Quasi-Extended Asymptotic Functions*

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The class F of "quasi-extended asymptotic functions" introduced in the present paper contains all extended asymptotic functions [8, (3.1)] (in particular, all examples constructed in [9, Sec. 1]). But F contains also some new asymptotic functions very similar to the Schwartz distributions. On the other hand, every two quasi-extended asymptotic functions can be multiplied as opposed to the Schwartz distributions; in particular, the square δ^2 of an asymptotic function δ similar to Dirac's delta-function is constructed as an example. The connection with the asymptotic functions introduced in [2] and [4] is established.

1. The Class of Quasi-Extended Asymptotic Functions

The class of the quasi-extended asymptotic functions F we are going to introduce in this paper contains all extended asymptotic functions [8, Sec. 3] but it contains also some new asymptotic functions, which cannot be obtained as an extension of any ordinary function. The important thing is that the class F is closed with respect to the addition and multiplication, which is essential for the further applications.

(1.1) Definition (Quasi-Extended Functions). An asymptotic function

$$f: D \to A^*, \tag{1.2}$$

where $D \subseteq A$ will be called a quasi-extended asymptotic function (or simply, a quasi-extended function) if it can be represented as

$$f(a) = \varphi_{as}(a, b) + o^{\nu(a)}, \qquad a \in D, \tag{1.3}$$

where $\nu(a)$ is the order of $f(a)$, φ is some continuous ordinary function of two real variables and b is a fixed asymptotic number. In other words, there exists a continuous function φ of the type

$$\varphi: X \times Y \to C, \tag{1.4}$$

where X and Y are two open subsets of R, such that (see [8, (2.1)])

$$X \subseteq D \subset X_{as} \tag{1.5}$$

and there exists an asymptotic number

$$b \in Y_{as} \tag{1.6}$$

for which the asymptotic extension $\varphi_{as}(a, b)$ [8, (3.1)] exists for all $a \in D$ and (1.3) holds. The set of all quasi-extended asymptotic functions will be denoted by F.

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(1.7) Definition (Generating Couples). If $f \in F$ and (1.3) holds, we will call the couple (φ, b) a generating couple of f. We shall say also that f is generated from (φ, b). The set of all generating couples of f will be denoted by $\operatorname{Gen} f$, i. e.

$$\operatorname{Gen} f = \{(\varphi, b): \text{ (1.3) holds}\}. \tag{1.8}$$

(1.9) Definition (Regular and Singular Functions). A quasi-extended function of the type (1.2) will be called regular if there exists a continuous ordinary function of one real variable:

$$\varphi: X \to C, \tag{1.10}$$

where X is an open subset of R such that (1.5) holds and

$$f(a) = \varphi_{as}(a) + o^{\nu(a)}, \qquad a \in D, \tag{1.11}$$

is valid, where $\nu(a)$ is the order of $f(a)$. The quasi-extended functions which are not regular will be called singular.

(1.12) Remark: We have just defined the quasi-extended functions of one variable by means of ordinary functions of two variables. However, Definition (1.1) (together with Definition (1.7) and Definition (1.9)) can be naturally generalized in order to introduce quasi-extended functions of n variables. Instead of (1.3) the representation

$$f(a_1, \ldots, a_n) = \varphi_{as}(a_1, \ldots, a_n, b) + o^{\nu(a_1, \ldots, a_n)}, \quad (a_1, \ldots, a_n) \in D, \tag{1.13}$$

must be used, where φ is on ardinary function of $n+1$ variables of the type discussed in [8, Definition (3.1), (*ii*)]. We are going to consider the case $n=1$ only, but all results in this section can be easily generalized to the case $n>1$.

(1.14) Remark: Every extended asymptotic function [8, Definition (3.1)] is a regular quasi-extended function. Indeed, the representation

$$f(a) = \varphi_{as}(a) = \varphi_{as}(a) + o^{\nu(a)} \tag{1.15}$$

holds, where $\nu(a)$ is the order of $f(a) = \varphi_{as}(a)$, corresponding to [5, Theorem 6] (here φ_{as} is an asymptotic extension of φ [8, (3.1)]).

(1.16) Remark: Every asymptotic function of the type

$$f(a) = \varphi_{as}(a, b), \qquad a \in D, \tag{1.17}$$

where φ is a continuous ordinary function and b is a fixed asymptotic number, is also a quasi-extended function. However, if (1.17) is a singular function, then it is not an extended-asymptotic function, i. e. it is not an asymptotic extension of any ordinary function of one real variable. We see that F is richer of functions than the set of the extended functions is.

(1.18) Remark: The $\mathcal{O}$-valued function, i.e. the asymptotic function of the type

$$o^{\nu(a)}, \qquad a \in D,$$

as well as the I-valued functions, i. e. the functions of the type

$$1^{\lambda(a)}, \qquad a \in D$$

where ν and λ are mappings of the type

$$\nu: D \to Z \cup \{\infty\}$$

and

$$\lambda: D \to N_0 \cup \{\infty\}$$

respectively, are also quasi-extended functions. In fact, they are regular asymptotic functions.

(1.19) Remark: Finally, the asymptotic functions of the type

$$(1.20) \qquad f(a)=c+o^{\nu(a)}, \qquad a \in D,$$

where $c \in C$, are also regular quasi-extended asymptotic functions. We shall call them quasi-constant asymptotic functions (because, strictly speaking, they are not constant functions).

(1.21) Lema: The constant asymptotic function

$$(1.22) \qquad f(a)=b, \qquad a \in D,$$

where $b \in A^*$ is a quasi-extended function if and only if the number b is of the type

$$(1.23) \qquad b=c+o^{\nu},$$

where $c \in C$ and $\nu \in Z \cup \{\infty\}$. By the way, f is a special type of a quasi-constant function (see (1.20)) in this case.

Proof: The important fact here is that

$$(1.24) \qquad (\mathrm{const})_{as}=\mathrm{const}.$$

The following three lemmas follow directly from the fact that the values of any quasi-extended asymptotic functions are asymptotic numbers (just like the values of any asymptotic function). More precisely, these three lemmas follow directly from [5, Theorem 4]. That is why we are not going to give their proofs.

(1.25) Lemma: Let f be a quasi-extended asymptotic function and let (1.3) hold. Then

$$(1.26) \qquad f(a)=1^{\lambda(a)}\varphi_{as}(a, b), \qquad a \in D_0,$$

is valid, where

$$(1.27) \qquad D_0=\{a : a \in D, \ f(a) \notin \mathcal{O}\}$$

and $\lambda(a)$ is the relative order of $f(a)$ [5, Definition 5 (*iii*)].

Proof: The lemma follows directly from [5, Theorem 25].

(1.28) Definition (Additive and Multiplicative Forms). Let f be a quasi-extended function and let (1.3) and (1.26) hold. Then (1.3) will be called an additive form of f and (1.26) will be called a multiplicative form of f.

(1.29) Remark: We would like to stress that $\nu(a)$ in any additive form (1.3) of f is the order of $f(a)$ and $\lambda(a)$ in any multiplicative form (1.26) of f is the relative order of $f(a)$.

(1.30) Lemma: The asymptotic function of the type

$$(1.31) \qquad f(a)=\varphi_{as}(a, b)+o^{\nu(a)}, \qquad a \in D,$$

where φ is a continuous ordinary function of two real variables, $b \in A$ (is fixed) and $O^{\nu_0(a)}$, $a \in D$, is an arbitrary $\mathcal{O}$-valued function, is a quasi-extended function. Let $\tilde{\nu}(a)$ be the order of $\varphi_{as}(a, b)$. Then

$$(1.32) \qquad \nu(a)=\min\left[\tilde{\nu}(a), \nu_0(a)\right]$$

is the order of $f(a)$ and consequently, (1.3) for $\nu(a)$ determined by (1.32) is an additive form of f.

(1.33) Remark: (1.31) is not (in general) an additive form of f because $\nu_0(a)$ is not (in general) the order of $f(a)$.

(1.34) Lemma: Let f be a quasi-extended function and let (1.3) be its additive form. Then the formulae

$$(1.35) \qquad \mu(a)+\lambda(a)=\nu(a), \qquad a \in D,$$

$$(1.36) \qquad \widetilde{\mu}(a)+\widetilde{\lambda}(a)=\widetilde{\nu}(a), \qquad a \in D,$$

$$(1.37) \qquad \mu(a)=\min[\widetilde{\mu}(a), \ \nu(a)], \qquad a \in D,$$

as well as the inequalities:

$$(1.38) \qquad \mu(a)\leq\widetilde{\mu}(a), \qquad a \in D,$$

$$(1.39) \qquad \nu(a)\leq\widetilde{\nu}(a), \qquad a \in D,$$

$$(1.40) \qquad \lambda(a)\leq\widetilde{\lambda}(a), \qquad a \in D,$$

are valid, where $\mu(a)$, $\nu(a)$ and $\lambda(a)$ are the power, the order and the relative order of $f(a)$ respectively and $\widetilde{\mu}(a)$, $\widetilde{\nu}(a)$ and $\widetilde{\lambda}(a)$ are the power, the order and the relative order of $\varphi_{\mathrm{as}}(a, b)$ respectively (b is fixed).

Proof: An immediate consequence from [5, Theorem 4]. We are going to use the inequalities (1.38)—(1.40) very often in future and in particular during the proof of Theorem (4.1).

2. The generating set

In the theory of quasi-extended functions it is important to know the connection which may exist between two couples (φ, b) and (ψ, c), which generate the same quasi-extended function f, i. e. $(\varphi, b), (\psi, c) \in \mathrm{Gen} f$ (see (1.8)). In other words, we must describe somehow the generating set $\mathrm{Gen} f$ of any f provided one of its elements is known. The following two theorems deal with this question.

(2.1) Theorem: Let f be a quasi-extended asymptotic function and let (1.2)—(1.6) hold for some generating couple (φ, b) of f, i. e. $(\varphi, b) \in \mathrm{Gen} f$. Let $\nu(a)$ be the order of $f(a)$ (i. e. (1.3) be the additive form of f). Let, finally, ψ be a continuous ordinary function defined on $X \times Y$ and $c \in Y_{\mathrm{as}}$. Then $(\psi, c) \in \mathrm{Gen} f$, i. e.

$$(2.2) \qquad f(a)=\psi_{\mathrm{as}}(a, \ c)+o^{\nu(a)}, \qquad a \in D,$$

if and only if the following condition (denoted by $\otimes$) is valid: $\otimes$ For each $a \in D$, each $\alpha \in a$, each $\beta \in b$ and each $\gamma \in c$

$$(2.3) \qquad \lim_{s\to 0} s^{-n}[\varphi(\alpha(s), \ \beta(s))-\psi(\alpha(s), \ \gamma(s))]=0$$

for all $n \in Z$ such that $n\leq\nu(a)$.

(2.4) Remark: In the cases $\nu(a) \in Z$ (but not $\nu(a)=\infty$) (2.3) could be replaced by

$$(2.5) \qquad \lim_{s\to 0} s^{-\nu(a)}[\varphi(\alpha(s), \ \beta(s))-\psi(\alpha(s), \ \gamma(s))]=0$$

leaving out the expression "for all $n \in Z$ such that $n\leq\nu(a)$". We would like to notify that the proof of the theorem is based on the inequalities (1.38)—(1.40), as well as on some results of [5] and [6].

Proof: Let $\nu_1(a)$, $\nu_2(a)$ and $\nu_0(a)$ be the orders of $\varphi_{as}(a, b)$, $\psi_{as}(a, b)$ and $\varphi_{as}(a, b)-\psi_{as}(a, c)$ resp. Then (1.3) implies

$$(2.6) \qquad \nu(a)\leq\nu_1(a), \qquad a\in D,$$

corresponding to Lemma (1.34). Let (2.2) hold (together with (1.3)—(1.6), of course). If we subtract (1.3) and (2.2), we shall obtain

$$(2.7) \qquad o^{\nu(a)}=\varphi_{as}(a, b)-\psi_{as}(a, c)+o^{\nu(a)}, \quad a\in D,$$

which is equivalent to

$$(2.8) \qquad \varphi_{as}(a, b)-\psi_{as}(a, c)\subseteq o^{\nu(a)}, \quad a\in D.$$

On the other hand, (2.8) implies:

$$(2.9) \qquad \lim_{s\to 0} s^{-n}[\varphi(\alpha(s), \beta(s))=\psi(\alpha_1(s), \gamma(s))]=0$$

for all $a\in D$, all α, $\alpha_1\in a$, all $\beta\in b$, all $\gamma\in c$ and all $n\in Z$ such that $n\leq\nu(a)$. But (2.9) reduces to ⊗ in the case $\alpha_1=\alpha$; (*ii*) Let ⊗ hold (together with (1.3)—(1.6)). We must show that (2.2) holds, too. First of all ⊗ implies

$$(2.10) \qquad \nu(a)\leq\nu_2(a), \qquad a\in D.$$

Indeed, if we assume $\nu(a_0)>\nu_2(a_0)$ for some $a_0\in D$, we shall obtain (bearing in mind (2.6))

$$(2.11) \qquad \lim_{s\to 0} s^{-1}[\varphi(\alpha(s), \beta(s))-\psi(\alpha(s), \gamma(s))]\neq 0$$

for some $\alpha\in a_0$, some $\beta\in b$, some $\gamma\in c$ and all $n\in Z$, such that $\nu_2(a_0)<n\leq\nu(a_0)$, which contradicts ⊗ But (2.10) is equivalent to

$$(2.12) \qquad o^{\nu(a)}+o^{\nu_2(a)}=o^{\nu(a)}, \qquad a\in D,$$

corresponding to [5, (10)]. Moreover, (2.6) and (2.10) implies

$$(2.13) \qquad \nu(a)\leq\nu_0(a), \qquad a\in D,$$

which is equivalent to

$$(2.14) \qquad o^{\nu(a)}+o^{\nu_0(a)}=o^{\nu(a)}, \qquad a\in D,$$

since

$$(2.15) \qquad \nu_0(a)=\min[\nu_1(a), \nu_2(a)], \qquad a\in D,$$

corresponding to [5, (10)]. On the other hand, ⊗ means

$$(2.16) \qquad \varphi(\alpha, \beta)-\psi(\alpha, \gamma)\in o^{\nu(a)}, \qquad a\in D,$$

for all $\alpha\in a$, all $\beta\in b$ and all $\gamma\in c$. Consequently,

$$(2.17) \qquad \mathrm{as}\{\varphi(\alpha, \beta)-\psi(\alpha, \gamma)\}\subseteq o^{\nu(a)}, \qquad a\in D,$$

corresponding to [5, Lema 1]. Adding $o^{\nu(a)}$ to both sides of (2.17), bearing in mind [6, Theorem 14], we obtain

$$(2.18) \qquad o^{\nu(a)}+as\{\varphi(\alpha, \beta)-\psi(\alpha, \gamma)\}=o^{\nu(a)}, \qquad a\in D,$$

for all $\alpha\in a$, all $\beta\in b$ and all $\gamma\in c$. On the other hand (on the ground of [8, Lemma (3.22]) we have

$$(2.19) \qquad \varphi_{as}(a, \beta)-\psi_{as}(a, b)=\mathrm{as}\{\varphi(\alpha, \beta)-\psi(\alpha, \gamma)\}+o^{\nu_0(a)}, \qquad a\in D,$$

for any $\alpha \in a$, any $\beta \in b$ and any $\gamma \in c$. Let us add $o^{\nu(a)}$ to both sides of (2.19), bearing in mind (2.14) and (2.18):

$$(2.20) \qquad \varphi_{as}(a,\ b)-\psi_{as}(a,\ c)+o^{\nu(a)}=o^{\nu(a)}, \qquad a \in D.$$

Adding $\psi_{as}(a, c)$ to both sides of (2.20), bearing in mind (2.12), we obtain

$$(2.21) \qquad \varphi_{as}(a,\ b)+o^{\nu(a)}=\psi_{as}(a,\ c)+o^{\nu(a)}, \qquad a \in D,$$

which coincides with (2.2). The proof is finished.

The next theorem is quite similar to the above one. It deals with the same question in the special case $b=c$. We shall often use this theorem instead of the previous one because of its simplicity.

(2.22) Theorem: Let f be a quasi-extended function, let $\nu(a)$ be the order of $f(a)$ and let (1.3)—(1.6) hold for some generating couple (φ, b), i. e. $(\varphi, b) \in \operatorname{Gen} f$. Let ψ be another continuous ordinary function defined on $X \times Y$. Then $(\psi, b) \in \operatorname{Gen} f$, i. e.

$$(2.23) \qquad f(a)=\psi_{as}(a,\ b)+o^{\nu(a)}, \qquad a \in D,$$

if and only if the following condition (denoted by $**$) is valid: $**$ For each $a \in D$, each $\alpha \in a$ and each $\beta \in b$

$$(2.24) \qquad \lim_{s\to 0} s^{-n}[\varphi(\alpha(s),\ \beta(s)-\psi(\alpha(s),\ \beta(s))]=0$$

for all $n \in Z$ such that $n \leq \nu(a)$.

(2.25) Remark: If $\nu(a) \in Z$ (but not $\nu(a)=\infty$) (2.24) could be replaced by

$$(2.26) \qquad \lim_{s\to 0} s^{-\nu(a)}[\varphi(\alpha(s),\ \beta(s))-\psi(\alpha(s),\ \beta(s))]=0$$

leaving out the expression "for all $n \in Z \dots$ etc.".

(2.27) Remark: The above theorem could be formulated as follows: $(\psi, b) \in \operatorname{Gen} f$, i. e. (2.23) is valid, if and only if ψ can be represented in the form

$$(2.28) \qquad \psi(x,\ y)=\varphi(x,\ y)+\Delta(x,\ y), \qquad x \in X,\ y \in Y,$$

where Δ has the property: For each $a \in D$, each $\alpha \in a$ and each $\beta \in b$

$$(2.29) \qquad \lim_{s\to 0} s^{-n}\Delta(\alpha(s),\ \beta(s))=0$$

for all $n \in Z$ such that $n \leq \nu(a)$, where $\nu(a)$ be the order of $f(a)$.

Proof: The proof is quite analogous to the proof of Theorem (2.1); in fact, it coincides almost with it. We omit it.

3. Standard asymptotic number

Let f and g be two quasi-extended asymptotic functions. The following question arises: In which cases can f and g be represented in their additive or multiplicative forms for *the same asymptotic number b?* In other words, in which cases is there (the same) $b \in A$ for which $(\varphi, b) \in \operatorname{Gen} f$ and $(\psi, b) \in \operatorname{Gen} g$ for some (appropriately chosen) ordinary functions φ and ψ? The answer is "In all cases". Moreover, it turns out that there exists a standard asymptotic number b, namely, $b=s$ [8, (3.26)] such that every quasi-extended function can be generated by a couple (φ, s) for some ordinary function φ. The following theorem deals with that question.

(3.1) Theorem: Let f be a quasi-extended function, $\nu(a)$ be the order of $f(a)$ and let (1.3)—(1.6) hold for some generating couple $(\varphi, b) \in \operatorname{Gen} f$. Then $(\psi, s) \in \operatorname{Gen} f$, too, i. e.

$$f(a)=\psi_{as}(a, s)+o^{\nu(a)}, \qquad a \in D, \tag{3.2}$$

where

$$\psi(x, y)=\varphi(x, \beta(y)), \qquad x \in X, y \in o, \varepsilon) \tag{3.3}$$

for some (sufficiently small) $\varepsilon \in \mathfrak{R}$, $\varepsilon>0$ and any (arbitrarily chosen and fixed) $\beta \in b$.

Proof: The theorem is an immediate consequence of [8, Lemma (3.22) and Corollary (3.24)].

(3.4) Remark: Corresponding to Lemma (1.25), (3.2) implies

$$f(a)=1^{\lambda(a)}\psi_{as}(a, s), \qquad a \in D, \tag{3.5}$$

where $\lambda(a)$ is the relative order of $f(a)$ and D_0 is given by (1.27). In other words, (3.5) is a multiplicative form of $f(a)$ [Definition (1.28)].

(3.6) Corollary: Every two quasi-extended functions f_1 and f_2 can be represented in the forms (their additive forms)

$$f_1(a)=\varphi_{1as}(a, b)+o^{\nu_1(a)}, \qquad a \in D_1, \tag{3.7}$$

$$f_2(a)=\varphi_{2as}(a, b)+o^{\nu_2(a)}, \qquad a \in D_2, \tag{3.8}$$

respectively, *for the same* asymptotic number b and some ordinary functions φ_1 and φ_2. In other words, for every $f_1, f_2 \in F$ there exist the same $b \in A$ such that $(\varphi_1, b) \in \operatorname{Gen} f_1$ as well as $(\varphi_2, b) \in \operatorname{Gen} f_2$ for some φ_1 and φ_2.

Proof: Indeed, (3.7) and (3.8) are valid (at least) for $b=s$ corresponding to Theorem (3.1.)

(3.9) Remark: The above theorem suggests the following question: Why we did not fix $b=s$ at the very beginning of our approach (in Definition (1.1)) but used different (in general) b for the different quasi-extended asymptotic functions? The answer is: "For the sake of covenience". We mean that in some cases the choice of the ordinary function φ (which together with the asymptotic number b generates a given quasi-extended function f) can be done more easily if "b" is different from "s". For instance, in all examples of extended asymptotic functions constructed in [9, Sec. 1] (which are, at the same time, quasi-extended functions) the number b was chosen as $b=s+o^1$ (but not $b=s$). Otherwise, the definition [9, (1.24)]:

$$\delta(a)=\Delta_{as}(a, s+o^1), \qquad a \in \Omega,$$

of the asymptotic function δ where Δ is given in [9, (1.20)] (and $b=s+o^1$) must be replaced by the more complicated one (by $b=s$):

$$\delta(a)=\Delta_{as}(a, s)+o^{\nu(a)}, \qquad a \in \Omega,$$

where the values of $\nu(a)$ are given in [9, (1.23)].

4. Algebraic operations in F

(4.1) Theorem (Algebraic Operations in F). (i) The class of all quasi-extended asymptotic functions F is closed with respect to the addition (subtraction) and multiplication. Namely, if $f_1, f_2 \in F$ and

(4.2) $\qquad (\varphi_1, b) \in \operatorname{Gen} f_1; \qquad (\varphi_2, b) \in \operatorname{Gen} f_2$

then

(4.3) $$(\varphi_1 \pm \varphi_2, b) \in \operatorname{Gen}(f_1 \pm f_2)$$

(4.4) $$(\varphi_1 \cdot \varphi_2, b) \in \operatorname{Gen}(f_1 \cdot f_2)$$

are valid; (*ii*) Moreover, let

(4.5) $$f_1(a) = \varphi_{1as}(a, b) + o^{\nu_1(a)}, \qquad a \in D_1,$$

(4.6) $$f_2(a) = \varphi_{2as}(a, b) + o^{\nu_2(a)}, \qquad a \in D_2,$$

be additive forms of f_1 ang f_2 respectively (for the same b, corresponding Corollary (3.6)) and let

(4.7) $$f_1(a) = 1^{\lambda_1(a)} \cdot \varphi_{1as}(a, b), \qquad a \in D_1,$$

(4.8) $$f_2(a) = 1^{\lambda_2(a)} \cdot \varphi_{2as}(a, b), \qquad a \in D_2,$$

be multiplicative forms of f_1 and f_2, respectively. Then:
a) The representation

(4.9) $$f_1(a) \pm f_2(a) = (\varphi_1 \pm \varphi_2)_{as}(a, b) + o^{\nu(a)}, \quad a \in D,$$

is an additive form of $f_1 \pm f_2$, where

(4.10) $$D = D_1 \cap D_2,$$

(4.11) $$\nu(a) = \min[\nu_1(a), \nu_2(a)], \qquad a \in D;$$

b) The representations

(4.12) $$f_1(a) \cdot f_2(a) = (\varphi_1 \cdot \varphi_2)_{as}(a, b) + o^{\nu_0(a)}, \qquad a \in D,$$

(4.13) $$f_1(a) \cdot f_2(a) = 1^{\lambda_0(a)} \cdot (\varphi_1 \cdot \varphi_2)_{as}(a, b), \qquad a \in D,$$

are the additive and multiplicative forms of $f_1 \cdot f_2$ respectively. The order $\nu_0(a)$ and the relative order $\lambda_0(a)$ of $f_1(a) \cdot f_2(a)$ are given by

(4.14) $$\nu_0(a) = \min[\mu_1(a) + \nu_2(a), \mu_2(a) + \nu_1(a)]$$
$$= \min[\tilde{\mu}_1(a) + \nu_2(a), \tilde{\mu}_2(a) + \nu_1(a), \nu_1(a) + \nu_2(a)], \qquad a \in D,$$

and

(4.15) $$\lambda_0(a) = \min[\lambda_1(a), \lambda_2(a)], \qquad a \in D,$$

respectively, where $\mu_1(a)$ and $\mu_2(a)$ are the powers of $f_1(a)$ and $f_2(a)$ and $\tilde{\mu}_1(a)$ and $\tilde{\mu}_2(a)$ are the powers of $\varphi_{1as}(a, b)$ and $\varphi_{2as}(a, b)$ respectively;
c) The representations

(4.16) $$f_1(a)/f_2(a) = 1^{\lambda_0(a)}(\varphi_1/\varphi_2)_{as}(a, b), \qquad a \in D_1 \cap D_2^0,$$

are additive and multiplicative forms of the ratio f_1/f_2, respectively. Here

(4.17) $$D_2^0 = \{a \in D_2 : f_2(a) \notin \mathcal{O}\},$$

corresponding to [8, Definition (1.7)].

Proof: The proof of this theorem is quite analogous to the proof of [9], Theorem (2.9)]. In fact, this theorem is a consequence of [9, Theorem (2.9) and Lemma (1.34). Let us add (and subtract) (4.5) and (4.6):

$$f_1(a)\pm f_2(a)=\varphi_{1as}(a,\ b)+o^{\nu_1(a)}\pm\varphi_{2as}(a,\ b)+o^{\nu_2(a)}$$
$$=\varphi_{1as}(a,\ b)\pm\varphi_{2as}(a,\ b)+o^{\nu(a)}=(\varphi_1\pm\varphi_2)_{as}(a,\ b)+o^{\widetilde{\nu}(a)}+o^{\nu(a)}$$
$$=(\varphi_1\pm\varphi_2)_{as}(a,\ b)+o^{\nu(a)},\qquad a\in D$$

where $\widetilde{\nu}(a)$ is the order of $\varphi_{1as}(a,\ b)\pm\varphi_{2as}(a,\ b)$, corresponding to [9, Theorem (2.9)], $\nu(a)$ is given by (4.11) and D is the set (4.10). We have just used the identity

(4.18) $$o^{\widetilde{\nu}(a)}+o^{\nu(a)}=o^{\nu(a)},\qquad a\in D,$$

which is equivalent to

(4.19) $$\nu(a)\leq\widetilde{\nu}(a),\qquad a\in D.$$

On its part, the latter inequality follows directly from Lemma (1.34). The case of multiplication is treated exactly in the same way bearing in mind [9, Theorem (2.9)] as well as Lema (1.4). That is all.

Now let us remind the comments of the beginning of Sec. 2 of [9] and especially [9, (2.5), (2.6), Example (1.9), (1.26)]. We see that the n-th power of the asymptotic function δ defined by [9, (1.24)] is a quasi-extended function, i. e

(4.20) $$\delta^n\in F,\qquad n=1,\ 2,\ 3,\ldots$$

That follows directly from the theorem just proved. Moreover,

(4.21) $$\delta^n(a)=(\Delta^n)_{as}(a,\ s+o^1)+o^{\nu_0(a)},\qquad a\in A$$

is an additive form of δ^n, where the ordinary function Δ is defined in [9, (1.20) Δ^n is the n-th power of Δ, $(\Delta^n)_{as}$ is the asymptotic extension of Δ^n and

(4.22) $$\nu_0(a)=\begin{cases}-n, & a=sx+sh,\quad x\in\mathfrak{R},\quad h\in\Omega_0,\\ -2n, & a=o^0,\\ n, & a=x+h,\quad x\in\mathfrak{R},\quad x\neq0,\quad x\in\Omega_0,\end{cases}$$

is the order of $\delta^n(a)$, corresponding to [9, (1.26)].

We kindly suggest the reader to remind [9, Example (1.19)] and to keep in mind (4.20)—(4.22).

(4.23) Definition: Let V be a class of continuous ordinary functions of two real variables of the type

(4.24) $$\varphi:\ X\times Y\to C,$$

where X and Y are two open subsets of $\mathfrak{R}$, which are the same for all functions from V. Let b be an asymptotic number from Y_{as}, i. e. $b\in Y_{as}$. Then the set of all quasi-extended asymptotic functions f for which there exists a generating couple $(\varphi,\ b)\in\operatorname{Gen}f$ such that $\varphi\in V$ will be denoted by $V_{as}(b)$. In other words, $f\in V_{as}(b)$ if f can be represented in the form

(4.25) $$f(a)=\varphi_{as}(a,\ b)+o^{\nu(a)},\qquad a\in D,$$

for some $\varphi\in V$.

(4.26) Theorem: Let V be a set of ordinary functions described in Definition (4.23) and let b be an asymptotic number from Y_{as}, i. e. $b\in Y_{as}$. Then: *(i)* $V_{as}(b)$ is closed with respect to the addition if and only if V is closed with respect to the addition; *(ii)* $V_{as}(b)$ is closed with respect to the multiplication if and only if V is closed with respect to multiplication; *(iii)* In the

cases when $V_{as}(b)$ is closed with respect to the algebraic operations (addition or multiplication or addition and multiplication) $V_{as}(b)$ has the same algebraic properties as A and A^* have (we mean that the identities [5, Theorem 6] are valid in $V_{as}(b)$) with respect to the corresponding algebraic operations.

Proof: (*i*) and (*ii*) follow directly from Theorem (4.1) and (*iii*) follows from [8, Lemma (1.13)].

(4.27) Corollary: Let C^n, $n=0, 1, \ldots, \infty$, $\mathfrak{D}$ and S be the well-known classes of ordinary functions (defined on $\mathfrak{R}$). Then the corresponding (according to (4.23)) classes of quasi-extended asymptotic functions $(C^n)_{as}(s)$, $n=0, 1, \ldots, \infty$, $\mathfrak{D}_{as}(s)$ and $S_{as}(s)$ are closed with respect to the addition and multiplication. Moreover, these classes have the same algebraic structure as A and A^*.

Notice that the extended asymptotic functions given in [9, (1.15), (1.12)] which are, at the same time, quasi-extended functions, are examples of functions from $(C^\infty)_{as}(s)$, $\mathfrak{D}_{as}(s)$ and $S_{as}(s)$, respectively.

(4.28) Remark (The Role of $V_{as}(b)$): The classes of quasi-extended asymptotic functions of the type $V_{as}(b)$ where V is some (arbitrarily chosen) class of ordinary functions [Definition (4.23)] will play an important role in our approach in the future. In the next section we shall set $V=F(x, y)$ where $F(x, y)$ is the class of ordinary functions defined in [4, 1, Sec. 3]. The corresponding class of quasi-extended asymptotic functions $[F(x, y)]_{as}(s)$ turns out to be isomorphic to the class $F(x)$ of asymptotic functions introduced in [4]. In the next paper of our series we shall set $V=\Phi$ where Φ is another class of ordinary functions closely connected with the analytic functions. The asymptotic functions from the corresponding class $\Phi_{as}(s)$ will be called quasi-distributions because they are realizations, in a certain sense, of Schwartz distributions.

(4.29) Change of a notation: Instead of the notation "$F(x, y)$" just used the notation "$F(x, s)$" is used in [4, 1]. Recall as well [5, Definition 12] that "s" is the short notation for the asymptotic number "$s+o^\infty$", i. e. $s\equiv s+o^\infty$ (see (5.3)).

(4.30) Theorem (Composition): Let $f(a)$, $a\in D$, be a quasi-extended asymptotic function, i. e. $f\in F$, and let $g(a)$, $a\in D_1$, be a regular quasi-extended function. If the condition

$$\{g(a):\ a\in D_1\}\subseteq D \tag{4.31}$$

holds, then the composition

$$(f_0 g)(a)=f(g(a)),\qquad a\in D_1, \tag{4.32}$$

is also a quasi-extended asymptotic function. Moreover, if (1.2)—(1.6) hold for some $(\varphi, b)\in \operatorname{Gen} f$ and

$$g(a)=\psi_{as}(a)+o^{\nu_1(a)},\qquad a\in D_1, \tag{4.33}$$

holds for some continuous ordinary function ψ of the type

$$\psi\colon X_1\to X \tag{4.34}$$

(where $X\times Y$ is the domain of φ corresponding to (1.2)—(1.6)), then

$$(f\circ g)(a)=(\varphi\circ\psi)_{as}(a, b)+o^{\nu_0(a)},\qquad a\in D_1, \tag{4.35}$$

is valid where

$$(\varphi\circ\psi)(x, y)=\varphi(\psi(x), y),\qquad x\in X_1,\ \ y\in Y, \tag{4.36}$$

and

$$(4.37) \qquad \nu_0(a)=\nu(g(a)), \qquad a \in D_1$$

is the order of $(fog)(a)$.

Proof: The proof is analogous to this of Theorem (4.1) and we shall omit it.

5. The connection with the asymptotic functions introduced in [4]

The notion of "asymptotic function" was introduced for the first time in [2] and a series of works [4] has appeared based on this notion. The definition of the asymptotic functions given in [4, I] is different from the one used here [8, (1.1)]. Namely, the asymptotic functions in [4] are not mappings from the set of the asymptotic numbers A into itself; they are equivalence classes of sequences of ordinary smooth functions of a particular type [4, I, Sec. 3]. An asymptotic function δ, similar, in a certain sense, to Dirac's delta-function is constructed in the framework of this approach [2], [4]. What is more interesting, it was shown that every two asymptotic functions of this type can be multiplied; in particular, several expressions for δ^2 were established in [2] and [4, III]. The following question arises: Is there any connection between the asymptotic functions as defined in [4], on one hand, and the asymptotic functions considered in the present paper (together with [8] and [9], of course), on the other. The answer is "yes" and we are going to discuss briefly this connection:

(*i*) Let $F(x, y)$ be the class of ordinary functions defined in [4, I, Sec. 3] (see (4.29)). In this reference the reader may find the exact definition of this class. We shall notice only that $F(x, y)$ is a class of complex-valued smooth (respect to "x") functions of two real variables of the type

$$(5.1) \qquad f(x, y), \qquad x \in \mathcal{R}, \quad y \in (o, s_1),$$

where s_1 is an arbitrarily fixed real positive number. Besides, $F(x, y)$ is closed with respect to the addition and multiplication ($F(x, y)$ is a ring of functions). About the other defining properties of $F(x, y)$ we refer the reader to [4, I];

(*ii*) Let us consider the class

$$(5.2) \qquad F_0 \overset{\text{def}}{=} [F(x, y)]_{\text{as}}(s)$$

of quasi-extended asymptotic functions obtained according to Definition (4.23) for $V=F(x, y)$ and $b=s$. Recall [8, (3.26)] that s is the following asymptotic number:

$$(5.3) \qquad s \equiv s+o^{\infty}=\{s+\Delta\colon \Delta \in A_s, \lim_{s\to 0} \Delta(s)/s^n = \text{for all } n \in Z\}.$$

According to Theorem (4.26), F_0 is closed with respect to the addition and multiplication and has the same algebraic structure as A and A^*;

(*iii*) It is easy to see that

$$(5.4) \qquad \mathfrak{D}_{\text{as}}(s) \subset S_{\text{as}}(s) \subset F_0$$

where $\mathfrak{D}_{\text{as}}(s)$ and $S_{\text{as}}(s)$ are discussed in (4.27);

(*iv*) *Integration in* F_0: Let $f \in F_0$ and let Δ be a Lebesgue measurable subset of $\mathcal{R}$ (an interval of $\mathcal{R}$, for example). Let us set

$$(5.5)\qquad J^*=\left\{\int_{\Delta}\varphi(x,\chi)_{dx}:\ (\varphi,h)\in \mathrm{Gen}\,f,\ \ \varphi\in F(x,y),\ h\in(o,s_1)_{as},\ \chi\in h\right\}.$$

The asymptotic cover [5, Definition 7] as J^* of J^* (which is an asymptotic number, i. e. as $J^*\in A^*$) will be called the integral of f on Δ and the following notations will be used:

$$(5.6)\qquad J=\text{as}\,J^*=\int_{\Delta} f(x)dx.$$

It can be proved that every asymptotic function Δf from F_0 is locally integrable, i. e.

$$(5.7)\qquad \int_{x_1}^{x_2} f(x)dx$$

exists (and belongs to A^*) for every $f\in F_0$ and every $x_1, x_2\in\mathfrak{R}$. The other analytic operations (differentiation, Fourier-transformation, convolution, etc.) can be introduced in an analogous way.

(*v*) *Example of Dirac's delta function:* Let $\varrho\in S$ and

$$(5.8)\qquad \int_{-\infty}^{\infty}\varrho(x)dx=1.$$

To consider the function

$$(5.9)\qquad \varphi(x,y)=\frac{1+\sqrt{y}}{y}\,\varrho\left(\frac{x}{y}\right),\qquad x\in\mathfrak{R},\ \ y\in(o,s_1),$$

which belongs to $F(x, y)$. The asymptotic extension $\varphi_{as}(a, b)$ of φ exists for every $a\in A$ and every $b\in(o,s_1)_{as}$ [8, (2.15)]. Let us put (for $b=s=s+o^\infty$)

$$(5.10)\qquad \delta(a)=\varphi_{as}(a,s),\qquad a\in A.$$

The values of δ are given by

$$(5.11)\qquad \delta(a)=\begin{cases}(s^{-1}+o^{-1})\varrho(x), & a=sx+sh,\ \ x\in\mathfrak{R},\ \ h\in\Omega_0,\\ o^{-2}, & a\in\{o^{-n}:\ n=0,1\ldots\},\\ o, & \text{for all other}\ \ a\in A.\end{cases}$$

It is clear that $\delta\in F_0$. Moreover,

$$(5.12)\qquad \int_{-\infty}^{\infty}\delta(x)dx=1+o^0=1^0,$$

and

$$(5.13)\qquad \int_{-\infty}^{\infty}\delta(x)\psi(x)dx=\psi(0)+o^0$$

for every $\psi\in\mathfrak{D}_{as}(s)$ (or $\psi\in S_{as}(s)$). Bearing in mind the isomorphism $\mathfrak{R}\simeq\mathfrak{R}^0$ [5, Theorem 20], we see that δ is a realisation of the Dirac's delta-function. The values of the square δ^2 (every two fun~'ions from F_0 can be multiplied) are given by

$$(5.14)\qquad \delta^2(a)=\begin{cases}(s^{-2}+o^{-2})\varrho^2(x), & a=sx+sh,\ x\in\Re,\ h\in\Omega_0,\\ o^{-4}, & a\in\{o^{-n}:\ n=1,2,\ldots\},\\ 0, & \text{for all other } a\in A.\end{cases}$$

Moreover, we obtain

$$(5.15)\qquad \int_{-\infty}^{\infty}\delta^2(x)\psi(x)dx=M\psi(0),\qquad \psi\in\mathfrak{D}_{as}(s),$$

where

$$(5.16)\qquad M=\frac{m}{s}+o^{-1},\qquad m=\int_{-\infty}^{\infty}\delta^2(x)dx.$$

Notice that M is an infinitely large asymptotic number (constant) [6, Definition 8], i. e. $r<M$ for any real number r. Moreover, M does not depend on choice of ψ;·

(*vi*) It can be shown that the class F_0 of quasi-extended asymptotic functions is isomorphic to the class $F(x)$ of asymptotic functions defined in [4, I, Sec. 4], i. e.

$$(5.17)\qquad F_0\cong F(x)$$

and the isomorphism preserves also the analytic operations (differentiation, integration and so on).

The class F_0, respectively $F(x)$, has several interesting properties which are discussed in detail in the series [4].

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