State-feedback control of non-linear systems

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A design method for state-feedback controllers for single-input non-linear systems is proposed. The method makes use of the transformations of the non-linear system into 'controllable-like' canonical forms. The resulting non-linear state feedback is designed in such a way that the eigenvalues of the linearized closed-loop model are invariant with respect to any constant operating point. The method constitutes an alternative approach to the design methodology recently proposed by Baumann and Rugh. Also a review of different transformation methods for non-linear systems is presented. An example and simulation results of different control strategies are provided to illustrate the design technique.

1. Introduction

A common method of controlling a non-linear system involves linearizing the system about an operating point, and then using linear feedback control methods to design the controller. This approach is successful when the operation of the system is restricted to a small region about this chosen operating point. When a wider range of operation is desired, this technique may fail. In such a case an alternative method is often used. This alternative method involves linearizing the system about a series of operating points. This approach, referred to as gain scheduling, involves varying the controller parameters in a way that pieces together several linear controllers.

Recently, a number of papers that suggest new approaches to control of non-linear systems have appeared. First, Sommer (1980) proposed a method that transforms a class of non-linear time-varying systems into a phase-variable canonical form. Subsequently, Su (1982) and Hunt *et al.* (1983) developed a procedure for global linearization. This procedure consists of transforming a non-linear system into a linear one in the whole state space. Dualization of these results allowed Krener and Respondek (1985) to devise a new design method for asymptotic observers (with linearizable error dynamics) for a class of nonlinear systems. The method of Hunt *et al.* (1983) was generalized by Reboulet and Champetier (1984). They proposed a technique for the transformation of a non-linear model into a linear one which is independent of the operating point. A similar idea can be traced in the approach taken by Baumann and Rugh (1984). Their method consists of finding a family of linearizations of the non-linear system, parameterized by constant operating points. Then a non-linear state feedback is computed such that the eigenvalues of the linearized closed-loop system are invariant for all closed-loop-constant operating points. This technique is an interesting alternative to the gain-scheduling method. The non-linear feedback obtained by using the Baumann and Rugh's algorithm provides scheduling of its own linearizations according to the closed-loop operating points.

2. System description and problem statement

The class of systems considered in this paper can be described by the following equation:

$$\dot{x}(t) = a(x(t)) + b(x(t))u(t) \tag{1}$$

where $a \in \mathbb{R}^n$ is analytic in a neighbourhood of the origin, with $a(0) = 0$, $b \in \mathbb{R}^n$ is also analytic in a neighbourhood of the origin. In other words, we assume that a and b are C^∞ vector fields on an open set in $\mathbb{R}^n$ containing the origin $a(0) = 0$.

The problem of interest to us is finding sufficient conditions on a and b so that there exists a C^∞ transformation $x^* = T(x): \mathbb{R}^n \to \mathbb{R}^n$ such that system (1) can be transformed into the *global non-linear controller form:*

$$\frac{d}{dt}\begin{bmatrix} x_1^* \\ \vdots \\ x_{n-1}^* \\ x_n^* \end{bmatrix} = \begin{bmatrix} x_2^* \\ \vdots \\ x_n^* \\ f(x^*) \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \tag{2}$$

We will also investigate other types of quasi-canonical forms if a reduction to form (2) is impossible.

The next issue we discuss in the paper is an application of the transformation of the non-linear system to the design of a state-feedback controller. In particular we are seeking a feedback law with the property that the eigenvalues of the linearized closed-loop system are placed at prescribed values and are invariant for all closed-loop-constant operating points. An example and the computer simulation of different control strategies are provided to illustrate the design technique.

3. Transformations of non-linear systems into quasicontroller canonical forms

It is well known that if a linear system, represented by a triple $\{A, b, c\}$ is completely controllable, then it can be reduced via a non-singular transformation to an equivalent controllable form (Kailath 1980). For a number of reasons, such as ease of determining a control law, it is often advantageous to work with such an equivalent system rather than with the original one.

The purpose of this section is to present constructive procedures for reducing non-linear systems into 'equivalent' forms which resemble controllable canonical forms known from linear-systems theory.

Specifically, we review three different approaches to simplification of non-linear systems. We begin by considering the approach inspired by Sommer (1980). In this discussion, notation proposed by Su (1982) and Hunt *et al.* (1983) and their results are utilized.

3.1. *Method* 1

Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function defined in a region Ω of $\mathbb{R}^n$. We may picture f as a vector field.

For two vector fields f and g on $\mathbb{R}^n$, the *Lie bracket* $[f, g]$ is a vector field defined by

$$[f, g] \triangleq \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$$

where $\partial f/\partial x$ *and* $\partial g/\partial x$ are the jacobians of f and g respectively. The Lie bracket is also denoted as

$$[f, g] = (\mathrm{ad}^1 f, g)$$

We define

$$(\mathrm{ad}^k f, g) = [f, (\mathrm{ad}^{k-1} f, g)]$$

where

$$(\mathrm{ad}^0 f, g) = g$$

Thus

$$(\mathrm{ad}^2 f, g) = [f, [f, g]]$$

For a *scalar* field h and a *vector* field $f = (f_1, \ldots, f_n)^{\mathrm{T}}$ the *Lie derivative* of h with respect to f is

$$\langle dh, f \rangle = \frac{\partial h}{\partial x_1} f_1 + \ldots + \frac{\partial h}{\partial x_n} f_n$$

Note that $\langle dh, f \rangle = \dfrac{\partial h}{\partial x} \cdot f = \nabla h \cdot f$. In further considerations, the following identity (Su 1982), which can be proven by verification, will be useful:

$$\langle dT, [f, g] \rangle = \langle d\langle dT, g \rangle, f \rangle - \langle d\langle dT, f \rangle, g \rangle$$

Let

$$x^* = T(x), \quad x \in \Omega \subseteq \mathbb{R}^n$$

be a one-to-one mapping and let

$$x = \tilde{T}(x^*)$$

be the inverse mapping. Differentiating x^* with respect to t gives

$$\frac{d}{dt} x^* = \frac{\partial T}{\partial x} \dot{x} = \frac{\partial T}{\partial x} (a(x(t)) + b(x(t))u(t)) \tag{3}$$

Comparison of (3) with (2) gives

$$\frac{\partial T}{\partial x} a(x) = \begin{bmatrix} x_2^* \\ x_3^* \\ \vdots \\ x_n^* \\ f(x^*) \end{bmatrix} = \begin{bmatrix} T_2 \\ T_3 \\ \vdots \\ T_n \\ f(x^*) \end{bmatrix} \tag{4}$$

and

$$\frac{\partial T}{\partial x} b(x) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{5}$$

From (4) we conclude that

$$\frac{\partial T_i}{\partial x} a(x) = T_{i+1}, \quad i = 1, \ldots, n-1$$

or equivalently,

$$\langle dT_i, a \rangle = T_{i+1}, \quad i = 1, \ldots, n-1.$$

Thus, for example,

$$T_2 = \langle dT_1, a \rangle$$

and

$$T_3 = \langle dT_2, a \rangle = \langle d\langle dT_1, a \rangle, a \rangle$$

etc. Therefore the desired transformation has the form

$$T = \begin{bmatrix} T_1 \\ \langle dT_1, a \rangle \\ \langle d\langle dT_1, a \rangle, a \rangle \\ \vdots \end{bmatrix} \tag{6}$$

Thus the problem of constructing the desired T is reduced to finding appropriate T_1.

In order to specify T_1, let us consider (5). It can be written in the following form:

$$\frac{\partial T_i}{\partial x} b(x) = 0, \quad i = 1, \ldots, n-1$$

$$\frac{\partial T_n}{\partial x} b(x) = 1$$

or

$$\langle dT_i, b \rangle = 0, \quad i = 1, \ldots, n-1$$

$$\langle dT_n, b \rangle = 1$$

Thus we have

$$\frac{\partial T_1}{\partial x} b = \frac{\partial T_1}{\partial x} (\mathrm{ad}^0\ a,\ b) = 0$$

Next, consider the equation

$$\langle dT_2, b \rangle = 0$$

Observe that

$$T_2 = \langle dT_1, a \rangle$$

Hence

$$\langle dT_2, b \rangle = \langle d\langle dT_1, a \rangle, b \rangle = 0$$

After some manipulations, we see that

$$\langle dT_2, b \rangle = -\langle dT_1, (\mathrm{ad}^1\ a,\ b) \rangle = -\frac{\partial T_1}{\partial x} (\mathrm{ad}^1\ a,\ b) = 0$$

and

$$\langle dT_3, b\rangle = +\frac{\partial T_1}{\partial x}(\mathrm{ad}^2\ a, b) = 0$$

$$\vdots$$

$$\langle dT_{n-1}, b\rangle = (-1)^{n-2}\frac{\partial T_1}{\partial x}(\mathrm{ad}^{n-2}\ a, b) = 0$$

$$\langle dT_n, b\rangle = (-1)^{n-1}\frac{\partial T_1}{\partial x}(\mathrm{ad}^{n-1}\ a, b) = 1$$

In the matrix form the above set of equations can be represented as follows:

$$\frac{\partial T_1}{\partial x}[b \quad (\mathrm{ad}^1\ a, b) \quad (\mathrm{ad}^2\ a, b) \quad \ldots \quad (\mathrm{ad}^{n-1}\ a, b)] = [0 \quad 0 \quad \ldots \quad 0 \quad 1]$$

The matrix

$$\mathscr{C} \triangleq [b \quad (\mathrm{ad}^1\ a, b) \quad (\mathrm{ad}^2\ a, b) \quad \ldots \quad (\mathrm{ad}^{n-1}\ a, b)]$$

will be designated as the *controllability matrix* of system (1). If $\mathscr{C}^{-1}$ exists then the last row of $\mathscr{C}^{-1}$, denoted by q, satisfies the equation

$$q\mathscr{C} = [0 \quad 0 \quad \ldots \quad 0 \quad 1]$$

Therefore we may make the following association:

$$\frac{\partial T_1}{\partial x} = q(x) \tag{7}$$

A vector field $q(x)$ for which there is a real-valued function T_1 such that (7) holds is called a *conservative field* or *gradient field*. In such a case T_1 is called the *field potential* of $q(x)$.

It is known that if $q:\mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable gradient field then the jacobian matrix of q is symmetric. However, the converse is false. But the following is true. Let Ω be an open coordinate rectangle in $\mathbb{R}^n$ and let $q(x)$ be a continuously differentiable vector field on Ω. If $q^1(x)$, the jacobian matrix of $q(x)$, is symmetric on Ω, then $q(x)$ is a gradient field.

In order to illustrate this algorithm for reducing a non-linear system into the non-linear global controller form, consider the following example.

Example 1

Consider an inverted pendulum with DC motor control as illustrated in Fig. 1. Assume that the DC motor is armature-controlled, and that motor inertia is negligible when compared with the pendulum inertia. The DC motor may be modelled as shown in Fig. 2. The torque delivered by the motor is

$$T_m = K_m I$$

Thus the torque applied to the pendulum is

$$T_p = 10T_m = 10K_m I$$

and

$$V = L\dot{I} + RI + K_b 10\dot{\theta}$$

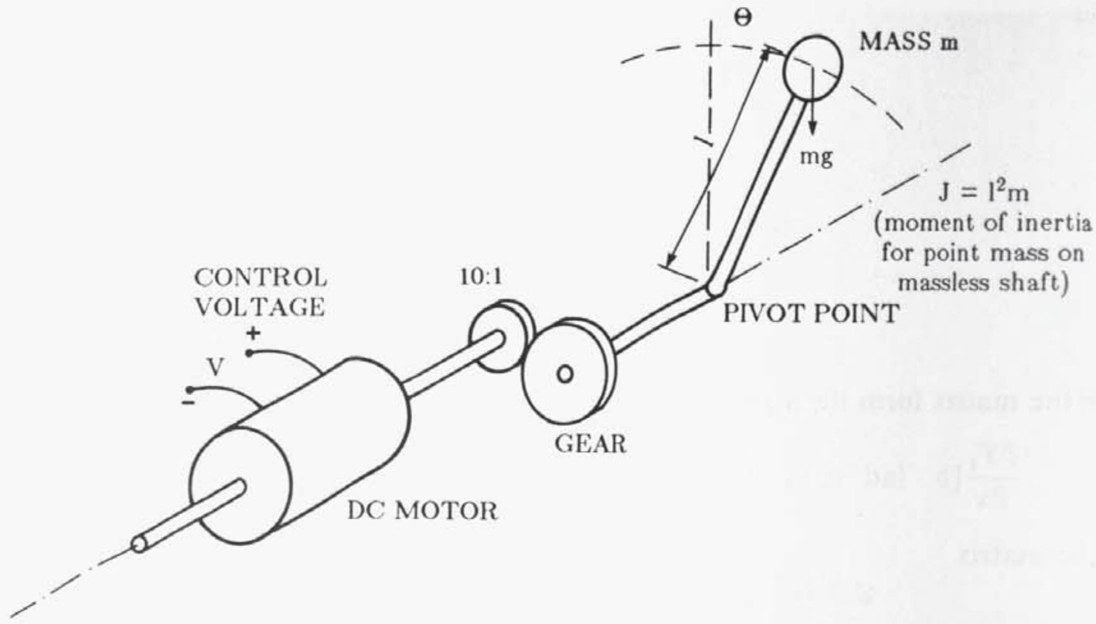


Figure 1. Inverted pendulum controlled by a DC motor.

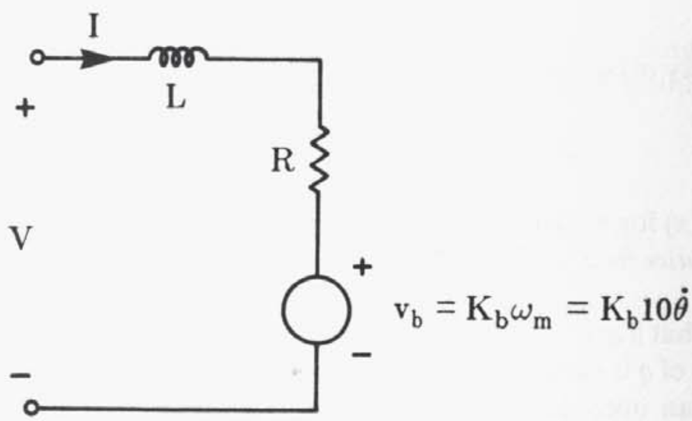


Figure 2. Model of an armature-controlled DC motor.

The pendulum kinematics can be described by the equation

$$T_p = -l^2 m\ddot{\theta} + lmg \sin\theta$$

Now, if we introduce the state variables

$$x_1 = \theta, \quad x_2 = \dot{\theta} = \omega, \quad x_3 = I$$

we arrive at the equations that describe our system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ \dfrac{g}{l}\sin x_1 + \dfrac{10K_m}{l^2 m}x_3 \\ -\dfrac{10K_b}{L}x_2 - \dfrac{R}{L}x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{L} \end{bmatrix} u$$

Let

$$K_1 = \frac{g}{l}, \quad K_2 = \frac{10K_m}{l^2 m}, \quad K_3 - \frac{10K_b}{L}, \quad K_4 = -\frac{R}{L}, \quad K_5 = \frac{1}{L}$$

Then, in terms of the above notation, the system equations take the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ K_1 \sin x_1 + K_2 x_3 \\ K_3 x_2 + K_4 x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ K_5 \end{bmatrix} u \tag{8}$$

The controllability matrix is

$$\begin{aligned} \mathscr{C} &= [b \quad [a, b] \quad [a, [a, b]]] = [b \quad (\mathrm{ad}^1\, a, b) \quad (\mathrm{ad}^2\, a, b)] \\ &= \begin{bmatrix} 0 & 0 & K_2 K_5 \\ 0 & -K_2 K_5 & K_2 K_4 K_5 \\ K_5 & -K_4 K_5 & K_2 K_3 K_5 + K_4^2 K_5 \end{bmatrix} \end{aligned}$$

The last row of $\mathscr{C}^{-1}$ is

$$q = \begin{bmatrix} \frac{1}{K_2 K_5} & 0 & 0 \end{bmatrix}$$

It is required that

$$\frac{\partial T_1}{\partial x} = q$$

Hence a natural choice of T_1 is

$$T_1 = \frac{1}{K_2 K_5} x_3$$

Therefore

$$T_2 = \langle dT_1, a \rangle = \frac{1}{K_2 K_5} x_2$$

and

$$T_3 = \langle dT_2, a \rangle = \frac{1}{K_2 K_5} (K_1 \sin x_1 + K_2 x_3)$$

Thus the transformation has the form

$$T = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \frac{1}{K_2 K_5} \begin{bmatrix} x_1 \\ x_2 \\ K_1 \sin x_1 + K_2 x_3 \end{bmatrix}$$

In the new coordinates, our system is represented by the following equations:

$$\begin{bmatrix} \dot{x}_1^* \\ \dot{x}_2^* \\ \dot{x}_3^* \end{bmatrix} = \begin{bmatrix} x_2^* \\ x_3^* \\ (K_1 \cos (K_2 K_5 x_1^*) + K_2 K_3) x_2^* + K_4 x_3^* - \dfrac{K_1 K_4}{K_2 K_5} \sin (K_2 K_5 x_1^*) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

It is worthwhile to note that a transformation constructed as such preserves the eigenvalues of the linearized system about the origin (Su *et al.* 1983). In our example the linearized model (about the origin) of the system in the original coordinates has the form

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ K_1 & 0 & K_2 \\ 0 & K_3 & K_4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ K_5 \end{bmatrix} u \tag{9}$$

It is easy to verify that the characteristic equation of the linearized model (9) is

$$s^3 - s^2 K_4 - (K_1 + K_2 K_3)s + K_1 K_4 = 0 \tag{10}$$

The linearized model of the system in the new coordinates is

$$x^* = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -K_1 K_4 & K_1 + K_2 K_3 & K_4 \end{bmatrix} x^* + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \tag{11}$$

Obviously the characteristic equations of the two systems (9) and (11) are identical.

The construction of the transformation that brings a non-linear system to the non-linear controller canonical form involves the solution of the exact differential equation represented by (7). In other words, the last row of the inverse of the controllability matrix must be a gradient field for (7) to be solvable. Often (7) is not exact. Fortunately, in some cases, it can be made exact by the use of an integrating factor. It is known that if one integrating factor exists then there are an infinite number of integrating factors (Moon and Spencer 1969). Let us consider the implications when (7) is not exact. Suppose we were able to find an integrating factor $\mu(x)$ such that the following equation is satisfied:

$$\frac{\partial T_1(x)}{\partial x} = \mu(x) q(x) = \tilde{q}(x) \tag{12}$$

Thus we have

$$\tilde{q}\mathscr{C} = [0 \quad 0 \quad \dots \quad 0 \quad \mu(x)]$$

The above implies that

$$\frac{\partial T_1}{\partial x}(\mathrm{ad}^{n-1}\ a,\ b) = \mu(x)$$

Therefore application of the transformation (6) to (1) with $T_1(x)$ as a solution of (12) brings the non-linear system into the following form:

$$\frac{d}{dt}\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{bmatrix} = \begin{bmatrix} \bar{x}_2 \\ \bar{x}_3 \\ \vdots \\ \bar{x}_n \\ f(\bar{x}) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \pm\mu(\bar{x}) \end{bmatrix} u \tag{13}$$

This form was designated by Sommer (1980) as the *non-linear phase-variable controller form.*

Example 2

To illustrate the implications of the non-exactness of (7), we analyse the following non-linear system on $\mathbb{R}^2$ (Su *et al.* 1983):

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \sin x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

The controllability matrix is

$$\mathscr{C} = [b \quad (\mathrm{ad}^1\, a, b)] = \begin{bmatrix} 0 & -\cos x_2 \\ 1 & 0 \end{bmatrix}$$

The last row of $\mathscr{C}^{-1}$ has the form

$$q(x) = \begin{bmatrix} -\dfrac{1}{\cos x_2} & 0 \end{bmatrix}$$

One can easily verify that $q(x)$ is not a conservative field, but it can be made conservative by the use of an integrating factor, e.g. $\mu = -\cos x_2$. Then

$$\tilde{q} = \mu q = [1 \quad 0] = \frac{\partial T_1}{\partial x}$$

Integration gives the solution

$$T_1 = x_1$$

i.e. T_1 is the field potential of $\tilde{q}$. Thus

$$T = \begin{bmatrix} T_1 \\ \langle dT_1, a \rangle \end{bmatrix} = \begin{bmatrix} x_1 \\ \sin x_2 \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

Note that

$$\frac{d}{dt}\bar{x} = \frac{\partial T}{\partial x}(a(x) + b(x)u) = \begin{bmatrix} \sin x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \cos x_2 \end{bmatrix} u$$

But $\bar{x}_2 = \sin x_2$, yielding the transformed system

$$\frac{d}{dt}\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \cos(\sin^{-1} \bar{x}_2) \end{bmatrix} u$$

3.2. *Method* 2

The second method of the simplification of non-linear systems by a non-linear transformation was discussed by Kuntsevich and Lychak (1977), and it is attributed to Korobov. Method 2 works for the systems described by the following class of non-linear equations:

$$\left.\begin{aligned} \dot{x}_i &= f_i(x_1, x_2, \ldots, x_i, x_{i+1}), \quad i = 1, \ldots, n-1 \\ \dot{x}_n &= f_n(x_1, x_2, \ldots, x_n, u) \end{aligned}\right\} \tag{14}$$

It can be shown that the transformation

$$\bar{x} = T(x) = \begin{bmatrix} x_1 \\ f_1(x_1, x_2) \\ \frac{d}{dt} f_1(x_1, x_2) \\ \vdots \\ \frac{d^{(n-2)}}{dt^{(n-2)}} f_1(x_1, x_2) \end{bmatrix} \tag{15}$$

puts the original system (14) into the form

$$\frac{d}{dt}\begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \dot{\bar{x}}_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} v(\tilde{T}(\bar{x}), u)$$

where $\tilde{T}(\bar{x})$ is the inverse mapping of $T(x)$.

Example 3

Consider the inverted pendulum controlled by a DC motor as described by (8). Note that this system belongs to a class described by (14). The transformation (15) will have the form

$$T = \begin{bmatrix} x_1 \\ f_1(x_1, x_2) \\ \frac{d}{dt} f_1(x_1, x_2) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ K_1 \sin x_1 + K_2 x_3 \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} \tag{16}$$

Note that this transformation differs from the one we obtained using Method 1 only by a factor $1/K_2K_5$. The transformed system using (16) can be represented as follows:

$$\frac{d}{dt}\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \begin{bmatrix} \bar{x}_2 \\ \bar{x}_3 \\ (K_1 \cos \bar{x}_1 + K_2 K_3)\bar{x}_2 + K_4(\bar{x}_3 - K_1 \sin \bar{x}_1) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ K_2 K_5 \end{bmatrix} u \tag{17}$$

3.3. *Method* 3

The third class of transformations of non-linear systems considered in the paper has the important property of preserving the eigenvalues of the linearized model at any of the operating points. Method 3 is based on the ideas of Reboulet and Champetier (1984), and applies to a general class of the non-linear systems described by the equation

$$\dot{x} = f(x, u)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^1$, $f: \mathbb{R}^{n+1} \to \mathbb{R}$, and $f \in C^1$.

Method 3 assumes that $f(0, 0) = 0$, i.e. $x = 0$, $u = 0$, is an operating point. We are also interested in constant operating points corresponding to non-zero constant inputs $u = \varepsilon$. These points form a set

$$O = \{(x_\varepsilon, \varepsilon) | f(x_\varepsilon, \varepsilon) = 0\}$$

In the neighbourhood of an operating point $(x_\varepsilon, \varepsilon)$ the dynamic behavior of the system can be described by a linear equation of the form

$$\delta\dot{x} = A(x_\varepsilon, \varepsilon)\, \delta x + b(x_\varepsilon, \varepsilon)\, \delta u \tag{18}$$

where

$$A \triangleq \left.\frac{\partial f}{\partial x}\right|_{(x_\varepsilon, \varepsilon)}, \quad b \triangleq \left.\frac{\partial f}{\partial u}\right|_{(x_\varepsilon, \varepsilon)}$$

Assume that at any operating point $(x_\varepsilon, \varepsilon)$ the pair $\{A(x_\varepsilon, \varepsilon), b(x_\varepsilon, \varepsilon)\}$ is completely controllable. This then implies that

$$\det [b(x_\varepsilon, \varepsilon) \quad A(x_\varepsilon, \varepsilon)b(x_\varepsilon, \varepsilon) \quad \dots \quad A^{n-1}(x_\varepsilon, \varepsilon)b(x_\varepsilon, \varepsilon)] \neq 0$$
$$\forall (x_\varepsilon, \varepsilon) \in O = \{(x_\varepsilon, \varepsilon) | f(x_\varepsilon, \varepsilon) = 0\}$$

Therefore we can find a similarity transformation $T(x_\varepsilon, \varepsilon)$ such that at any operating point of interest

$$TAT^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ x & x & x & x & \dots & x & x \end{bmatrix}, \quad Tb = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The transformation T is constructed in the usual way (Kailath 1980), i.e.

$$T = \begin{bmatrix} q \\ qA \\ \vdots \\ qA^{n-2} \\ qA^{n-1} \end{bmatrix}$$

where q is the last row of the controllability matrix $[b \quad Ab \quad \dots \quad A^{n-1}b]$.

Note that the rows of $T(x_\varepsilon, \varepsilon)$ can be viewed as differential 1-forms over the set $0_x = \{x_\varepsilon | \exists\, \varepsilon \text{ s.t. } f(x_\varepsilon, \varepsilon) = 0\}$. Integrating $T_i(x_\varepsilon, \varepsilon)$ along this set results in the mappings T_i, $i = 1, \dots, n$, such that

$$\nabla T_i|_{O_x} = q_i A^{i-1}, \quad i = 1, \dots, n$$

In other words,

$$\left.\frac{\partial T}{\partial x}\right|_{(x_\varepsilon, \varepsilon)} = \begin{bmatrix} q \\ qA \\ \vdots \\ qA^{n-1} \end{bmatrix}$$

Observe that the transformation constructed via Method 3 preserves the eigenvalues of the linearized system at any constant operating point.

Let us illustrate this in the following example.

Example 4

Consider again the inverted pendulum controlled by a DC motor. The system equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ K_1 \sin x_1 + K_2 x_3 \\ K_2 x_2 + K_4 x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ K_5 \end{bmatrix} u$$

The set of constant operating points is

$$O = \left\{ x_{1\varepsilon} = \sin^{-1}\left(\frac{K_2 K_5}{K_1 K_4}\varepsilon\right),\ x_{2\varepsilon} = 0,\ x_{3\varepsilon} = -\frac{K_5}{K_4}\varepsilon,\ u = \varepsilon \right\}$$

The linearized model at any $(x_\varepsilon, \varepsilon) \in O$ has the form

$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \\ \delta\dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ K_1 \cos x_{1\varepsilon} & 0 & K_2 \\ 0 & K_3 & K_4 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ K_5 \end{bmatrix} \delta u$$

In order to find the desired transformation we compute

$$q = \begin{bmatrix} \dfrac{1}{K_2 K_5} & 0 & 0 \end{bmatrix}$$

Integrating q along O_x, we obtain

$$T_1 = \frac{x_1}{K_2 K_5}$$

Now

$$qA = \begin{bmatrix} 0 & \dfrac{1}{K_2 K_5} & 0 \end{bmatrix}$$

The integration of qA along O_x results in

$$T_2 = \frac{x_2}{K_2 K_5}$$

Finally,

$$qA^2 = \begin{bmatrix} \dfrac{K_1 \cos x_{1\varepsilon}}{K_2 K_5} & 0 & \dfrac{1}{K_5} \end{bmatrix}$$

Integrating qA^2 along O_x yields

$$T_3 = \frac{K_1}{K_2 K_5} \sin x_1 + \frac{1}{K_5} x_3$$

As a result of the above manipulations, we have the desired transformation which transforms locally and globally the non-linear system (8) to the controllable form.

Moreover, this transformation preserves the constant operating points of the linearized models.

Example 4 illustrates the main purpose of this section. The goal is to find an algorithm for the construction of a transformation that would exhibit the property of both global and local transformations. Specifically, we are seeking a transformation that brings a non-linear system into its global non-linear controller canonical form while simultaneously preserving the eigenvalues of the linearized model at any constant operating point. Such a transformation may be obtained by combining the discussed methods together. The algorithm for constructing such a transformation is as follows.

Step 1. Using Methods 1 or 2, find a transformation

$$T(x) = \begin{bmatrix} T_1(x) \\ T_2(x) \\ \vdots \\ T_n(x) \end{bmatrix}$$

bringing a non-linear system to non-linear global controller form.

Step 2. Linearize the non-linear system about a generic constant operating point, i.e. find a parameterized family of linearized models of the non-linear system.

Step 3. Form the controllability matrix of the linearized model.

Step 4. Find

$$T(x_\varepsilon, \varepsilon) = \begin{bmatrix} q \\ qA \\ \vdots \\ qA^{n-1} \end{bmatrix}$$

such that the following relations are satisfied:

$$\nabla T_i(x)|_{(x_\varepsilon, \varepsilon)} = q_i A^{i-1}$$

In § 4 we will examine the application of these results to state-feedback design for non-linear systems.

4. State-variable feedback design

Recall that the systems considered in this paper can be described by the following equation:

$$\dot{x}(t) = a(x(t)) + b(x(t))u(t)$$

The state-feedback control laws to be examined have the form

$$u = v + k(x)$$

where $k(\cdot): \mathbb{R}^n \to \mathbb{R}$, $u(0) = 0$, and v is an external input.

Thus the closed-loop system can be described by the equation

$$\dot{x} = (a(x) + b(x)k(x)) + b(x)v$$

Note that $v = 0$, $x = 0$ is a constant operating point for both the open- and the closed-loop systems. Consider also the constant operating points corresponding to non-zero constant inputs, say $v = \beta$. Assume that

$$\frac{\partial a}{\partial x}(0) + b(0)\frac{\partial k}{\partial x}(0)$$

is invertible; then

$$a(x_\beta) + b(x_\beta)k(x_\beta) + b(x_\beta)\beta = 0$$

has a unique solution for $x_\beta(\cdot)$ as an analytic function of β in some neighbourhood of $\beta = 0$. It is possible to view the closed-loop operating point as a function of $u = \varepsilon$ (see Baumann and Rugh 1984).

Our goal here is to find an analytic feedback $k(\cdot)$ such that the eigenvalues of the closed-loop system are invariant with respect to any particular closed-loop operating point.

One approach to achieving this goal was proposed by Baumann and Rugh (1984). We suggest an alternative approach which utilizes a transformation of a non-linear system to a simpler form.

Specifically, using the algorithm outlined in § 3, we transform an open-loop system into the global non-linear controller form while preserving constant operating points. An appropriate feedback is found which provides the desired system eigenvalues at any constant operating point. Finally, the transformation $x^* = T(x)$ provides a path back to the original state variables.

Example 5

Consider again the inverted pendulum controlled by the armature-controlled DC motor.

The transformation that brings this system into a global non-linear controller form while simultaneously preserving the operating points was found in § 3 (Method 3) and has the form

$$\bar{x} = T(x) = \frac{1}{K_2K_5}\begin{bmatrix} x_1 \\ x_2 \\ K_1 \sin x_1 + K_2x_3 \end{bmatrix} \tag{19}$$

The transformed system is

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \end{bmatrix} = \begin{bmatrix} \bar{x}_2 \\ \bar{x}_3 \\ (K_1 \cos(K_2K_5\bar{x}_1) + K_2K_3)\bar{x}_2 + K_4\bar{x}_3 - \dfrac{K_1K_4}{K_2K_5}\sin(K_2K_5\bar{x}_1) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

Suppose we want the eigenvalues of the closed-loop linearized system to be the same at any constant closed-loop operating point. Denote these eigenvalues as $-e_1$, $-e_2$, $-e_3$. Then the characteristic equation of the linearized closed-loop system has the form

$$(s+e_1)(s+e_2)(s+e_3) \triangleq s^3+\alpha_2 s^2+\alpha_1 s+\alpha_0 \tag{20}$$

The state feedback assigning the desired eigenvalues to the linearized closed-loop system (in the new coordinates) is

$$\begin{aligned}\bar{k}(\bar{x}) = &-\alpha_0\bar{x}_1-\alpha_1\bar{x}_2-\alpha_2\bar{x}_3-(K_1 \cos(K_2K_5\bar{x}_1)+K_2K_3)\bar{x}_2\\ &-K_4\bar{x}_3+\frac{K_1K_4}{K_2K_5}\sin(K_2K_5\bar{x}_1)\end{aligned}$$

And the desired feedback, in the old coordinates, is

$$\begin{aligned}k(x) = &-\frac{\alpha_0 x_1}{K_2K_5}-\frac{\alpha_1 x_2}{K_2K_5}-\alpha_2\left(\frac{K_1}{K_2K_5}\sin x_1+\frac{1}{K_5}x_3\right)\\ &-(K_1\cos x_1+K_2K_3)\frac{x_2}{K_2K_5}-\frac{K_4}{K_5}x_3\end{aligned} \tag{21}$$

Application of this state feedback to the system (8) results in the closed-loop system whose linearized model at any constant (closed-loop) operating point has the characteristic equation of the form (20).

5. Example

In this section we present simulation results for different control strategies applied to the inverted pendulum controlled by the armature-controlled DC motor as described by (8). Reasonable parameters describing our system are $l=1$ m, $m=1$ kg, $g=9{\cdot}8$ m/s^2, $J=1$ N ms^2/rad, $K_m=0{\cdot}1$ Nm/A, $K_b=0{\cdot}1$ Vs/rad, $R=1\,\Omega$, $L=100$ mH. In terms of these parameters, the coefficients entering (8) are $K_1=9{\cdot}8$, $K_2=1$, $K_3=-10$, $K_4=-10$, $K_5=10$.

We analyse three different control laws:

(i) linear state feedback (L);

(ii) globally linearizing state feedback (GL);

(iii) partially linearizing state feedback (PL).

In order to derive a linear state feedback, we linearize our system about the origin, and then employ linear-systems pole-placement techniques (Kailath 1980) to find the appropriate control law. Suppose that the desired characteristic equation of the linearized closed-loop system has the form $s^3+\alpha_2 s^2+\alpha_1 s+\alpha_0=0$. Then, after some manipulations, we find the linear control law

$$u=kx=\left[\frac{-\alpha_0-K_1\alpha_2}{K_2K_5} \quad \frac{-K_1-K_2K_3-\alpha_1}{K_2K_5} \quad \frac{-K_4-\alpha_2}{K_5}\right]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

We require the closed-loop linearized system to have its eigenvalues located at $\{-e_1, -e_2, -e_3\}$, where $e_1=e_2=1\pm j\,1{\cdot}33$ ans $e_3=100$.

Globally (or fully) linearizing feedback was derived in § 4 and is given by (21).

Partially linearizing feedback (non-unique) has the form

$$u = -\frac{1}{K_2 K_5}\Bigg[K_2 K_3 x_2 + K_1 x_2 \cos x_1 - K_4 K_2 x_3 - \alpha_0 x_1 + \alpha_1 x_2 + \alpha_2(K_1 \sin x_1 + K_2 x_3) + 10^3 x_1 x_2^2 + \frac{10^3}{3} x_2^3\Bigg] \tag{22}$$

Observe that the closed-loop systems with globally and partially linearizing state feedbacks, when linearized about any constant operating point, have the same eigenvalues, invariant with respect to the operating point. The partially linearizing feedback was obtained by the addition of the non-linear term $g(x) = \left(10^3 x_1 x_2^2 + \frac{10^3}{3} x_2^3\right)\Big/ K_2 K_5$ to the globally linearizing feedback (21). This term can be thought of as modifying the characteristic equation of the linearized system as a function of state. The linear control law can be obtained by linearizing either (21) or (22) about the origin. By comparison, one can check that the eigenvalues of the closed-loop system with the linear feedback are different at different constant operating points.

The following figures contain simulation results for the discussed three different control strategies. First, in Fig. 3, the response of the open loop system (8) is depicted. In Figs. 4(a–c) a comparison of the responses of the closed-loop systems when subjected to different non-zero initial conditions is depicted. In all cases, the reference input is zero.

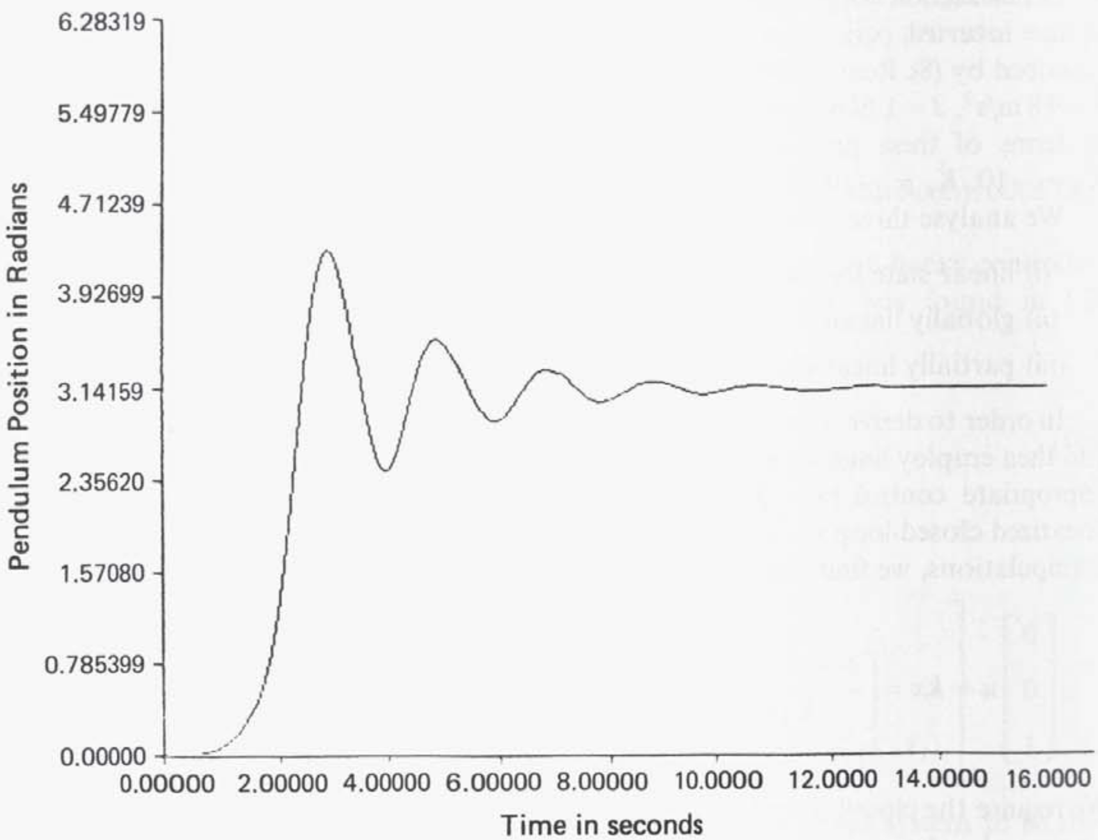
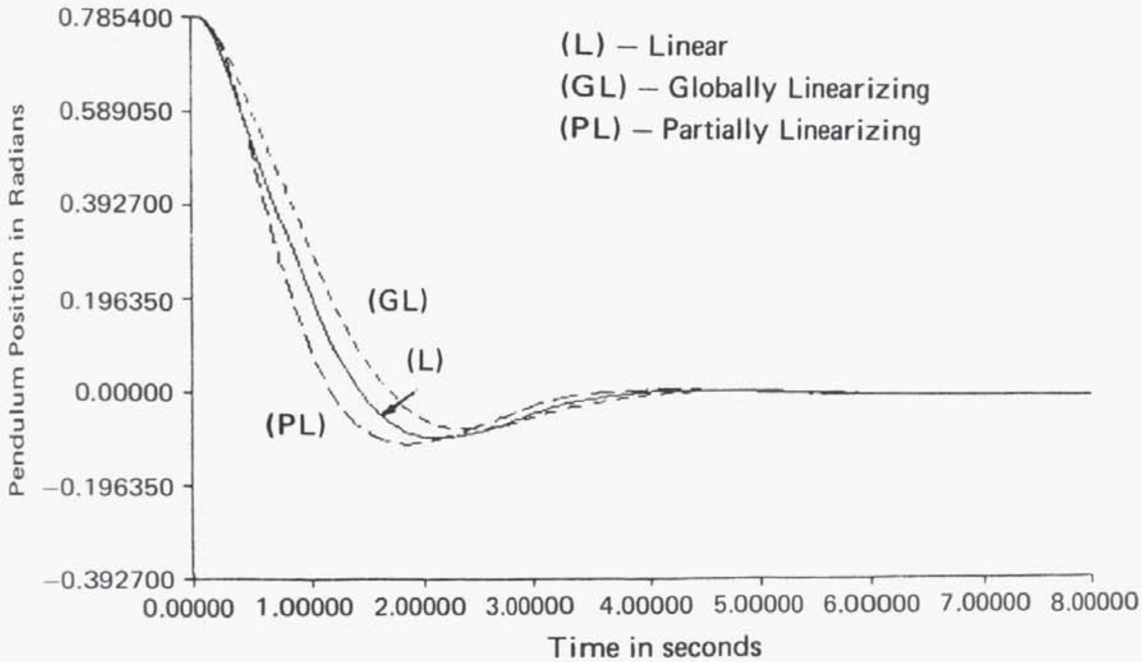
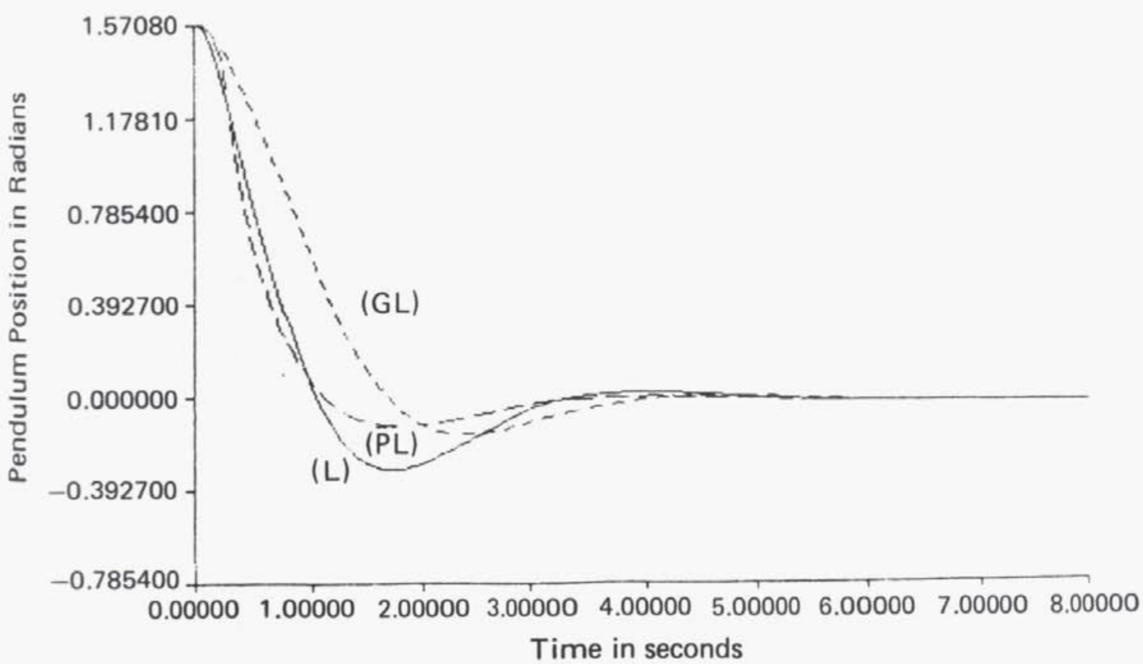


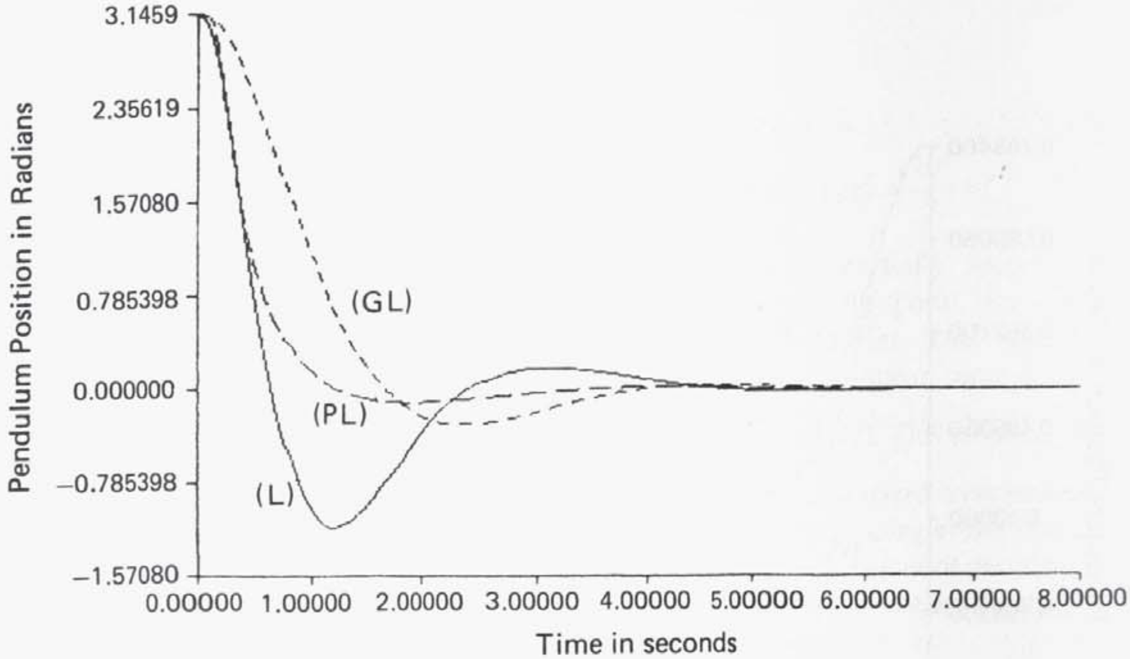
Figure 3. The response of the uncontrolled system (8) with the control voltage $u = V = 0$.


0.785400
0.589050
0.392700
0.196350
0.00000
−0.196350
−0.392700
Pendulum Position in Radians
0.00000 1.00000 2.00000 3.00000 4.00000 5.00000 6.00000 7.00000 8.00000
Time in seconds
(L) – Linear
(GL) – Globally Linearizing
(PL) – Partially Linearizing
(GL)
(L)
(PL)

(*a*)


1.57080
1.17810
0.785400
0.392700
0.000000
−0.392700
−0.785400
Pendulum Position in Radians
0.00000 1.00000 2.00000 3.00000 4.00000 5.00000 6.00000 7.00000 8.00000
Time in seconds
(GL)
(PL)
(L)

(*b*)



(*c*)

Figure 4. The zero-input responses of the closed-loop system with three different control laws for different initial conditions: (*a*) $x_1 = \theta = \pi/4$, $x_2 = x_3 = 0$; (*b*) $x_1 = \theta = \pi/2$, $x_2 = x_3 = 0$; (*c*) $x_1 = \theta = \pi$, $x_2 = x_3 = 0$.

Observe that the system with the globally linearizing (GL) non-linear control maintains a 'linear-like' constant response regardless of the magnitude of $x_1(0)$. This verifies the linearizing action of the fully linearizing non-linear feedback. By contrast, the shape of the response of the linear-feedback system (L) varies with the magnitude of $x_1(0)$. It appears that the linear feedback exhibits larger (more-negative) eigenvalues than those intended when the system is displaced outside the neighbourhood of $x_1 = 0$. This results in a highly 'underdamped-like' response, even though we designed for only a slightly underdamped response. The compromise between the two extremes is partially linearizing feedback. The response of the closed-loop system with this type of feedback is 'better' than the other two, in the sense that it provides a fast response without excessive overshoot. Furthermore, this feedback, as globally linearizing feedback does, maintains the closed-loop eigenvalues at any constant operating point.

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