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Lie symmetry and approximate Hojman conserved quantity of Appell equations for a weakly nonholonomic system

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Abstract For a weakly nonholonomic system, the Lie symmetry and approximate Hojman conserved quantity of Appell equations are studied. Based on the Appell equations for a weakly nonholonomic system under special infinitesimal transformations of a group in which the time is invariable, the definition of the Lie symmetry of the weakly nonholonomic system and its first-degree approximate holonomic system are given. With the aid of the structure equation that the gauge function satisfies, the exact and approximate Hojman conserved quantities deduced directly from the Lie symmetry are derived. Finally, an example is given to study the exact and approximate Hojman conserved quantity of the system.

Keywords Weakly nonholonomic system · Appell equations · Lie symmetry · Approximate Hojman conserved quantity

1 Introduction

In recent years, research on the dynamics of nonholonomic system has made great progress [1–7]. A special nonholonomic system whose constraint equations contain a small parameter is called a weakly nonholonomic system. This system becomes a holonomic system when the small parameter equals zero. Researchers have already discussed the equations of motion, approximate solution, canonical transformation, and stability of the weakly nonholonomic system [8–10]. A previous study [11] examined the special Mei symmetry and approximate conserved quantity of Appell equations for a weakly nonholonomic system. Appell equations are important in analytical mechanics and belong to one of the three types of mechanical systems in the theory of analytical mechanics. In the recent 20 years, Chinese researchers have gained fruitful achievements in research, promotion, and application of Appell equations [12–17]. Since 2000, Chinese researchers have made some achievements in this research area, especially in Lie symmetry [35–47] of constrained mechanical systems [18–34]. However, there are fewer results on Appell equations. To solve Appell equations, Mei Feng-Xiang first derived the Noether conserved quantity deduced indirectly from the Noether symmetry using Mei symmetry [48]. In the present study, we examined Lie symmetry and approximate Hojman conserved quantity for Appell equations of a weakly nonholonomic system. First, the Appell equations and their first-degree approximation formulas were established for a weakly nonholonomic system. Subsequently, the definition and criterion of Lie symmetry were obtained. Then, the exact and approximate Hojman conserved quantities were directly deduced from the Lie symmetry. Finally, an example

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has been given to illustrate the application of the theoretical results of this study.

2 Differential equations of motion of a weakly nonholonomic system

Suppose that the position of a mechanical system is determined by the n generalized coordinates, q_s ($s = 1, 2, \ldots, n$), and whose motion is subject to the g ideal bilateral homogeneous nonholonomic constraints of Chetaev type

$$f_\beta = \dot{q}_{\varepsilon+\beta} - bB_{\varepsilon+\beta,\delta}(t, q_s)\dot{q}_\delta = 0$$
$$(\beta = 1, 2, \ldots, g;\ \delta = 1, 2, \ldots, \varepsilon;\ \varepsilon = n - g;$$
$$s = 1, 2, \ldots, n), \tag{1}$$

where b is a small parameter. When $b = 0$, the constraint equation (1) becomes a constraint equation of the holonomic system. The Chetaev condition of the constraint equation (1) imposed on the virtual displacement δq_s is

$$\frac{\partial f_\beta}{\partial \dot{q}_s}\delta q_s = 0. \tag{2}$$

The Appell function of the system is $S = S(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$, and the generalized forces are $Q_s = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$; thus, the Appell equations of the system are

$$\frac{\partial S}{\partial \ddot{q}_s} = Q_s + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s}, \tag{3}$$

where λ_β denotes the constraint multipliers in (3). If the system is nonsingular, one can solve all the constraint multipliers $\lambda_\beta(t, b, \boldsymbol{q}, \dot{\boldsymbol{q}})$ using (1)–(2). Equation (3) can also be expressed as

$$\frac{\partial S}{\partial \ddot{q}_s} = Q_s + \Lambda_s, \tag{4}$$

where

$$\Lambda_s = \Lambda_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \tag{5}$$

are the generalized constraint forces. Equation (4) is the equation of the holonomic system, corresponding to those of the weakly nonholonomic system (1) and (3). It has been proved that the solution of the relevant holonomic system (4) corresponding to the weakly nonholonomic system (1) and (3) will give the motion if the initial conditions of motion satisfy the constraint equation (1).

To discuss the approximate solution of the weakly nonholonomic system, we can expand the generalized constraint forces Λ_s as a power series

$$\Lambda_s = \Lambda_{s0}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + b\Lambda_{s1}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$$
$$+ b^2\Lambda_{s2}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \cdots. \tag{6}$$

Then, the first-degree approximation of (4) can be expressed as

$$\frac{\partial S}{\partial \ddot{q}_s} = Q_s + \Lambda_{s0} + b\Lambda_{s1}. \tag{7}$$

Also, all generalized accelerations can be solved from (4) and can be written as

$$\ddot{q}_s = \alpha_s(t, b, \boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{8}$$

From (7) we obtain

$$\ddot{q}_s = \alpha_{s0}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + b\alpha_{s1}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{9}$$

3 Determining equation and definition of Lie symmetry

Considering the Lie symmetry for (4) and (7), let us take the special infinitesimal transformations of group in which the time is invariable as

$$t^* = t, \qquad q_s^*(t^*) = q_s(t) + \Delta q_s, \quad (s = 1, \ldots, n). \tag{10}$$

Equation (10) can also be extended into

$$t^* = t, \qquad q_s^*(t^*) = q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{11}$$

where ε is an infinitesimal parameter, and ξ_s denotes the generation functions of the infinitesimal transformations. By introducing the infinitesimal generator vector $X^{(0)}$ and its first and second extended infinitesimal generators $\tilde{X}^{(1)}$ and $\tilde{X}^{(2)}$, we get

$$X^{(0)} = \xi_s \frac{\partial}{\partial q_s},$$
$$\tilde{X}^{(1)} = X^{(0)} + \frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s}, \tag{12}$$
$$\tilde{X}^{(2)} = \tilde{X}^{(1)} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t}\frac{\partial}{\partial \ddot{q}_s}.$$

Since $\frac{\partial S}{\partial \ddot{q}_s} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s}$, the determining equation of Lie symmetry (4) can be written as

$$\tilde{X}^{(2)}\left(\frac{\partial S}{\partial \ddot{q}_s}\right) = \tilde{X}^{(1)}(Q_s) + \tilde{X}^{(1)}(\Lambda_s). \tag{13}$$

Equation (8) can be written as

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s = \frac{\partial \alpha_s}{\partial q_k}\xi_k + \frac{\partial \alpha_s}{\partial \dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k. \tag{14}$$

Equation (7) can be obtained as

$$\tilde{X}^{(2)}\left(\frac{\partial S}{\partial \ddot{q}_s}\right) = \tilde{X}^{(1)}(Q_s) + \tilde{X}^{(1)}(\Lambda_{s0}) + \tilde{X}^{(1)}(b\Lambda_{s1}), \tag{15}$$

and (9) can be rewritten as

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s = \frac{\partial(\alpha_{s0} + b\alpha_{s1})}{\partial q_k}\xi_k + \frac{\partial(\alpha_{s0} + b\alpha_{s1})}{\partial \dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k. \tag{16}$$

Definition 1 If the infinitesimal generator ξ_s satisfies (13) or (14), then the relevant symmetry is Lie symmetry of the weakly nonholonomic system (1) and (3) or holonomic system (4) corresponding to the weakly nonholonomic system (1) and (3).

The restriction equation of Lie symmetry for the weakly nonholonomic constraints (1) under the special infinitesimal transformations (11) can be expressed as

$$\tilde{X}^{(1)}\{f_\beta(t, b, \boldsymbol{q}, \dot{\boldsymbol{q}})\} = 0. \tag{17}$$

Definition 2 If the infinitesimal generator ξ_s satisfies (13) or (14) and the restriction equation (17), then the relevant symmetry is weakly Lie symmetry of the weakly nonholonomic system (1) and (3) or holonomic system (4) corresponding to the weakly nonholonomic system (1) and (3).

Considering the Appell–Chetaev condition (2) imposed on the virtual displacement δq_s, we have the following additional restriction equation:

$$\frac{\partial f_\beta}{\partial \dot{q}_s}\xi_s = 0, \quad (\beta = 1, \ldots, g;\ s = 1, \ldots, n). \tag{18}$$

Definition 3 If the infinitesimal generator ξ_s satisfies (13) or (14), the restriction equation (17), and the additional restriction equation (18), then the relevant symmetry is strong Lie symmetry of the weakly nonholonomic system (1) and (3) or holonomic system (4) corresponding to the weakly nonholonomic system (1) and (3).

Definition 4 If the infinitesimal generator ξ_s satisfies (15) or (16), then the relevant symmetry is Lie symmetry of the first-degree approximate holonomic system (7) corresponding to the weakly nonholonomic system (1) and (3).

Definition 5 If the infinitesimal generator ξ_s satisfies (15) or (16) and the restriction equation (17), then the relevant symmetry is weakly Lie symmetry of the first-degree approximate holonomic system (7) corresponding to the weakly nonholonomic system (1) and (3).

Definition 6 If the infinitesimal generator ξ_s satisfies (15) or (16), the restriction equation (17), and the additional restriction equation (18), then the relevant symmetry is strong Lie symmetry of the first-degree approximate holonomic system (7) corresponding to the weakly nonholonomic system (1) and (3).

4 Hojman conserved quantity deduced from the special Lie symmetry

Proposition 1 [49] *If the infinitesimal generator ξ_s satisfies the determining equation (13) or (14) and if the function $\mu = \mu(t, q, \dot{q})$ satisfies the equation*

$$\frac{\partial \alpha_s}{\partial \dot{q}_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu = 0, \tag{19}$$

then the exact Hojman conserved quantity can be deduced from the Lie symmetry of Appell equations for the weakly nonholonomic system (1) and (3), or the corresponding holonomic system (4) is

$$I_H = \frac{1}{\mu}\frac{\partial}{\partial q_s}(\mu\xi_s) + \frac{1}{\mu}\frac{\partial}{\partial \dot{q}_s}\left(\mu\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) = \text{const}. \tag{20}$$

Proposition 2 *If the infinitesimal generator ξ_s satisfies (13) or (14) and the restriction equation (17), and if the function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfies (19), then the exact Hojman conserved quantity (20) can be deduced from weakly Lie symmetry of Appell equations for the weakly nonholonomic system (1) and (3), or the corresponding holonomic system (4).*

Proposition 3 *If the infinitesimal generator ξ_s satisfies (13) or (14), the restriction equation (17), and the additional restriction equation (18), and if the function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfies (19), then the exact Hojman conserved quantity (20) can be deduced from strong Lie symmetry of Appell equations for the weakly nonholonomic system (1) and (3) or the corresponding holonomic system (4).*

Proposition 4 *If the infinitesimal generator ξ_s satisfies (15) or (16), and if a function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfies*

$$\frac{\partial(\alpha_{s0} + b\alpha_{s1})}{\partial \dot{q}_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \ln \mu = 0, \tag{21}$$

then the approximate Hojman conserved quantity (20) can be deduced from Lie symmetry of Appell equations for the first-degree approximate holonomic system (7).

Proof Taking the derivative of Eq. (20) with respect to t, we can obtain

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} I_H = \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial q_s}\xi_s\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial \xi_s}{\partial q_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left[\frac{1}{\mu}\frac{\partial \mu}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s + \frac{\partial}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right]. \tag{22}$$

Furthermore, we can easily prove that

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s = \frac{\partial}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) - \frac{\partial}{\partial q_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \frac{\partial}{\partial \dot{q}_k}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right), \tag{23}$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial \xi_s}{\partial q_s} = \frac{\partial}{\partial q_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \frac{\partial(\alpha_{k0} + b\alpha_{k1})}{\partial q_s}\frac{\partial \xi_s}{\partial \dot{q}_k}.$$

By taking the partial derivative of (16) with respect to $\dot{q}_s$ and the summation of terms with subscript s, we have

$$\frac{\partial}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) = \frac{\partial}{\partial \dot{q}_s}\left(\frac{\partial(\alpha_{s0} + b\alpha_{s1})}{\partial q_k}\xi_k\right) + \frac{\partial}{\partial \dot{q}_s}\left(\frac{\partial(\alpha_{s0} + b\alpha_{s1})}{\partial \dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k\right). \tag{24}$$

By substituting (23) and (24) into (22), we have

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} I_H = \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial q_s}\xi_s\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) + \frac{\partial^2(\alpha_{s0} + b\alpha_{s1})}{\partial q_k \partial \dot{q}_s}\xi_k + \frac{\partial^2(\alpha_{s0} + b\alpha_{s1})}{\partial \dot{q}_k \partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k. \tag{25}$$

By using (21) we obtain

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial q_s}\xi_s\right) = -\frac{\partial^2(\alpha_{s0} + b\alpha_{s1})}{\partial q_k \partial \dot{q}_s}\xi_k + \frac{1}{\mu}\frac{\partial \mu}{\partial q_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \frac{1}{\mu}\frac{\partial \mu}{\partial \dot{q}_k}\frac{\partial(\alpha_{k0} + b\alpha_{k1})}{\partial q_s}\xi_s, \tag{26}$$

$$\begin{aligned}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) &= -\frac{\partial^2(\alpha_{s0} + b\alpha_{s1})}{\partial \dot{q}_k \partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k \\ &\quad - \frac{1}{\mu}\frac{\partial(\alpha_{k0} + b\alpha_{k1})}{\partial \dot{q}_s}\frac{\partial \mu}{\partial \dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s \\ &\quad + \frac{1}{\mu}\frac{\partial \mu}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) - \frac{1}{\mu}\frac{\partial \mu}{\partial q_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s.\end{aligned} \tag{27}$$

By substituting (26) and (27) into (25) we have

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} I_H = \frac{1}{\mu}\frac{\partial \mu}{\partial \dot{q}_s}\left\{\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) - \frac{\partial(\alpha_{s0} + b\alpha_{s1})}{\partial \dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k - \frac{\partial(\alpha_{s0} + b\alpha_{s1})}{\partial q_k}\xi_k\right\} = 0.$$

Thus, our proof is completed. □

Proposition 5 *If the infinitesimal generator ξ_s satisfies (15) or (16) and the restriction equation (17), and if the function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfies (21), then the approximate Hojman conserved quantity (20) can be deduced from weakly Lie symmetry of Appell equations for the first-degree approximate holonomic system (7).*

Proposition 6 *If the infinitesimal generator ξ_s satisfies (15) or (16), the restriction equation (17), and the additional restriction equation (18), and if the function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfies (21), then the approximate Hojman conserved quantity (20) can be deduced from strong Lie symmetry of Appell equations for the first-degree approximate holonomic system (7).*

5 An illustrative example

In the following, we have given an example to illustrate the application of the above-mentioned results.

Let us consider a weakly nonholonomic system where a particle of unit quality moves in the two-dimensional space that is perpendicular to the Earth's surface and whose Appell function, generalized force, and constraint equations are

$$S = \frac{1}{2}(\ddot{q}_1^2 + \ddot{q}_2^2), \tag{28}$$

$$Q_1 = 0, \qquad Q_2 = -g, \tag{29}$$

$$f = \dot{q}_2 - bt\dot{q}_1 = 0. \tag{30}$$

Let us now study the Lie symmetry and Hojman conserved quantity of the system.

By substituting (28)–(30) into (3) we can get

$$\ddot{q}_1 = -\lambda bt, \qquad \ddot{q}_2 = -g + \lambda. \tag{31}$$

From (30) and (31) we obtain

$$\lambda = \frac{b\dot{q}_1 + g}{1 + b^2t^2}. \tag{32}$$

Thus, we have

$$\ddot{q}_1 = -\frac{b^2t\dot{q}_1 + btg}{1 + b^2t^2}, \qquad \ddot{q}_2 = \frac{b\dot{q}_1 - b^2t^2g}{1 + b^2t^2}. \tag{33}$$

According to the determining equation (14) of Lie symmetry, we can obtain

$$\ddot{\xi}_1 = -\frac{b^2t}{1 + b^2t^2}\dot{\xi}_1, \qquad \ddot{\xi}_2 = \frac{b}{1 + b^2t^2}\dot{\xi}_1. \tag{34}$$

The above-mentioned equations (34) have the following solutions:

$$\xi_1 = \xi_2 = 1, \tag{35}$$

$$\xi_1 = 1, \qquad \xi_2 = \left(\dot{q}_1 + bt\dot{q}_2 - bq_2 + \frac{1}{2}gbt^2\right)^2. \tag{36}$$

The restriction equation (17) gives

$$\dot{\xi}_2 - bt\dot{\xi}_1 = 0. \tag{37}$$

From the additional restriction equation (18) we get

$$\xi_2 - bt\xi_1 = 0. \tag{38}$$

Apparently, the infinitesimal generators (35) and (36) satisfy the restriction equation (37) but do not satisfy the additional restriction equation (38), respectively. Therefore, they are the respective infinitesimal generators of Lie symmetry and weakly Lie symmetry for the weakly nonholonomic system.

Furthermore, from (18) we obtain

$$-\frac{b^2t}{1 + b^2t^2} + \frac{\bar{d}}{dt}\ln\mu = 0. \tag{39}$$

The above-mentioned equation (39) has the following solutions:

$$\mu = \sqrt{1 + b^2t^2}, \tag{40}$$

$$\mu = \sqrt{1 + b^2t^2}\left(\dot{q}_1 + bt\dot{q}_2 - bq_2 + \frac{1}{2}gbt^2\right). \tag{41}$$

From (35), (40), and (20) we obtain

$$I_{H_1} = 0 = \text{const.} \tag{42}$$

From (35), (41), and (20) we get

$$I_{H_2} = -b\left(\dot{q}_1 + bt\dot{q}_2 - bq_2 + \frac{1}{2}gbt^2\right)^{-1} = \text{const.} \tag{43}$$

From (36), (40), and (20) we obtain

$$I_{H_3} = -2b\left(\dot{q}_1 + bt\dot{q}_2 - bq_2 + \frac{1}{2}gbt^2\right) = \text{const.} \tag{44}$$

Equations (36), (41), and (20) give

$$I_{H_4} = -3b\left(\dot{q}_1 + bt\dot{q}_2 - bq_2 + \frac{1}{2}gbt^2\right) = \text{const.} \tag{45}$$

Based on Propositions 1 and 2, we know that I_{H_1}, I_{H_2}, I_{H_3}, and I_{H_4} are the exact Hojman conserved quantity of Lie symmetry and weakly Lie symmetry for the weakly nonholonomic system (31) or the corresponding holonomic system (33), where I_{H_1} is the common conserved quantity. As the infinitesimal generators (35) and (36) do not satisfy the additional restriction equation (38), from Proposition 3 we know that these conserved quantities are not the exact Hojman conserved quantity of strong Lie symmetry for the weakly nonholonomic system (31) or the corresponding holonomic system (33).

The first-degree approximate equations of (33) are

$$\ddot{q}_1 = -btg, \qquad \ddot{q}_2 = b\dot{q}_1. \tag{46}$$

The determining equation (16) of Lie symmetry gives

$$\ddot{\xi}_1 = 0,\qquad \ddot{\xi}_2 = b\dot{\xi}_1. \tag{47}$$

The above-mentioned equations (47) have the following solutions:

$$\xi_1 = \xi_2 = 1, \tag{48}$$

$$\xi_1 = (\dot{q}_2 - bq_1)^2, \qquad \xi_2 = 0, \tag{49}$$

$$\xi_1 = \left(\dot{q}_1 + \dot{q}_2 - bq_1 + \frac{1}{2}gbt^2\right)^2, \qquad \xi_2 = 0. \tag{50}$$

From the restriction equation (17) we get

$$\dot{\xi}_2 - bt\dot{\xi}_1 = 0. \tag{51}$$

Furthermore, from the additional restriction equation (18) we get

$$\xi_2 - bt\xi_1 = 0. \tag{52}$$

The infinitesimal generators (48)–(50) satisfy the restriction equation (51) but do not satisfy the additional restriction equation (52). Therefore, they can be considered as the infinitesimal generators of Lie symmetry and weakly Lie symmetry for the first-degree approximate holonomic system.

From (21) we obtain

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} \ln \mu = 0. \tag{53}$$

The above-mentioned equation (53) has the following solutions:

$$\mu = 1, \tag{54}$$

$$\mu = \dot{q}_2 - bq_1, \tag{55}$$

$$\mu = \dot{q}_1 + \dot{q}_2 - bq_1 + \frac{1}{2}gbt^2. \tag{56}$$

Therefore, from (48), (54), and (20) we obtain

$$I_{H_5} = 0. \tag{57}$$

From (48), (55), and (20) we get

$$I_{H_6} = -b(\dot{q}_2 - bq_1)^{-1} = \text{const.} \tag{58}$$

From (48), (56), and (20) we can obtain

$$I_{H_7} = -2b\left(\dot{q}_1 + \dot{q}_2 - bq_1 + \frac{1}{2}gbt^2\right)^{-1} = \text{const.} \tag{59}$$

Equations (49), (54), and (20) give

$$I_{H_8} = -2b(\dot{q}_2 - bq_1) = \text{const.} \tag{60}$$

From (49), (55), and (20) we get

$$I_{H_9} = -3b(\dot{q}_2 - bq_1) = \text{const.} \tag{61}$$

From (50), (54), and (20) we obtain

$$I_{H_{10}} = -2b\left(\dot{q}_1 + \dot{q}_2 - bq_1 + \frac{1}{2}gbt^2\right) = \text{const.} \tag{62}$$

From (50), (56), and (20) we can acquire

$$I_{H_{11}} = -3b\left(\dot{q}_1 + \dot{q}_2 - bq_1 + \frac{1}{2}gbt^2\right) = \text{const.} \tag{63}$$

According Propositions 4 and 5, we know that I_{H_5}, I_{H_6}, I_{H_7}, I_{H_8}, I_{H_9}, $I_{H_{10}}$, and $I_{H_{11}}$ are the approximate Hojman conserved quantities of Lie symmetry and weakly Lie symmetry for the first-degree approximate system (46), where I_{H_5} is the common conserved quantity. As the infinitesimal generators (48)–(50) do not satisfy the additional restriction equation (52), from Proposition 6 we know that these conserved quantities are not the approximate Hojman conserved quantities of strong Lie symmetry for the first-degree approximate system (46).

By taking the derivatives of expressions (57)–(63) with respect to time t, according to (31), we respectively obtain

$$\dot{I}_{H_5} = 0,$$

$$\dot{I}_{H_6} = -b\frac{\ddot{q}_2 - b\dot{q}_1}{(\dot{q}_2 - bq_1)^2} = \frac{b^4t^2\dot{q}_1 + b^3t^2g}{(1 + b^2t^2)(\dot{q}_2 - bq_1)^2} = \mathrm{O}(b^3),$$

$$\dot{I}_{H_7} = -2b\frac{\ddot{q}_1 + \ddot{q}_2 - b\dot{q}_1 + gbt}{(\dot{q}_1 + \dot{q}_2 - bq_1 + \frac{1}{2}gbt^2)^2} = \frac{2(b^3t\dot{q}_1 + b^4t^2\dot{q}_1 + b^3t^2g - b^4t^3g)}{(1 + b^2t^2)(\dot{q}_1 + \dot{q}_2 - bq_1 + \frac{1}{2}gbt^2)^2} = \mathrm{O}(b^3),$$

$$\dot{I}_{H_8} = -2b(\ddot{q}_2 - b\dot{q}_1) = \frac{2(b^4t^2\dot{q}_1 + gb^3t^2)}{1 + b^2t^2} = \mathrm{O}(b^3),$$

$$\dot{I}_{H_9} = -3b(\ddot{q}_2 - b\dot{q}_1) = \frac{3(b^4t^2\dot{q}_1 + gb^3t^2)}{1 + b^2t^2}$$
$$= \mathrm{O}(b^3),$$
$$\dot{I}_{H_{10}} = -2b(\ddot{q}_1 + \ddot{q}_2 - b\dot{q}_1 + gbt)$$
$$= \frac{2(b^3t\dot{q}_1 + b^4t^2\dot{q}_1 + b^3t^2g - b^4t^3g)}{(1 + b^2t^2)}$$
$$= \mathrm{O}(b^3),$$
$$\dot{I}_{H_{11}} = -3b(\ddot{q}_1 + \ddot{q}_2 - b\dot{q}_1 - gbt)$$
$$= \frac{3(b^3t\dot{q}_1 + b^4t^2\dot{q}_1 + b^3t^2g - b^4t^3g)}{(1 + b^2t^2)}$$
$$= \mathrm{O}(b^3).$$

Thus, the above-mentioned quantities are the approximate Hojman conserved quantities of the weakly nonholonomic system.

6 Conclusions

This study examined the theory of Lie symmetry and Hojman conserved quantity of Appell equations for a weakly nonholonomic system, and analyzed the exact and approximate Hojman conserved quantities. The theoretical results apply not only to the weakly nonholonomic system, when the small parameter b is zero, but also to the holonomic system. The method through which the approximate conserved quantity can be obtained by expanding the generalized constraint force Λ_s as a power series in small parameter b can also be applied to other mechanical and physical systems with other small parameters. Thus, the results of this study have a great significance in improving and developing Lie symmetry and conserved quantity of mechanical system.

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