TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 348, Number 2, February 1996

AN EXISTENCE RESULT FOR LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH C^∞ COEFFICIENTS IN AN ALGEBRA OF GENERALIZED FUNCTIONS

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ABSTRACT. We prove the existence of solutions for essentially all linear partial differential equations with C^∞-coefficients in an algebra of generalized functions, defined in the paper. In particular, we show that H. Lewy's equation has solutions whenever its right-hand side is a classical C^∞-function.

1. INTRODUCTION

The present paper is related to [22] and [23], but we shall not assume familiarity with them. Our framework is A. Robinson's [16] nonstandard analysis in a form close to the one presented in T. Lindstrøm [13]. For convenience of the reader a short introduction to the nonstandard analysis is presented in an Appendix at the end of this paper. For a discussion of the localization properties of the generalized functions (in terms of "restrictions and sheaves") we refer to A. Kaneko [11].

The main result of the paper states that the equations of the type

$$P(x,\partial)U(x) = F(x), \qquad x \in \Omega, \tag{1.1}$$

have solutions U in $\mathcal{A}(\Omega)$ for any choice of the right-hand side F also in $\mathcal{A}(\Omega)$, in particular, whenever F is a classical C^∞-function on Ω. Here Ω is an open set of $\mathbb{R}^d$ (d is a natural number), $\mathcal{A}(\Omega)$ is an algebra of localizable generalized functions, larger than the class $\mathcal{E}(\Omega) = C^\infty(\Omega)$ of the smooth complex valued functions (C^∞-functions) on Ω and

$$P(x,\partial) = \sum_{|\alpha|\le m} a_\alpha(x)\partial^\alpha \tag{1.2}$$

is a linear partial differential operator (m is a natural number) with coefficients a_α in $\mathcal{E}(\Omega)$, satisfying the condition:

$$\sum_{|\alpha|\le m} |a_\alpha(x)| \neq 0, \qquad x \in \Omega. \tag{1.3}$$

Received by the editors March 7, 1994 and, in revised form, November 28, 1994.
1991 *Mathematics Subject Classification.* Primary 35A05, 35D05, 35E20, 46S10, 46S20.
Key words and phrases. Existence of generalized solutions, Schwartz distribution, nonstandard function, nonstandard functional analysis, nonstandard extension, transfer principle, saturation principle.

In particular, we show that H. Lewy's [12] equation

$$\frac{\partial U(x)}{\partial x_1} + i\frac{\partial U(x)}{\partial x_2} - 2i(x_1 + ix_2)\frac{\partial U(x)}{\partial x_3} = F(x), \qquad x \in \mathbb{R}^3, \tag{1.4}$$

has a solution U in $\mathcal{A}(\mathbb{R}^3)$ for any choice of F in $\mathcal{A}(\mathbb{R}^3)$, in particular, whenever F is a classical smooth function on $\mathbb{R}^3$.

The generalized functions in $\mathcal{A}(\Omega)$ are "localizable in Ω" in the sense that the family $\{\mathcal{A}(\Omega)\}_{\Omega\in\tau}$ is a sheaf of differential algebras in $\mathbb{R}^d$ (A. Kaneko [11]), where τ denotes the usual Euclidean topology on $\mathbb{R}^d$. This property justifies both the usage of $\mathcal{A}(\Omega)$ for spaces of solutions and the name "generalized functions" for their elements. The algebra $\mathcal{A}(\Omega)$ is constructed in the paper as a factor space of the class of nonstandard functions ${}^*\mathcal{E}(\Omega)$.

The result of this paper is a generalization of a similar result in [23], where the existence of solutions for equations of the type (1.1) has been established for a more restricted class of differential operators $P(x, \partial)$ with smooth coefficients (still including H. Lewy's equation (1.4)) in the class of "generalized distributions" $\widehat{\mathcal{E}}(\mathbb{R}^d)$, defined also in [23].

In addition to the notations introduced above, we denote by $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ the class of C^∞-functions with compact supports in Ω and by $\mathcal{D}'(\Omega)$ and $\mathcal{E}'(\Omega)$ we denote the class of Schwartz distributions on Ω and the class of Schwartz distributions with compact supports in Ω, respectively L. Schwartz [21]. We shall write $\operatorname{supp} T$ for the support of $T \in \mathcal{D}'(\Omega)$ and we shall sometimes write $T(x)$ instead of the more correct T even when T is not a classical function. For integration in $\mathbb{R}^d$ we use the Lebesgue integral. As usual, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ will be the systems of the natural, real and complex numbers, respectively, and we use also the notation $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. For the partial derivatives we write ∂^α, $\alpha \in \mathbb{N}_0^d$. If $\alpha = (\alpha_1, \dots, \alpha_d)$ for some $\alpha \in \mathbb{N}_0^d$, we write $|\alpha| = \alpha_1 + \cdots + \alpha_d$. We also use the notation:

$$\Omega_P = \{x \in \Omega : \mathop{\qquad}_{|\alpha|\le m} |a_\alpha(x)| \neq 0\}, \tag{1.5}$$

where a_α are the coefficients in $P(x, \partial)$.

Recall that any linear partial differential equation with constant coefficients

$$P(\partial)U = F \tag{1.6}$$

has solutions U in $\mathcal{D}'(\Omega)$ for any choice of F also in $\mathcal{D}'(\Omega)$ (L. Ehrenpreis [10], B. Malgrange [14]). A general existence result for the linear partial differential equations with smooth coefficients was first conjectured, then proved to be false in the settings of distributions (H. Lewy [12]) and hyperfunctions (P. Schapira [20]). In particular, H. Lewy's equation (1.4) is famous for not having (even local) solutions in $\mathcal{D}'(\mathbb{R}^3)$, nor in the class of Sato's hyperfunctions $\mathcal{B}(\mathbb{R}^3)$ for a large choice of F even in $\mathcal{D}(\mathbb{R}^3)$. That explains why we are looking for solutions in a class of generalized functions different from the usual classes of classical functions, hyperfunctions and distributions.

The result presented in this paper is related to F. Treves's work [24], where the local existence and uniqueness of the Cauchy problem for the equation (1.1) is proved for operators $P(x, \partial)$ with analytic coefficients, where the right-hand side F, the Cauchy data and the solution are, in general, analytic functionals. In addition to the restriction on the coefficients however, this result does not include the case when F is in $\mathcal{E}(\Omega) - \mathcal{D}(\Omega)$ and moreover, the analytic functionals are not localizable objects in the sense of the sheaf theory (A. Kaneko [11]).

The present paper is related to some results in N. Aronszajn [1] and M. S. Baouendi [2] where the solvability of some particular partial differential equations with polynomial coefficients has been established in the space of the "traces of the analytic solutions of the heat equation". For comparison we shall mention that our result is more general and moreover, the "traces" are not localizable objects in the sense of the sheaf theory (A. Kaneko [11]).

Our result is related also to J. F. Colombeau's work [4], where a general existence result for the linear PDE's with smooth coefficients,

$$P^h(x, \partial)U(x) \sim F(x), \tag{1.7}$$

has been established in the class of the "new generalized functions" $\mathcal{G}(\Omega)$ (J. F. Colombeau [3]). Here $P^h(x, \partial)$ is a regularization of the original operator $P(x, \partial)$ (depending on a function h) and $\sim$ is an equivalence relation in $\mathcal{G}(\Omega)$ called "association". Later this result was improved in J. F. Colombeau, A. Heibig, M. Oberguggenberger [5]–[6], where the uniqueness of the Cauchy problem for the equation (1.7) was proved in the class of generalized functions $\mathcal{G}_g(\Omega)$ and the association $\sim$ in (1.7) was replaced by the strict equality in $\mathcal{G}_g(\Omega)$. As in the previous two references however, the functions in $\mathcal{G}_g(\Omega)$ are not localizable objects in the sense of the sheaf theory (M. Oberguggenberger [18, Chapter V, §19]).

Finally, we should mention the global solvability of arbitrary analytic partial differential equations in the framework of E. E. Rosinger [17, Chapter 2] and the existence results for continuous partial differential equation, obtained by means of the Dedekind order completion method in M. Oberguggenberger and E. E. Rosinger [19].

(1.8) *Remark.* We shall briefly explain the philosophy of our paper:

(1) We believe that any naturally defined class of partial differential equations, in particular, the equations of the type (1.1), should be solvable in a suitable class of (classical or generalized) functions $S(\Omega)$.

(2) The functions in $S(\Omega)$ should be "localizable in Ω" in the sense that the family $\{S(\Omega)\}_{\Omega\in\tau}$ is a sheaf in $\mathbb{R}^d$ (A. Kaneko [11], §2). Among other things this property guarantees that any function in $S(\Omega)$ has a support which is a closed set of Ω and that the differential operators with smooth coefficients act "locally" in $S(\Omega)$ in the sense that they are sheaf endomorphisms in $\{S(\Omega)\}_{\Omega\in\tau}$.

(3) For the equations of the type (1.1), the class $S(\Omega)$ should be a differential module over the class of smooth functions $\mathcal{E}(\Omega)$.

Recall that the classes of smooth functions $\mathcal{E}(\Omega)$, Schwartz distributions $\mathcal{D}'(\Omega)$ and J. F. Colombeau's new generalized functions $\mathcal{G}(\Omega)$ satisfy both (2) and (3), while the space of the analytic functionals (used in F. Treves [24]), the space of the "traces of the analytic solutions of the heat equation" (used in N. Aronszajn [1] and M. S. Baouendi [2]) and the algebra of "global generalized functions" $\mathcal{G}_g(\Omega)$ (used in J. F. Colombeau, A. Heibig, M. Oberguggenberger [5]–[6]) fail to satisfy condition (2).

In contrast to the above, the algebra of generalized functions $\mathcal{A}(\Omega)$, constructed in this paper, satisfies more than the conditions (1)–(3) require: in addition to the properties (1)–(3), $\mathcal{A}(\Omega)$ satisfies the following:

(4) $\mathcal{A}(\Omega)$ is a differential algebra over the field of the nonstandard complex numbers ${}^*\mathbb{C}$ (hence $\mathcal{A}(\Omega)$ is an algebra over $\mathbb{C}$). In addition, $\mathcal{A}(\Omega)$ contains the

class of smooth functions $\mathcal{E}(\Omega)$ as a differential subalgebra (which is more than to be simply a "module over $\mathcal{E}(\Omega)$").

(5) The functions in $\mathcal{A}(\Omega)$ are pointwise functions from $\widetilde{\Omega}$ into ${}^*\mathbb{C}$ where the domain $\widetilde{\Omega}$ is a set larger than Ω.

(1.9) *Remark* (J. F. Colombeau's New Generalized Functions). We should mention that there is a strong similarity between the algebra of generalized functions $\mathcal{A}(\Omega)$ and its generalized scalars ${}^*\mathbb{C}$, discussed in this paper, and the algebra of "new generalized functions" $\mathcal{G}(\Omega)$ and their generalized scalars $\overline{\mathbb{C}}$, introduced recently by J. F. Colombeau [3] in the framework of standard analysis (for a similar construction see also Yu. V. Egorov [8]–[9]). For that reason we view this work as an attempt to establish a connection between the nonlinear theory of generalized functions and nonstandard analysis with hope that the interaction between these two theories will prove fruitful for both. We should mention that the involvement of nonstandard analysis has resulted in some improvements of the corresponding standard counterparts; we shall mention two of them:

(a) An improvement of the algebraic properties of the generalized scalars: ${}^*\mathbb{C}$ constitutes a field in contrast to $\overline{\mathbb{C}}$ which is a ring with zero divisors.

(b) The possibility to apply the powerful methods of the nonstandard analysis, in particular, the transfer and saturation principles (the latter is the key in the proof of the existence results; see the Appendix at the end of this paper).

2. Some preliminary results

In what follows Ω will be an open subset of $\mathbb{R}^d$ and $P(x, \partial)$ will be an arbitrary linear partial differential operator (1.2) with coefficients a_α in $\mathcal{E}(\Omega)$.

(2.1) **Definition.** We define the mapping $\xi \to P_\xi$ from Ω into $\mathcal{D}'(\Omega)$, by the formula:

$$\langle P_\xi, \varphi \rangle = (P(x, \partial)\varphi(x))|_{x=\xi}, \qquad \varphi \in \mathcal{D}(\Omega). \tag{2.2}$$

(2.3) **Lemma.** *For P_ξ we have the following representation*:

$$P_\xi(x) = \sum_{|\alpha|\le m} (-1)^{|\alpha|} a_\alpha(\xi) \partial_x^\alpha \delta(x - \xi),$$

where $\delta(x-\xi)$ is Dirac's delta distribution concentrated at $\{\xi\}$ and $\partial_x^\alpha \delta(x-\xi)$ is its α-derivative with respect to x.

Proof. For any $\varphi \in \mathcal{D}(\Omega)$ we have:

$$\left\langle \sum_{|\alpha|\le m} (-1)^{|\alpha|} a_\alpha(\xi) \partial_x^\alpha \delta(x-\xi), \varphi(x) \right\rangle = \sum_{|\alpha|\le m} (-1)^{|\alpha|} a_\alpha(\xi) \left\langle \partial_x^\alpha \delta(x-\xi), \varphi(x) \right\rangle$$
$$= \sum_{|\alpha|\le m} a_\alpha(\xi)(\partial^\alpha \varphi(x))|_{x=\xi} = (P(x,\partial)\varphi(x))|_{x=\xi} = \langle P_\xi, \varphi \rangle. \quad \triangle$$

(2.4) **Lemma.** *Let $\xi \in \Omega$. Then $P_\xi = 0$ iff $\xi \in \Omega_P$ (see (1.5)).*

Proof. ($\Rightarrow$) is obvious. ($\Leftarrow$) $P_\xi = 0$ implies $\sum_{|\alpha|\le m} (-1)^{|\alpha|} a_\alpha(\xi) \partial_x^\alpha \delta(x-\xi) = 0$, by Lemma (2.3), which implies $a_\alpha(\xi) = 0$ for all $\alpha \in \mathbb{N}_0^d$, $|\alpha| \le m$, since $\partial_x^\alpha \delta(x-\xi)$, $\alpha \in \mathbb{N}_0^d$, $|\alpha| \le m$, are linearly independent in $\mathcal{D}'(\Omega)$. $\triangle$

(2.5) **Lemma.** *Let $n \in \mathbb{N}$ and ξ_i, $i = 1, 2, \ldots, n$, be n distinct points in Ω_P (see (1.5)). Then P_{ξ_i}, $i = 1, 2, \ldots, n$, are linearly independent in $\mathcal{D}'(\Omega)$.*

Proof. We have $P_{\xi_i} = 0$ for all $i = 1, 2, \ldots, n$, by Lemma (2.4), which implies $\operatorname{supp}(P_{\xi_i}) = \{\xi_i\}$, by Lemma (2.3). Now, the result follows, since $\operatorname{supp}(P_{\xi_i})$ are mutually disjoint.

3. Existence result in ${}^*\mathcal{E}(\Omega)$

We shall temporarily give up the requirements (2) imposed in Remark (1.8) in the Introduction, and prove an existence result for equations of the type (1.1) in the algebra of the nonstandard functions ${}^*\mathcal{E}(\Omega)$. The localization property (2) will be achieved in the next section by an appropriate factorization of ${}^*\mathcal{E}(\Omega)$.

In what follows, we shall work in a nonstandard model with a set of individuals that contains the complex numbers $\mathbb{C}$ and degree of saturation larger than $\operatorname{card}\mathbb{R}$ (Appendix, Axiom 3). In particular, any polysaturated nonstandard model of $\mathbb{C}$ will suffice (Appendix, Definition (A.5)). If X is a set of complex numbers or a set of (standard) functions, then ${}^*X$ will be its nonstandard extension and if $f : X \to Y$ is a (standard) mapping, then ${}^*f : {}^*X \to {}^*Y$ will be its nonstandard extension (Appendix, Definition (A.4), (i)). For integration over ${}^*\mathbb{R}^d$ we use the *-Lebesgue integral. We shall systematically apply the saturation and transfer principles in the form presented in (Appendix, Axiom 2 and Axiom 3).

Let ${}^*\mathcal{E}(\Omega)$ be the nonstandard extension of $\mathcal{E}(\Omega)$. Recall that all functions f in ${}^*\mathcal{E}(\Omega)$ are pointwise functions of the type $f : {}^*\Omega \to {}^*\mathbb{C}$, where ${}^*\Omega$ is the nonstandard extension of Ω and ${}^*\mathbb{C}$ is the field of the nonstandard complex numbers. Consequently, the restrictions of f on Ω are mappings of the type $f|_\Omega : \Omega \to {}^*\mathbb{C}$. We consider ${}^*\mathcal{E}(\Omega)$ as a differential algebra over ${}^*\mathbb{C}$ with respect to pointwise addition, multiplication, multiplication by scalars in ${}^*\mathbb{C}$ and internal partial differentiation of any standard order. Recall that $\mathcal{E}(\Omega) \subset {}^*\mathcal{E}(\Omega)$ through the mapping $f \to {}^*f$, where ${}^*f$ is the nonstandard extension of f. The class $\mathcal{E}(\Omega)$ (with the usual operations) is a differential subalgebra of ${}^*\mathcal{E}(\Omega)$. Notice that ${}^*\mathcal{D}(\Omega)$ is also a differential subalgebra of ${}^*\mathcal{E}(\Omega)$. Finally, it can be shown that $\mathcal{D}'(\Omega)$ (with the usual distributional operations) can be imbedded in ${}^*\mathcal{E}(\Omega)$ as a differential linear subspace over $\mathbb{C}$ (for a proof in the case $\Omega = \mathbb{R}^d$ we refer to [22]), but we are not going to use this imbedding in what follows.

For the nonstandard extension ${}^*P_\xi : {}^*\mathcal{D}(\Omega) \to {}^*\mathbb{C}$ of the functional P_ξ, defined in the previous section, we have the formula:

$$\langle {}^*P_\xi, \varphi\rangle = ({}^*P(x, \partial)\varphi(x))|_{x=\xi}, \qquad \varphi \in {}^*\mathcal{D}(\Omega),$$

where ${}^*P(x, \partial) : {}^*\mathcal{E}(\Omega) \to {}^*\mathcal{E}(\Omega)$ is the nonstandard extension of $P(x, \partial)$ and $\xi \in \Omega$. Notice that $\langle {}^*P_\xi, {}^*\varphi\rangle = \langle P_\xi, \varphi\rangle$ for all φ in $\mathcal{D}$, where ${}^*\varphi$ is the nonstandard extension of φ.

(3.1) **Lemma.** *Let $P(x, \partial)$ be a linear partial differential operator (1.2) with coefficients a_α in $\mathcal{E}(\Omega)$, $f \in {}^*\mathcal{E}(\Omega)$ and $n \in \mathbb{N}$. Then for any choice of the distinct points $\xi_i, i = 1, 2, \ldots, n$, in Ω_P (see (1.5)) the system of equations*

$$\langle {}^*P_{\xi_i}, \varphi\rangle = f(\xi_i), \qquad i = 1, 2, \ldots, n, \tag{3.2}$$

has a solution φ in ${}^*\mathcal{D}(\Omega)$ which satisfies the inequalities:

$$\sup_{x \in {}^*\Omega} |\partial^\alpha \varphi(x)| \le L(\max_{i=1,2,\ldots,n} |f(\xi_i)|), \qquad \alpha \in \mathbb{N}_0^d, \tag{3.3}$$

for any infinitely large number L in ${}^*\mathbb{R}_+$.

Proof. By Lemma (2.5), $P_{\xi_i}, i = 1, 2, \dots, n$, are linearly independent in $\mathcal{D}'(\Omega)$ and hence, the sets

$$\Phi_i = \bigcap_{j=1}^{k} N(P_{\xi_j}) - N(P_{\xi_i}), \qquad i = 1, 2, \dots, n,$$

are non-empty (N. Dunford and J. T. Schwartz [7, V.3., Lemma 10, p. 421]), where $N(P_{\xi_i})$ denotes the null space of $P_{\xi_i} : \mathcal{D}(\Omega) \to \mathbb{C}$. It is easy to verify now that the function

$$\varphi = \sum_{i=1}^{n} \frac{f(\xi_i)}{\langle P_{\xi_i}, \varphi_i \rangle} {}^*\varphi_i \tag{3.4}$$

is a solution of (3.2) for any choice of $\varphi_i \in \Phi_i$, where ${}^*\varphi_i$ are the nonstandard extensions of φ_i. The function φ in (3.4) obviously satisfies (3.3) since both $\sup_{X \in {}^*\Omega} |{}^*\partial^\alpha \varphi_i(x)|$ and $|\langle P_{\xi_i}, \varphi_i \rangle|$ are standard real numbers.

(3.5) **Theorem.** *Let $P(x, \partial)$ be a linear partial differential operator* (1.2) *with coefficients a_α in $\mathcal{E}(\Omega)$ and $f \in {}^*\mathcal{E}(\Omega)$. Then*:

(i) *There exists u in ${}^*\mathcal{D}(\Omega)$ such that*

$${}^*P(x, \partial)u(x) = f(x) \tag{3.6}$$

for all x in Ω_P, where Ω_P is defined in (1.5).

(ii) *If, in addition, $f(x) = 0$ for all $x \in \Omega - \Omega_P$, then* (3.6) *holds for all $x \in \Omega$.*

Proof. (i) Define the family of internal sets

$$\mathcal{M}_\xi = \{\varphi \in {}^*\mathcal{D}(\Omega) : \langle {}^*P_\xi, \varphi \rangle = f(\xi)\}, \qquad \xi \in \Omega_P,$$

and observe that, by Lemma (3.1), it has the finite interaction property. Hence, by the saturation principle (Appendix, Axiom 3), the interaction

$$\mathcal{M} = \bigcap_{\xi \in \Omega_P} \mathcal{M}_\xi$$

is not empty. Thus, any u in $\mathcal{M}$ satisfies (3.6) for all $x \in \Omega_P$.

(ii) follows immediately from (i) since, by assumption, the left- and right-hand sides of (3.6) are both 0 for all x in $\Omega - \Omega_P$. The proof is complete.

(3.7) **Theorem.** *Let $P(x, \partial)$ be a linear partial differential operator* (1.2) *with coefficients a_α in $\mathcal{E}(\Omega)$, satisfying the condition* (1.3). *Then for any choice of f in ${}^*\mathcal{E}(\Omega)$, in particular, for any f in $\mathcal{E}(\Omega)$, there exists u in ${}^*\mathcal{D}(\Omega)$ such that*

$${}^*P(x, \partial)u(x) = f(x), \qquad x \in \Omega. \tag{3.8}$$

Proof. The result follows immediately from Theorem (3.5) since $\Omega_P = \Omega$, by assumption.

The next result shows that if the right-hand side in (3.8) and its solution u happen to be classical smooth functions, then (3.8) holds in $\mathcal{E}(\Omega)$ in the usual sense.

(3.9) **Lemma.** *Let $f, u \in \mathcal{E}(\Omega)$. Then ${}^*u$ satisfies the equation*

$$(3.10) \qquad {}^*P(x,\partial)\,{}^*u(x) = {}^*f(x), \qquad x \in \Omega,$$

if and only if u satisfies the equation

$$(3.11) \qquad P(x,\partial)u(x) = f(x), \qquad x \in \Omega.$$

Proof. The result follows immediately from the fact that ${}^*f$, ${}^*u$ and ${}^*P(x,\partial)$ are extensions of f, u and $P(x,\partial)$, respectively; hence we can drop the asterisks in (3.10).

(3.12) **Corollary.** *For any choice of f in ${}^*\mathcal{E}(\mathbb{R}^3)$, in particular, for any choice of f in $\mathcal{E}(\mathbb{R}^3)$, the H. Lewy [12] equation*

$$(3.13) \qquad \frac{\partial u(x)}{\partial x_1} + i\frac{\partial u(x)}{\partial x_2} - 2i(x_1 + ix_2)\frac{\partial u(x)}{\partial x_3} = f(x), \qquad x \in \mathbb{R}^3,$$

has a solution u in ${}^*\mathcal{D}(\mathbb{R}^3)$.

Proof. In this case we have $\Omega = \mathbb{R}^3$, $m = 1$, and the condition (1.3) reduces to:

$$\underset{|\alpha|\leq 1}{} |a_\alpha(x)| = 2 + 2\sqrt{x_1^2 + x_2^2} = 0, \qquad x \in \mathbb{R}^3,$$

which is (obviously) true. Thus, $\Omega = \Omega_P = \mathbb{R}^3$ and the result follows from Theorem (3.7).

(3.14) **Corollary.** *Let $P(\partial)$ be a linear partial differential operator with constant coefficients. Then for any choice of f in ${}^*\mathcal{E}(\Omega)$, in particular, for any f in $\mathcal{E}(\Omega)$, the equation*

$$(3.15) \qquad {}^*P(\partial)u(x) = f(x), \qquad x \in \Omega,$$

has a solution u in ${}^*\mathcal{D}(\Omega)$.

Proof. The result follows from Theorem (3.7) since the operators with constant coefficients satisfy (unless they are trivial) the condition (1.3).

What follows is an estimate for the solutions and their derivatives:

(3.16) **Theorem.** *Let $P(x,\partial)$ be a linear partial differential operator* (1.2) *with coefficients a_α in $\mathcal{E}(\Omega)$, satisfying the condition* (1.3). *Let $f \in {}^*\mathcal{E}(\Omega)$ and $f|_\Omega$ be bounded in ${}^*\mathbb{R}$ in the sense that*

$$|f(x)| \leq M, \qquad x \in \Omega,$$

holds for some $M \in {}^*\mathbb{R}_+$. Then for any choice of the infinitely large constant L in ${}^*\mathbb{R}_+$ there exists u in ${}^*\mathcal{D}(\Omega)$ which satisfies both the equation (3.8) *and the estimates*:

$$(3.17) \qquad |\partial^\alpha u(x)| \leq LM, \qquad x \in {}^*\Omega, \alpha \in \mathbb{N}_0^d.$$

Proof. Let $\gamma : \Omega \to \mathbb{R}$ be an unbounded real function on Ω. For any $\xi \in \Omega$, we define the internal set:

$$\mathcal{L}_\xi = \{\varphi \in {}^*\mathcal{D}(\Omega) : \langle {}^*P_\xi, \varphi\rangle = f(\xi), \sup_{x\in{}^*\Omega} |\partial^\alpha\varphi(x)| \leq LM, \alpha \in \mathbb{N}_0^d, |\alpha| \leq |\gamma(\xi)|\}.$$

The family $\{\mathcal{L}_\xi\}_{\xi\in\Omega}$, has the finite intersection property, by Lemma (3.1) since $\Omega = \Omega_P$, by assumption. Hence, by the saturation principle (Appendix, Axiom 3), the intersection

$$\mathcal{L} = \bigcap_{\xi\in\Omega} \mathcal{L}_\xi$$

is not empty. Thus, any u in $\mathcal{L}$ satisfies both (3.8) and (3.17). The proof is complete.

We turn now to the localization property of the classes ${}^*\mathcal{E}(\Omega)$. Denote $S = \{\mathcal{E}(\Omega) : \Omega \in \tau\}$, where τ is the usual Euclidean topology on $\mathbb{R}^d$. Recall that the mapping $\mathcal{E} : \tau \to S$, defined by $\Omega \to \mathcal{E}(\Omega)$, is a sheaf in $\mathbb{R}^d$ of differential algebras over $\mathbb{C}$ with respect to the usual (pointwise) restriction (A. Kaneko [11, §2]). Let ${}^*\tau$, ${}^*S$ and ${}^*\mathcal{E}$ be the nonstandard extentions of τ, S and $\mathcal{E}$, respectively. We have the following result:

(3.18) **Proposition.** *The mapping ${}^*\mathcal{E}\colon {}^*\tau \to {}^*S$ is a sheaf in ${}^*\mathbb{R}^d$ of differential algebras over ${}^*\mathbb{C}$ with respect to the pointwise restriction in ${}^*\mathbb{R}^d$.*

Proof. The result follows immediately by transfer principle and the fact that the mapping $\mathcal{E} : \tau \to S$ is a sheaf in $\mathbb{R}^d$.

Notice that the above result guarantees that the functions in ${}^*\mathcal{E}(\Omega)$ are localizable in ${}^*\Omega$ but not necessarily in Ω. It is easy to check that ${}^*\mathcal{E}$ is a presheaf in $\mathbb{R}^d$ of differential algebras over ${}^*\mathbb{C}$ (A. Kaneko [11, p. 16]). However, the next example shows that ${}^*\mathcal{E}$ is not a sheaf in $\mathbb{R}^d$ and thus, it does not satisfy the requirement (2) imposed in Remark (1.8).

(3.19) **Example.** Let $\Omega = \mathbb{R}$ and $f \in {}^*\mathcal{E}(\mathbb{R})$ be defined by $f(x) = {}^*\varphi(x - \nu)$ for $x \in {}^*\mathbb{R}$, where $\varphi \in \mathcal{D}(\mathbb{R})$, $\varphi = 0$, ${}^*\varphi$ is the nonstandard extension of φ and ν is a fixed infinitely large number in ${}^*\mathbb{R}$. Then $f(x) = 0$ for all finite points x in ${}^*\mathbb{R}$, in particular, for all $x \in \mathbb{R}$ and still $f = 0$ in ${}^*\mathcal{E}(\mathbb{R})$.

One consequence of the above results is that the supports of the functions in ${}^*\mathcal{E}(\Omega)$ are closed subsets in ${}^*\Omega$ instead of being closed subsets in Ω, as required by (2) in Remark (1.8). We shall "improve" this property of ${}^*\mathcal{E}(\Omega)$ in the next section by an appropriate factorization.

4. Algebra of generalized functions $\mathcal{A}(\Omega)$

In this section we define an algebra of generalized functions $\mathcal{A}(\Omega)$ as a factor space of the type $\mathcal{A}(\Omega) = {}^*\mathcal{E}(\Omega)/\mathcal{I}(\Omega)$, where $\mathcal{I}(\Omega)$ is an ideal in ${}^*\mathcal{E}(\Omega)$ defined below. In contrast to ${}^*\mathcal{E}(\Omega)$, the algebra $\mathcal{A}(\Omega)$ satisfies the localization property (2), imposed in Remark (1.8) in the Introduction, but this topic will be discussed in the next section.

Let τ be, as before, the usual Euclidean topology on $\mathbb{R}^d$ and Ω be an open set of $\mathbb{R}^d$. By $\widetilde{\Omega}$ we denote the set of the nearstandard points of ${}^*\Omega$; i.e.

$$\widetilde{\Omega} = \bigcup_{x\in\Omega} \mu(x), \tag{4.1}$$

where $\mu(x)$, $x \in \mathbb{R}^d$, is the system of monads of the topological space $(\mathbb{R}^d, \tau)$ (Appendix, Definition (A.9)).

(4.2) **Lemma.** *Let* $\{\Omega_i\}_{i\in I}$ *be an open covering of* Ω. *Then*:

$$\widetilde{\Omega} = \bigcup_{i\in I} \widetilde{\Omega}_i.$$

Proof.

$$\bigcup_{i\in I} \widetilde{\Omega}_i = \bigcup_{i\in I} \bigcup_{x\in\Omega_i} \mu(x) = \bigcup_{x\in\Omega} \mu(x) = \widetilde{\Omega}.$$

(4.3) **Definition.** (i) We define the factor space $\mathcal{A}(\Omega) = {}^*\mathcal{E}(\Omega)/\mathcal{I}(\Omega)$, where

$$\mathcal{I}(\Omega) = \{f \in {}^*\mathcal{E}(\Omega) : f(x) = 0, x \in \widetilde{\Omega}\}.$$

We supply $\mathcal{A}(\Omega)$ with addition, multiplication, multiplication by scalars in ${}^*\mathbb{C}$ and partial differentiation of any standard order, inherited from ${}^*\mathcal{E}(\Omega)$. The elements of $\mathcal{A}(\Omega)$ will be called "generalized functions on Ω".

(ii) We define the inclusion $\mathcal{E}(\Omega) \subset \mathcal{A}(\Omega)$, by $f \to Q_\Omega({}^*f)$, where $Q_\Omega : {}^*\mathcal{E}(\Omega) \to \mathcal{A}(\Omega)$ is the quotient mapping and ${}^*f$ is the nonstandard extension of f.

(iii) For any generalized function $F = Q_\Omega(f)$ in $\mathcal{A}(\Omega)$, we define values (graph in $\widetilde{\Omega} \times {}^*\mathbb{C}$) $F : \widetilde{\Omega} \to {}^*\mathbb{C}$, by $F(x) = f(x)$, $x \in \widetilde{\Omega}$.

It is clear that $\mathcal{I}(\Omega)$ is a proper differential ideal in ${}^*\mathcal{E}(\Omega)$ (e.g. the function f, defined in Example (3.19), belongs to $\mathcal{I}(\Omega)$). Thus, $\mathcal{A}(\Omega)$ is a differential algebra over the scalars ${}^*\mathbb{C}$. Also, if $f \in \mathcal{E}(\Omega)$, then ${}^*f \in \mathcal{I}(\Omega)$ iff $f = 0$, so that the mapping $f \to Q_\Omega({}^*f)$ is injective. It preserves the usual operations in $\mathcal{E}(\Omega)$, since the mapping $f \to {}^*f$ from $\mathcal{E}(\Omega)$ into ${}^*\mathcal{E}(\Omega)$ preserves them. Finally, it is clear that the graph of $Q_\Omega(f)$ in $\widetilde{\Omega} \times {}^*\mathbb{C}$ is correctly defined in the sense that it does not depend on the choice of f. Notice that if $F(x) = 0$ in ${}^*\mathbb{C}$ for all $x \in \widetilde{\Omega}$, then $F = 0$ in $\mathcal{A}(\Omega)$. Also from the definition of values it is clear that the algebraic operations in $\mathcal{A}(\Omega)$ coincide with the pointwise operations with the values of the corresponding functions. Thus, the generalized functions in $\mathcal{A}(\Omega)$ can be identified with their graphs in $\widetilde{\Omega} \times {}^*\mathbb{C}$. In particular, if $f \in \mathcal{E}(\Omega)$, then: "$Q_\Omega({}^*f)(x) = 0$ in ${}^*\mathbb{C}$ for all $x \in \Omega$" $\Leftrightarrow$ "$Q_\Omega({}^*f) = 0$ in $\mathcal{A}(\Omega)$" $\Leftrightarrow$ "$f = 0$ in $\mathcal{E}(\Omega)$" and in this sense the graph in $\mathcal{A}(\Omega)$ generalizes the usual graph in $\mathcal{E}(\Omega)$.

Our next goal is to extend the differential operators from $\mathcal{E}(\Omega)$ to $\mathcal{A}(\Omega)$.

(4.4) **Definition.** Let $P(x, \partial)$ be a linear partial differential operator (1.2) with coefficients a_α in $\mathcal{E}(\Omega)$. We define $\widehat{P}(x, \partial) : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$, by

$$\widehat{P}(x, \partial) Q_\Omega(f) = Q_\Omega({}^*P(x, \partial) f), \tag{4.5}$$

where ${}^*P(x, \partial)$ is the nonstandard extension of $P(x, \partial)$.

The next results follow immediately from the above definitions:

(4.6) **Lemma.** *Let* $P(x, \partial)$ *be a linear partial differential operator* (1.2) *with coefficients* a_α *in* $\mathcal{E}(\Omega)$. *Then*:

(i) $\widehat{P}(x, \partial)$ *has the symbol*

$$\widehat{P}(x, \partial) = \underset{|\alpha|\leq m}{} c_\alpha(x)\partial^\alpha, \tag{4.7}$$

where all operations (*addition, multiplication and differentiation*) *are in the sense of* $\mathcal{A}(\Omega)$.

(ii) *The operator* $\widehat{P}(x,\partial)$ *is an extension of* $P(x,\partial)$ *in the sense that for any* $f \in \mathcal{E}(\Omega)$ *we have*:

$$\widehat{P}(x,\partial)Q_\Omega({}^*f\,) = Q_\Omega({}^*(P(x,\partial)f\,)). \tag{4.8}$$

(4.9) **Simpler notations.** (i) We shall often write simply $P(x,\partial)$ instead of the more precise $\widehat{P}(x,\partial)$ when no confusion could arise; e.g. the action of $P(x,\partial)$ in $\mathcal{A}(\Omega)$ will be written as:

$$P(x,\partial)Q_\Omega(f\,) = Q_\Omega({}^*P(x,\partial)f), \tag{4.10}$$

where $f \in {}^*\mathcal{E}(\Omega)$.

(ii) We shall sometimes write simply f instead of $Q_\Omega({}^*f\,)$ for $f \in \mathcal{E}(\Omega)$, identifying the standard function f with its image $Q_\Omega({}^*f\,)$ in $\mathcal{A}(\Omega)$.

5. Localization properties and integral in $\mathcal{A}(\Omega)$

We show that the algebra of generalized functions $\mathcal{A}(\Omega)$, introduced in the previous section, satisfies the localization property (2) and, hence, all requirements (1)–(5), imposed in Remark (1.8) in the Introduction. We also define an integral in $\mathcal{A}(\Omega)$ which generalizes the usual Lebesgue integral in $\mathbb{R}^d$. For the concepts of "sheaf", used in the following discussion, we refer to A. Kaneko [11, §2].

(5.1) **Definition** (Restriction, Support)**.** Let Ω and G be two open sets in $\mathbb{R}^d$, $G \subseteq \Omega$, and $Q_\Omega(f\,) \in \mathcal{A}(\Omega)$. Then:

(i) We define the restriction $Q_\Omega(f\,)|_G \in \mathcal{A}(G)$ of $Q_\Omega(f\,)$ on G by $Q_\Omega(f\,)|_G = Q_G(f\,|_{{}^*G})$, where $f\,|_{{}^*G}$ is the pointwise restriction of $f \in {}^*\mathcal{E}(\Omega)$ on ${}^*G$.

(ii) $F \in \mathcal{A}(\Omega)$ is said to vanish on G if $Q_\Omega(f\,)|_G = 0$ in $\mathcal{A}(G)$. The support $\operatorname{supp} F$ of F is the complement of the largest open subset of Ω where F vanishes.

The above definition is justified by the following result:

(5.2) **Proposition.** *Let* $\mathcal{S} = \{\mathcal{A}(\Omega) : \Omega \in \tau\}$, *where* τ *is the usual Euclidean topology on* $\mathbb{R}^d$. *Let* $P(x,\partial)$ *be a linear partial differential operator* (1.2) *with coefficients* a_α *in* $\mathcal{E}(\Omega)$ *and* $\widehat{P}(x,\partial)$ *be its extension to* $\mathcal{A}(\Omega)$ *in the sense of Definition* (4.4). *Then*:

(i) *The mapping* $\mathcal{A} : \tau \to \mathcal{S}$, *defined by* $\Omega \to \mathcal{A}(\Omega)$, *is a sheaf in* $\mathbb{R}^d$ *of differential algebras over* ${}^*\mathbb{C}$.

(ii) *The mapping* $f \to Q_\Omega({}^*f\,)$ *from* $\mathcal{E}(\Omega)$ *into* $\mathcal{A}(\Omega)$ *is a sheaf homomorphism* (*of differential algebras over* $\mathbb{C}$).

(iii) $\widehat{P}(x,\partial)$ *is a sheaf endomorphism in* $\mathcal{S}$.

Proof. (i) $\mathcal{A}$ is obviously a presheaf on $\mathbb{R}^d$. To show that $\mathcal{A}$ is actually a sheaf in $\mathbb{R}^d$, we have to take an open covering $\{\Omega_i\}_{i\in I}$ of Ω and to check the properties FI and FII in A. Kaneko [11, p. 17]. The proof is almost identical to the proof that the family $\{\mathcal{D}'(\Omega) : \Omega \in \tau\}$ is a sheaf in $\mathbb{R}^d$ and we shall skip it (still we have to apply the transfer principle, Appendix, Axiom 2, and to involve Lemma (4.2) at some points of the proof).

(ii) For any $f \in \mathcal{E}(\Omega)$ and any open $G \subseteq \Omega$, we have ${}^*f|_{{}^*G} = {}^*(f\,|_G)$, by the transfer principle (Appendix, Axiom 2). Hence, $Q_\Omega({}^*f\,)|_G = Q_G({}^*f\,|_{{}^*G}) = Q_G({}^*(f|_G))$, as required.

(iii) For any $f \in {}^*\mathcal{E}(\Omega)$ and any open $G \subseteq \Omega$, we have

$$ {}^*P(x,\partial)(f|_{{}^*G}) = ({}^*P(x,\partial)f\,)|_{{}^*G},$$

by the transfer principle (Appendix, Axiom 2). Hence, we obtain

$$\begin{aligned}\widehat{P}(x,\partial)(Q_\Omega(f)|_G) &= \widehat{P}(x,\partial)Q_G(f|_{^*G})\\ &= Q_G(^*P(x,\partial)(f|_{^*G})) = Q_G((^*P(x,\partial)f)|_{^*G}).\end{aligned}$$

On the other hand, we have

$$(\widehat{P}(x,\partial)Q_\Omega(f))|_G = Q_\Omega(^*P(x,\partial)f)|_G = Q_G((^*P(x,\partial)f)|_{^*G}).$$

Thus, $\widehat{P}(x,\partial)(Q_\Omega(f)|_G) = (\widehat{P}(x,\partial)Q_\Omega(f))|_G$, as required. The proof is complete.

(5.3) **Definition** (Integral and Pairing)**.** (i) Let $Q_\Omega(f) \in \mathcal{A}(\Omega)$ and X be a Lebesgue measurable set of $\mathbb{R}^d$ whose closure $\overline{X}$ in $(\mathbb{R}^d,\tau)$ is a compact subset of Ω. Then we define the integral of $Q_\Omega(f)$ over X with values in $^*\mathbb{C}$ by:

$$\int_X Q_\Omega(f)\,dx = \int_{^*X} f(x)\,dx.$$

(ii) Let $Q_\Omega(f)$ have a compact support in Ω. Then we define the integral of $Q_\Omega(f)$ over the whole domain Ω with values in $^*\mathbb{C}$ by:

$$\int_\Omega Q_\Omega(f)\,dx = \int_{^*G} f(x)\,dx,$$

where G is a Lebesgue measurable set of $\mathbb{R}^d$ whose closure $\overline{G}$ in $(\mathbb{R}^d,\tau)$ is a compact subset of Ω such that $\operatorname{supp} Q_\Omega(f) \subset G$.

(iii) We define the pairing between $\mathcal{A}(\Omega)$ and $\mathcal{D}(\Omega)$ by

$$\langle Q_\Omega(f),\varphi\rangle = \int_\Omega Q_\Omega(f)Q_\Omega(^*\varphi)\,dx,$$

where $f \in {}^*\mathcal{E}(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$.

The correctness of the above definitions follows from the fact that in both cases the nonstandard integrals (on the right-hand sides) are over some (internal) subsets of $\widetilde{\Omega}$ and hence, the result of integration does not depend on the choice of the representative f.

(5.4) **Proposition.** *The integral in $\mathcal{A}(\Omega)$ is a generalization of the usual* (*Lebesgue*) *integral in $\mathbb{R}^d$ in the sense that*

(i) *We have*

$$\int_X Q_\Omega(^*f)\,dx = \int_X f(x)\,dx,$$

for all f in $\mathcal{E}(\Omega)$ and any Lebesgue measurable $X \subset \mathbb{R}^d$ whose closure $\overline{X}$ is a compact subset of Ω.

(ii) *We have*

$$\int_\Omega Q_\Omega(^*f)\,dx = \int_\Omega f(x)\,dx,$$

for all f in $\mathcal{D}(\Omega)$.

(iii) $\langle Q_\Omega(^*f),\varphi\rangle = \langle f,\varphi\rangle$ *for any $f \in \mathcal{E}(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$, where $\langle f,\varphi\rangle$ is the usual pairing between $\mathcal{E}(\Omega)$ and $\mathcal{D}(\Omega)$.*

Proof. Both results follow immediately from the fact that for any standard (measurable) function f and any standard (measurable) set $X \subseteq \mathbb{R}^d$ we have

$$\int_{{}^*X} {}^*f(x)\, dx = \int_X f(x)\, dx$$

whenever the integrals are (simultaneously) convergent.

6. Existence result in $\mathcal{A}(\Omega)$

In this section we prove the existence of solutions for the equations (1.1) in the algebra of generalized functions $\mathcal{A}(\Omega)$.

All existence results, obtained in Section 6, can be "transferred" from ${}^*\mathcal{E}(\Omega)$ to the factor space $\mathcal{A}(\Omega)$ through the quotient mapping Q_Ω. If u is a solution of a given equation in ${}^*\mathcal{E}(\Omega)$, then $U = Q_\Omega(u)$ will be a solution of the same equation in $\mathcal{A}(\Omega)$. Notice that all solutions U belong to $Q_\Omega[{}^*\mathcal{D}(\Omega)]$ since u belongs to ${}^*\mathcal{D}(\Omega)$. On the other hand, $Q_\Omega[{}^*\mathcal{D}(\Omega)]$ is a differential subalgebra of $\mathcal{A}(\Omega)$, i.e.

$$Q_\Omega[{}^*\mathcal{D}(\Omega)] \subset \mathcal{A}(\Omega), \tag{6.1}$$

since ${}^*\mathcal{D}(\Omega)$ is a differential subalgebra of ${}^*\mathcal{E}(\Omega)$. All equations are satisfied (by a given U in $\mathcal{A}(\Omega)$) in the sense that the values of the left- and right-hand sides of the equations are equal in ${}^*\mathbb{C}$ (pointwise) for all $x \in \Omega$.

Here are the existence results in $\mathcal{A}(\Omega)$:

(6.2) **Theorem.** *Let $P(x, \partial)$ be a linear partial differential operator* (1.2) *with coefficients a_α in $\mathcal{E}(\Omega)$, satisfying the condition* (1.3). *Then for any choice of F in $\mathcal{A}(\Omega)$, in particular, for any F in $\mathcal{E}(\Omega)$, there exists U in $Q_\Omega[{}^*\mathcal{D}(\Omega)]$ such that*

$$P(x, \partial)U(x) = F(x), \qquad x \in \Omega. \tag{6.3}$$

(6.4) **Lemma.** *Let $F,\ U \in \mathcal{E}(\Omega) \subset \mathcal{A}(\Omega)$ (i.e. F and U are classical smooth functions considered as generalized functions in $\mathcal{A}(\Omega)$). Then* (6.3) *holds in $\mathcal{A}(\Omega)$ if and only if it holds in $\mathcal{E}(\Omega)$ in the usual sense.*

(6.5) **Corollary.** *For any choice of F in $\mathcal{A}(\mathbb{R}^3)$, in particular, for any choice of F in $\mathcal{E}(\mathbb{R}^3)$, H. Lewy's equation* (1.4) *has a solution U in $Q_\Omega[{}^*\mathcal{D}(\mathbb{R}^3)]$.*

(6.6) **Corollary.** *Let $P(\partial)$ be a linear partial differential operator with constant coefficients. Then for any choice of F in $\mathcal{A}(\Omega)$, in particular, for any F in $\mathcal{E}(\Omega)$, the equation*

$$P(\partial)U(x) = F(x), \qquad x \in \Omega, \tag{6.7}$$

has a solution U in $Q_\Omega[{}^*\mathcal{D}(\Omega)]$.

(6.8) **Theorem.** *Let $P(x, \partial)$ be a linear partial differential operator* (1.2) *with coefficients a_α in $\mathcal{E}(\Omega)$, satisfying the condition* (1.3). *Let the generalized function $F \in \mathcal{A}(\Omega)$ be bounded on Ω in ${}^*\mathbb{R}_+$ in the sense that*

$$|F(x)| \leq M, \qquad x \in \Omega, \tag{6.9}$$

for some $M \in {}^*\mathbb{R}_+$. Then for any choice of the infinitely large constant L in ${}^*\mathbb{R}_+$ there exists U in $Q_\Omega[{}^*\mathcal{D}(\Omega)]$ which satisfies both the equation (6.3) *and the estimates*:

$$|\partial^\alpha U(x)| \leq LM, \qquad x \in \widetilde{\Omega}, \quad \alpha \in \mathbb{N}_0^d. \tag{6.10}$$

Common Proof for (6.2), (6.4), (6.5), (6.6) *and* (6.8). These results follow immediately from their counterparts in Section 3 and the simple fact that for any $F = Q_\Omega(f)$ in $\mathcal{A}(\Omega)$ we have $F(x) = f(x)$ (pointwise) for all $x \in \Omega$ since $\Omega \subset \widetilde{\Omega}$.

Acknowledgments

The author thanks Tom Boehme and Kent Morrison for the useful discussion of the manuscript.

Appendix: A short introduction to nonstandard analysis

We present the A. Robinson [16] Nonstandard Analysis by means of three axioms known as the Extension, Transfer and Saturation Principles. Some readers might find it easier to read this text starting from part 4°, where Nonstandard Analysis has been presented as a sequential construction within the framework of Standard Analysis, and then returning to parts 1°–3°. For a further study we recommend Tom Lindstrøm [13], where the reader will find other references on the subject.

1°. **Preparation of a standard theory.** In any standard theory the mathematical objects can be classified into two groups: abstract points which we shall refer to as "standard individuals" (or just "individuals") and "sets" (sets of individuals, sets of sets of individuals, sets of sets of sets of individuals, etc.). In what follows S denotes the set of the individuals of the standard theory under consideration and we shall restrict our discussion to the case when S is an infinite set. For example, in analysis we choose $S = \mathbb{R}$ or $S = \mathbb{C}$ in general topology $S = X \cup \mathbb{R}$, where (X, T) is a topological space, in functional analysis $S = \mathcal{V} \cup \mathbf{K}$, where $\mathcal{V}$ is a vector space over the scalars $\mathbf{K}$, etc. The superstructure $V(S)$ on S is the union

$$V(S) = \bigcup_{k \in \mathbb{N}_0} V_k(S), \tag{A.1}$$

where $V_k(S)$ are defined inductively by $V_0(S) = S$ and $V_{k+1}(S) = V_k(S) \cup \mathcal{P}(V_k(S))$ and $\mathcal{P}(X)$ denotes the power set of X. If $A \in V(S)$, then we define the type $t(A)$ of A by $t(A) = \min\{k \in \mathbb{N}_0 : A \in V_k(S)\}$. The superstructure $V(S)$ consists of all mathematical objects of the theory: the individuals are in $V_0(S)$; the ordered pairs $\langle x, y\rangle$ in $S \times S$ belong to $V_2(S)$ since they can be perceived as sets of the type $\{x, \{x, y\}\}$; the functions $f : S \to S$ and, more generally, the relations in S are subsets of $V_2(S)$ and hence, belong to $V_3(S)$; the algebraic operations in S are perceived as subsets of $S \times S \times S$ and hence also belong to $V(S)$, etc.

2°. **Formal language.** For the study of $V(S)$ we use a formal language $L(V(S))$ based on bounded quantifier formulas only, i.e. formulas of the type $\Phi(A_1, \dots, A_q)$ with constants A_i in $V(S)$, that can be made by:

(a) the symbols: $=, \in, \rceil$ (not) $, \wedge$ (and), $\vee$ (or), $\forall, \exists, \Rightarrow, \Leftrightarrow,$ ( );

(b) countably many variables: x, x_1, $x_2, \dots, y$, z;

(c) bounded quantifiers of the type $(\forall x \in A)$ and $(\exists x \in A)$, where $A \in V(S)$.

For example, let $f : \mathbb{R} \to \mathbb{R}$ be a real function in real analysis and let $x_0 \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}_+$. For the set of individuals we choose $S = \mathbb{R}$. Then:

$$\begin{aligned} &\Phi(\varepsilon, x_0, f(x_0), \mathbb{R}_+, \mathbb{R}, f, <, |\cdot|, -) \\ &\quad = (\exists \delta \in \mathbb{R}_+)(\forall x \in \mathbb{R})(|x - x_0| \leq \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon) \end{aligned} \tag{A.2}$$

is a bounded quantifier formula in $L(V(\mathbb{R}))$, with constants: ε, x_0, $f(x_0)$, $\mathbb{R}_+$,$\mathbb{R}$, f, $<, |\ |$, "$-$", where $<$, $|\ |$ and "$-$" are the order relation, absolute value and subtraction in $\mathbb{R}$, respectively (perceived as elements in $V(\mathbb{R})$). The bounded quantifier formulas are interpreted as statements about $V(S)$. Notice that the usage of unbounded quantifiers such as $(\forall x)(\exists y)$ is forbidden in $L(V(S))$. For a more detailed exposition of the formal language $L(V(S))$ associated with $V(S)$ we refer to Tom Lindstrøm [13, Chapter IV], but we believe that the reader can successfully proceed further without a special background in mathematical logic. After this preparation of the standard theory we are ready to involve the nonstandard methods:

3°. **Axiomatic approach to nonstandard analysis.** Let S be an infinite set (of standard individuals of the standard theory under consideration) and $V(S)$ be its superstructure (A.1). A nonstandard model of S consists of the superstructure $V({}^*S)$ on a set of "nonstandard individuals" ${}^*S$, and an injective mapping $A \to {}^*A$ from $V(S)$ into $V({}^*S)$, called an "extension mapping", which satisfies the following two axioms:

Axiom 1 (Extension Principle)**.** (a) $s = {}^*s$ for all (standard individuals) $s \in S$;
(b) S is a proper subset of ${}^*S$, i.e. $S \subset {}^*S$ and $S = {}^*S$.

Axiom 2 (Transfer Principle)**.** A formula $\Phi(A_1, \dots, A_q)$ is true in $L(V(S))$ iff its nonstandard counterpart ${}^*\Phi({}^*A_1, \dots, {}^*A_q)$ is true in $L(V({}^*S))$, where ${}^*\Phi({}^*A_1, \dots, {}^*A_q)$ is obtained from $\Phi(A_1, \dots, A_q)$ by replacing all constants $A_1, \dots,$ A_q by their *-images ${}^*A_1, \dots, {}^*A_q$, respectively.

(A.3) *Remark.* Notice that ${}^*S$ is the image of S under the mapping *. Once ${}^*S$ is found, the superstructure $V({}^*S)$ is determined by (A.1), where S is replaced by ${}^*S$. The formal language $L(V({}^*S))$ differs from $L(V(S))$ only by its constants: they belong to $V({}^*S)$. Hence the formula ${}^*\Phi({}^*A_1, \dots, {}^*A_q)$ is interpreted as a statement about $V({}^*S)$. For example, if Φ is the formula in $V(\mathbb{R})$ given by (A.2), then its nonstandard counterpart in $L(V({}^*\mathbb{R}))$ is given by:

$$\begin{aligned}&{}^*\Phi(\varepsilon, x_0, f(x_0), {}^*\mathbb{R}_+, {}^*\mathbb{R}, {}^*f, <, |\ |, -)\\&\qquad\qquad = (\exists \delta \in {}^*\mathbb{R}_+)(\forall x \in {}^*\mathbb{R})(|x - x_0| < \delta \Rightarrow |{}^*f(x) - f(x_0)| < \varepsilon),\end{aligned}$$

where ${}^*\mathbb{R}$ and ${}^*\mathbb{R}_+$ are (by definition) the sets of the nonstandard real numbers and positive nonstandard real numbers, respectively, the $*$-image ${}^*f$ of f is (by definition) the "nonstandard extension" of f, the asterisks in front of the standard reals are skipped since $\varepsilon = {}^*\varepsilon$, $x_0 = {}^*x_0$ and $f(x_0) = {}^*f(x_0)$, by the Extension Principle and, in addition, the asterisks in front of ${}^*<$, ${}^*|\ |$, ${}^*-$ are also skipped, by convention, although these symbols now mean the order relation, absolute value and subtraction in ${}^*\mathbb{R}$, respectively.

(A.4) **Definition** (Classification)**.** (i) The objects (individuals or sets) in the range of the $*$-mapping are called "standard" (although they are actually images of standard objects). If $A \in V(S)$, then ${}^*A$ is called the "nonstandard extension" of A (since A can be imbedded in ${}^*A$ by the mapping $a \to {}^*a$ in the cases when A is a set).

(ii) An object (individual or set) in $V({}^*S)$ is called "internal" if it is an element of a standard set of $V({}^*S)$. The set of all internal objects is denoted by ${}^*V(S)$, i.e. ${}^*V(S) = \{\mathcal{A} \in V({}^*S) : \mathcal{A} \in {}^*A \text{ for some } A \in V(S)\}$. The sets in $V({}^*S) - {}^*V(S)$ are called "external".

Notice that the nonstandard individuals in ${}^*S$ are internal objects. Moreover, if $s \in {}^*S$, then s is standard (in the sense of the above definition) iff $s \in S$, which justifies the above terminology.

Let κ be an infinite cardinal number such that $\kappa \geq \aleph_1$, where $\aleph_1$ is the successor of $\aleph_0 = \operatorname{card} \mathbb{N}$. The next (and last) axiom depends on the choice of κ.

Axiom 3 (Saturation Principle: κ-saturation)**.** $V({}^*S)$ is κ-saturated in the sense that

$$\bigcap_{\gamma \in \Gamma} \mathcal{A}_\gamma = \emptyset$$

for any family of internal sets $\{\mathcal{A}_\gamma\}_{\gamma \in \Gamma}$ in $V({}^*S)$ with the finite intersection property and index set Γ such that $\operatorname{card} \Gamma < \kappa$.

(A.5) **Definition.** $V({}^*S)$ is called polysaturated if it is κ-saturated for some cardinal number κ such that $\kappa \geq \operatorname{card} V(S)$.

(A.6) *Remark* (The choice of κ). We should mention that a given set of standard individuals S has actually many nonstandard models $V({}^*S)$ although they can be shown to be isomorphic under some extra set-theoretical assumptions at least in the case when they have the same degree of saturation κ. The choice of κ, however, is in our hands and depends on the standard theory and our specific goals. For example, in our paper the saturation principle has always been applied to families of internal sets with index set $\Gamma \subseteq \Omega$. Since $\operatorname{card} \Omega = c$, where $c = \operatorname{card} \mathbb{R}$, we chose a c^+-saturated nonstandard model with a set of individuals $S = \mathbb{C}$, where c^+ is the successor of c. In particular, any polysaturated model of $\mathbb{C}$ will do.

(A.7) *Remark* (E. Nelson's approach). The axiomatic approach presented above is an up-to-date version of A. Robinson's nonstandard analysis (A. Robinson [16]). There is another axiomatic formulation of nonstandard analysis due to E. Nelson [15] known also as "Internal Set Theory".

(A.8) **Consistence Theorem.** *For any infinite set S and any infinite cardinal κ such that $\kappa \geq \aleph_1$, there exists a κ-saturated* (*polysaturated*) *nonstandard model $V({}^*S)$.*

A sketch of the proof in the particular case $\kappa = \aleph_1$ (where $\aleph_1$ is the successor of $\aleph_0 = \operatorname{card} \mathbb{R}$) is presented in 4° below. For the general proof we refer to Tom Lindstrøm [13, Chapter III–IV].

(A.9) **Definition** (Monads)**.** Let (X, T) be a topological space and $x \in X$. Then the set $\mu(x) \subset {}^*X$, defined by

$$\mu(x) = \bigcap_{x \in G \in T} {}^*G,$$

is called the monad of x in (X, T).

4°. **Sequential approach to nonstandard analysis.** Although Nonstandard Analysis arose historically in close connection with model theory and mathematical logic, it is completely possible to construct it in the framework of Standard Analysis, i.e. assuming the axioms of Standard Analysis only (along with the Axiom of Choice). The method is known as "ultrapower construction" or "constructive nonstandard analysis". This part of our exposition can be viewed either as a proof of the consistence theorem above (in the particular case $\kappa = \aleph_1$) or as an independent "sequential approach" to Nonstandard Analysis:

(A.10) **Definition** (Ultrapower Construction)**.** (i) Let $p : \mathcal{P}(\mathbb{N}) \to \{0, 1\}$ be a finitely additive measure such that $p(A) = 0$ for all finite $A \subset \mathbb{N}$ and $p(\mathbb{N}) = 1$. To see that there exist measures with these properties, take a free ultrafilter $\mathcal{U} \subset \mathcal{P}(\mathbb{N})$ on $\mathbb{N}$ (here the Axiom of Choice is involved) and define $p(A) = 0$ for $A \notin \mathcal{U}$ and $p(A) = 1$ for $A \in \mathcal{U}$. We shall keep p fixed in what follows.

(ii) Let $S^{\mathbb{N}}$ be the set of all sequences in S. Define an equivalence relation $\sim$ in $S^{\mathbb{N}}$ by: $\{a_n\} \sim \{b_n\}$ if $a_n = b_n$ a.e. (where "a.e." stands for "almost everywhere"), i.e. if $p(\{n : a_n = b_n\}) = 1$. Then the factor space ${}^*S = S^{\mathbb{N}}/\sim$ defines a set of nonstandard individuals. (Notice that ${}^*S$ depends on the choice of the measure p.) We shall denote by $\langle a_n \rangle$ the equivalence class determined by the sequence $\{a_n\}$. The inclusion $S \subset {}^*S$ is defined by $s \to \langle s, s, \dots \rangle$. We can determine now the superstructure $V({}^*S)$ by (A.1), where S is replaced by ${}^*S$, and the latter is treated as a set of individuals (although it is, actually, a set of sets of sequences).

(iii) Let $V(S)^{\mathbb{N}}$ be the set of all sequences in $V(S)$ (i.e. sequences of points in S, sequences of subsets of S, sequences of functions, sequences of "mixture of points and functions",..., sequences of "everything"). A sequence $\{A_n\}$ in $V(S)^{\mathbb{N}}$ is called "tame" if there exists m in $\mathbb{N}_0$ such that $A_n \in V_m(S)$ for all $n \in \mathbb{N}$. If $\{A_n\}$ is a tame sequence in $V(S)^{\mathbb{N}}$, then its type $t(\{A_n\})$ is defined as the (unique) $k \in \mathbb{N}_0$ such that $t(A_n) = k$ a.e., where $t(A_n)$ is the type of A_n in $V(S)$ defined in 1°. To any tame sequence $\{A_n\}$ in $V(S)^{\mathbb{N}}$ we associate an element $\langle A_n \rangle$ in $U({}^*S)$ by induction on the type of $\{A_n\}$: If $t(\{A_n\}) = 0$, then $\langle A_n \rangle$ is the element in ${}^*S$, defined in (ii) above. If $\langle B_n \rangle$ is already defined for all tame sequences $\{B_n\}$ in $V(S)^{\mathbb{N}}$ with $t(\{B_n\}) < k$ and $t(\{A_n\}) = k$, then

$$\langle A_n \rangle = \{\langle B_n \rangle : \{B_n\} \in V(S)^{\mathbb{N}}; t(\{B_n\}) < k; B_n \in A_n \text{ a.e.}\}.$$

The element $\mathcal{A} \in V({}^*S)$ is called "internal" if it is of the type $\mathcal{A} = \langle A_n \rangle$ for some tame sequence $\{A_n\}$ in $V(S)^{\mathbb{N}}$. The elements of $V({}^*S)$ of the type ${}^*A = \langle A, A, \dots \rangle$ for some $A \in V(S)$ are called "standard". Now we define the extension $*$-mapping from $V(S)$ into $V({}^*S)$ by $A \to {}^*A$ and the construction of the nonstandard model is complete. We shall leave to the reader to check that this model satisfies Axiom 1, Axiom 2 and Axiom 3 for $\kappa = \aleph_1$ treated now as theorems.

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