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# On Lie rings of torsion groups

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**Abstract** We prove that the Lie ring associated to the lower central series of a finitely generated residually-p torsion group is graded nil.

**Keywords** Torsion group · Nil Lie algebra

**Mathematics Subject Classification** 20F40 · 20F45 · 20F50

## 1 Introduction

Let $G$ be a group. A descending sequence of normal subgroups $G = G_1 > G_2 > \cdots$ is called a central series if $[G_i, G_j] \subseteq G_{i+j}$ for all $i, j \geq 1$. The direct sum of abelian groups $L(G) = \oplus_{i\geq 1} G_i/G_{i+1}$ is a graded Lie ring with Lie bracket $[a_i G_{i+1}, b_j G_{j+1}] = [a_i, b_j]G_{i+j+1}$; $a_i \in G_i, b_j \in G_j$.

Of particular interest are the lower central series: $G_1 = G, G_{i+1} = [G_i, G], i \geq 1$, and, for a fixed prime number $p$, the Zassenhaus series (see [7,8]).

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Let $p$ be a prime number. We say that a group $G$ is residually-$p$ if the intersection of all normal subgroups of indices $p^i$, $i \geq 1$, is trivial.

A graded Lie ring $L = L_1 + L_2 + \cdots$ is called graded nil if for an arbitrary homogeneous element $a \in L_i$ the adjoint operator $ad(a) : x \to [a, x]$ is nilpotent.

The main instrument in the study of the Burnside problem in the class of residually $p$-groups is the connection between torsion in the group $G$ and graded nilness in the Lie algebra $L(G)$ of the Zassenhaus series (see [6–9]).

In this paper we prove this connection for the lower central series and for an arbitrary central series of $G$.

**Theorem 1** *Let $\Gamma$ be a finitely generated residually-$p$ torsion group. Let $\Gamma = \Gamma_1 > \Gamma_2 > \cdots$ be the lower central series of $\Gamma$. Then the Lie ring $L(\Gamma) = \oplus_{i\geq 1}\Gamma_i/\Gamma_{i+1}$ is graded nil.*

Note that the known important classes of torsion groups: Golod–Shafarevich groups (see [2,3]), Grigorchuck groups [4] and Gupta–Sidki groups [5] are residually—$p$.

We say that a (possibly infinite) group $\Gamma$ is a $p$-group if for an arbitrary element $g \in \Gamma$ there exists $k \geq 1$ such that $g^{p^k} = 1$. Clearly, for a residually $p$-group being torsion and being a $p$-group are equivalent.

**Theorem 2** *Let $\Gamma$ be a $p$-group. Let $\Gamma = \Gamma_1 > \Gamma_2 > \cdots$ be a central series. Then the Lie ring $L(\Gamma) = \oplus_{i\geq 1}\Gamma_i/\Gamma_{i+1}$ is locally graded nil.*

In other words, we claim that an arbitrary finitely generated graded subalgebra of $L(\Gamma)$ is graded nil.

## 2 Definitions and results

Let $\Gamma$ be a residually-$p$ group and let $G$ be its pro-$p$ completion (see [1]). For an element $y \in G$ let $\langle y^G \rangle$ denote the closed normal subgroup of $G$ generated by $y$. Let $[y^G, y^G]$ denote the closed commutator subgroup of $\langle y^G \rangle$.

For elements $g_1, g_2, \ldots, g_n \in G$ let $[g_1, g_2, \ldots, g_n] = [g_1, [g_2, [\ldots, g_n]]\ldots]$ be their left-normed commutator.

We will need the following equalities which can be found in [1]:

(1) For an arbitrary integer $k \geq 1$ we have $[y, x]^k = [y^k, x] \bmod [y^G, y^G]$;

(2) $[y, x^k] = [y, x]^{\binom{k}{1}}[y, x, x]^{\binom{k}{2}} \ldots [y, \underbrace{x, \ldots x}_{k}]^{\binom{k}{k}} \bmod [y^G, y^G]$.

Let $p$ be a prime number. The equalities (1) and (2) imply that

(3) $c = [y, x^p][y, \underbrace{x, \ldots, x}_{p}]^{-1} = [y^{\binom{p}{1}}, x][y^{\binom{p}{2}}, x, x] \ldots [y^{\binom{p}{p-1}}, \underbrace{x, \ldots x}_{p-1}] \bmod [y^G, y^G]$.

Hence $[y^p, x] = c[[y^p, x], x]^{-\binom{p}{2}/p} \ldots [[y^p, x], \underbrace{x, \ldots x}_{p-2}]^{-\binom{p}{p-1}/p} \bmod [y^G, y^G]$.

Iterating this process and the use of equalities (1) and (2) we conclude that there exists an infinite sequence of nonnegative integers $k_i \geq 0$ such that

(4) $[y^p, x] = c[c, x]^{k_1}[c, x, x]^{k_2} \ldots \mod [y^G, y^G]$.

Let $\rho$ be a left-normed group commutator, $\rho = [g_1, \ldots g_m]$, where each element $g_i$ is either equal to y or to $x^{p^k}$ for some $k \geq 1$. Let $d_x(\rho)$ denote the sum of powers of $x$ involved in $\rho$ and $d_y(\rho)$ the number of times the element $y$ occurs in $\rho$.

**Lemma 1** *Let* $\rho = [y, x^{p^{i_1}}, x^{p^{i_2}}, \ldots, x^{p^{i_k}}]$, $d_y(\rho) = 1$, $0 \leq i_1, \ldots, i_k \leq l-1$. *Then* $\rho$ *is a converging product of commutators* $\sigma$ *of types:*

(i) $\sigma = [y, x^{p^{j_1}}, x^{p^{j_2}}, \ldots, x^{p^{j_s}}]$, *where* $d_y(\sigma) = 1$, *no more than one integer of* $j_1, \ldots, j_s$ *is different from 0 and* $d_x(\sigma) \geq d_x(\rho)$;
(ii) $\sigma = [y, x^{p^{j_1}}, \ldots, y, \ldots]$, *where* $d_y(\sigma) \geq 2$ *and* $(d_y(\sigma) - 1)p^l + d_x(\sigma) \geq d_x(\rho)$.

*Proof* Suppose that $i_\alpha, i_\beta \geq 1$, $1 \leq \alpha \neq \beta \leq k$. We will represent $\rho$ as a product of commutators of types (i) and (ii) and of commutators of type

(iii) $\rho' = [y, x^{p^{j_1}}, x^{p^{j_2}}, \ldots, x^{p^{j_s}}]$, where $0 \leq j_1, \ldots, j_s \leq l-1$ and $0 \leq d_x(\rho') > d_x(\rho)$

and type

(iv) $\rho'' = [y, x^{p^{j_1}}, x^{p^{j_2}}, \ldots, x^{p^{j_s}}]$, where $0 \leq j_1, \ldots, j_s \leq l-1$ and $d_x(\rho'') = d_x(\rho)$, $s > k$.

Iterating we will get rid of commutators (iii) and (iv).

Without loss of generality we will assume that $\alpha = 1$, $\beta = 2$ and $1 \leq i_1 \leq i_2$.

By (3) we have

$[y, x^{p^{i_1}}] = [y^{\binom{p}{1}}, x^{p^{i_1-1}}] \ldots [y^{\binom{p}{p-1}}, \underbrace{x^{p^{i_1-1}}, \ldots, x^{p^{i_1-1}}}_{p-1}][y, \underbrace{x^{p^{i_1-1}}, \ldots, x^{p^{i_1-1}}}_{p}]$

mod $[y^G, y^G]$.

Now $\rho$ is a product of commutators of the form $[y^{\binom{p}{t}}, \underbrace{x^{p^{i_1-1}}, \ldots, x^{p^{i_1-1}}}_{t}, x^{p^{i_2}}, \ldots, x^{p^{i_k}}]$, $1 \leq t \leq p-1$; of the commutator $[y, \underbrace{x^{p^{i_1-1}}, \ldots, x^{p^{i_1-1}}}_{p}, x^{p^{i_2}}, \ldots, x^{p^{i_k}}]$ and of commutators that involve at least two occurrences of $y$ and the powers $x^{p^{i_1-1}}, x^{p^{i_2}}, \ldots, x^{p^{i_k}}$ (see [7]).

The latter commutators are commutators of type (ii). The commutator $[y, \underbrace{x^{p^{i_1-1}}, \ldots, x^{p^{i_1-1}}}_{p}, x^{p^{i_2}}, \ldots, x^{p^{i_k}}]$ satisfies condition (iv). Hence it remains to consider commutators of the form $[y^{\binom{p}{t}}, \underbrace{x^{p^{i_1-1}}, \ldots, x^{p^{i_1-1}}}_{t}, x^{p^{i_2}}, \ldots, x^{p^{i_k}}]$.

Since $p | \binom{p}{t}$, $1 \leq t \leq p-1$, we will consider the commutator

$$[y^p, \underbrace{x^{p^{i_1-1}}, \ldots, x^{p^{i_1-1}}}_{t}, x^{p^{i_2}}, \ldots, x^{p^{i_k}}].$$

Modulo longer commutators we can move the power $x^{p^{i_2}}$ to the left. By (4) we get

$$[y^p, x^{p^{i_2}}] = \sigma[\sigma, x]^{s_1}[\sigma, x, x]^{s_2}\dots$$

where $\sigma = [y, x^{p^{i_2+1}}][y, \underbrace{x^{p^{i_2}}, \dots, x^{p^{i_2}}}_{p}]^{-1} \bmod [y^G, y^G]$.

The commutators $[y, x^{p^{i_2+1}}, \underbrace{x^{p^{i_1-1}}, \dots, x^{p^{i_1-1}}}_{t}, \dots x^{p^{i_k}}]$ and

$[y, \underbrace{x^{p^{i_2}}, \dots, x^{p^{i_2}}}_{p}, \underbrace{x^{p^{i_1-1}}, \dots, x^{p^{i_1-1}}}_{t}, \dots x^{p^{i_k}}]$ are of type (iii) since $i_2 \geq i_1$ and therefore $p^{i_2+1} + p^{i_1-1} > p^{i_2} + p^{i_1}$.

This finishes the proof of the lemma.

Consider again the commutator $\rho = [y, x^{p^{i_1}}, \dots, x^{p^{i_k}}]$. Suppose that $x^{p^l} = 1$. Consider the $l$-tuple $ind(\rho) = (k_{l-1}, \dots, k_0)$, $k_i \in Z_{\geq 0}$, where $k_i$ is the number of times $i$ occurs among $i_1, \dots, i_k$. Clearly, $k_0 + k_1 + \cdots k_{l-1} = k$.

Consider the length-lex order in $Z^l{}_{\geq 0}$: $(\alpha_1, \dots, \alpha_l) > (\beta_1, \dots, \beta_l)$ if either $\sum \alpha_i > \sum \beta_i$ or $\sum \alpha_i = \sum \beta_i$ and $(\alpha_1, \dots, \alpha_l) > (\beta_1, \dots, \beta_l)$ lexicographically.

**Lemma 2** *Let* $x, y \in G$, $x^{p^l} = 1$, $y^{p^s} = 1$. *A commutator* $\rho = [y, x^{p^{i_1}}, \dots, x^{p^{i_k}}]$ *such that* $d_x(\rho) \geq (s+1)p^l$ *can be represented as a product of commutators* $\sigma = [y, x^{p^{j_1}}, \dots, y, \dots, x^{p^{j_q}}]$, *where* $d_y(\sigma) \geq 2$ *and* $(d_y(\sigma) - 1)p^l + d_x(\sigma) \geq d_x(\rho)$.

*Proof* We will show that $\rho$ is a (converging) product of commutators of the form $\sigma_1$ and $\sigma_2$, where $d_y(\sigma_1) \geq 2$, $(d_y(\sigma_1) - 1)p^l + d_x(\sigma_1) \geq d_x(\rho)$ for commutators of the form $\sigma_1$, whereas commutators of the form $\sigma_2$ look as $\sigma_2 = [y, x^{p^{j_1}}, \dots, x^{p^{j_t}}]$ with $d_x(\sigma_2) > d_x(\rho)$ or $d_x(\sigma_2) = d_x(\rho)$ and $ind(\sigma_2) > ind(\rho)$. Then, applying this assertion to commutators of the form $\sigma_2$ and iterating we will get rid of commutators $\sigma_2$.

We claim that at least one $i$, $0 \leq i \leq l-1$, occurs in $i_1, \dots, i_k$ not less than $p$ times. Indeed, otherwise $d_x(\rho) \leq (p-1)(1 + p + \cdots + p^{l-1})$, which contradicts our assumption that $d_x(\rho) \geq (s+1)p^l$.

Suppose that $i$ occurs in $i_1, \dots, i_k$ not less than p times and $i$ is the smallest in $\{i_1, \dots, i_k\}$ with this property. Moving the occurrences of $i$ to the left, modulo longer commutators, we assume $i_1 = \cdots = i_p = i$.

By (2) we have

$$[y, \underbrace{x^{p^i}, \dots, x^{p^i}}_{p}] = [y, x^{p^{i+1}}][y^{\binom{p}{1}}, x^{p^i}]^{-1} \dots [y^{\binom{p}{p-1}}, \underbrace{x^{p^i}, \dots, x^{p^i}}_{p-1}]^{-1}\tau_1 \dots \tau_q,$$

where $\tau_j$ are commutators that involve $y$ at least twice.

The commutator $\sigma' = [y, x^{p^{i+1}}, x^{p^{i_{p+1}}}, \dots, x^{p^{i_k}}]$ has greater index than $\rho$. Indeed, $d_x(\sigma') = d_x(\rho)$, but $ind(\sigma')$ is lexicographically greater than $ind(\rho)$.

For a commutator $\tau_j' = [\tau_j, x^{p^{i_{p+1}}}, \ldots, x^{p^{i_k}}]$, we have

$$d_x(\tau_j') \geq d_x(\rho) - (p-1)p^i.$$

Hence, $p^l(d_y(\tau_j') - 1) + d_x(\tau_j') \geq p^l + d_x(\rho) - (p-1)p^i > d_x(\rho)$.

Consider now the commutator $\rho' = [y^p, x^{p^i}, x^{p^{i_{p+1}}}, \ldots, x^{p^{i_k}}]$.

We claim that there exists $j \in \{i_{p+1}, \ldots, i_k\}$ such that $j \geq i$. Indeed, otherwise all integers in $\{i_{p+1}, \ldots, i_k\}$ are smaller than $i$ and therefore occur $\leq (p-1)$ times. Hence,

$$d_x(\rho) \leq pp^i + (p-1)(1 + p + \cdots + p^{i-1}) = p^{i+1} + p^i - 1 < 2p^l \leq (s+1)p^l,$$

which contradicts the assumption of the lemma.

Moving $x^{p^j}$ to the right end in $\rho'$ modulo longer commutators we will assume that $i_k = j \geq i$.

Consider the commutator $\rho'' = [y^p, x^{p^i}, x^{p^{i_{p+1}}}, \ldots, x^{p^{i_{k+1}}}]$. We have $d_x(\rho'') = d_x(\rho) - (p-1)p^i - p^j \geq sp^l$. By the induction assumption on $s$ the commutator $\rho''$ is a product of commutators $w$ in $y^p$ and $x$, each commutator involves $\mu = \mu(w) \geq 2$ elements $y^p$ and $(\mu-1)p^l + d_x(w) \geq d_x(\rho'')$. We will assume that $w = [w_1, \ldots, w_\mu]$, $w_j = [y^p, \ldots]$, $1 \leq j \leq \mu$.

*Remark* Any commutator that has degree $\geq \mu + 1$ in $y$ and degree $\geq d_x(w)$ in $x$ fits the requirements of the lemma since $\mu p^l + d_x(w) \geq d_x(\rho'') + p^l \geq d_x(\rho)$.

The commutator $[w_1, \ldots, w_\mu, x^{p^j}]$ is equal to a product

$$[[w_1, x^{p^j}], w_2, \ldots, w_\mu][w_1, [w_2, x^{p^j}], \ldots] \ldots [w_1, \ldots, [w_\mu, x^{p^j}]]$$

modulo longer commutators (see the Remark above).

Consider $[w_1, \ldots, [w_\nu, x^{p^j}], \ldots, w_\mu]$.

In $[w_\nu, x^{p^j}]$ move $x^{p^j}$ to the left position next to $y^p$ modulo longer commutators (see the Remark above).

By (4), $[y^p, x^{p^j}] = c[c, x]^{k_1}[c, x, x]^{k_2} \ldots \tau_1 \ldots \tau_q$, where
$c = [y, x^{p^{j+1}}][y, \underbrace{x^{p^j}, \ldots, x^{p^j}}_{p}]^{-1}$; $\tau_1, \ldots \tau_q \in [y^G, y^G]$; $d_x(\tau_1), \ldots, d_x(\tau_q) \geq p^j$.

If the commutator $[y^p, x^{p^j}]$ is replaced by one of $\tau_1, \ldots, \tau_q$ then see the Remark.

If $[y^p, x^{p^j}]$ is replaced by $c$ then

$$\begin{aligned} &d_x([w_1, \ldots, w_{\nu-1}, c, w_{\nu+1}, \ldots, w_\mu]) \geq \\ &\quad d_x([w_1, \ldots, w_\mu, x^{p^j}]) + (p-1)p^j d_x(w) + pj + 1. \end{aligned}$$

Hence, $(\mu-1)p^l + d_x([w_1, \ldots, w_\mu, x^{p^j}]) \geq (\mu-1)p^l + d_x(\mu) + pj + 1 \geq d_x(\rho'') + p^{j+1} = d_x(\rho) - (p-1)p^i - p^j + p^{j+1} = d_x(\rho) + (p-1)(p^j - p^i) \geq d_x(\rho)$.

Since $d_y([w_1, \ldots, w_\mu, x^{p^j}]) \geq 2$, this commutator satisfies the requirements of the lemma. If the commutators $[y^p, x^{p^j}]$ is replaced by $[c, \underbrace{x, \ldots, x}_{t}]^{k_t}$, then $(\mu - 1)p^l + d_x([w_1, \ldots, w_{\nu-1}, [c, \underbrace{x, \ldots, x}_{t}]^{k_1}, w_{\nu+1}, \ldots, w_\mu]) > d_x(\rho)$.

This finishes the proof of the lemma.

**Lemma 3** *Let $x \in G_i$, $x^{p^l} = 1$, $y \in G_j$, $y^{p^s} = 1$. Suppose that $j \geq 2ip^l$. Then $(yG_{j+1})ad(xG_{i+1})^{(s+1)p^l} = 0$ in the Lie algebra $L = \sum_{k=1}^{\infty} G_k/G_{k+1}$.*

*Proof* By Lemma 2 the group commutator $\rho = [y, \underbrace{x, \ldots, x}_{(s+1)p^l}]$ can be represented as a product of commutators $w = [w_1, \ldots, w_\mu]$, $\mu \geq 2$, where each $w_k$ is a commutator of the type $w_k = [y, x^{p^{j_1}}, \ldots, x^{p^{j_r}}]$, $(\mu - 1)p^l + d_x(w) \geq d_x(\rho) = (s+1)p^l$.

By Lemma 1 each $w_k$ is a product of commutators of type (i) or (ii). A commutator of type (ii) just increases the degree in $y$. Let $[y, x^{p^{j_1}}, \ldots, x^{p^{j_r}}]$ be a commutator of type (i). So all $j_1, \ldots, j_r$, except possibly one, are equal to 0. This implies that

$$[y, x^{p^{j_1}}, \ldots, x^{p^{j_r}}] \in G_{j+i(p^{j_1}+\cdots+p^{j_r}-(p^{l-1}-1))}.$$

Hence, $w \in G_d$, where $d = \mu j + id_x(w) - \mu i(p^{l-1} - 1) \geq j + (\mu - 1)ip^l + (\mu - 1)ip^l + id_x(w) - \mu i(p^{l-1} - 1) \geq j + id_x(\rho) + i[(\mu - 1)p^l - \mu(p^{l-1} - 1)]$.

Now it remains to notice that $(\mu - 1)p^l - \mu(p^{l-1} - 1) > 0$. We showed that $d > j + id_x(\rho)$, which implies the lemma.

**Lemma 4** *The Lie ring $L(\Gamma)$ is weakly graded nil, i.e., for arbitrary homogeneous elements $a, b \in L(\Gamma)$ there exists $n(a, b) \geq 1$ such that $bad(a)^{n(a,b)} = 0$.*

*Proof* Let $a \in \Gamma_i$, $a^{p^l} = 1$. Let $n(a) = 2ip^l$. By Lemma 3, for an arbitrary element $b \in \Gamma_j$, $j \geq n(a)$, there exists an integer $n(a, b) \geq 1$ such that $[b, \underbrace{a, a, \ldots, a}_{n(a,b)}] \in G_{j+in(a,b)+1}$.

Since $\Gamma$ is a torsion group it follows that for an arbitrary $k \geq 1$ the subgroup $\Gamma_k$ has finite index in $\Gamma$, hence $\Gamma_k$ is open in $\Gamma$. The subgroup $G_k$ is the completion of $\Gamma_k$. Hence $\Gamma \cap G_k = \Gamma_k$.

We proved that $bad(a)^{n(a,b)} = 0$ in $L(\Gamma)$. Now let $b$ be an arbitrary homogeneous element from $L(\Gamma)$. Then the degree of the element $b' = bad(a)^{n(a)}$ is greater than $n(a)$. Hence, $bad(a)^{n(a)+n(a,b')} = b'ad(a)^{n(a,b')} = 0$, which finishes the proof of the lemma.

**Lemma 5** *Let $L$ be a Lie algebra over a field $\mathbb{Z}/p\mathbb{Z}$ generated by elements $x_1, \ldots, x_m$. Let $a \in L$ be an element such that $x_i ad(a)^{p^k} = 0$, $1 \leq i \leq m$. Then $Lad(a)^{p^k} = (0)$.*

*Proof* The algebra $L$ is embeddable in its universal associative enveloping algebra $U(L)$. Let $a^{p^k}$ be the power of the element $a$ in $U(L)$. For an arbitrary element $b \in L$ we have $bad(a)^{p^k} = [b, a^{p^k}]$. If the element $a^{p^k}$ commutes with all generators $x_1, \ldots, x_m$ then $[L, a^{p^k}] = Lad(a)^{p^k} = (0)$, which finishes the proof of the lemma.

**Lemma 6** *Let $L$ be a Lie ring generated by elements $x_1, \ldots, x_m$. Suppose that $p^l L = (0)$. Let $a \in L$ be an element such that $x_i ad(a)^{p^k} = 0$, $1 \le i \le m$. Then $Lad(a)^{p^k l} = (0)$.*

*Proof* By Lemma 5 we have $Lad(a)^{p^k} \subseteq pL$. Hence $L(ad(a)^{p^k})^l \subseteq p^l L = (0)$, which proves the lemma.

*Proof of Theorem 1* Let $x_1, \ldots, x_m$ be generators of the group $\Gamma$. Then the elements $x_i\Gamma_2$, $1 \le i \le m$, generate the Lie ring $L(\Gamma)$. Let $p^l$ be the maximum of orders of the elements $x_1, \ldots, x_m$, so $x_i^{p^l} = 1$, $1 \le i \le m$. Then $p^l(x_i\Gamma_2) = 0$ in the Lie ring $L(\Gamma)$. Hence $p^l L(\Gamma) = (0)$.

Let $a$ be a homogeneous element of $L(\Gamma)$. By Lemma 4 there exists $k \ge 1$ such that $(x_i\Gamma_2)ad(a)^{p^k} = 0$ for $i = 1, \ldots, m$. Now Lemma 6 implies that $L(\Gamma)ad(a)^{p^k \cdot l} = (0)$, which finishes the proof of Theorem 1.

*Proof of Theorem 2* Without loss of generality we assume that $\cap_i \Gamma_i = (1)$. We view the subgroups $\{\Gamma_i | i \ge 1\}$ as a basis of neighborhoods of 1 thus making $\Gamma$ a topological group. Let $G$ be a completion of $\Gamma$ in this topology. Let $G_i$ be the closure of $\Gamma_i$ in $G$. Then $G_i \cap \Gamma = \Gamma_i$ and $G = G_1 > G_2 > \cdots$ is a central series of the group $G$. Arguing as in Lemmas 3, 4 we conclude that the Lie ring $L(\Gamma) = \oplus_{i \ge 1}\Gamma_i/\Gamma_{i+1}$ is weakly graded nil. Choose homogeneous elements $a_1, \ldots, a_m \in L(\Gamma)$. Since $\Gamma$ is a $p$-group it follows that there exists $l \ge 1$ such that $p^l a_i = 0$, $1 \le i \le m$. Consider the subring $L'$ of $L(\Gamma)$ generated by $a_1, \ldots, a_m$, $p^l L' = (0)$. If $a$ is a homogeneous element from $L'$ and $a_i ad(a)^{p^k} = 0$, $1 \le i \le m$, then by Lemma 6 we have $L'ad(a)^{p^k \cdot l} = (0)$, which finishes the proof of Theorem 2.



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