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Algebra Universalis

Defining subdirect product closed classes in infinitary logic

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ABSTRACT. We note that a class of models is subdirect product closed if and only if it is definable by a class of $L_{\infty\infty}$ -sentences of a special form.

1. Introduction

By a *weak implication* we mean a sentence in $L_{\infty\infty}$ of the form

$$(\forall \bar{x})(\exists y) \left[[y \in \bar{x}] \wedge \bigwedge_{i \in I} \varrho_i(\bar{x}) \rightarrow \delta(\bar{x}) \right], \quad (\omega)$$

where ϱ_i for $i \in I$, and δ are atomic first-order formulas, $\bar{x} = (x_k)_{k \in K}$, and $[y \in \bar{x}]$ is an abbreviation of $\bigvee_{k \in K} [y = x_k]$. Note that (ω) is equivalent to

$$(\forall \bar{x}) \left[[(\forall y)[y \in \bar{x}] \wedge \bigwedge_{i \in I} \varrho_i(\bar{x})] \rightarrow \delta(\bar{x}) \right]$$

and $\mathbf{M} \models (\forall y)[y \in \bar{a}]$ if and only if the valuation $x_i \mapsto a_i$ is onto M . Thus, \mathbf{M} satisfies (ω) if and only if $\mathbf{M} \models \bigwedge_{i \in I} \varrho_i(\bar{a})$ yields $\mathbf{M} \models \delta(\bar{a})$, for every valuation $x_i \mapsto a_i$ onto M .

A model \mathbf{M} is a *subdirect product* of models \mathbf{M}_s , $s \in S$, if \mathbf{M} is a submodel of the direct product $\prod_{s \in S} \mathbf{M}_s$ and all projections of the product restricted to \mathbf{M} are onto [2]. We write then $\mathbf{M} \leq_{SD} \prod_{s \in S} \mathbf{M}_s$. A class \mathcal{C} of models is *subdirect product closed* if every subdirect product of models from \mathcal{C} also belongs to \mathcal{C} . (All considered classes are assumed to be classes of models in the same fixed first-order language L and are closed under taking isomorphic images.)

In this note we would like to present the following observation.

Theorem. *A class is subdirect product closed if and only if it is definable by a class of weak implications.*

The proof of this fact is obtained by appropriate modifications in the proof of A. Shafaat's theorem [11], which states that the class of models is submodel

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and direct product closed (is a prevariety) if and only if it is definable by possibly infinite implications.

In Section 3, we describe an example showing that we cannot obtain an axiomatization by a set of infinite sentences of a bit more general form than weak implications even for the class $\mathbf{P}_S(\mathbf{M})$ of all subdirect products of a given finite model \mathbf{M} . This is contrary to the situation for prevarieties.

Classes typically considered in universal algebra, such as prevarieties, quasivarieties, and varieties, are subdirect product closed. But the fact that they are closed under additional operations, such as taking subalgebras, ultraproducts, and homomorphic images, makes their axiomatizations simpler. Classes of models closed solely under taking subdirect products appear in abstract algebraic logic as classes of Suszko reduced matrices for deductive systems [5], in particular as classes of reduced matrices for protoalgebraic deductive systems [4]. In the protoalgebraic case though, we also have a simpler axiomatization. Indeed, by [3, Theorem 13.10], such classes may be defined by finite implications (quasi-identities) and one possibly infinite sentence of the form $(\forall x, y)(\exists \bar{z})[\bigwedge_{i \in I} \varrho_i(x, y, \bar{z}) \rightarrow x \approx y]$, where ϱ_i are atomic first-order formulas and there are only countable many variables in \bar{z} . However, no particular axiomatization is known in the general case.

The problem of the existence of a first-order axiomatization for $\mathbf{P}_S(\mathbf{M})$, where \mathbf{M} is finite, was considered by G. Grätzer in [7], and later revisited by J. T. Baldwin and M. A. Samhan in [1], where certain sufficient conditions were presented.

Let us recall an important result of R. C. Lyndon [10, 9]. He showed that subdirect product closed classes that are first-order definable are in fact definable by first-order special Horn sentences.

Finally, note that there is an easy semantical characterization of subdirect product closed classes. For a class \mathcal{C} and a model \mathbf{M} let $\text{Con}_{\mathcal{C}}(\mathbf{M})$ be the ordered set of congruences α of \mathbf{M} such that $\mathbf{M}/\alpha \in \mathcal{C}$. For algebras the order is set inclusion, and for models it is pairwise set inclusion (see [6] for the definition of congruence for models). Then \mathcal{C} is subdirect product closed if and only if $\text{Con}_{\mathcal{C}}(\mathbf{M})$ is a complete lattice, for every \mathbf{M} .

2. Proof of the theorem

We start by verifying the easier forward direction of the Theorem.

Lemma 2.1. *The subdirect product operation preserves the satisfaction of weak implications.*

Proof. Let σ be a weak implication of the form (ω) , let \mathbf{M}_s , $s \in S$, be models satisfying σ , let $\mathbf{M} \leq_{SD} \prod_{s \in S} \mathbf{M}_s$ be a subdirect product, and let $x_i \rightarrow a_i = (a_i^s)_{s \in S}$ be a valuation onto \mathbf{M} such that $\mathbf{M} \models \bigwedge_{i \in I} \varrho_i(\bar{a})$. Then

$$\mathbf{M}_s \models \left[\bigwedge_{i \in I} \varrho_i(\bar{a}^s) \rightarrow \varrho(\bar{a}^s) \right] \wedge \left[\bigwedge_{i \in I} \varrho_i(\bar{a}^s) \right],$$

and hence $\mathbf{M}_s \models \delta(\bar{a}^s)$. This proves that $\mathbf{M} \models \delta(\bar{a})$. \square

Let us now fix a subdirect product closed class \mathcal{C} . For every model \mathbf{M} (not necessarily in \mathcal{C}) define

$$\Phi_{\mathbf{M}} = \{\varphi : \mathbf{M} \rightarrow \mathbf{M}_{\varphi} \mid \varphi \text{ is onto, } M_{\varphi} \subseteq M, \text{ and } \mathbf{M}_{\varphi} \in \mathcal{C}\}.$$

Let $\eta_{\mathbf{M}} : \mathbf{M} \rightarrow \prod_{\varphi \in \Phi_{\mathbf{M}}} \mathbf{M}_{\varphi}$ be the homomorphism given by $\eta_{\mathbf{M}}(a)(\varphi) = \varphi(a)$. Put $\mathbf{R}(\mathbf{M}) = \eta_{\mathbf{M}}(\mathbf{M})$. Next let $\bar{x} = (x_a)_{a \in M}$ be a sequence of distinct variables and $x_a \mapsto a$ be the valuation onto M . Define $\Omega_{\mathbf{M}}$ to be the set of all weak implications of the form

$$(\forall \bar{x})(\exists y) [\bigwedge \{\varrho(\bar{x}) \mid \varrho \text{ is atomic, } \mathbf{M} \models \varrho(\bar{a})\} \wedge [y \in \bar{x}] \rightarrow \delta(\bar{x})],$$

where $\mathbf{R}(\mathbf{M}) \models \delta(\eta_{\mathbf{M}}(\bar{a}))$. Note that $\mathbf{M} \models \Omega_{\mathbf{M}}$ yields $\mathbf{M} \cong \mathbf{R}(\mathbf{M})$. Finally, put $\Omega = \bigcup_{\mathbf{M}} \Omega_{\mathbf{M}}$.

Lemma 2.2. *The class of weak implications Ω defines \mathcal{C} .*

Proof. Assume that $\mathbf{M} \models \Omega$. Then in particular, $\mathbf{M} \models \Omega_{\mathbf{M}}$, and we have $\mathbf{M} \cong \mathbf{R}(\mathbf{M}) \leq_{SD} \prod_{\varphi \in \Phi_{\mathbf{M}}} \mathbf{M}_{\varphi}$. Hence, $\mathbf{M} \in \mathcal{C}$.

Now suppose that $\mathbf{M} \in \mathcal{C}$. Consider a model \mathbf{N} . Let $\bar{x} = (x_b)_{b \in N}$ be a sequence of distinct variables and $x_b \mapsto a_b$ a valuation onto M . Assume that $\mathbf{M} \models \varrho(\bar{a})$ whenever $\mathbf{N} \models \varrho(\bar{b})$ for every atomic $\varrho(\bar{x})$ (here we employ the valuation $x_b \mapsto b$). This means that the mapping $\varphi : b \mapsto a_b$ is a homomorphism from \mathbf{N} onto \mathbf{M} . Without loss of generality, we may assume that $M \subseteq N$, and hence $\varphi \in \Phi_{\mathbf{N}}$. Let $\pi : \mathbf{R}(\mathbf{N}) \rightarrow \mathbf{M}$ be the restriction of the φ -th projection. Then $\pi \circ \eta_{\mathbf{N}} = \varphi$. This yields that whenever $\delta(\bar{x})$ is an atomic formula such that $\mathbf{R}(\mathbf{N}) \models \delta(\eta_{\mathbf{N}}(\bar{b}))$, then $\mathbf{M} \models \delta(\bar{a})$. This proves that $\mathbf{M} \models \Omega_{\mathbf{N}}$. \square

3. Example

If we can define a subdirect product closed class \mathcal{C} by weak implications in which the number of universal quantifiers is bounded by a cardinal κ , then all models of cardinality greater than κ belong to \mathcal{C} . Thus, we cannot expect in general an axiomatization by a set of weak implications. The following example shows that it is also the case for finitely generated subdirect product closed classes even if we allow sentences of a bit more general form than weak implications.

Let $\forall^{\kappa} \exists^{\infty} - L_{\infty\infty}$ be the language consisting of formulas in $L_{\infty\infty}$ of the form $(\forall \bar{x})(\exists \bar{y}) \gamma(\bar{x}, \bar{y})$, where \bar{x} consists of less than κ variables and γ is quantifier free.

Lemma 3.1. *Let σ be a sentence in $\forall^{\kappa} \exists^{\infty} - L_{\infty\infty}$. If*

$$(\forall C \subseteq M) [|C| \leq \kappa \text{ implies } (\exists \mathbf{N} \leq \mathbf{M}) [C \subseteq N \text{ and } \mathbf{N} \models \sigma]],$$

then $\mathbf{M} \models \sigma$.

Let $\mathbf{A} = (\{0, 1, 2\}, +)$ be the algebra where the operation is given by the table

$+$	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

R. McKenzie showed that $P_S(\mathbf{A})$ is not definable in first-order logic [8, Chapter 7, Exercises 55–58]. We slightly modify his argument in order to obtain the following fact.

Fact 3.2. *The class $P_S(\mathbf{A})$ is not definable in $\forall^\kappa\exists^\infty\text{-}L_{\infty\infty}$ for any cardinal κ .*

Proof. We may assume that κ is infinite. Let λ be a cardinal greater than κ . For $f \in A^\lambda$, define $\text{supp}(f) = \{i \in \lambda \mid f(i) \neq 0\}$ and put

$$B = \{f \in A^\lambda \mid f(0) \in \{0, 1\} \text{ and } |\text{supp}(f)| \leq \kappa\}.$$

Clearly, B is a carrier of the subalgebra \mathbf{B} of \mathbf{A}^λ . Note that for $f, g \in B$,

$$\text{supp}(f) \cap \text{supp}(g) = \emptyset \quad \Rightarrow \quad f + g = f. \tag{L0}$$

Let us distinguish two elements f_0, f_1 in B by $f_0(0) = 0, f_1(0) = 1$, and $f_0(i) = f_1(i) = 0$ for $i \neq 0$. Then for every $f \in B$, we have $f + f = f_0$ and f_0 is the only idempotent element in \mathbf{B} .

We will obtain the aim by verifying that every sentence in $\forall^\kappa\exists^\infty\text{-}L_\infty$ valid in $P_S(\mathbf{A})$ is also valid in \mathbf{B} and showing that $\mathbf{B} \notin P_S(\mathbf{A})$. The first fact follows from the following claim and Lemma 3.1.

Claim. *For every subset C of B with $|C| \leq \kappa$, there exists $\mathbf{D} \leq \mathbf{B}$ such that $C \subseteq D$ and $\mathbf{D} \in P_S(\mathbf{A})$.*

Let $I = \bigcup_{f \in C} \text{supp}(f)$. As $|C| \leq \kappa$, so $|I| \leq \kappa$ and there exists $i^* \in \lambda - I$. Let h be the element of B such that $h(i^*) = 1$ and $h(i) = 0$ for $i \neq i^*$. Put

$$E = \{f \in B \mid \text{supp } f \subseteq I\}$$

and $D = E \cup \{h\}$. Clearly, $\mathbf{E} \leq \mathbf{B}$, and by (L0), $\mathbf{D} \leq \mathbf{B}$. Observe that for a homomorphism $\psi: \mathbf{E} \rightarrow \mathbf{A}$ and an element $a \in A - \psi(E)$, the mapping given by $\varphi(g) = \psi(g)$ for $g \in E$ and $\varphi(h) = a$ is a homomorphism from \mathbf{D} into \mathbf{A} . In order to show that $\mathbf{D} \in P_S(\mathbf{A})$, we have to find an onto homomorphism $\varphi: \mathbf{D} \rightarrow \mathbf{A}$ separating g_1 and g_2 for every pair g_1, g_2 of distinct elements in D . If $g_1(i) \neq g_2(i)$ for some $i \in I - \{0\}$, then as φ we can take the projection on the i -th coordinate. If $g_1(0) \neq g_2(0)$ or $\{g_1, g_2\} = \{f_0, h\}$, let $\psi: \mathbf{E} \rightarrow \mathbf{A}$ be the projection on the 0-th coordinate and define $\varphi(g) = \psi(g)$ for $g \in E$ and $\varphi(h) = 2$.

Claim. $\mathbf{B} \notin P_S(\mathbf{A})$.

Let $\varphi: \mathbf{B} \rightarrow \mathbf{A}$ be any homomorphism separating f_0 and f_1 . The element f_0 , as the only idempotent one in \mathbf{B} , must be mapped by φ onto 0. Assume that $\varphi(f_1) = a \neq 0$. For $f \in B$ satisfying $f(0) = 1$, the equality $f_1 + f = f_0$

yields $\varphi(f) = a$. Now for $g \in B$ such that $g(0) = 0$, let $g'(0) = 1$ and $g'(i) = g(i)$ for $i \neq 0$. Since κ is infinite, $g' \in B$. Then $\varphi(g') = a$, $g + g' = f_0$, and hence $\varphi(g) \in \{0, a\}$. Thus, φ is not onto. \square

Note however that for a finite model \mathbf{M} in a finite language, the class $\mathsf{P}_S(\mathbf{M})$ is definable in $L_{\aleph_1 \aleph_1}$ by a sentence of the form $(\forall \bar{x})(\exists \bar{y})(\forall \bar{z}) \gamma(\bar{x}, \bar{y}, \bar{z})$, where γ is quantifier free and \bar{x}, \bar{y} are finite [7, Point 3.3].

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