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M. M. Mohie EL-Din · M. M. Amein · Nahed S. A. Ali · M. S. Mohamed

Measures of information for concomitants of generalized order statistics from subfamilies of Farlie–Gumbel–Morgenstern distributions

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Abstract In this paper, we study Shannon's entropy and Fisher information number for concomitants of generalized order statistics from subfamilies of Farlie–Gumbel–Morgenstern when the marginal distributions are Weibull, exponential, Pareto and power function. Also, we provide some numerical results of Shannon entropy and Fisher information number for concomitants of order statistics.

Mathematics Subject Classification 62B10 · 62H99 · 62P99

الملخص
في هذا البحث، ندرس كون شانون والعدد فيشر للمعلومات لمصاحبات الرتب الإحصائية المعممة من عائلات فرعية لفارلي-جمبل-مورجسترن عندما تكون التوزيعات الهامشية هي وايبل، اسية، باريتو ودالة قوى. نعطي ايضا بعض النتائج العددية لكون شانون والعدد فيشر للمعلومات لمصاحبات الرتب الإحصائية.

1 Introduction

Kamps [9] has introduced GOS's as a unified model of ordered random variables such as ordinary order statistics, sequential order statistics, progressive type-II censoring, record values and Pfeifers records. The joint density function of the GOS's $X(1,n,m,k), X(2,n,m,k), \ldots, X(n,n,m,k)$ is given by:

$$f^{X(1,n,m,k),\ldots,X(n,n,m,k)}(x_1,\ldots,x_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}(1-F(x_i))^m f(x_i)\right) \times (1-F(x_n))^{k-1} f(x_n),$$
$$F^{-1}(0) \le x_1 \le x_2 \le \cdots \le x_n \le F^{-1}(1),$$

with parameters $n \in \mathbb{N}$, $k > 0$, $m \in \mathbb{R}$, such that $\gamma_r = k + (n-r)(m+1) > 0$, for all $1 \le r \le n$.

M. M. Mohie EL-Din · M. M. Amein
Department of Mathematics, Faculty of Science, Al-Azhar University, Cairo, Egypt

M. M. Amein
Department of Mathematics and Statistics, Faculty of Science, Taif University, Hawia, Kingdom of Saudi Arabia

N. S. A. Ali · M. S. Mohamed (✉)
Department of Mathematics, Faculty of Education, Ain Shams University, Cairo, Egypt
E-mail: jin_tiger123@yahoo.com

Morgenstern [12] has introduced FGM distributions; Gumbel [8] has studied FGM for exponential distribution. Farlie [5] has considered this family in the general form. Let $F_X(x)$ and $F_Y(y)$ be the distribution functions of the random variables X and Y, respectively. Then the probability density function (pdf) of the bivariate FGM distributions is given by:

$$f_{X,Y}(x, y) = f_X(x) f_Y(y)[1 + \alpha(2F_X(x) - 1)(2F_Y(y) - 1)], \quad -1 \le \alpha \le 1. \tag{1.1}$$

Here, $f_X(x)$ and $f_Y(y)$ are the marginal pdf's of X and Y, respectively. The parameter α is known as the dependence parameter of the random variables X and Y. If α is zero, then X and Y are independent. For the FGM family with pdf given by (1.1), the density function of the concomitant of r-th GOS's $Y_{[r,n,m,k]}$, $1 \le r \le n$, is given by Beg and Ahsanullah [1], as follows:

$$g_{[r,n,m,k]}(y) = f_Y(y)\left[1 + \alpha C^*(r, n, m, k)(2F_Y(y) - 1)\right], \tag{1.2}$$

where $C^*(r, n, m, k) = 1 - \frac{2\prod_{j=1}^{r}\gamma_j}{\prod_{i=1}^{r}(\gamma_i+1)}$. Entropy is an index that is used to measure dispersion, volatility risk and uncertainty. This concept was formerly introduced by Shannon [13] in the information theory literature. The Shannon entropy of a random variable X is a mathematical measure of information which measures the average reduction of uncertainty of X. Tahmasebi and Behboodian [15] have introduced the Shannon entropy for concomitants of GOS's of FGM family, the Shannon entropy for a continuous random variable X with pdf $f_X(x)$ is defined as:

$$H(X) = -E(\ln f_X(X)) = -\int_{-\infty}^{\infty} f_X(x) \ln f_X(x) \mathrm{d}x. \tag{1.3}$$

Tahmasebi and Jafari [16] have introduced the Fisher information number for concomitants of GOS's of FGM family, the Fisher information number for a continuous random variable X with pdf $f_X(x)$ is defined as:

$$I(X) = \int_{-\infty}^{\infty} \left[\frac{\partial \ln f_X(x)}{\partial x}\right]^2 f_X(x) \mathrm{d}x, \tag{1.4}$$

this is Fisher information number for location parameter, and also called shift-invariant Fisher information number. Furthermore, it has been used to develop a unifying theory physical law called the principle of "extreme physical information" (see Frieden [6,7]). Noting that, it is different than what was introduced by BuHamra and Ahsanullah [2].

Remark 1.1 In the computation of this paper, we use some important formulas as follow:

1. Let $T(t) = \int_{-\infty}^{\infty}[f(x)]^t \mathrm{d}x$. Then $[-\frac{\partial T(t)}{\partial t}]_{t=1} = H(X)$, $t \ge 1$.
2. Let $A(X) = \int_{-\infty}^{\infty} u(x) f(x) \ln f(x) \mathrm{d}x$, $U(t) = \int_{-\infty}^{\infty} u(x)[f(x)]^t \mathrm{d}x$. Then $[\frac{\partial U(t)}{\partial t}]_{t=1} = A(X)$, $t \ge 1$.

The rest of this article is organized as follows. In Sect. 2, we derive Shannon entropy for concomitants of GOS's of FGM family for some well-known distributions such as Weibull, Pareto and power function distributions. In Sect. 3, we develop Fisher information number for concomitants of GOS's of FGM family for some well-known distributions such as exponential, Pareto and power function distributions. In Sect. 4, we compute numerical values of our results for concomitants of order statistics.

2 Shannon entropy for concomitants of GOS's from subfamilies of FGM family

Tahmasebi and Behboodian [15] have introduced the Shannon entropy for concomitants of GOS's of FGM family by the following theorem:

Theorem 2.1 *If $Y_{[r,n,m,k]}$ is the concomitant of r-th GOS's from (1.1), then, from (1.3), the Shannon entropy of $Y_{[r,n,m,k]}$ for $1 \le r \le n$, $\alpha \ne 0$, $-1 \le \alpha \le 1$ is given by:*

$$H(Y_{[r,n,m,k]}) = W(r, \alpha, n, m, k) + H(Y)(1 - \alpha C^*(r, n, m, k)) - 2\alpha C^*(r, n, m, k)\phi_f(y), \tag{2.1}$$

where

$$W(r, \alpha, n, m, k) = \frac{1}{4\alpha C^*(r, n, m, k)}\left[(1 - \alpha C^*(r, n, m, k))^2 \ln(1 - \alpha C^*(r, n, m, k)) - (1 + \alpha C^*(r, n, m, k))^2 \ln(1 + \alpha C^*(r, n, m, k))\right] + \frac{1}{2}, \tag{2.2}$$

$$\phi_f(y) = \int_{-\infty}^{\infty} F_Y(y) f_Y(y) \ln f_Y(y) \mathrm{d}y. \tag{2.3}$$

In the following subsections, we will apply the last theorem to some subfamilies of FGM family such as Weibull, Pareto and power function distributions.

2.1 Weibull distribution

The pdf and cdf for Weibull distribution are given by, respectively:

$$f(y) = cy^{c-1}e^{-y^c}, \tag{2.4}$$

$$F(y) = 1 - e^{-y^c}, \quad 0 \le y < \infty, \ c > 0. \tag{2.5}$$

Theorem 2.2 *If $Y_{[r,n,m,k]}$ is the concomitant of r-th GOS's for Weibull distribution from (1.1) and (2.5) then, from (2.1), the Shannon entropy of $Y_{[r,n,m,k]}$ for $1 \le r \le n$, $\alpha \ne 0$, $-1 \le \alpha \le 1$ is given by:*

$$\begin{aligned} H(Y_{[r,n,m,k]}) = W(r,\alpha,n,m,k) + (1+\alpha C^*(r,n,m,k)) \left[\frac{c-1}{c}\nu - \ln c + 1\right] \\ -\alpha C^*(r,n,m,k) \left[\frac{c-1}{c}(\nu + \ln 2) - \ln c + \frac{1}{2}\right], \end{aligned} \tag{2.6}$$

where $\nu = -\Gamma'(1)$ is the Euler's constant and $W(r,\alpha,n,m,k)$ is defined by (2.2).

Proof From (2.1), (2.4) and (2.5), we have

$$\begin{aligned} H(Y_{[r,n,m,k]}) &= W(r,\alpha,n,m,k) - (1-\alpha C^*(r,n,m,k)) \int_0^\infty cy^{c-1}\exp(-y^c)\ln(cy^{c-1}\exp(-y^c))\mathrm{d}y \\ &\quad - 2\alpha C^*(r,n,m,k) \int_0^\infty cy^{c-1}\exp(-y^c)(1-\exp(-y^c))\ln(cy^{c-1}\exp(-y^c))\mathrm{d}y, \\ &= W(r,\alpha,n,m,k) - (1+\alpha C^*(r,n,m,k)) \int_0^\infty cy^{c-1}\exp(-y^c)\ln(cy^{c-1}\exp(-y^c))\mathrm{d}y \\ &\quad + 2\alpha C^*(r,n,m,k) \int_0^\infty cy^{c-1}\exp(-2y^c)\ln(cy^{c-1}\exp(-y^c))\mathrm{d}y, \\ &= W(r,\alpha,n,m,k) - (1+\alpha C^*(r,n,m,k))\, I + 2\alpha C^*(r,n,m,k)\, II. \end{aligned} \tag{2.7}$$

To find

$$I = -H(Y) = \int_0^\infty cy^{c-1}\exp(-y^c)\ln(cy^{c-1}\exp(-y^c))\mathrm{d}y,$$

first, we want to obtain

$$\begin{aligned} T(t) = \int_0^\infty [f(y)]^t \mathrm{d}y &= \int_0^\infty c^t y^{t(c-1)} \exp(-ty^c)\mathrm{d}y \\ &= c^{t-1} t^{-\left(\frac{t(c-1)+1}{c}\right)} \Gamma\left(\frac{t(c-1)+1}{c}\right). \end{aligned}$$

$$\begin{aligned} \Longrightarrow \frac{\partial T(t)}{\partial t} = T'(t) = Qc^{t-1}(\ln c)\Gamma\left(\frac{t(c-1)+1}{c}\right) + Q'c^{t-1}\Gamma\left(\frac{t(c-1)+1}{c}\right) \\ +Qc^{t-1}\left(\frac{c-1}{c}\right)\Gamma'\left(\frac{t(c-1)+1}{c}\right), \end{aligned}$$

where $Q = t^{-\left(\frac{t(c-1)+1}{c}\right)}$, $Q' = Q\left(-\left(\frac{t(c-1)+1}{tc}\right) - \left(\frac{c-1}{c}\right)\ln t\right)$.

$$\Longrightarrow T'(1) = I = \ln c - \frac{c-1}{c}\nu - 1. \tag{2.8}$$

To find

$$II = \int_0^\infty cy^{c-1}\exp(-2y^c)\ln(cy^{c-1}\exp(-y^c))\mathrm{d}y,$$

first, we want to obtain

$$\begin{aligned} U(t) &= \int_0^\infty \exp(-y^c)[f(y)]^t \mathrm{d}y = \int_0^\infty c^t y^{t(c-1)}\exp(-(t+1)y^c)\mathrm{d}y \\ &= c^{t-1}(t+1)^{-\left(\frac{t(c-1)+1}{c}\right)}\Gamma\left(\frac{t(c-1)+1}{c}\right). \end{aligned}$$

$$\Longrightarrow U'(1) = II = \frac{\ln c}{2} - \frac{c-1}{c}\left(\frac{\nu + \ln 2}{2}\right) - \frac{1}{4}, \tag{2.9}$$

where $\nu = -\Gamma'(1) = 0.57722$ is the Euler's constant. By substituting (2.8) and (2.9) in (2.9), the result follows. □

From Weibull distribution, we can get the Shannon entropy of other related distributions such as exponential and Rayleigh distributions by changing the parameters.

2.2 Pareto distribution

The pdf and cdf for Pareto distribution are given by, respectively:

$$f(y) = cy^{-(c+1)}, \tag{2.10}$$

$$F(y) = 1 - y^{-c}, \quad y \geq 1,\ c > 0. \tag{2.11}$$

Theorem 2.3 *If $Y_{[r,n,m,k]}$ is the concomitant of r-th GOS's for Pareto distribution from (1.1) and (2.11) then, from (2.1), the Shannon entropy of $Y_{[r,n,m,k]}$ for $1 \leq r \leq n$, $\alpha \neq 0$, $-1 \leq \alpha \leq 1$ is given by:*

$$\begin{aligned} H(Y_{[r,n,m,k]}) &= W(r,\alpha,n,m,k) + (1 + \alpha C^*(r,n,m,k))\left[\ln\frac{1}{c} + \frac{1}{c} + 1\right] \\ &\quad - \alpha C^*(r,n,m,k)\left[\frac{c+1}{2c} - \ln c\right], \end{aligned} \tag{2.12}$$

where $W(r,\alpha,n,m,k)$ is defined by (2.2).

Proof The proof is similar to the proof of Theorem 2.2. □

2.3 Power distribution function

The pdf and cdf for Power distribution function are given by, respectively:

$$f(y) = cy^{c-1}, \tag{2.13}$$

$$F(y) = y^c, \quad 0 \leq y \leq 1,\ c > 0. \tag{2.14}$$

Theorem 2.4 *If $Y_{[r,n,m,k]}$ is the concomitant of r-th GOS's for Power distribution function from (1.1) and (2.14) then, from (2.1), the Shannon entropy of $Y_{[r,n,m,k]}$ for $1 \leq r \leq n$, $\alpha \neq 0$, $-1 \leq \alpha \leq 1$ is given by:*

$$H(Y_{[r,n,m,k]}) = W(r,\alpha,n,m,k) - (1-\alpha C^*(r,n,m,k))\left[\ln c - \frac{c-1}{c}\right] - \alpha C^*(r,n,m,k)\left[\ln c - \frac{c-1}{2c}\right], \tag{2.15}$$

where $W(r,\alpha,n,m,k)$ is defined by (2.2).

Proof The proof is similar to the proof of Theorem 2.2. □

3 Fisher information number for concomitants of GOS's from subfamilies of FGM family

Tahmasebi and Jafari [16] have introduced the Fisher information number for concomitants of GOS's of FGM family by the following theorem:

Theorem 3.1 *If $Y_{[r,n,m,k]}$ is the concomitant of r-th GOS's from (1.1), then, from (1.4), the Fisher information number of $Y_{[r,n,m,k]}$ for $1 \leq r \leq n$, $\alpha \neq 0$, $-1 \leq \alpha \leq 1$ is given by:*

$$I(Y_{[r,n,m,k]}) = \int_{-\infty}^{\infty}\left[\frac{\partial \ln f_Y(y)}{\partial y}\right]^2 f_Y(y)\left[1+\alpha C^*(r,n,m,k)(2F_Y(y)-1)\right]\mathrm{d}y + 4\alpha C^*(r,n,m,k)\int_{-\infty}^{\infty} f_Y'(y) f_Y(y)\mathrm{d}y + 4[\alpha C^*(r,n,m,k)]^2\int_{-\infty}^{\infty}\frac{f_Y^3(y)}{1+\alpha C^*(r,n,m,k)(2F_Y(y)-1)}\mathrm{d}y. \tag{3.1}$$

In the following subsections, we will apply the last theorem for some subfamilies of FGM family such as exponential, Pareto and power function distributions.

3.1 Exponential distribution

The pdf and cdf for exponential distribution are given by, respectively:

$$f(y) = e^{-y}, \tag{3.2}$$

$$F(y) = 1 - e^{-y}, \quad 0 \leq y < \infty. \tag{3.3}$$

Theorem 3.2 *If $Y_{[r,n,m,k]}$ is the concomitant of r-th GOS's for exponential distribution function from (1.1) and (3.3) then, from (3.1), the Fisher information number of $Y_{[r,n,m,k]}$ for $1 \leq r \leq n$, $\alpha \neq 0$, $-1 \leq \alpha \leq 1$ is given by:*

$$I(Y_{[r,n,m,k]}) = \frac{(1+\alpha C^*(r,n,m,k))^2}{2\alpha C^*(r,n,m,k)}\left[\ln(-1-\alpha C^*(r,n,m,k)) - \ln(-1+\alpha C^*(r,n,m,k))\right] - 4\alpha C^*(r,n,m,k). \tag{3.4}$$

Proof Let $d = \alpha C^*(r,n,m,k)$. From (3.1), (3.2) and (3.3), we have

$$\begin{aligned} I(Y_{[r,n,m,k]}) &= \int_0^{\infty}\left[e^{-y}\left[1+d(1-2e^{-y})\right] - 4de^{-2y} + 4d^2\frac{e^{-3y}}{1+d(1-2e^{-y})}\right]\mathrm{d}y \\ &= \int_0^{\infty}\left[-8de^{-2y} + \frac{(1+d)^2}{-2d+(1+d)e^{y}}\right]\mathrm{d}y \\ &= -4d + \frac{(1+d)^2}{2d}\left[\ln(-1-d) - \ln(-1+d)\right]. \end{aligned}$$

□

3.2 Pareto distribution

Theorem 3.3 *If $Y_{[r,n,m,k]}$ is the concomitant of r-th GOS's for Pareto distribution from (1.1) and (2.11) then, from (3.1), the Fisher information number of $Y_{[r,n,m,k]}$ for $1 \le r \le n$, $\alpha \ne 0$, $-1 \le \alpha \le 1$ is given by:*

$$I(Y_{[r,n,m,k]}) = c(c+1)^2\left[\frac{1}{c+2} + \alpha C^*(r,n,m,k)\left(\frac{1}{c+2} - \frac{1}{c+1}\right)\right] - 2\alpha C^*(r,n,m,k)c^2 - \frac{\alpha C^*(r,n,m,k)c^3}{c+1} - \frac{(1+\alpha C^*(r,n,m,k))c^3}{c+2} + \frac{(1+\alpha C^*(r,n,m,k))^2 c^3}{4\alpha C^*(r,n,m,k)}\left[{}_2F_1\left(\frac{2}{c}, 1; 1+\frac{2}{c}; \frac{2\alpha C^*(r,n,m,k)}{1+\alpha C^*(r,n,m,k)}\right) - 1\right]. \tag{3.5}$$

Proof Let $d = \alpha C^*(r,n,m,k)$. From (3.1), (2.10) and (2.11), we have

$$\begin{aligned} I(Y_{[r,n,m,k]}) &= c(c+1)^2 \int_1^\infty \left[y^{-c-3} + d(y^{-c-3} - 2y^{-2c-3})\right] \mathrm{d}y - 4dc^2(c+1)\int_1^\infty y^{-2c-3}\mathrm{d}y \\ &\quad + 4d^2c^3 \int_1^\infty \frac{y^{-3c-3}}{1+d(1-2y^{-c})}\mathrm{d}y \\ &= c(c+1)^2\left[\frac{1}{c+2} + d\left(\frac{1}{c+2} - \frac{1}{c+1}\right)\right] - 2dc^2 + 4d^2c^3\int_1^\infty \frac{y^{-3c-3}}{1+d(1-2y^{-c})}\mathrm{d}y \\ &= c(c+1)^2\left[\frac{1}{c+2} + d\left(\frac{1}{c+2} - \frac{1}{c+1}\right)\right] - 2dc^2 + 4d^2c^3 III. \end{aligned} \tag{3.6}$$

To find

$$\begin{aligned} III = \int_1^\infty \frac{y^{-3c-3}}{1+d(1-2y^{-c})}\mathrm{d}y &= \int_1^\infty \left[-\frac{y^{-2c-3}}{2d} - \frac{1+d}{4d^2}y^{-c-3} - \frac{(1+d)^2}{8d^3}y^{-3} + \frac{(1+d)^3}{8d^3}\frac{1}{y^3((1+d)-2dy^{-c})}\right]\mathrm{d}y \\ &= -\frac{1}{4d(1+c)} - \frac{(1+d)}{4d^2(2+c)} - \frac{(1+d)^2}{16d^3} + \frac{(1+d)^3}{8d^3} IV, \end{aligned} \tag{3.7}$$

we want to obtain

$$IV = \int_1^\infty \frac{1}{y^3((1+d)-2dy^{-c})}\mathrm{d}y = -\int_1^\infty y^{-3}(2dy^{-c} - (1+d))^{-1}\mathrm{d}y,$$

let $z = y^{-c} \Rightarrow \mathrm{d}y = \frac{-1}{c}z^{-\frac{1}{c}-1}\mathrm{d}z$. Then

$$\begin{aligned} IV &= \frac{1}{c}\int_1^\infty z^{\frac{2}{c}-1}(2dz - (1+d))^{-1}\mathrm{d}z \\ &= \frac{1}{c}\int_1^\infty z^{\frac{2}{c}-1}\sum_{n=0}^\infty \binom{-1}{n}(2dz)^n(-(1+d))^{-1-n}\mathrm{d}z \\ &= \frac{1}{c}\sum_{n=0}^\infty \binom{-1}{n}\left(\frac{2d}{1+d}\right)^n(-1)^{-1-n}(1+d)^{-1}\int_1^\infty z^{n+\frac{2}{c}-1}\mathrm{d}z \\ &= \left[\frac{-1}{2(1+d)}\sum_{n=0}^\infty \binom{-1}{n}(-1)^{-n}\frac{1}{n+\frac{2}{c}}\left(\frac{2}{c}\right)\left(\frac{2d}{1+d}z\right)^n z^{\frac{2}{c}}\right]_1^\infty \\ &= \frac{1}{2(1+d)}\,{}_2F_1\left(\frac{2}{c}, 1; 1+\frac{2}{c}; \frac{2d}{1+d}\right), \end{aligned} \tag{3.8}$$

where ${}_2F_1(a, b; c; z)$ is the Gaussian or ordinary hypergeometric function. By substituting (3.8) in (3.7) and (3.7) in (3.6), the result follows. □

Table 1 $H(Y_{[r,n,m,k]})$ for Weibull distribution and $I(Y_{[r,n,m,k]})$ for exponential distribution based on order statistics with $c = 20$

n	r	$H(Y_{[r,n,m,k]})$						$I(Y_{[r,n,m,k]})$					
		$\alpha = -1$	−0.5	−0.25	0.25	0.5	1	$\alpha = -1$	−0.5	−0.25	0.25	0.5	1
5	1	−1.63092	−1.51894	−1.47843	−1.42561	−1.41328	−1.4196	0.686329	0.515059	0.707262	1.36765	1.79543	2.80079
5	2	−1.51894	−1.47843	−1.46174	−1.43533	−1.42561	−1.41328	0.515059	0.707262	0.843006	1.17556	1.36765	1.79543
5	3	–	–	–	–	–	–	–	–	–	–	–	–
5	4	−1.41328	−1.42561	−1.43533	−1.46174	−1.47843	−1.51894	1.79543	1.36765	1.17556	0.843006	0.707262	0.515059
5	5	−1.4196	−1.41328	−1.42561	−1.47843	−1.51894	−1.63092	2.80079	1.79543	1.36765	0.707262	0.515059	0.686329
10	1	−1.69815	−1.5406	−1.4868	−1.42196	−1.41092	−1.4388	1.37896	0.472589	0.653507	1.45998	2.00724	3.31924
10	2	−1.61884	−1.51486	−1.47682	−1.42639	−1.414	−1.41713	0.618968	0.527356	0.718638	1.34955	1.75432	2.70172
10	3	−1.55461	−1.49205	−1.46754	−1.43152	−1.42001	−1.41053	0.464475	0.623875	0.791021	1.24359	1.51681	2.13918
10	4	−1.50309	−1.47209	−1.45896	−1.43735	−1.42887	−1.41664	0.57098	0.753962	0.870058	1.14236	1.296	1.63357
10	5	−1.46316	−1.45493	−1.45107	−1.44386	−1.44052	−1.43435	0.829741	0.911911	0.955242	1.04614	1.0936	1.19237
10	6	−1.43435	−1.44052	−1.44386	−1.45107	−1.45493	−1.46316	1.19237	1.0936	1.04614	0.955242	0.911911	0.829741
10	7	−1.41664	−1.42887	−1.43735	−1.45896	−1.47209	−1.50309	1.63357	1.296	1.14236	0.870058	0.753962	0.57098
10	8	−1.41053	−1.42001	−1.43152	−1.46754	−1.49205	−1.55461	2.13918	1.51681	1.24359	0.791021	0.623875	0.464475
10	9	−1.41713	−1.414	−1.42639	−1.47682	−1.51486	−1.61884	2.70172	1.75432	1.34955	0.718638	0.527356	0.618968
10	10	−1.4388	−1.41092	−1.42196	−1.4868	−1.5406	−1.69815	3.31924	2.00724	1.45998	0.653507	0.472589	1.37896
15	1	−1.72675	−1.54926	−1.49006	−1.42072	−1.41059	−1.4494	1.94028	0.465824	0.634748	1.49538	2.08929	3.52418
15	2	−1.6664	−1.53059	−1.48297	−1.42354	−1.41172	−1.42867	0.9729	0.48757	0.677039	1.41806	1.91065	3.08108
15	3	−1.61442	−1.51335	−1.47622	−1.42669	−1.41429	−1.41631	0.597637	0.532266	0.722956	1.34279	1.73901	2.66496
15	4	−1.56941	−1.49748	−1.4698	−1.43017	−1.41824	−1.41092	0.471878	0.59633	0.772279	1.26965	1.57468	2.27465
15	5	−1.53059	−1.48297	−1.4637	−1.43399	−1.42354	−1.41172	0.48757	0.677039	0.824812	1.19871	1.41806	1.91065
15	6	−1.49748	−1.4698	−1.45793	−1.43812	−1.43017	−1.41824	0.59633	0.772279	0.88038	1.13005	1.26965	1.57468
15	7	−1.4698	−1.45793	−1.45249	−1.44259	−1.43812	−1.43017	0.772279	0.88038	0.938823	1.06378	1.13005	1.26965
15	8	–	–	–	–	–	–	–	–	–	–	–	–
15	9	−1.43017	−1.43812	−1.44259	−1.45249	−1.45793	−1.4698	1.26965	1.13005	1.06378	0.938823	0.88038	0.772279

Table 1 continued

n	r	$H(Y_{[r,n,m,k]})$						$I(Y_{[r,n,m,k]})$					
		$\alpha=-1$	−0.5	−0.25	0.25	0.5	1	$\alpha=-1$	−0.5	−0.25	0.25	0.5	1
15	10	−1.41824	−1.43017	−1.43812	−1.45793	−1.4698	−1.49748	1.57468	1.26965	1.13005	0.88038	0.772279	0.59633
15	11	−1.41172	−1.42354	−1.43399	−1.4637	−1.48297	−1.53059	1.91065	1.41806	1.19871	0.824812	0.677039	0.48757
15	12	−1.41092	−1.41824	−1.43017	−1.4698	−1.49748	−1.56941	2.27465	1.57468	1.26965	0.772279	0.59633	0.471878
15	13	−1.41631	−1.41429	−1.42669	−1.47622	−1.51335	−1.61442	2.66496	1.73901	1.34279	0.722956	0.532266	0.597637
15	14	−1.42867	−1.41172	−1.42354	−1.48297	−1.53059	−1.6664	3.08108	1.91065	1.41806	0.677039	0.48757	0.9729
15	15	−1.4494	−1.41059	−1.42072	−1.49006	−1.54926	−1.72675	3.52418	2.08929	1.49538	0.634748	0.465824	1.94028

Table 2 $H(Y_{[r,n,m,k]})$ and $I(Y_{[r,n,m,k]})$ for Pareto distribution based on order statistics with $c = 20$

n	r	$H(Y_{[r,n,m,k]})$						$I(Y_{[r,n,m,k]})$					
		$\alpha = -1$	−0.5	−0.25	0.25	0.5	1	$\alpha = -1$	−0.5	−0.25	0.25	0.5	1
5	1	−1.67362	−1.78946	−1.86287	−2.03787	−2.13946	−2.37362	253.395	198.525	280.141	550.681	723.581	1126.36
5	2	−1.78946	−1.86287	−1.90314	−1.99064	−2.03787	−2.13946	198.525	280.141	336.4	472.607	550.681	723.581
5	3	–	–	–	–	–	–	–	–	–	–	–	–
5	4	−2.13946	−2.03787	−1.99064	−1.90314	−1.86287	−1.78946	723.581	550.681	472.607	336.4	280.141	198.525
5	5	−2.37362	−2.13946	−2.03787	−1.86287	−1.78946	−1.67362	1126.36	723.581	550.681	280.141	198.525	253.395
10	1	−1.63728	−1.75934	−1.84535	−2.06012	−2.18889	−2.49637	521.317	178.986	257.661	588.102	808.808	1332.46
10	2	−1.68225	−1.79574	−1.86644	−2.03348	−2.12983	−2.35043	227.992	203.922	284.88	543.338	707.012	1086.86
10	3	−1.74229	−1.83507	−1.88823	−2.00755	−2.0737	−2.21956	174.074	245.195	314.922	500.297	611.102	861.79
10	4	−1.81504	−1.87725	−1.91071	−1.9823	−2.02043	−2.10141	222.748	299.564	347.55	459.076	521.596	658.292
10	5	−1.89938	−1.92221	−1.93389	−1.95775	−1.96994	−1.99484	330.926	364.77	382.563	419.788	439.183	479.452
10	6	−1.99484	−1.96994	−1.95775	−1.93389	−1.92221	−1.89938	479.452	439.183	419.788	382.563	364.77	330.926
10	7	−2.10141	−2.02043	−1.9823	−1.91071	−1.87725	−1.81504	658.292	521.596	459.076	347.55	299.564	222.748
10	8	−2.21956	−2.0737	−2.00755	−1.88823	−1.83507	−1.74229	861.79	611.102	500.297	314.922	245.195	174.074
10	9	−2.35043	−2.12983	−2.03348	−1.86644	−1.79574	−1.68225	1086.86	707.012	543.338	284.88	203.922	227.992
10	10	−2.49637	−2.18889	−2.06012	−1.84535	−1.75934	−1.63728	1332.46	808.808	588.102	257.661	178.986	521.317
15	1	−1.62706	−1.74859	−1.8389	−2.06859	−2.20797	−2.54581	741.537	175.251	249.777	602.43	841.763	1413.59
15	2	−1.65214	−1.77264	−1.85317	−2.05005	−2.16639	−2.43964	363.34	186.16	267.521	571.119	769.972	1237.92
15	3	−1.6856	−1.79811	−1.86778	−2.03184	−2.12623	−2.34185	220.019	206.063	286.678	540.597	700.841	1072.19
15	4	−1.72602	−1.82497	−1.88272	−2.01397	−2.08747	−2.25102	175.259	233.543	307.16	510.891	634.502	916.111
15	5	−1.77264	−1.85317	−1.89798	−1.99642	−2.05005	−2.16639	186.16	267.521	328.891	482.032	571.119	769.972
15	6	−1.82497	−1.88272	−1.91357	−1.9792	2.01397	−2.08747	233.543	307.16	351.8	454.057	510.891	634.502
15	7	−1.88272	−1.91357	−1.92949	−1.9623	−1.9792	−2.01397	307.16	351.8	375.825	427.002	454.057	510.891
15	8	–	–	–	–	–	–	–	–	–	–	–	–
15	9	−2.01397	−1.9792	−1.9623	−1.92949	−1.91357	−1.88272	510.891	454.057	427.002	375.825	351.8	307.16

Table 2 continued

n	r	$H(Y_{[r,n,m,k]})$						$I(Y_{[r,n,m,k]})$					
		$\alpha = -1$	−0.5	−0.25	0.25	0.5	1	$\alpha = -1$	−0.5	−0.25	0.25	0.5	1
15	10	−2.08747	−2.01397	−1.9792	−1.91357	−1.88272	−1.82497	634.502	510.891	454.057	351.8	307.16	233.543
15	11	−2.16639	−2.05005	−1.99642	−1.89798	−1.85317	−1.77264	769.972	571.119	482.032	328.891	267.521	186.16
15	12	−2.25102	−2.08747	−2.01397	−1.88272	−1.82497	−1.72602	916.111	634.502	510.891	307.16	233.543	175.259
15	13	−2.34185	−2.12623	−2.03184	−1.86778	−1.79811	−1.6856	1072.19	700.841	540.597	286.678	206.063	220.019
15	14	−2.43964	−2.16639	−2.05005	−1.85317	−1.77264	−1.65214	1237.92	769.972	571.119	267.521	186.16	363.34
15	15	−2.54581	−2.20797	−2.06859	−1.8389	−1.74859	−1.62706	1413.59	841.763	602.43	249.777	175.251	741.537

Table 3 $H(Y_{[r,n,m,k]})$ and $I(Y_{[r,n,m,k]})$ for power function distribution based on order statistics with $c = 20$

n	r	$H(Y_{[r,n,m,k]})$						$I(Y_{[r,n,m,k]})$					
		$\alpha = -1$	−0.5	−0.25	0.25	0.5	1	$\alpha = -1$	−0.5	−0.25	0.25	0.5	1
5	1	−2.44028	−2.2228	−2.12954	−1.97121	−1.90613	−1.80695	1115.59	714.253	545.156	288.058	216.343	299.546
5	2	−2.2228	−2.12954	−2.08647	−2.00731	−1.97121	−1.90613	714.253	545.156	469.704	340.202	288.058	216.343
5	3	–	–	–	–	–	–	–	–	–	–	–	–
5	4	−1.90613	−1.97121	−2.00731	−2.08647	−2.12954	−2.2228	216.343	288.058	340.202	469.704	545.156	714.253
5	5	−1.80695	−1.90613	−1.97121	−2.12954	−2.2228	−2.44028	299.546	216.343	288.058	545.156	714.253	1115.59
10	1	−2.55546	−2.26843	−2.14989	−1.95558	−1.8798	−1.77819	1324.38	798.397	581.543	267.626	202.123	586.307
10	2	−2.41862	−2.21392	−2.12553	−1.97439	−1.91165	−1.81407	1075.83	697.949	538.032	292.401	220.741	270.952
10	3	−2.29683	−2.16234	−2.10186	−1.99391	−1.94643	−1.86502	850.93	603.971	496.385	320.16	256.444	200.68
10	4	−2.18777	−2.11361	−2.07889	−2.01412	−1.98406	−1.92868	650.124	516.969	456.701	350.66	305.923	236.693
10	5	−2.09029	−2.06767	−2.05661	−2.03502	−2.02449	−2.00393	476.291	437.631	419.095	383.697	366.873	335.081
10	6	−2.00393	−2.02449	−2.03502	−2.05661	−2.06767	−2.09029	335.081	366.873	383.697	419.095	437.631	476.291
10	7	−1.92868	−1.98406	−2.01412	−2.07889	−2.11361	−2.18777	236.693	305.923	350.66	456.701	516.969	650.124
10	8	−1.86502	−1.94643	−1.99391	−2.10186	−2.16234	−2.29683	200.68	256.444	320.16	496.385	603.971	850.93
10	9	−1.81407	−1.91165	−1.97439	−2.12553	−2.21392	−2.41862	270.952	220.741	292.401	538.032	697.949	1075.83
10	10	−1.77819	−1.8798	−1.95558	−2.14989	−2.26843	−2.55546	586.307	202.123	267.626	581.543	798.397	1324.38
15	1	−2.60206	−2.28609	−2.15765	−1.94984	−1.87047	−1.77081	1407.29	831.054	595.509	260.539	200.531	815.38
15	2	−2.50214	−2.24764	−2.14067	−1.96255	−1.89139	−1.78964	1228.31	759.996	565.013	276.55	206.835	419.166
15	3	−2.4106	−2.21061	−2.12403	−1.97559	−1.91373	−1.81685	1061.09	691.882	535.373	294.051	222.513	261.821
15	4	−2.32602	−2.17497	−2.10772	−1.98897	−1.93747	−1.85102	904.959	626.834	506.616	312.954	246.118	205.559
15	5	−2.24764	−2.14067	−2.09173	−2.00267	−1.96255	−1.89139	759.996	565.013	478.775	333.179	276.55	206.835
15	6	−2.17497	−2.10772	−2.07607	−2.0167	−1.98897	−1.93747	626.834	506.616	451.884	354.655	312.954	246.118
15	7	−2.10772	−2.07607	−2.06074	−2.03105	−2.0167	−1.98897	506.616	451.884	425.982	377.318	354.655	312.954
15	8	–	–	–	–	–	–	–	–	–	–	–	–
15	9	−1.98897	−2.0167	−2.03105	−2.06074	−2.07607	−2.10772	312.954	354.655	377.318	425.982	451.884	506.616
15	10	−1.93747	−1.98897	−2.0167	−2.07607	−2.10772	−2.17497	246.118	312.954	354.655	451.884	506.616	626.834
15	11	−1.89139	−1.96255	−2.00267	−2.09173	−2.14067	−2.24764	206.835	276.55	333.179	478.775	565.013	759.996

Table 3 continued

n	r	$H(Y_{[r,n,m,k]})$						$I(Y_{[r,n,m,k]})$					
		$\alpha=-1$	−0.5	−0.25	0.25	0.5	1	$\alpha=-1$	−0.5	−0.25	0.25	0.5	1
15	12	−1.85102	−1.93747	−1.98897	−2.10772	−2.17497	−2.32602	205.559	246.118	312.954	506.616	626.834	904.959
15	13	−1.81685	−1.91373	−1.97559	−2.12403	−2.21061	−2.4106	261.821	222.513	294.051	535.373	691.882	1061.09
15	14	−1.78964	−1.89139	−1.96255	−2.14067	−2.24764	−2.50214	419.166	206.835	276.55	565.013	759.996	1228.31
15	15	−1.77081	−1.87047	−1.94984	−2.15765	−2.28609	−2.60206	815.38	200.531	260.539	595.509	831.054	1407.29

3.3 Power distribution function

Theorem 3.4 *If $Y_{[r,n,m,k]}$ is the concomitant of r-th GOS's for Power distribution function from (1.1) and (2.14) then, from (3.1), the Fisher information number of $Y_{[r,n,m,k]}$ for $1 \le r \le n$, $\alpha \ne 0$, $-1 \le \alpha \le 1$ is given by:*

$$I(Y_{[r,n,m,k]}) = c(c-1)^2 \left[\frac{1}{c-2} - \alpha C^*(r,n,m,k) \left(\frac{1}{(c-1)(c-2)} \right) \right] + 2\alpha C^*(r,n,m,k) c^2 + \frac{c^3}{2\alpha C^*(r,n,m,k)} \left[\frac{-(-1+\alpha C^*(r,n,m,k))^2}{2} + \frac{2\alpha C^*(r,n,m,k)(-1+\alpha C^*(r,n,m,k))}{c-2} + \frac{2(\alpha C^*(r,n,m,k))^2}{c-1} + \frac{(-1+\alpha C^*(r,n,m,k))^2}{2} {}_2F_1\left(\frac{-2}{c}, 1; 1-\frac{2}{c}; \frac{-2\alpha C^*(r,n,m,k)}{1-\alpha C^*(r,n,m,k)} \right) \right], \tag{3.9}$$

$c \ne 1, 2$.

Proof The proof is similar to the proof of Theorem 3.3. □

4 Numerical results

Tables 1, 2 and 3 provide $H(Y_{[r,n,m,k]})$ and $I(Y_{[r,n,m,k]})$ values of the concomitants of order statistics ($\gamma_i = n-i+1$) when the marginal distributions are Weibull, exponential, Pareto and power function for $1 \le r \le n$, $n = 5, 10, 15$ and $\alpha = -1, -0.5, -0.25, 0.25, 0.5, 1$. We can find some properties from the numerical results as follow:

1. For n is odd, $r = \frac{n+1}{2}$ we find that $H(Y_{[r,n,m,k]})$ and $I(Y_{[r,n,m,k]})$ are indeterminate.
2. Let $H(Y_{[r,n,m,k]})=H_\alpha(Y_{[r]})$ and $I(Y_{[r,n,m,k]})=I_\alpha(Y_{[r]})$. Then we have $H_\alpha(Y_{[r]})=H_{-\alpha}(Y_{[n-r+1]})$ and $I_\alpha(Y_{[r]}) = I_{-\alpha}(Y_{[n-r+1]})$, $1 \le r \le n$.
3. For fixed n, r, at $c > n, r < \frac{n+1}{2}$ ($r > \frac{n+1}{2}$) then we have, for Pareto distribution $H(Y_{[r,n,m,k]})$ is decreasing (increasing) in α, for power function distribution $H(Y_{[r,n,m,k]})$ is increasing (decreasing) in α.
4. For fixed n, α, at $c > n$, $r \ne n$, $-1 \le \alpha < 0$ ($0 < \alpha \le 1$) then we have, for Weibull distribution $H(Y_{[r,n,m,k]})$ is increasing (decreasing) in r, for Pareto distribution $H(Y_{[r,n,m,k]})$ is decreasing (increasing) in r, for power function distribution $H(Y_{[r,n,m,k]})$ is increasing (decreasing) in r.
5. For fixed n, α, at $c > n$, $-0.5 \le \alpha < 0$ ($0 < \alpha \le 0.5$) then we have, for Weibull distribution $I(Y_{[r,n,m,k]})$ is increasing (decreasing) in r, for Pareto distribution $I(Y_{[r,n,m,k]})$ is increasing (decreasing) in r, for power function distribution $I(Y_{[r,n,m,k]})$ is decreasing (increasing) in r.
6. For fixed r, α, at $c > n$, $-1 \le \alpha < 0$ ($0 < \alpha \le 1$) then we have, for Weibull distribution $H(Y_{[r,n,m,k]})$ is decreasing (increasing) in n, for Pareto distribution $H(Y_{[r,n,m,k]})$ is increasing (decreasing) in n, for power function distribution $H(Y_{[r,n,m,k]})$ is decreasing (increasing) in n.
7. For fixed r, α, at $c > n$, $-0.5 \le \alpha < 0$ ($0 < \alpha \le 0.5$) then we have, for Weibull distribution $I(Y_{[r,n,m,k]})$ is decreasing (increasing) in n, for Pareto distribution $I(Y_{[r,n,m,k]})$ is decreasing (increasing) in n, for power function distribution $I(Y_{[r,n,m,k]})$ is increasing (decreasing) in n.
8. For GOS's, we find that $-1 \le C^*(r,n,m,k) \le 1$ for all possible values of r, n, m, k.

5 Conclusion

We derived an analytical expression of Shannon entropy and Fisher information number from subfamilies of FGM family such as Weibull, exponential, Pareto and power distributions, based on concomitants of GOS's. Applications of these results are applied based on order statistics as a special case of GOS's. We find some important relations in entropy and Fisher information number at some values of the parameters. Conditions for decreasing (increasing) uncertainty and Fisher information number are obtained. We also observed the limit values of the constant $C^*(r,n,m,k)$ for GOS's. The proposed procedures may be considered for other models (such as dual generalized order statistics and case-II of generalized order statistics which are introduced by Burkschat et al. [3] and Kamps and Cramer [10], respectively), and for some other distributions.

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References

1. Beg, M.I.; Ahsanullah, M.: Concomitants of generalized order statistics from Farlie–Gumbel–Morgensterm distributions. Stat. Methodol. **5**, 1–20 (2008)
2. BuHamra, S.S.; Ahsanullah, M.: Fisher information in concomitants of generalized order statistics in Farlie–Gubmel–Morgenstern distributions. J. Stat. Theory Appl. **4**(4), 387–399 (2005)
3. Burkschat, M.; Kamps, U.; Cramer, E.: Dual generalized order statistics. Metron **LXI**(1), 13–26 (2003)
4. David, H.A., O'Connell, M.J.; Yang, S.S.: Distribution and expected value of the rank of a concomitant of an order statistic. Ann. Stat. **5**, 216–223 (1977)
5. Farlie, D.J.G.: The performance of some correlation coefficients for a general bivariate distribution. Biometrika **47**, 307–323 (1960)
6. Frieden, B.R.: Applications to optics and wave mechanics of the criterion of maximum Cramer-Rao bound. J. Mod. Opt. **35**, 1297–1316 (1988)
7. Frieden, B.R.: Physics from Fisher Information—A Unification. Cambridge University Press, Cambridge (1998)
8. Gumbel, E.J.: Bivariate exponential distributions. J. Am. Stat. Assoc. **55**, 698–707 (1960)
9. Kamps, U.: A concept of generalized order statistics. J. Stat. Plan. Inference **48**, 1–23 (1995)
10. Kamps, U.; Cramer, E.: On distribution of generalized order statistics. Statistics **35**, 269–280 (2001)
11. Mohie El-Din, M.M.; Amein, M.M.; Mohamed, M.S.: Concomitants of case-II of generalized order statistics from Farlie–Gumbel–Morgenstern distributions. J. Stat. Appl. Probab. **3**, 345–353 (2015)
12. Morgentern, D.: Einfache beispiele zweidimensionaler verteilunngen. Mitteilungsblatt fur Mathemtische Statistik **8**, 234–235 (1956)
13. Shannon, C.E.: A mathematical theory of communication. Bell Syst. Tech. J. **27**, 379–423 and 623–656 (1948)
14. Tahmasebi, S.; Behboodian, J.: Entropy for concomitants of order statistics in generalized Morgenstern (GM) sub-family and Pseudo-Weibull distribution. World Appl. Sci. J. **8**, 789–791 (2010)
15. Tahmasebi, S.; Behboodian, J.: Shannon information for concomitants of generalized order statistics in Farlie–Gumbel–Morgenstern (FGM) family. Bull. Malays. Math. Sci. Soc. **35**, 975–981 (2012)
16. Tahmasebi, S.; Jafari, A.A.: Fisher information number for concomitants of generalized order statistics in Morgenstern family. J. Inf. Math. Sci. **5**, 15–20 (2013)