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Combinatorial scheme of finding minimal number of periodic points for smooth self-maps of simply connected manifolds

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Abstract. Let M be a closed smooth connected and simply connected manifold of dimension m at least 3, and let r be a fixed natural number. The topological invariant $D_r^m[f]$, defined by the authors in [*Forum Math.* **21** (2009), 491–509], is equal to the minimal number of r-periodic points in the smooth homotopy class of f, a given self-map of M. In this paper, we present a general combinatorial scheme of computing $D_r^m[f]$ for arbitrary dimension $m \geq 4$. Using this approach we calculate the invariant in case r is a product of different odd primes. We also obtain an estimate for $D_r^m[f]$ from below and above for some other natural numbers r.

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1. Introduction

Let f be a self-map of a compact manifold M. The problem of minimizing the number of fixed or periodic points in a homotopy class of f is one of the important challenges in modern periodic point theory. In this paper, we consider the smooth version of this question, asking about minimal number of r-periodic points in the smooth homotopy class of f, i.e., for

$$\min\left\{\#\operatorname{Fix}(g^r) : g \overset{s}{\sim} f\right\}, \tag{1.1}$$

where $\overset{s}{\sim}$ means that the maps g and f are C^1-homotopic.

It is known since 2006 that for continuous category the minimum in (1.1) is given by the classical invariant $NF_r(f)$. This invariant was originally introduced by Jiang [18] in 1983 as a lower bound for the number of r-periodic points in the homotopy class, and it was proved in 2006 that $NF_r(f)$ is the best such lower bound, i.e., it is equal to the minimum in (1.1); see [16]. During the last decade, $NF_r(f)$ was computed in many particular cases; see [13, 14, 15, 19, 20, 21].

Recent investigations of the authors showed that the smooth and continuous theories do not coincide. In [6, 9] two counterparts of $NF_r(f)$ were defined for smooth category: $D_r^m[f]$ for simply connected manifolds and its generalization $NJD_r^m[f]$ for non-simply connected ones.

The difference between continuous and smooth categories is clearly noticeable for the simply connected case. In such situation $NF_r(f) \in \{0,1\}$ but $D_r^m[f]$ is usually greater than 1. It turned out that then the only obstacle (as the fundamental group is trivial) to minimize the number of periodic points comes from their fixed point indices. By the classical Poincaré–Lefschetz theorem, for each n the Lefschetz number $L(f^n)$ is equal to the sum of fixed point indices of f^n at points that are fixed by f^n. On the other hand, the sequence of fixed point indices at an isolated fixed or periodic point for a smooth map has a very special form. As a result, to obtain the sequence $\{L(f^n)\}_{n|r}$ as a sum of indices, one usually needs many periodic points (unlike in a continuous case, where the forms of sequences of indices are more arbitrary, and thus Lefschetz numbers can be realized by one such sequence [16]).

The invariant $D_r^m[f]$ is equal to the minimal number of sequences in the decomposition of Lefschetz numbers of iterations $\{L(f^n)\}_{n|r}$ into sequences, each of which can be realized as fixed point indices at a periodic orbit of a smooth local map. As a consequence, to find the value of $D_r^m[f]$, one needs to know all possible forms of local fixed point indices of a smooth map in the given dimension m. All such forms were described for three-dimensional maps in [12] which allowed us to find $D_r^3[f]$ for $S^2 \times I$ [6], S^3 [7], two-holed three-dimensional closed ball [5] and also $NJD_r^3[f]$ for $\mathbb{R}P^3$ [10].

Recently, the complete list of all sequences of local indices of iterations in arbitrary dimension has been found [11], which enabled us to calculate $D_r^m[f]$ in dimension 4 [8].

The main goal of this paper is to provide the effective methods of computing $D_r^m[f]$ for arbitrary higher-dimensional manifolds. In order to do that, at first we show that finding the value of the invariant may be simplified in higher-dimensional case (cf. Theorem 4.2). This observation is also an answer to the question (asked during a discussion in the conference *Nielsen Theory and Related Topics*, St. John's, Newfoundland, Canada, 2009) about the differences between three- and higher-dimensional cases in smooth category. Namely, we prove that for $m > 3$ one may find smooth g homotopic to f such that $\mathrm{Fix}(g^r) = D_r^m[f]$ and all r-periodic points of g are fixed points, while for $m = 3$ in addition to fixed points some 2-periodic orbits for g may remain irreducible (Theorem 4.3, Remark 4.4). For $m \geq 4$ this finding enables us to describe purely combinatorial scheme of the calculation of $D_r^m[f]$, which we introduce in Section 5. The scheme makes it possible to determine $D_r^m[f]$

for odd r and maps with fast growth of the Lefschetz numbers of iterations, i.e., satisfying our Standing Assumptions 5.1 (Theorem 5.5). This class of maps covers, for example, self-maps of S^m with degree d, where $|d| > 1$.

Finally, in Section 6, we demonstrate our method in action, calculating $D_r^m[f]$ in case r is a product of different odd primes (Theorem 6.6) and we apply this result in Section 7 to obtain an estimate for $D_r^m[f]$ from below and above for some other natural numbers r (Theorem 7.3).

2. The invariant $D_r^m[f]$

2.1. Sketch of the construction

At first we sketch the definition of $D_r^m[f]$ to provide the general topological background of our idea; for further details the reader may consult [6].

Problem 2.1. We are given a smooth self-map $f : M \to M$ of a smooth closed connected and simply connected manifold of dimension $m \geq 4$ and a number $r \in \mathbb{N}$. We seek the minimal number of r-periodic points in the smooth homotopy class of f:

$$MF_r^{\text{diff}}(f) = \min \left\{ \# \operatorname{Fix}(g^r) : g \overset{s}{\sim} f \right\},$$

where $\overset{s}{\sim}$ means that the maps g and f are C^1-homotopic.

We will briefly describe in the items below how this question reduces to a calculation of our combinatorial-type invariant denoted as $D_r^m[f]$.

(1) Let us consider an isolated periodic point $x \in \operatorname{Fix}(f^p)$. Then the integer sequence $\{c_k\}_k = \{\operatorname{ind}(f^k, x)\}_k$ must satisfy strong restrictions found by Chow, Mallet-Paret and Yorke [3]. We will call each integer sequence that satisfies such conditions $DD^m(p)$ sequence.

(2) Assume now for simplicity that the minimal number of r-periodic points can be realized by fixed points: there is a smooth map g smoothly homotopic to f satisfying

$$\# \operatorname{Fix}(g^r) = MF_r^{\text{diff}}(f)$$

and

$$\operatorname{Fix}(g^r) = \operatorname{Fix}(g).$$

(In fact, one of our results, i.e., Theorem 4.2, states that this is true for $m \geq 4$.)

(3) Consider the above map g. Now

$$\operatorname{Fix}(g^r) = \operatorname{Fix}(g) = \{x_1, \dots, x_u\},$$

where $u = MF_r^{\text{diff}}(f)$. This implies that

$$L(f^k) = L(g^k) = \sum_{i=1}^{u} \operatorname{ind}(g^k, x_i).$$

Thus the (finite) sequence $\{L(f^k)\}_{k|r}$ is the sum of u $DD^m(1)$ sequences.

(4) In [6] we proved, using advanced Nielsen techniques, that the inverse is also true. If $\{L(f^k)\}_{k|r}$ is the sum of u $DD^m(1)$ sequences, then f is homotopic to a smooth map g with $\#\operatorname{Fix}(g^r) = u$.

(5) Finally, for a smooth map $f : M \to M$, the minimal number of r-periodic points $MF_r^{\text{diff}}(f)$ is equal to the minimal number of summands in a decomposition of $\{L(f^k)\}_{k|r}$ into the sum of $DD^m(1)$ sequences, which is the value of $D_r^m[f]$ and gives the answer to Problem 2.1.

(6) Effective computations of $D_r^m[f]$ are possible because we know all the forms of $DD^m(1)$ sequences in arbitrary dimension (cf. Section 3).

2.2. Definitions and theorems

Now we give more information concerning the invariant $D_r^m[f]$.

Definition 2.2. A sequence of integers $\{c_n\}_{n=1}^{\infty}$ is called $DD^m(p)$ sequence if there are: a C^1 map $\phi : U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^m$ is open; and P, an isolated p-orbit of ϕ, such that $c_n = \operatorname{ind}(\phi^n, P)$ (notice that $c_n = 0$ if n is not a multiple of p). The finite sequence $\{c_n\}_{n|r}$ will be called $DD^m(p|r)$ sequence if this equality holds only for $n \mid r$, where r is fixed.

Let us fix an integer $r \geq 1$. The value of the invariant $D_r^m[f]$ is given as the minimal decomposition of the sequence of Lefchetz numbers of iterations into $DD^m(p|r)$ sequences.

Definition 2.3. Let $\{L(f^n)\}_{n|r}$ be a finite sequence of Lefschetz numbers. We decompose $\{L(f^n)\}_{n|r}$ into the sum

$$L(f^n) = c_1(n) + \cdots + c_s(n), \tag{2.1}$$

where c_i is a $DD^m(l_i|r)$ sequence for $i = 1, \ldots, s$. Each such decomposition determines the number $l = l_1 + \cdots + l_s$. We define the number $D_r^m[f]$ as the smallest l which can be obtained in this way.

The invariant $D_r^m[f]$ was defined in [6] and it is equal to the minimal number of r-periodic points in the smooth homotopy class of f.

Theorem 2.4 (see [6]). *Let M be a closed smooth connected and simply connected manifold of dimension $m \geq 3$ and $r \in \mathbb{N}$ a fixed number. Then,*

$$D_r^m[f] = MF_r^{\text{diff}}(f).$$

The convenient way of writing down sequences of indices of iterations is to represent each of them as an integral combination of some basic periodic sequences $\{\operatorname{reg}_k(n)\}_n$.

Definition 2.5. For a given k we define the *basic sequence* as

$$\operatorname{reg}_k(n) = \begin{cases} k & \text{if } k \mid n, \\ 0 & \text{if } k \nmid n. \end{cases}$$

Any sequence of indices of iterations (and also Lefchetz numbers of iterations) may be represented in the form of *periodic expansion* (cf. [17]), namely

$$\operatorname{ind}(f^n, x_0) = \sum_{k=1}^{\infty} a_k \operatorname{reg}_k(n), \tag{2.2}$$

where $a_n = \frac{1}{n}\sum_{k|n} \mu(k)\operatorname{ind}(f^{(n/k)}, x_0)$, μ is the classical Möbius function, i.e., $\mu : \mathbb{N} \to \mathbb{Z}$ is defined by the following three properties:

(i) $\mu(1) = 1$,
(ii) $\mu(k) = (-1)^s$ if k is a product of s different primes,
(iii) $\mu(k) = 0$ otherwise.

Moreover, all coefficients a_k in (2.2) are integers, which was proved by Dold [4].

The invariant $D_r^m[f]$ is defined by a use of $DD^m(p)$ sequences, but it turns out that it is enough to know only the forms of $DD^m(1)$ sequences, because the complete list of all $DD^m(p)$ sequences can be obtained from the list of $DD^m(1)$ ones, by replacing each reg_k by reg_{pk} (see Definition 2.6 and Theorem 2.7 below for the formal explanation of this statement). As a consequence, the forms of $DD^m(1)$ sequences that are given in Theorem 3.2 in Section 3 allow one to easily determine all forms of $DD^m(p)$ sequences.

Definition 2.6. We will say that the $DD^m(p)$ sequence $\{\tilde{c}_n\}_n$ comes from the given $DD^m(1)$ sequence $\{c_n\}_n$ with the periodic expansion

$$c_n = \sum_{d=1}^{\infty} a_d \operatorname{reg}_d(n)$$

if the periodic expansion of $\{\tilde{c}_n\}_n$ has the form

$$\tilde{c}_n = \sum_{d=1}^{\infty} a_d \operatorname{reg}_{pd}(n).$$

Theorem 2.7 (see [6]). *Every $DD^m(p)$ sequence comes from some $DD^m(1)$ sequence.*

3. Indices of iterations in $\mathbb{R}^m$

In this section, we give the complete list of all forms of indices of iterations of smooth maps in a given dimension $m \geq 3$.

Let us remark here that the problem of finding the forms of indices of iterations of particular class of maps is difficult in general. Nevertheless, last years brought some important results concerning planar homeomorphism [23], $\mathbb{R}^3$-homeomorphisms [2, 24] and holomorphic maps [25, 26, 27].

Definition 3.1. Let H be a finite subset of natural numbers. We introduce the following notation.

By $\mathrm{LCM}(H)$ we mean the least common multiple of all elements in H with the convention that $\mathrm{LCM}(\emptyset) = 1$. We define the set $\overline{H}$ by $\overline{H} = \{\mathrm{LCM}(Q) : Q \subset H\}$.

For natural s we denote by $L(s)$ any set of natural numbers of the form $\overline{L}$ with $\#L = s$ and $1, 2 \notin L$.

By $L_2(s)$ we denote any set of natural numbers of the form $\overline{L}$ with $\#L = s + 1$ and $1 \notin L$, $2 \in L$.

Theorem 3.2 (Main Theorem I [11]**).** *Let $U \subset \mathbb{R}^m$, where $m \geq 3$, be an open neighborhood of 0 and let $f : U \to \mathbb{R}^m$ be a C^1 map having 0 as an isolated fixed point for each iteration. Then the sequence of local indices of iterations $\{\mathrm{ind}(f^n, 0)\}_{n=1}^{\infty}$ has one of the following forms.*

(I) *For m odd,*

$$(A^o) \qquad \mathrm{ind}(f^n, 0) = \sum_{k \in L_2(\frac{m-3}{2})} a_k \, \mathrm{reg}_k(n);$$

$$(B^o), (C^o), (D^o) \qquad \mathrm{ind}(f^n, 0) = \sum_{k \in L(\frac{m-1}{2})} a_k \, \mathrm{reg}_k(n),$$

where

$$a_1 = \begin{cases} 1 & \text{in the case } (B^o), \\ -1 & \text{in the case } (C^o), \\ 0 & \text{in the case } (D^o); \end{cases}$$

$$(E^o), (F^o) \qquad \mathrm{ind}(f^n, 0) = \sum_{k \in L_2(\frac{m-1}{2})} a_k \, \mathrm{reg}_k(n),$$

where $a_1 = 1$ and

$$a_2 = \begin{cases} 0 & \text{in the case } (E^o), \\ -1 & \text{in the case } (F^o). \end{cases}$$

(II) *For m even,*

$$(A^e) \quad \mathrm{ind}(f^n, 0) = \sum_{k \in L_2(\frac{m-4}{2})} a_k \, \mathrm{reg}_k(n);$$

$$(B^e) \quad \mathrm{ind}(f^n, 0) = \sum_{k \in L(\frac{m-2}{2})} a_k \, \mathrm{reg}_k(n);$$

$$(C^e), (D^e), (E^e) \quad \mathrm{ind}(f^n, 0) = \sum_{k \in L_2(\frac{m-2}{2})} a_k \, \mathrm{reg}_k(n),$$

where

$$a_1 = \begin{cases} 1 & \text{in the case } (C^e), \\ -1 & \text{in the case } (D^e), \\ 0 & \text{in the case } (E^e); \end{cases}$$

$$(F^e) \qquad \mathrm{ind}(f^n, 0) = \sum_{k \in L(\frac{m}{2})} a_k \, \mathrm{reg}_k(n),$$

where $a_1 = 1$.

4. Minimal number of periodic points can be realized at fixed points

Let M be a closed smooth connected and simply connected manifold of dimension $m \geq 4$ and $r \in \mathbb{N}$ a fixed number. In this section, based on the knowledge of the forms of indices of iterations, we will prove that it is always possible to find in a given smooth homotopy class a map g with the minimal number of r-periodic points such that g has only fixed points (up to the rth iteration).

Lemma 4.1. *For a finite subset $G \subset \mathbb{N}$ we have*

$$p\,\overline{G} = \overline{p\,G} \setminus \{1\} \cup \{p\}.$$

Proof. By the relation $p\,\mathrm{LCM}(K) = \mathrm{LCM}(pK)$ which holds for $K \neq \emptyset$, we get

$$\begin{aligned} p\,\overline{G} &= p\,\mathrm{LCM}\{K : K \subset G\} = p\,\mathrm{LCM}\{K : K \subset G, K \neq \emptyset\} \cup \{p\} \\ &= \mathrm{LCM}\{pK : K \subset G, K \neq \emptyset\} \cup \{p\} = \overline{p\,G} \setminus \{1\} \cup \{p\}. \end{aligned}$$ □

Theorem 4.2. *For $m \geq 4$ in Definition* 2.3 *of $D^m_r[f]$ we may equivalently use only $DD^m(1|r)$ sequences.*

Proof. We will show that every $DD^m(p)$ sequence with $p \geq 2$ is a sum of at most two $DD^m(1)$ sequences, which proves our theorem.

By Theorem 2.7 every $DD^m(p)$ sequence can be represented in the form

$$\sum_{k \in p \cdot \overline{G}} a_k \,\mathrm{reg}_k(n), \tag{4.1}$$

where the forms of $\overline{G}$ are described in Theorem 3.2, with perhaps some additional restrictions on coefficients. We will prove that the sequence (4.1) with arbitrary coefficients a_k is always a sum of at most two $DD^m(1)$ sequences.

The dimension m is fixed, we will consider two cases in dependence on the parity of m.

Case I (m is odd). Here we consider two subcases:

(IA) $\overline{G} = L(\frac{m-1}{2})$; i.e., every $DD^m(p)$ sequence comes from some $DD^m(1)$ sequence of the types (B^o), (C^o), (D^o).

(IB) $\overline{G} = L_2(\frac{m-1}{2})$; i.e., every $DD^m(p)$ sequence comes from some $DD^m(1)$ sequence of the types $(E^o), (F^o)$. This case covers also (A^o) where $\overline{G} = L_2(\frac{m-3}{2})$ (remind that we ignored the influence of the restrictions for a_1, a_2).

We will consider each of the above subcases separately.

(IA) $\overline{G} = L(\frac{m-1}{2})$. Then $G = \{d_1, \ldots, d_s\}$ is an arbitrary set of different integers $d_i > 2$, where $s = \frac{m-1}{2}$.

By Lemma 4.1, $p\,\overline{G} = \overline{p\,G} \setminus \{1\} \cup \{p\}$. Thus we can realize all $a_k\,\mathrm{reg}_k$ with $k \in \overline{p\,G} \setminus \{1\}$ by one sequence of the type (D^o) (for which coefficient a_1

at reg_1 disappears) and the remaining part, i.e., $a_p \operatorname{reg}_p$ also by one sequence of the type (D^o) if $p > 2$ or by one sequence of the type (A^o) if $p = 2$. As a result, in this case, independently of the value of p, each $DD^m(p)$ sequence is a sum of two $DD^m(1)$ sequences.

(IB) $\overline{G} = L_2(\frac{m-1}{2})$. Similarly, we may represent G in the following form: $G = \{d_1, \dots, d_s, 2\}$, an arbitrary set consisting of $s + 1$ elements ($s = \frac{m-1}{2}$), with different integers $d_i > 2$.

Again using Lemma 4.1 we obtain

$$\begin{aligned} p\,\overline{G} &= p\overline{\{d_1, \dots, d_s, 2\}} = \overline{\{pd_1, \dots, pd_s, 2p\}} \setminus \{1\} \cup \{p\} \\ &= \overline{\{pd_1, \dots, pd_s, 2\}} \setminus \{1, 2\} \cup \{p, 2p\}. \end{aligned}$$

Thus we can realize all $a_k \operatorname{reg}_k$ with $k \in \overline{\{pd_1, \dots, pd_s, 2\}} \setminus \{1, 2\}$ by one sequence of the type (E^o), which gives the contribution to a_1 equal to 1. The remaining expression has the form

$$-\operatorname{reg}_1 + a_p \operatorname{reg}_p + a_{2p} \operatorname{reg}_{2p}, \tag{4.2}$$

and, since $m \geq 5$, it can be realized either by one sequence of the type (C^o) (if $p > 2$) or by one sequence of the type (A^o) (if $p = 2$). This completes the proof for m odd.

Case II (m is even). There are also two subcases:

(IIA) $\overline{G} = L(\frac{m}{2})$; i.e., every $DD^m(p)$ sequence comes from some $DD^m(1)$ sequence of the type (F^e), this case covers also (B^e) where $\overline{G} = L(\frac{m-2}{2})$.

(IIB) $\overline{G} = L_2(\frac{m-2}{2})$; i.e., every $DD^m(p)$ sequence comes from some $DD^m(1)$ sequence of the types (C^e), (D^e), (E^e); this case covers also (A^e) where $\overline{G} = L_2(\frac{m-4}{2})$.

Analyzing each of the subcases separately we obtain

(IIA) $\overline{G} = L(\frac{m}{2})$. In the same way as in the subcase (IA) we show that a given $DD^m(p)$ sequence is a sum of at most two $DD^m(1)$ sequences.

(IIB) $\overline{G} = L_2(\frac{m-2}{2})$. We have $G = \{d_1, \dots, d_s, 2\}$, an arbitrary set consisting of $s + 1$ elements ($s = \frac{m-2}{2}$), with different integers $d_i > 2$.

By Lemma 4.1 we get

$$p\,\overline{G} = p\overline{\{d_1, \dots, d_s, 2\}} = \overline{\{pd_1, \dots, pd_s, 2p\}} \setminus \{1\} \cup \{p\}.$$

Notice that the set $\{pd_1, \dots, pd_s, 2p\}$ consists of $s + 1 = \frac{m}{2}$ elements and 2 does not belong to the set. Thus we can realize all $a_k \operatorname{reg}_k$ with $k \in \overline{\{pd_1, \dots, pd_s, 2p\}}$ by one sequence of the type (F^e), which gives the contribution to a_1 equal to 1. The remaining expression has the form

$$-\operatorname{reg}_1 + a_p \operatorname{reg}_p, \tag{4.3}$$

and can be realized either by one sequence of the type (D^e) for $p > 2$, or by one sequence of the type (A^e) for $p = 2$.

Finally, in each subcase we are able to realize the sum (4.1) by no more than two sequences. This completes the proof for m even and the proof of the whole theorem. $\square$

Assume we have a given decomposition of Lefschetz numbers of iterations into $DD^m(p|r)$ sequences. Then, by the construction described in [9], one can find in the smooth homotopy class of f a map g for which p-periodic orbits are in the one-to-one correspondence with $DD^m(p|r)$ sequences. The above fact and Theorems 2.4 and 4.2 imply the following result.

Theorem 4.3. *Let f be a smooth self-map of M, a closed smooth connected and simply connected manifold of dimension $m \geq 4$ and let $r \in \mathbb{N}$ be a fixed number. Then, it is always possible to find g smoothly homotopic to f such that all its r-periodic points are fixed points and* $\mathrm{Fix}(g^r) = MF_r^{\mathrm{diff}}(f)$.

Remark 4.4. Let us notice that in three-dimensional case, in the computation of $D_r^3[f]$, in addition to $DD^3(1)$ sequences also some $DD^3(2)$ sequences are needed [6].

5. Combinatorial scheme of finding $D_r^m[f]$ for maps with nonvanishing coefficients of periodic expansion

We fix the natural number r. For the divisors of r we represent the sequence of Lefschetz numbers of iterations in the form of periodic expansion:

$$L(f^n) = \sum_{k|r} b_k \mathrm{reg}_k(n). \tag{5.1}$$

In the rest of the paper we will work under the following assumptions.

Standing Assumptions 5.1.

(I) $f : M \to M$ is a smooth self-map of a smooth closed connected and simply connected m-manifold, where $m \geq 4$.

(II) r is odd and $b_k \neq 0$ for all $k \neq 1$ dividing r.

Remark 5.2. The class of maps satisfying our Standing Assumptions contains maps with fast grow of Lefschetz numbers of iterations. The simplest example is a self-map of the m-dimensional sphere S^m with degree d such that $|d| > 1$. Other simple examples, described in terms of eigenvalues of homology groups, are provided in [22] for self-maps of manifolds M such that $H_j(M;Q) \approx \mathbb{Q}$ if $j \in J \cup \{0\}$, $H_j(M;\mathbb{Q}) \approx \{0\}$ otherwise, where J is a subset of the set of natural numbers $\mathbb{N}$ with cardinality 1, 2 or 3.

First, it is convenient to find the minimal decomposition of the sum $L(f^n) = \sum_{k|r} b_k \mathrm{reg}_k(n)$ into $DD^m(p|r)$ sequences modulo reg_1; i.e., we require that equality (2.1) holds only for all divisors $i|r$ different from 1 (thus we temporarily ignore the coefficient at reg_1).

Let us remind that, by Definition 2.3 and Theorem 4.2, $D^m_r[f]$ is equal to the minimal number v of $DD^m(1|r)$ sequences which give in sum Lefschtetz numbers of iterations:

$$L(f^n) = \sum_k b_k \operatorname{reg}_k(n) = c_1(n) + \cdots + c_v(n) \quad \text{for } n|r, \tag{5.2}$$

where each c_i is $DD^m(1|r)$ sequences, $1 \leq i \leq v$.

5.1. Finding $D^m_r[f]$ modulo reg_1

Let $\operatorname{Div}(r)$ denote the set of all divisors of r different from 1. We will show that finding the minimal decomposition is equivalent to finding a minimal family of subsets of $\operatorname{Div}(r)$ satisfying some simple conditions.

Let us consider a decomposition of Lefschetz numbers

$$\sum_{k|r} b_k \operatorname{reg}_k = c_1 + \cdots + c_h \tag{5.3}$$

into $DD^m(1)$ sequences for $k|r$.

As we consider the case of odd r and ignore the coefficient b_1, the only sequences $\{c_i\}_i$ that may appear in (5.3) are one of the types (B^o)–(D^o) of Theorem 3.2 (in the case of odd m); or (F^e) (in the case of even m), with possibly some coefficients a_k equal to zero. This means that for any such $DD^m(1)$ sequence $\{c_i\}_i$, there exists a set A_i with (at most) s nontrivial divisors of r such that

$$c_i = \sum_{k \in \overline{A}_i} a_k \operatorname{reg}_k,$$

(remind that $s = \frac{m-1}{2}$ for odd m and $s = \frac{m}{2}$ for even m).

Since the $DD^m(1)$ sequences $\{c_i\}_i$ realize all $b_k \operatorname{reg}_k$ for $k|r$, we obtain the following lemma.

Lemma 5.3. $\sum_{k|r} b_k \operatorname{reg}_k$ *can be represented as the sum of h $DD^m(1|r)$ sequences* mod reg_1 *if and only if there exists a family of subsets of $A_1, \ldots, A_h \subset$* $\operatorname{Div}(r)$ *satisfying*

(i) $\#A_i \leq s$ *for* $i = 1, \ldots, h$;
(ii) $\bigcup_i \overline{A}_i = \operatorname{Div}(r)$.

Notice that condition (ii) is equivalent to

(ii)′ for each $k|r$, $k \neq 1$, there exist an $i = 1, \ldots, h$ and a subset $K \subset A_i$ such that $k = \operatorname{LCM}(K)$.

As a consequence, we get the following lemma.

Lemma 5.4. *Let us consider a minimal family of subsets $A_1, \ldots, A_{v_0} \subset \operatorname{Div}(r)$ satisfying*

$$\#A_i \leq s, \tag{5.4}$$

$$\forall_{1 \neq k|r} \exists_i \exists_{K \subset A_i} \operatorname{LCM}(K) = k. \tag{5.5}$$

Then

$$v_0 = D^m_r[f] \mod \operatorname{reg}_1 .$$

5.2. Finding $D_r^m[f]$

Now we may take into account also the coefficient at reg_1.

Theorem 5.5. *Let $f : M \to M$ and assume that our Standing Assumptions* 5.1 *are satisfied. Let v_0 be a minimal number for which there exist sets $A_1, \dots, A_{v_0}$ satisfying conditions* (5.4) *and* (5.5).

Then, for even m there is

$$D_r^m[f] = \begin{cases} v_0 & \text{if } b_1 = v_0 \text{ or there exists a decomposition} \\ & A_1, \dots, A_{v_0} \text{ in which } \#A_i < \frac{m}{2} \text{ for some } i, \\ v_0 + 1 & \text{otherwise.} \end{cases} \tag{5.6}$$

While for odd m there is

$$D_r^m[f] = \begin{cases} v_0 & \text{if } |b_1| \le v_0 \text{ or there exists a decomposition} \\ & A_1, \dots, A_{v_0} \text{ in which } \#A_i < \frac{m-1}{2} \text{ for some } i, \\ v_0 + 1 & \text{otherwise.} \end{cases} \tag{5.7}$$

Proof. Consider the case of odd m first. If $\#A_i < \frac{m-1}{2}$, then we may replace, in a minimal decomposition realizing Lefschetz numbers modulo reg_1, a sequence of the types (B^o)–(D^o) by (A^o) with a prescribed coefficient at reg_1. Thus we can realize also b_1.

Now assume that $\#A_i = \frac{m-1}{2}$ for all i. We have v_0 sequences of the types (B^o)–(D^o) and we would like to adjust them in such a way that the sum of their coefficients at reg_1 gives b_1. In other words, we can use t_B, t_C, t_D sequences of the types (B^o), (C^o), (D^o), respectively, where t_B, t_C, t_D are prescribed nonnegative integers satisfying $t_B + t_C + t_D = v_0$. Since the contribution of each of these sequences to b_1 is $+1$, -1, 0, respectively, we may force them to obtain b_1 in sum if and only if $-b_1 \le v_0 \le b_1$. Then we need no extra sequences, hence $D_r^m[f] = v_0$.

If none of the conditions in (5.7) is satisfied, we have to use one sequence more of the type (A^o) with the coefficient $a_1 = b_1$. If m is even, the proof is analogous, with the difference that we can use only sequences of the type (F^e). □

Remark 5.6. In the first part of our Standing Assumption (II) we restrict ourselves to the simpler case of odd r. Our aim is to describe the essence of the introduced method rather than use it to find the exact formulas in every case. For even r it could be complicated, however also possible, for example for any self-map f of S^3 the value of $D_r^3[f]$ was found also for even r in [7].

Remark 5.7. Notice that in case the second part of Standing Assumption (II) is not satisfied, i.e., there are some $b_k = 0$ in the periodic expansion of Lefschetz numbers in (5.1), then the right-hand sides of equalities (5.6) and (5.7) give the upper bound for the number of $DD^m(1)$ sequences in the decomposition of $\{L(f^n)\}_{n|r}$. As a consequence, we always get (independently of the map) the estimates from above for the minimal number of r-periodic points in the smooth homotopy class of a given map.

6. $D^m_r[f]$ in case r is a product of different odd primes

6.1. Reduction to the combinatorial problem

Remind that the dimension $m = 2s$ or $m = 2s + 1$, where $s \geq 2$, and that Standing Assumptions 5.1 are satisfied. We define $I_v = \{1, \dots, v\}$ and by 2^{I_v} we denote the collection of all subsets of I_v.

Lemma 6.1. *Assume that $r = p_1 \cdots p_v$ is a product of different odd primes. Then $D^m_r[f]$ modulo reg_1 is equal to the least number h such that there is a family of subsets $\mathcal{B}_1, \dots, \mathcal{B}_h \subset 2^{I_v}$ satisfying*

(1) *$\#\mathcal{B}_i \leq s$ for $i = 1, \dots, h$;*
(2) *for each $J \subset I_v$ $(J \neq \emptyset)$ there exist an $i = 1, \dots, h$ and a subfamily $\mathcal{B}'_i \subset \mathcal{B}_i$ such that J is the union of all sets contained in $\mathcal{B}'_i$.*

Condition (2) *may also be reformulated as*

$$\bigcup_{i=1}^{h} \bigcup_{\mathcal{B}'_i \subset \mathcal{B}_i} \bigcup_{B \in \mathcal{B}'_i} B = 2^{I_v} \setminus \{\emptyset\}. \tag{6.1}$$

Proof. We will show that conditions (1) and (2) of Lemma 6.1 are equivalent to conditions (5.4) and (5.5). As, by our assumption, $r = p_1 \cdots p_v$ is a product of v different odd primes, there is a natural bijection $D : \mathrm{Div}(r) \to 2^{I_v} \setminus \{\emptyset\}$ given by

$$D(p_{i_1} \cdots p_{i_t}) = \{i_1, \dots, i_t\}.$$

Furthermore, it can be extended to a bijection $\tilde{D} : 2^{\mathrm{Div}(r)} \to 2^{2^{I_v} \setminus \{\emptyset\}}$ by

$$\tilde{D}(\{r_1, \dots, r_s\}) = \{D(r_1), \dots, D(r_s)\}.$$

Now, condition (5.5), i.e.,

$$\forall_{1 \neq k | r} \exists_i \exists_{K \subset A_i} \mathrm{LCM}(K) = k,$$

may be translated into

$$\forall_{\emptyset \neq D(k) \subset 2^{I_v}} \exists_i \exists_{\tilde{D}(K) \subset \tilde{D}(A_i)} D(\mathrm{LCM}(K)) = D(k). \tag{6.2}$$

Let us denote $D(k) := J$, $\tilde{D}(K) := \mathcal{B}'_i$, $\tilde{D}(A_i) = \mathcal{B}_i$, and notice that if $K = \{r_1, \dots, r_s\}$, then the condition $D(\mathrm{LCM}(K)) = D(k)$ takes the form

$$\begin{aligned} J = D(k) = D(\mathrm{LCM}(K)) = D(\mathrm{LCM}\{r_1, \dots, r_s\}) \\ = D(r_1) \cup \cdots \cup D(r_s) = \bigcup_{D(r_i) \in \tilde{D}(K)} D(r_i) = \bigcup_{B \in \mathcal{B}'_i} B. \end{aligned}$$

Thus we obtained exactly condition (2) of Lemma 6.1.

By the equality $\tilde{D}(A_i) = \mathcal{B}_i$, condition (5.4) is obviously transformed into condition (1).

The inverse map D^{-1} gives the inverse transformation of the conditions, which shows that they are equivalent. □

6.2. The formula for $D^m_r[f]$ in case r is a multiple of different odd primes

By the previous section, $D^m_r[f]$ modulo reg_1 is equal to the number given by Lemma 6.1. As a consequence, its computation reduces to the following.

Problem 6.2. We fix a natural number $s \geq 2$. For a given number $v \geq s$ we denote by $h_s(v)$ the least natural number satisfying: *there exist families $\mathcal{B}_1, \ldots, \mathcal{B}_{h_s(v)}$, where $\mathcal{B}_i \subset 2^{I_v} \setminus \{\emptyset\}$ and moreover*

(1) *$\#\mathcal{B}_i \leq s$ for $i = 1, \ldots, h_s(v)$,*
(2) *for each nonempty $J \subset \{1, \ldots, v\}$ there exist an $i \in \{1, \ldots, h_s(v)\}$ and a subfamily $\mathcal{B}'_i \subset \mathcal{B}_i$ such that J is the union of all sets contained in $\mathcal{B}'_i$.*

Find the explicit formula for $h_s(v)$.

The next theorem gives a formula for the number $h_s(v)$. To make this formula uniform we will use the following convention. We will uniquely represent each natural number v as $v = k \cdot s + R$, where $k \in \mathbb{N} \cup \{0\}$ and $R = 1, \ldots, s$. In particular, if s divides v, then $v = k \cdot s + s$.

Theorem 6.3. *Let f be a self-map of m-dimensional manifold M ($m = 2s$ or $m = 2s + 1$) and let our Standing Assumptions 5.1 be satisfied. Let r be a product of v different odd primes, where $v \geq s$. We represent v in the form $v = k \cdot s + R$, where $R = 1, \ldots, s$ and $k \in \mathbb{N} \cup \{0\}$. Then*

$$D^m_r[f] \mod \mathrm{reg}_1 = h_s(v) = \frac{2^{sk+R} - 2^R}{2^s - 1} + 1. \tag{6.3}$$

In other words,

$$h_s(v) = h_s(sk + R) = \left(\textit{the least integer} \geq \frac{2^{sk+R} - 1}{2^s - 1}\right). \tag{6.4}$$

Before we give the proof of Theorem 6.3, we will prove some helpful lemmas.

Lemma 6.4. $\frac{2^{sk+R}-2^R}{2^s-1} + 1$ *is the least integer greater than or equal to* $\frac{2^{sk+R}-1}{2^s-1}$.

Proof. Let us notice that

$$\begin{aligned}\frac{2^{sk+R} - 2^R}{2^s - 1} + 1 &= \frac{2^{sk+R} + 2^s - 2^R - 1}{2^s - 1} \\ &= \frac{2^R(2^{sk} - 1) + (2^s - 1)}{2^s - 1} \\ &= 2^R \cdot \frac{(2^s)^k - 1}{2^s - 1} + 1\end{aligned}$$

is an integer. On the other hand,

$$\frac{2^{sk+R} - 2^R}{2^s - 1} + 1 = \frac{2^{sk+R} + 2^s - 2^R - 1}{2^s - 1} = \frac{2^{sk+R} - 1}{2^s - 1} + \frac{2^s - 2^R}{2^s - 1}.$$

To complete the proof, it remains to notice that $0 \leq \frac{2^s-2^R}{2^s-1} < 1$ for $R = 1, \ldots, s$. $\square$

The next lemma shows that the sequence expressed by the right-hand side of (6.3) can be given inductively.

Lemma 6.5. *The sequence* $a(sk+R) = \frac{2^{sk+R}-2^R}{2^s-1} + 1$, *where* $R = 1, \dots, s$, $k \in \mathbb{N} \cup \{0\}$ *(i.e, expressed by the right-hand side of* (6.3) *of Theorem* 6.3*) is given by the following recurrence (for* $v \geq s$*):* $a(s) = 1$,

$$a(sk+R+1) = \begin{cases} 2a(sk+R) - 1 & \text{when } R < s, \\ 2a(sk+R) + 1 & \text{when } R = s. \end{cases}$$

Proof. Since in our convention $s = s \cdot 0 + s$,

$$a(s) = \frac{2^{s \cdot 0 + s} + 2^s - 2^s - 1}{2^s - 1} = 1,$$

which proves the first inductive step.

Now, we assume that the formula holds for $sk+R$ and we will prove it for $sk+R+1$. We will consider two cases in the dependance on the value of R.

Case I $(R = s)$**.** Then,

$$\begin{aligned} 2 \cdot a(sk+s) + 1 &= 2 \cdot \left(\frac{2^{sk+s} - 2^s}{2^s - 1} + 1 \right) + 1 \\ &= 2 \cdot \frac{2^{sk+s} - 2^s + 2^s - 1}{2^s - 1} + 1 \\ &= \frac{2^{s(k+1)+1} - 2^1}{2^s - 1} + 1 \\ &= a(s(k+1)+1). \end{aligned}$$

Case II $(R \leq s-1)$**.** Then,

$$\begin{aligned} 2 \cdot a(sk+R) - 1 &= 2 \cdot \left(\frac{2^{sk+R} - 2^R}{2^s - 1} + 1 \right) - 1 \\ &= \frac{2^{sk+R+1} - 2^{R+1}}{2^s - 1} + 1 \qquad (6.5) \\ &= a(sk+R+1). \end{aligned}$$

□

Proof of Theorem 6.3.

($\geq$) We notice that a family containing s subsets realizes at most $2^s - 1$ nonempty subsets in I_{sk+R}. Thus to realize all subsets we need at least $\frac{2^{sk+R}-1}{2^s-1}$ such families. It remains to recall that (cf. (6.4)) $\frac{2^{sk+R}-2^R}{2^s-1} + 1$ is the least integer greater than or equal to $\frac{2^{sk+R}-1}{2^s-1}$.

($\leq$) We will write below for short $h(v)$ instead of $h_s(v)$. We show inductively that for each number $v = sk+R$, where $k \geq 0$, $R = 1, \dots, s$, there is a family $\mathcal{A}_{sk+R} = \{A_1, \dots, A_{h(sk+R)}\}$ realizing (in the sense defined in Problem 2.1) each subset in I_{sk+R}, where

- A_i is a family containing s subsets of I_{sk+R} for $i < h(sk+R)$,
- $A_{h(sk+R)} = \{\{sk+1\},\dots,\{sk+R\}\}$ (hence $A_{h(sk+R)}$ contains $R \le s$ subsets of I_{sk+R}).

We start the induction with the number s. Then each subset in $I_s = \{1,\dots,s\}$ is a sum of a family of subsets in $\{\{1\},\dots,\{s\}\}$, which agrees with $h(s) = 1$.

Now we assume that the theorem holds for $sk+R$. This means that all subsets in I_{sk+R} can be realized by a family $\mathcal{A}_{sk+R} = \{A_1,\dots,A_{h(sk+R)}\}$, where $\#A_i = s$ for $i < h(sk+R)$ and $A_{h(sk+R)} = \{\{sk+1\},\dots,\{sk+R\}\}$. Now we proceed the inductive step for two cases.

Case 1 ($R = 1,\dots,s-1$). We will show that I_{sk+R+1} can be realized by the family

$$\begin{aligned}\mathcal{A}_{sk+R+1} = \Big\{&A_1,\dots,A_{h(sk+R)-1},A'_1,\dots,A'_{h(sk+R)-1},\\ &\{\{sk+1\},\dots,\{sk+R+1\}\}\Big\},\end{aligned} \tag{6.6}$$

where A'_i is obtained from A_i by adding the element $sk+R+1$ to each set in A_i. In fact, for a subset $B \subset I_{sk+R+1}$ let us consider three subcases:

(i) $sk+R+1 \notin B$. Then $B \subset I_{sk+R}$, hence by inductive assumption B can be realized by the family

$$\big\{A_1,\dots,A_{h(sk+R)-1},\{\{sk+1\},\dots,\{sk+R\},\{sk+R+1\}\}\big\} \subset \mathcal{A}_{sk+R+1}.$$

(ii) $sk+R+1 \in B$ but $B \neq \{sk+R+1\}$. Here, by the same argument as above, $B \setminus \{sk+R+1\}$ can be realized by

$$\big\{A_1,\dots,A_{h(sk+R)-1},\{\{sk+1\},\dots,\{sk+R\},\{sk+R+1\}\}\big\} \subset \mathcal{A}_{sk+R+1}.$$

In consequence, B can be realized by

$$\big\{A'_1,\dots,A'_{h(sk+R)-1},\{\{sk+1\},\dots,\{sk+R\},\{sk+R+1\}\}\big\} \subset \mathcal{A}_{sk+R+1}.$$

(iii) $B = \{sk+R+1\}$. Now $B \in \{\{sk+1\},\dots,\{sk+R+1\}\} \in \mathcal{A}_{sk+R+1}$.

Summing up, the considered family (6.6) realizes all nonempty subsets of $I_{sk+R+1} = \{1,\dots,sk+R+1\}$. It remains to notice that counting the number of subfamilies in (6.6) we get

$$\begin{aligned}\#\mathcal{A}_{sk+R+1} &= 2(\#\mathcal{A}_{sk+R}-1)+1\\ &= 2(a(sk+R)-1)+1 = 2a(sk+R)-1\\ &= a(sk+R+1),\end{aligned} \tag{6.7}$$

where the last equality comes from Lemma 6.5.

Case 2 ($R = s$). Then by the inductive assumption I_{sk+s} can be realized by a family $\mathcal{A}_{sk+s} = \{A_1, \dots, A_{h(sk+R)}\}$, where $\#A_i = s$. In such a case, I_{sk+s+1} can be realized by

$$\mathcal{A}_{sk+s+1} = \{A_1, \dots, A_{h(sk+R)}, A'_1, \dots, A'_{h(sk+R)}, \{sk+s+1\}\}, \tag{6.8}$$

where A'_i is obtained by adding the element $sk + s + 1$ to each set in A_i. Again, counting the number of subfamilies in (6.8) and applying Lemma 6.5 we get

$$\#\mathcal{A}_{sk+s+1} = 2\#\mathcal{A}_{sk+s} + 1 = 2a(sk+s) + 1 = a(sk+s+1),$$

which completes the proof. □

In the final theorem below we take into account also the coefficient a_1 and find the value of $D_r^m[f]$.

Theorem 6.6. *Let f be a self-map of m-dimensional manifold M ($m = 2s$ or $m = 2s+1$) and let our Standing Assumptions* 5.1 *be satisfied. Let r be a product of v different odd primes, where $v \geq s$. We represent v in the form $v = k \cdot s + R$, where $R = 1, \dots, s$ and $k \in \mathbb{N} \cup \{0\}$. Then*

$$D_r^m[f] = \begin{cases} h_s(v) & \textit{if } (s \nmid v) \textit{ or } (L(f) = h_s(v)) \\ & \textit{or } (m \textit{ is odd and } |L(f)| < h_s(v)), \\ h_s(v) + 1 & \textit{otherwise.} \end{cases} \tag{6.9}$$

where $h_s(v) = h_s(sk+R) = \frac{2^{sk+R} - 2^R}{2^s - 1} + 1$.

Proof. Let us reformulate equalities (5.6) and (5.7) of Theorem 5.5, expressing them by a use of the equivalence given in Lemma 6.1, and taking into account that $v_0 = h_s(v)$ and $b_1 = L(f)$, $s = \frac{m}{2}$ for even m and $s = \frac{m-1}{2}$ for odd m. Then for even m we obtain (in terms described in Problem 6.2)

$$D_r^m[f] = \begin{cases} h_s(v) & \text{if } L(f) = h_s(v) \text{ or there exists a family} \\ & \mathcal{B}_1, \dots, \mathcal{B}_{h_s(v)} \text{ in which } \#\mathcal{B}_i < s \text{ for some } i, \\ h_s(v) + 1 & \text{otherwise.} \end{cases} \tag{6.10}$$

While for odd m there is

$$D_r^m[f] = \begin{cases} h_s(v) & \text{if } |L(f)| \leq h_s(v) \text{ or there exists a family} \\ & \mathcal{B}_1, \dots, \mathcal{B}_{h_s(v)} \text{ in which } \#\mathcal{B}_i < s \text{ for some } i, \\ h_s(v) + 1 & \text{otherwise.} \end{cases} \tag{6.11}$$

By the part ($\geq$) of the proof of Theorem 6.3 we get that in case $s|v$, every set of the family realizing all nonempty sets in I_v must contain s elements. On the other hand, if $s \nmid v$, then in part ($\leq$) it was shown that there exists a family realizing all nonempty sets in I_v with one set of the form $\{\{sk+1\},\dots,\{sk+R\}\}$. This set has less than s elements, because $s \nmid v = sk+R$, so $R < s$.

Summing up, we can replace the condition appearing in (6.10) and (6.11) namely,

there exists a decomposition $\mathcal{B}_1,\dots,\mathcal{B}_{h_s(v)}$ *in which* $\#\mathcal{B}_i < s$ *for some* i

by the statement $s \nmid v$, which gives us the conditions in (6.9). $\square$

Remark 6.7. Note that if we assume $v < s$, then obviously $D_r^m[f] = 1$, and that is the reason why we considered only the case of $v \geq s$ in Theorem 6.6.

Remark 6.8. Let us notice that under our Standing Assumptions 5.1 the value $D_r^m[f]$ depends only on the dimension ($m = 2s$ or $m = 2s+1$) and the value of r.

7. Estimation for $D_r^m[f]$

In this section, we extend Theorem 6.3 for a product of the primes that are not necessarily different. We will use the notation introduced in the previous section. Instead of giving a closed formula, which would be very complicated, we provide an estimation for $D_r^m[f]$ in case $r = p_1^{a_1}\cdots p_w^{a_w} p_{w+1}\cdots p_v$, where p_i are different odd primes and $w+s \leq v$. By $C(x)$ we will denote the smallest integer not less than x (so-called ceiling function).

Lemma 7.1. *Let f be a self-map of m-dimensional manifold M ($m = 2s$ or $m = 2s+1$) and let our Standing Assumptions* 5.1 *be satisfied. Let $r = p_1^{a_1}\cdots p_t^{a_t}$, where $p_1,\dots,p_t$ are different odd primes. Then*

$$(D_r^m[f] \bmod \operatorname{reg}_1) \geq C\left(\frac{(a_1+1)\cdots(a_t+1)-1}{2^s-1}\right). \tag{7.1}$$

Proof. The number r has $(a_1+1)\cdots(a_t+1)-1$ divisors different from 1. By Lemma 5.4, $D_r^m[f]$ is the minimal number of sets, having at most s elements, that produce every divisor as the least common multiplicity of some of their subsets. On the other hand, every set of divisors consisting of s elements has 2^s-1 nonempty subsets. Thus, every set of s elements produces at most 2^s-1 divisors different from 1. $\square$

Lemma 7.2. *For arbitrary real numbers $a_1,\dots,a_k$ the following inequality holds:*

$$C(a_1)+\cdots+C(a_k) \leq C(a_1+\cdots+a_k)+k-1.$$

Theorem 7.3. *Let f be a self-map of m-dimensional manifold M ($m = 2s$ or $m = 2s+1$) and let our Standing Assumptions* 5.1 *be satisfied. Let $r = p_1^{a_1}\cdots p_w^{a_w} p_{w+1}\cdots p_v$, where $w+s \leq v$. Then we have the following estimation:*

$$G \leq D_r^m[f] \mod \operatorname{reg}_1 \leq G+H, \tag{7.2}$$

where

$$G = C\left(\frac{2^w + (a_1+1)\cdots(a_w+1)\cdot(2^{v-w}-1)-1}{2^s-1}\right),$$

$$H = [(a_1+1)\cdots(a_w+1) - 2^w] + C\left(\frac{(a_1+1)\cdots(a_w+1)-2^w}{s}\right).$$

Proof. Obviously, the left-hand side of (7.2) holds by Lemma 7.1. We prove now the right-hand side of (7.2).

For the convenience of the reader we first prove the simpler case of $w = 1$.

By Lemma 5.4, to determine $D^m_r[f]$ one has to find the minimal number of sets A_i, each having no more than s elements, that provide the realization (in the sense described in Lemma 5.4) of the set of all nontrivial divisors. We will call below each such A_i s-set.

If $w = 1$, then $r = p_1^a p_2 \cdots p_v$ and each nontrivial divisor of r has the form

$$p_1^{\beta_1}\cdots p_v^{\beta_v},$$

where $0 \le \beta_1 \le a$ and $\exists_{1\le i\le v}\beta_i \ne 0$ and $0 \le \beta_i \le 1$ for $i = 2,\dots,v$ with integer values of β_i. We have to demonstrate that to realize these divisors one can use no more than

$$\begin{aligned} C\left(\frac{2^1+(a+1)\cdot(2^{v-1}-1)-1}{2^s-1}\right) + [(a+1)-2] + C\left(\frac{(a+1)-2}{s}\right) \\ = C\left(\frac{(a+1)\cdot 2^{v-1}-a}{2^s-1}\right) + (a-1) + C\left(\frac{a-1}{s}\right) \quad (7.3)\end{aligned}$$

s-sets.

Let us notice that the set $\mathrm{Div}(r)$ of all nontrivial divisors of $r = p_1^a \cdot p_2\cdots p_v$ splits into the following disjoint sum:

$$\mathrm{Div}(r) = \widetilde{D}_0 \cup D_1 \cup \cdots \cup D_a, \qquad (7.4)$$

where $D_i = \{p_1^i \cdot p_2^{\beta_2}\cdots p_v^{\beta_v} : \beta_j \in \{0,1\} \text{ for } j = 2,\dots,v\}$, $\widetilde{D}_0 = D_0 \setminus \{1\}$. We notice that

(1) to realize $\widetilde{D}_0 \cup D_1$, by Theorem 6.3, it is enough to take $h_s(v)$ s-sets;
(2) to realize elements in

$$D_i \setminus \{p_1^i\} = \{p_1^i p_2^{\beta_2}\cdots p_v^{\beta_v} : 0 \le \beta_j \le 1,\ \text{not all } \beta_2,\dots,\beta_v \text{ are equal to } 0\}$$

it is enough, again by Theorem 6.3, to use $h_s(v-1)$ s-sets for any fixed $i = 2,\dots,a$;
(3) the above families realize all nontrivial divisors of $r = p_1^a p_2\cdots p_v$ with the exception of $\{p^2, p^3, \dots, p^a\}$. To realize this set it is enough to use $C(\frac{a-1}{s})$ s-sets.

Summing up the number of s-sets needed to realize the above families and using (6.4) for $h_s(v)$ and Lemma 7.2, we obtain

$$\begin{aligned}
&(D_r^m[f] \bmod \mathrm{reg}_1)\\
&\quad \le h_s(v) + (a-1)\cdot h_s(v-1) + C\left(\frac{a-1}{s}\right)\\
&\quad = C\left(\frac{2^v-1}{2^s-1}\right) + (a-1)\cdot C\left(\frac{2^{v-1}-1}{2^s-1}\right) + C\left(\frac{a-1}{s}\right)\\
&\quad \le C\left(\frac{2^v-1}{2^s-1} + (a-1)\frac{2^{v-1}-1}{2^s-1}\right) + (a-1) + C\left(\frac{a-1}{s}\right)\\
&\quad = C\left(\frac{(a+1)\cdot 2^{v-1}-a}{2^s-1}\right) + (a-1) + C\left(\frac{a-1}{s}\right),
\end{aligned} \tag{7.5}$$

as required.

Now, we will prove the general case. We consider

$$r = p_1^{a_1}\cdots p_w^{a_w} p_{w+1}\cdots p_v,$$

where $w + s \le v$.

We define

$$D_{(\alpha_1,\ldots,\alpha_w)} = \left\{p_1^{\alpha_1}\cdots p_w^{\alpha_w}\cdot p_{w+1}^{\beta_{w+1}}\cdots p_v^{\beta_v} : \beta_j \in \{0,1\} \text{ for } j = w+1,\ldots,v\right\}.$$

Let us notice that now $\mathrm{Div}(r)$ is a disjoint sum

$$\begin{aligned}
&\left(\bigcup_{(\alpha_1,\ldots,\alpha_w)} D_{(\alpha_1,\ldots,\alpha_w)} \setminus \{1\}\right) \cup \left(\bigcup_{(\alpha_1,\ldots,\alpha_w)} \left(D_{(\alpha_1,\ldots,\alpha_w)} \setminus \{p_1^{\alpha_1}\cdots p_w^{\alpha_w}\}\right)\right)\\
&\qquad \cup \bigcup_{(\alpha_1,\ldots,\alpha_w)} \{p_1^{\alpha_1}\cdots p_w^{\alpha_w}\},
\end{aligned} \tag{7.6}$$

where the first term of the summation (7.6) runs over the set $\{0 \le \alpha_1,\ldots,\alpha_w \le 1\}$, while the second and the third run over the remaining part of $\{0 \le \alpha_i \le a_1,\ldots,0 \le \alpha_w \le a_w\}$.

We notice that

(1) to realize, in the sense described in Lemma 5.4, the set

$$\bigcup_{(\alpha_1,\ldots,\alpha_w)} D_{(\alpha_1,\ldots,\alpha_w)} \setminus \{1\},$$

where the summation extends only over indices $\alpha_i \in \{0,1\}$, it is enough, by Theorem 6.3, to use $h_s(v)$ s-sets. Furthermore, notice that there are $(a_1+1)\cdots(a_w+1) - 2^w$ of the other divisors of $p_1^{a_1}\cdots p_w^{a_w}$;

(2) to realize $D_{(\alpha_1,\ldots,\alpha_w)} \setminus \{p^{\alpha_1}\cdots p^{\alpha_w}\}$ it is enough to use $h_s(v-w)$ of s-sets for any fixed $(\alpha_1,\ldots,\alpha_w)$;

(3) the last summand of (7.6) contains $(a_1+1)\cdots(a_w+1) - 2^w$ elements, so to realize them one can use at most $C\left(\frac{(a_1+1)\cdots(a_w+1)-2^w}{s}\right)$ s-sets.

Finally, we get by (6.4) and Lemma 7.2 that the number of s-sets needed to realize the whole set $\mathrm{Div}(r)$ (which is an upper bound for $D_r^m[f] \mod \mathrm{reg}_1$) does not exceed

$$\begin{aligned}
& h_s(v) + ((a_1+1)\cdots(a_w+1) - 2^w) \cdot h_s(v-w) \\
& \quad + C\left(\frac{(a_1+1)\cdots(a_w+1)-2^w}{s}\right) \\
& = C\left(\frac{2^v-1}{2^s-1}\right) + ((a_1+1)\cdots(a_w+1)-2^w)\cdot C\left(\frac{2^{v-w}-1}{2^s-1}\right) \\
& \quad + C\left(\frac{(a_1+1)\cdots(a_w+1)-2^w}{s}\right) \\
& \leq C\left(\frac{2^v-1}{2^s-1} + ((a_1+1)\cdots(a_w+1)-2^w)\cdot\frac{2^{v-w}-1}{2^s-1}\right) \\
& \quad + ((a_1+1)\cdots(a_w+1)-2^w) + C\left(\frac{(a_1+1)\cdots(a_w+1)-2^w}{s}\right) \\
& = C\left(\frac{2^w+(2^{v-w}-1)(a_1+1)\cdots(a_w+1)-1}{2^s-1}\right) \\
& \quad + ((a_1+1)\cdots(a_w+1)-2^w) + C\left(\frac{(a_1+1)\cdots(a_w+1)-2^w}{s}\right). \qquad \square
\end{aligned}$$

Remark 7.4. Notice that under our Standing Assumptions 5.1, by Theorem 5.5, we get

$$D_r^m[f] \mod \mathrm{reg}_1 \leq D_r^m[f] \leq D_r^m[f] \mod \mathrm{reg}_1 + 1. \tag{7.7}$$

As a consequence, by Theorem 7.3 we get the following estimation for $D_r^m[f]$:

$$G \leq D_r^m[f] \leq G + H + 1, \tag{7.8}$$

where

$$\begin{aligned}
G &= C\left(\frac{2^w + (a_1+1)\cdots(a_w+1)\cdot(2^{v-w}-1)-1}{2^s-1}\right), \\
H &= [(a_1+1)\cdots(a_w+1)-2^w] + C\left(\frac{(a_1+1)\cdots(a_w+1)-2^w}{s}\right).
\end{aligned}$$

Remark 7.5. The obtained estimation for $D_r^m[f]$ gives the lower bound for the number of periodic points in the smooth homotopy class of f (left-hand side of inequality (7.8)) and states that one can always find in the smooth homotopy class of f a map with no more than $(G+H+1)$ r-periodic points (right-hand side of inequality (7.8)).

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