \end{aligned}$$

which together with (3.18) gives

$$\begin{aligned} &\sum_{n=-\infty}^{\infty} \frac{(6n+3)q^{3n(n+1)+2n+2}}{1+q^{6n+3}} - \sum_{n=-\infty}^{\infty} \frac{(6n-1)q^{3n(n+1)-2n}}{1+q^{6n-1}} \\ &= \frac{2(q^2;q^2)_\infty^2}{(-q^2;q^2)_\infty^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)+2n+2}}{1-q^{6n+3}} + \frac{q[q^2,-q^3,q^4;q^6]_\infty (q^6;q^6)_\infty^4}{[q,-q,-q^2,-q^2,-q^4;q^6]_\infty}. \end{aligned} \tag{3.19}$$

We note that (see [6, Eq. (2.1)])

$$\frac{q[q^2,-q^3,q^4;q^6]_\infty (q^6;q^6)_\infty^4}{[q,-q,-q^2,-q^2,-q^4;q^6]_\infty} = \frac{(q^2;q^2)_\infty^2}{(-q^2;q^2)_\infty^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+3)+1}}{1-q^{6n+3}}. \tag{3.20}$$

Substituting (3.20) into (3.19), we find that

$$\sum_{n=-\infty}^{\infty} \frac{(6n+3)q^{3n(n+1)+2n+2}}{1+q^{6n+3}} - \sum_{n=-\infty}^{\infty} \frac{(6n-1)q^{3n(n+1)-2n}}{1+q^{6n-1}}$$
$$= \frac{2\left(q^2;q^2\right)_\infty^2}{\left(-q^2;q^2\right)_\infty^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)+2n+2}}{1-q^{6n+3}} + \frac{\left(q^2;q^2\right)_\infty^2}{\left(-q^2;q^2\right)_\infty^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)+1}}{1-q^{6n+3}}$$
$$= \frac{\left(q^2;q^2\right)_\infty^2}{\left(-q^2;q^2\right)_\infty^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)+1}}{1-q^{2n+1}}, \tag{3.21}$$

where the last step follows from the fact that

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)+2n+1}}{1-q^{6n+3}} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)+4n+2}}{1-q^{6n+3}}$$

and

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1-q^{2n+1}} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}(1+q^{2n+1}+q^{4n+2})}{1-q^{6n+3}}.$$

By (3.21), we find that, to prove (3.17), it suffices to show that

$$\sum_{n=-\infty}^{\infty} \frac{(6n+3)q^{3n(n+1)+2n+2}}{1+q^{6n+3}} = -\sum_{n=-\infty}^{\infty} \frac{(6n-3)q^{3n^2+n-1}}{1+q^{6n-3}}$$

which is easily checked to be true by replacing n by $-n$ in the sum on the left-hand side. This completes the proof of (3.17) and thus (1.4).

Now, by taking the sum and difference of (1.5) and (1.6), respectively, we obtain the two equivalent formulas

$$\sum_{\substack{m\in\mathbb{Z}\\ 3m^2+2m+1\le n}} \left(m+\frac{1}{3}\right) c\left(\omega(q); n-3m^2-2m-1\right) = \sum_{\substack{a,b\in\mathbb{Z}\\ ab=n\\ 12|a-3b-2}} d\left(N,\tilde{N},\frac{1}{3},\frac{1}{3}\right) - \sum_{\substack{a,b\in\mathbb{Z}\\ ab=n\\ 12|a-3b-8}} d\left(N,\tilde{N},\frac{1}{3},\frac{1}{3}\right), \tag{3.22}$$

and

$$\sum_{\substack{m\in\mathbb{Z}\\ 3m^2+2m+1\le n}} \left(m+\frac{1}{3}\right) c\left(\omega(-q); n-3m^2-2m-1\right) = 2R_n - \sum_{\substack{a,b\in\mathbb{Z}\\ ab=n\\ 12|a-3b-8 \text{ or } 12|a-3b-2}} d\left(N,\tilde{N},\frac{1}{3},\frac{1}{3}\right). \tag{3.23}$$

From (3.10) and noting that

$$3\sum_{\substack{a,b\in\mathbb{Z}\\ ab=n\\ 12|a-3b-8}} d\left(N,\tilde{N},\frac{1}{3},\frac{1}{3}\right) = \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=n\\ 1\le a\le 3b-1\\ 12|a-3b-8 \text{ or } 12|a-3b+8}} a - \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=n\\ a\ge 3b+1\\ 12|a-3b-8 \text{ or } 12|a-3b+8}} 3b$$

and

$$3\sum_{\substack{a,b\in\mathbb{Z}\\ ab=n\\ 12|a-3b-2}} d\left(N,\tilde{N},\frac{1}{3},\frac{1}{3}\right)=\sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=n\\ 1\le a\le 3b-1\\ 12|a-3b-2\text{ or }12|a-3b+2}} a-\sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=n\\ a\ge 3b+1\\ 12|a-3b-2\text{ or }12|a-b+2}} 3b,$$

we see that (3.22) follows from

$$\frac{q\left(q^2;q^2\right)_\infty^3}{\left(-q;q^2\right)_\infty^2}\omega(q)=\sum_{n=-\infty}^{\infty}\frac{(3n+2)q^{3n^2+14n+8}}{1-q^{12n+8}}-3\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{nq^{3n^2+2n}}{1-q^{12n}}$$
$$-\sum_{n=-\infty}^{\infty}\frac{(3n+2)q^{3n^2+8n+4}}{1-q^{12n+8}}+3\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{nq^{3n^2+8n}}{1-q^{12n}} \tag{3.24}$$

or equivalently

$$\frac{q\left(q^2;q^2\right)_\infty^3}{\left(-q;q^2\right)_\infty^2}\omega(q)=-3\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{nq^{3n^2+2n}}{1+q^{6n}}-\sum_{n=-\infty}^{\infty}\frac{(3n+2)q^{3n^2+8n+4}}{1+q^{6n+4}}$$
$$=-3\sum_{n=-\infty}^{\infty}\frac{nq^{3n^2+2n}}{1+q^{6n}}+\sum_{n=-\infty}^{\infty}\frac{(3n+1)q^{3n^2+4n+1}}{1+q^{6n+2}}. \tag{3.25}$$

We now set $l=1, j=0$ and replace q by $-q$ in Theorem 2.3 to obtain

$$2\frac{\left[q,q^2;q^6\right]_\infty\left(q^6;q^6\right)_\infty^2}{\left[-1,-q,-q^2;q^6\right]_\infty}\sum_{m=0}^{\infty}\left(-\frac{q^{6m+1}}{1+q^{6m+1}}+\frac{q^{6m+5}}{1+q^{6m+5}}-\frac{q^{6m+1}}{1-q^{6m+1}}+\frac{q^{6m+5}}{1-q^{6m+5}}\right)$$
$$=-\sum_{n=-\infty}^{\infty}\frac{(6n+2)q^{3n^2+4n+1}}{1+q^{6n+2}}+\sum_{n=-\infty}^{\infty}\frac{6nq^{3n^2+2n}}{1+q^{6n}}+4\frac{\left(q^2;q^2\right)_\infty^2}{\left(-q;q^2\right)_\infty^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n^2+5n+2}}{1-q^{6n+3}}. \tag{3.26}$$

By ([8], Corollary 3.1), we have

$$\sum_{m=0}^{\infty}\left(-\frac{q^{6m+1}}{1+q^{6m+1}}+\frac{q^{6m+5}}{1+q^{6m+5}}-\frac{q^{6m+1}}{1-q^{6m+1}}+\frac{q^{6m+5}}{1-q^{6m+5}}\right)=-4\sum_{j=-\infty}^{\infty}\frac{q^{6j+1}}{1-q^{12j+2}}$$
$$=-\frac{4q\left[q^8;q^{12}\right]_\infty\left(q^{12};q^{12}\right)_\infty^2}{\left[q^2,q^6;q^{12}\right]_\infty}$$

which together with (3.26) gives

$$\sum_{n=-\infty}^{\infty}\frac{(6n+2)q^{3n^2+4n}}{1+q^{6n+2}}-\sum_{n=-\infty}^{\infty}\frac{6nq^{3n^2+2n-1}}{1+q^{6n}}=2\frac{\left(q^2;q^2\right)_\infty^3}{\left(-q;q^2\right)_\infty^2}\times\frac{\left[q^6;q^{18}\right]_\infty^3\left(q^{18};q^{18}\right)_\infty^3}{\left[q^2,q^3;q^6\right]_\infty\left(q^6;q^6\right)_\infty^2}$$
$$+4\frac{\left(q^2;q^2\right)_\infty^2}{\left(-q;q^2\right)_\infty^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(3n+5)+1}}{1-q^{6n+3}}.$$

This together with ([15], Eq. (5.8)) implies

$$2\frac{\left(q^2;q^2\right)_\infty^3}{\left(-q;q^2\right)_\infty^2}\omega(q)=\sum_{n=-\infty}^{\infty}\frac{(6n+2)q^{3n^2+4n}}{1+q^{6n+2}}-\sum_{n=-\infty}^{\infty}\frac{6nq^{3n^2+2n-1}}{1+q^{6n}}$$

which is equivalent to (3.25). Thus, (3.22) is proven.

For (3.23), it suffices to prove

$$\sum_{\substack{m\in\mathbb{Z}\\ 3m^2+2m+1\le n}} (3m+1)\, c\left(\omega(-q); n-3m^2-2m-1\right)$$
$$= 6R_n - \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=n\\ 1\le a\le 3b-1\\ a-3b\equiv 2,4,8,10 \pmod{12}}} a + \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=n\\ a\ge 3b+1\\ a-3b\equiv 2,4,8,10 \pmod{12}}} 3b$$
$$= 6R_n - \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=n\\ 1\le a\le 3b-1\\ a\equiv b \pmod 2}} a + \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=n\\ a\ge 3b+1\\ a\equiv b \pmod 2}} 3b.$$
$$= 6R_n - \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=n\\ 1\le a\le 3b\\ a\equiv b \pmod 2}} a + \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=n\\ a\ge 3b\\ a\equiv b \pmod 2}} 3b, \tag{3.27}$$

where in the second equality, we used the fact that

$$\sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=n\\ 1\le a\le 3b-1\\ a-3b\equiv 0 \pmod 6}} a = \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=n\\ 1\le 3a\le 3b-1\\ a\equiv b \pmod 2}} 3a = \sum_{\substack{a,b\in\mathbb{Z}^+\\ ba=n\\ 1\le 3b\le 3a-1\\ a\equiv b \pmod 2}} 3b = \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=n\\ a\ge 3b+1\\ 3\mid a\\ a\equiv b \pmod 2}} 3b = \sum_{\substack{a,b\in\mathbb{Z}^+\\ ab=n\\ a\ge 3b+1\\ a-3b\equiv 0 \pmod 6}} 3b.$$

Next, we examine the right-hand side of (2.9). Note that

$$\sum_{n=1}^{\infty} \frac{q^{2n-1}}{\left(1+q^{2n-1}\right)^2} - 2\sum_{n=1}^{\infty} \frac{q^{2n}}{\left(1-q^{2n}\right)^2} = \sum_{n=1}^{\infty}\left(\frac{(2n-1)q^{2n-1}}{1-q^{4n-2}} - \frac{4nq^{2n}}{1-q^{2n}} + \frac{2nq^{4n}}{1-q^{4n}}\right)$$
$$= \sum_{n=1}^{\infty} 6\tilde{R}_n q^n,$$

where

$$\tilde{R}_n := \begin{cases} \frac{1}{3}\left(\sigma\left(\frac{n}{4}\right) - 2\sigma\left(\frac{n}{2}\right)\right), & \text{if } n \text{ is even;}\\ \frac{1}{6}\sigma(n), & \text{if } n \text{ is odd,}\end{cases}$$

and

$$\sum_{\substack{n=-\infty\\ n\ne 0}}^{\infty} \frac{q^{3n^2}}{(1-q^{2n})^2} = \sum_{n=1}^{\infty} \frac{q^{3n^2}}{(1-q^{2n})^2} + \sum_{n=1}^{\infty} \frac{q^{3n^2+4n}}{(1-q^{2n})^2}$$
$$= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} q^{3n^2+2mn}(m+1) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} q^{3n^2+2mn}(m-1)$$
$$= \sum_{n=1}^{\infty} q^{3n^2} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} 2mq^{n(3n+2m)}.$$

Similarly,

$$\sum_{\substack{n=-\infty\\ n\ne 0}}^{\infty} \frac{(3n-1)q^{3n^2}}{1-q^{2n}} = \sum_{n=1}^{\infty}(3n-1)q^{3n^2} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} 6nq^{n(3n+2m)}.$$

Hence, identity (2.9) is equivalent to

$$
\begin{aligned}
q\frac{(q^2;q^2)_\infty^3}{(-q;q^2)_\infty^2}\omega(-q) &= \sum_{n=1}^{\infty} 6\tilde{R}_n q^n + \sum_{n=1}^{\infty} 3nq^{3n^2} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(6n+2m)q^{n(3n+2m)} \\
&= \sum_{n=1}^{\infty} 6\tilde{R}_n q^n + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(3n+2m)q^{n(3n+2m)} + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} 3nq^{n(3n+2m)} \\
&= \sum_{n=1}^{\infty} 6\tilde{R}_n q^n + \sum_{n=1}^{\infty} q^n \sum_{\substack{b,m\in\mathbb{Z}^+ \\ b(3b+2m)=n}} (3b+2m) + \sum_{n=1}^{\infty} q^n \sum_{\substack{b,m\in\mathbb{Z}^+ \\ b(3b+2m-2)=n}} 3b \\
&= \sum_{n=1}^{\infty} 6\tilde{R}_n q^n + \sum_{n=1}^{\infty} q^n \sum_{\substack{a,b\in\mathbb{Z}^+ \\ ab=n \\ a>3b \\ a\equiv b \pmod 2}} a + \sum_{n=1}^{\infty} q^n \sum_{\substack{a,b\in\mathbb{Z}^+ \\ ab=n \\ a>3b-2 \\ a\equiv b \pmod 2}} 3b. \qquad (3.28)
\end{aligned}
$$

Applying (3.10) while extracting the coefficient of q^n from both sides of (3.28), we obtain

$$
\sum_{\substack{m\in\mathbb{Z} \\ 3m^2+2m+1\le n}} (3m+1)\, c\left(\omega(-q); n-3m^2-2m-1\right) = 6\tilde{R}_n + \sum_{\substack{a,b\in\mathbb{Z}^+ \\ ab=n \\ a>3b \\ a\equiv b \pmod 2}} a + \sum_{\substack{a,b\in\mathbb{Z}^+ \\ ab=n \\ a>3b-2 \\ a\equiv b \pmod 2}} 3b.
$$

Note that

$$
\begin{aligned}
\sum_{n=1}^{\infty} 6(R_n-\tilde{R}_n)q^n &= \sum_{n=1}^{\infty} 2\sigma(n/2)q^{2n} + \sum_{n=1}^{\infty}\sigma(2n-1)q^{2n-1} \\
&= \sum_{n=1}^{\infty}\frac{2nq^{4n}}{1-q^{4n}} + \sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{4n-2}} \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} 2nq^{4mn} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(2n-1)q^{(2n-1)(2m-1)} \\
&= \sum_{n=1}^{\infty} q^n \sum_{\substack{a,b\in\mathbb{Z}^+ \\ 4ab=n}} 2a + \sum_{n=1}^{\infty} q^n \sum_{\substack{a,b\in\mathbb{Z}^+ \\ (2a-1)(2b-1)=n}} (2a-1) \\
&= \sum_{n=1}^{\infty} q^n \sum_{\substack{a,b\in\mathbb{Z}^+ \\ ab=n \\ a\equiv b \pmod 2}} a.
\end{aligned}
$$

Thus, (3.27) follows upon observing that

$$
\sum_{\substack{a,b\in\mathbb{Z}^+ \\ ab=n \\ a\equiv b \pmod 2}} a - \sum_{\substack{a,b\in\mathbb{Z}^+ \\ ab=n \\ 1\le a\le 3b \\ a\equiv b \pmod 2}} a + \sum_{\substack{a,b\in\mathbb{Z}^+ \\ ab=n \\ a\ge 3b \\ a\equiv b \pmod 2}} 3b = \sum_{\substack{a,b\in\mathbb{Z}^+ \\ ab=n \\ a>3b \\ a\equiv b \pmod 2}} a + \sum_{\substack{a,b\in\mathbb{Z}^+ \\ ab=n \\ a>3b-2 \\ a\equiv b \pmod 2}} 3b.
$$

This proves (3.23). Adding (3.22) and (3.23), then dividing by two yields (1.5) while subtracting (3.23) from (3.22), then dividing by two implies (1.6). □

4 Other applications of (1.1)

It is worth noting that (1.1) can also be used to obtain other identities involving mock theta functions. For example, in [3], Andrews, Rhoades and Zwegers consider the automorphic properties of the q-series

$$\nu_2(q) := \frac{1}{(q)_\infty^3}\left(\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{(-1)^{n+1}nq^{\frac{n(n+1)}{2}}}{1-q^n}-\frac{1}{4}-2\sum_{n=1}^{\infty}\frac{q^n}{(1+q^n)^2}\right),$$

which is related to the generating function for the number of concave compositions of n [1]. In particular, to show that $\nu_2(q)$ is a mock theta function, they require the following key identity (see Theorem 1.3 in [3])

$$\tilde{F}(q) := \frac{1}{(q)_\infty(-q)_\infty^2}\sum_{n=-\infty}^{\infty}\frac{q^{\frac{n(n+1)}{2}}}{1+q^n} = -2\nu_2(q). \tag{4.1}$$

We now prove a generalization of (4.1).

Theorem 4.1. *We have*

$$\frac{(q)_\infty^2}{[1/b]_\infty}\sum_{k=-\infty}^{\infty}\frac{(-b)^k q^{\frac{k(k+1)}{2}}}{1-bq^k} = -\sum_{n=0}^{\infty}\left(\frac{q^n b}{(1-bq^n)^2}+\frac{q^{n+1}/b}{(1-q^{n+1}/b)^2}\right)+\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{(-1)^n nq^{n(n+1)/2}}{1-q^n}. \tag{4.2}$$

Proof of Theorem 4.1. Setting $r = 1$, $s = 2$ and replacing a_1 by bb_1 in (1.1), after rearranging, we obtain

$$\frac{[bb_1, b_1/b]_\infty\,(q)_\infty^2}{[b, b_1, b_1]_\infty} = \sum_{k=-\infty}^{\infty}\frac{(-b)^k q^{\frac{k(k+1)}{2}}}{1-bq^k} - \frac{[b]_\infty}{b[b_1]_\infty}\sum_{k=-\infty}^{\infty}\frac{(-b_1)^{k+1}q^{\frac{k(k+1)}{2}}}{1-b_1q^k}. \tag{4.3}$$

Multiplying by $(1-b_1)^2$ and applying the operator $\frac{d^2}{d^2b_1}\Big|_{b_1=1}$, we obtain

$$\begin{aligned}&\frac{(qb, q/b)_\infty}{(1-b)(q)_\infty^2}+\frac{[1/b]_\infty}{(q)_\infty^2}\sum_{n=1}^{\infty}\left(\frac{2q^n}{(1-q^n)^2}-\frac{q^n b}{(1-bq^n)^2}-\frac{q^n/b}{(1-q^n/b)^2}\right)\\&=\sum_{k=-\infty}^{\infty}\frac{(-b)^k q^{\frac{k(k+1)}{2}}}{1-bq^k}+\frac{[b]_\infty}{b(q)_\infty^2}\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\left\{\frac{(-1)^n q^{n(n+1)/2}(1+q^n)}{(1-q^n)^2}+\frac{(-1)^n nq^{n(n+1)/2}}{1-q^n}\right\}.\end{aligned} \tag{4.4}$$

Letting $a, c, d, e \to 1$ and $b \to q$ in a limiting case of Watson's ${}_8\phi_7$ transformation, [5, Eq. (7.2), p. 16]

$$\sum_{n=0}^{\infty}\frac{(aq/bc, d, e; q)_n\left(\frac{aq}{de}\right)^n}{(q, aq/b, aq/c; q)_n} = \frac{(aq/d, aq/e; q)_\infty}{(aq, aq/de; q)_\infty}\sum_{n=0}^{\infty}\frac{(a, b, c, d, e; q)_n\left(1-aq^{2n}\right)\left(-a^2\right)^n q^{n(n+3)/2}}{(q, aq/b, aq/c, aq/d, aq/e; q)_n(1-a)(bcde)^n},$$

we find that

$$\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2} = -\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(n+1)/2}(1+q^n)}{(1-q^n)^2}. \tag{4.5}$$

Substituting (4.5) into (4.4) and rearranging, we obtain

$$\frac{-1}{b(1-b)^2} - \sum_{n=1}^{\infty}\left(\frac{q^n b}{(1-bq^n)^2} + \frac{q^n/b}{(1-q^n/b)^2}\right) = \frac{(q)_\infty^2}{[1/b]_\infty}\sum_{k=-\infty}^{\infty}\frac{(-b)^k q^{\frac{k(k+1)}{2}}}{1-bq^k} - \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-1)^n n q^{n(n+1)/2}}{1-q^n}$$

which implies (4.2). □

Multiplying by $\frac{1-1/b}{(q;q)_\infty^3}$ on both sides of (4.2) and setting $b=-1$, we obtain (4.1). Using Theorem 4.1, one can also show

$$q(q^4;q^4)_\infty^3 B(q) = \sum_{n=0}^{\infty}\left(\frac{q^{4n+1}}{(1-q^{4n+1})^2} + \frac{q^{4n+3}}{(1-q^{4n+3})^2}\right) - \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-1)^n n q^{2n(n+1)}}{1-q^{4n}} \tag{4.6}$$

where

$$B(q) := \sum_{n=0}^{\infty}\frac{q^n\left(-q;q^2\right)_n}{(q;q^2)_{n+1}}$$

is a 2nd order mock theta function (see [14]). To see this, replace q and b by q^4 and q, respectively in (4.2) to obtain

$$\frac{\left(q^4;q^4\right)_\infty^2}{\left[1/q;q^4\right]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(2n+3)}}{1-q^{4n+1}} = -\sum_{n=0}^{\infty}\left(\frac{q^{4n+1}}{(1-q^{4n+1})^2} + \frac{q^{4n+3}}{(1-q^{4n+3})^2}\right) + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-1)^n n q^{2n(n+1)}}{1-q^{4n}}.$$

By ([15], (3.2a), (3.2b), Eq. (5.2)), we have

$$B(q) = \frac{1}{(q,q^3,q^4;q^4)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(2n+3)}}{1-q^{4n+1}}$$

and thus (4.6) follows.

Extracting the coefficient of q^n on both sides of (4.6), we obtain the following corollary.

Corollary 4.2. *For a fixed positive integer n, we have*

$$\sum_{\substack{m\in\mathbb{Z}^+\\ 2m^2+m+1\leq n}}(-1)^m(2m+1)c(B,n-2m^2-2m-1) = \sum_{\substack{0<d|n\\ \frac{n}{d}\ \ odd}} d - \sum_{\substack{a,b\in\mathbb{Z}^+\\ 1\leq a<b\\ 2ab=n\\ a\not\equiv b\pmod 2}}(-1)^a a.$$

Similar results exist, for example, for the mock theta functions $\psi(q)$, $\rho(q)$ and $\lambda(q)$ of order 6 and $V_0(q)$ of order 8 as they can be written in terms of Appell-Lerch series (see Section 5 of [15]).

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