

Mean-square stability analysis of approximations of stochastic differential equations in infinite dimensions

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Abstract The (asymptotic) behaviour of the second moment of solutions to stochastic differential equations is treated in mean-square stability analysis. This property is discussed for approximations of infinite-dimensional stochastic differential equations and necessary and sufficient conditions ensuring mean-square stability are given. They are applied to typical discretization schemes such as combinations of spectral Galerkin, finite element, Euler–Maruyama, Milstein, Crank–Nicolson, and forward and backward Euler methods. Furthermore, results on the relation to stability properties of corresponding analytical solutions are provided. Simulations of the stochastic heat equation illustrate the theory.

Keywords Asymptotic mean-square stability · Numerical approximations of stochastic differential equations · Linear stochastic partial differential equations ·

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1 Introduction

An interesting property of a stochastic differential equation (SDE) or a stochastic partial differential equation (SPDE) is the qualitative behaviour of its second moment for large times. Both types of equations can be interpreted as SDEs on a (here separable) Hilbert space $(H, \langle \cdot, \cdot \rangle_H)$. More specifically, let us consider a complete filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t, t \geq 0), P)$ satisfying the “usual conditions” and the model problem

$$dX(t) = (AX(t) + FX(t)) dt + G(X(t)) dL(t) \quad (1.1)$$

with \mathcal{F}_0 -measurable, square-integrable initial condition $X(0) = X_0$. Here, the operator $A : \mathcal{D}(A) \rightarrow H$ is the generator of a C_0 -semigroup $S = (S(t), t \geq 0)$ on H and F is a linear and bounded operator on H , i.e., $F \in L(H)$. Furthermore, L denotes a U -valued Q -Lévy process that is assumed to be a square-integrable martingale as considered in [27] on the real separable Hilbert space $(U, \langle \cdot, \cdot \rangle_U)$ with covariance $Q \in L(U)$ of trace class and let $G \in L(H; L(U; H))$.

We recall from [24] that an equilibrium (solution) of (1.1) is the zero solution ($X_e(t) = 0, t \geq 0$). It is called *mean-square stable* if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathbb{E}[\|X(t)\|_H^2] < \varepsilon$ for all $t \geq 0$ whenever $\mathbb{E}[\|X_0\|_H^2] < \delta$. It is further *asymptotically mean-square stable* if it is mean-square stable and there exists $\delta > 0$ such that $\mathbb{E}[\|X_0\|_H^2] < \delta$ implies $\lim_{t \rightarrow \infty} \mathbb{E}[\|X(t)\|_H^2] = 0$. A lot of effort has been dedicated to the asymptotic mean-square stability analysis in finite and infinite dimensions, see e.g., [2, 17, 24, 26].

Since analytical solutions to SDEs are rarely available, approximations in time and possibly in space by numerical methods have to be considered. The main focus of research in recent years has been on strong and weak convergence when the discretization parameters Δt in time and h in space tend to zero. However, this property does not guarantee that the approximation shares the same (asymptotic) mean-square stability properties as the analytical solution. For finite-dimensional SDEs it is known that the specific choice of Δt is essential. The goal of this manuscript is to generalize the theory of asymptotic mean-square stability analysis to a Hilbert space setting. We develop a theory for approximation schemes that has a priori no relation to the original Eq. (1.1) and its properties. Later on, we will discuss which conditions on (1.1) and its approximation lead to similar behaviour. An important application of mean-square stability is in multilevel Monte Carlo methods, where combinations of approximations on different space and time grids are computed. If the solution is mean-square unstable on any of the included levels, this is enough for the estimator not to behave as it should, see, e.g., [1].

The mean-square stability analysis of numerical approximations of SDEs started by considering approximations of the one-dimensional geometric Brownian motion, see e.g., [14, 15, 29]. As it has been pointed out in [10, 11], the analysis of higher-dimensional systems and their approximations is also necessary, since the asymptotic behaviour of the corresponding mean-square processes of systems with commuting and non-commuting matrices often differs. The tools to perform mean-square stability analysis of SDE approximations presented in [11] could in principle be used for approximations of infinite-dimensional SDEs by a method of lines approach: After projection on an N_h -dimensional space the mean-square stability properties of the resulting finite-dimensional SDEs and their approximations can be determined by considering the eigenvalues of $N_h^2 \times N_h^2$ -dimensional matrices. However, due to the computational complexity as $N_h \rightarrow \infty$, neither the symbolic nor the numerical computation of these eigenvalues can be done for arbitrarily large systems. For this reason, we use an approach based on tensor-product-space-valued processes and properties of tensorized linear operators.

The outline of this article is as follows: Sect. 2 sets up a theory of mean-square stability analysis for discrete stochastic processes derived from recursions as they appear in approximations of infinite-dimensional SDEs. In the main result, necessary and sufficient conditions for asymptotic mean-square stability are shown. These results are then applied in Sect. 3 to numerical approximations of (1.1) based on spatial Galerkin discretization schemes and time discretizations with Euler–Maruyama and Milstein methods using backward/forward Euler and Crank–Nicolson as rational semigroup approximations. We conclude this work presenting simulations of stochastic heat equations with spectral Galerkin and finite element methods in Sect. 4 that illustrate the theory.

2 Asymptotic mean-square stability analysis

This section is devoted to the setup of asymptotic mean-square stability for families of stochastic processes in discrete time given by recursion schemes as they typically show up in approximations of (1.1). We derive necessary and sufficient conditions ensuring asymptotic mean-square stability that can be checked in practice as it is shown later in Sect. 3.

Let $(V_h, h \in (0, 1])$ be a family of finite-dimensional subspaces $V_h \subset H$ with $\dim(V_h) = N_h \in \mathbb{N}$ indexed by a refinement parameter h . With an inner product induced by $\langle \cdot, \cdot \rangle_H$, V_h becomes a Hilbert space with norm $\| \cdot \|_H$. For a linear operator $D : V_h \rightarrow V_h$, the operator norm $\|D\|_{L(V_h)}$ is therefore given by $\sup_{v \in V_h} \|Dv\|_H / \|v\|_H$ and can be seen to coincide with $\|DP_h\|_{L(H)}$, where P_h denotes the orthogonal projection onto V_h .

Let us further consider the time interval $[0, \infty)$ and for convenience equidistant time steps $t_j = j\Delta t$, $j \in \mathbb{N}_0$, with fixed step size $\Delta t > 0$. Hence, $t \rightarrow \infty$ is equivalent to $j \rightarrow \infty$. Assume that we are given a sequence of V_h -valued random variables $(X_h^j, j \in \mathbb{N}_0)$ determined by the linear recursion scheme

$$X_h^{j+1} = D_{\Delta t, h}^{\det} X_h^j + D_{\Delta t, h}^{\text{stoch}, j} X_h^j \tag{2.1}$$

with \mathcal{F}_0 -measurable initial condition $X_h^0 \in L^2(\Omega; V_h)$, i.e., $\mathbb{E}[\|X_h^0\|_{V_h}^2] < \infty$. Here $D_{\Delta t, h}^{\det} \in L(V_h)$ and $D_{\Delta t, h}^{\text{stoch}, j}$ is an $L(V_h)$ -valued random variable for all j .

In terms of SDE (1.1), one can think of $D_{\Delta t, h}^{\det}$ as the approximation of the solution operator of the deterministic part

$$dX(t) = (AX(t) + FX(t)) dt, \quad t \in [t_j, t_{j+1})$$

and $D_{\Delta t, h}^{\text{stoch}, j}$ approximates the stochastic part

$$dX(t) = G(X(t)) dL(t), \quad t \in [t_j, t_{j+1}).$$

Although, in general, any not necessarily equidistant time discretization $(t_j, j \in \mathbb{N}_0)$ that satisfies $t_j \rightarrow \infty$ if $j \rightarrow \infty$ would be sufficient for the following theory, we see in the given SDE example that $D_{\Delta t, h}^{\det}$ would be j -dependent in this case, which we want to omit for the sake of readability.

Inspired by properties of standard approximation schemes for (1.1), we put the following assumptions on the family $(D_{\Delta t, h}^{\text{stoch}, j}, j \in \mathbb{N}_0)$.

Assumption 2.1 Let $h, \Delta t > 0$ be fixed. The family $(D_{\Delta t, h}^{\text{stoch}, j}, j \in \mathbb{N}_0)$ is \mathcal{F} -compatible in the sense of [12, 21], i.e., $D_{\Delta t, h}^{\text{stoch}, j}$ is $\mathcal{F}_{t_{j+1}}$ -measurable and $\mathbb{E}[D_{\Delta t, h}^{\text{stoch}, j} | \mathcal{F}_{t_j}] = 0$ for all $j \in \mathbb{N}_0$. Furthermore, for all $j \in \mathbb{N}_0$, let

$$\|D_{\Delta t, h}^{\text{stoch}, j}\|_{L^2(\Omega; L(V_h))} = \mathbb{E}[\|D_{\Delta t, h}^{\text{stoch}, j}\|_{L(V_h)}^2]^{1/2} < \infty$$

and

$$\mathbb{E} \left[D_{\Delta t, h}^{\text{stoch}, j} \otimes D_{\Delta t, h}^{\text{stoch}, j} \mid \mathcal{F}_{t_j} \right] = \mathbb{E} \left[D_{\Delta t, h}^{\text{stoch}, j} \otimes D_{\Delta t, h}^{\text{stoch}, j} \right],$$

where \otimes denotes the tensor product.

For the recursion scheme (2.1) an *equilibrium* (solution) is given by the zero solution, which is defined as $X_{h, e}^j = 0$ for all $j \in \mathbb{N}_0$. We define mean-square stability of the zero solution of (2.1) in what follows.

Definition 2.1 Let $X_h = (X_h^j, j \in \mathbb{N}_0)$ be given by (2.1) for fixed h and Δt . The zero solution $(X_{h, e}^j = 0, j \in \mathbb{N}_0)$ of (2.1) is called *mean-square stable* if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathbb{E}[\|X_h^j\|_H^2] < \varepsilon$ for all $j \in \mathbb{N}_0$ whenever $\mathbb{E}[\|X_h^0\|_H^2] < \delta$.

It is called *asymptotically mean-square stable* if it is mean-square stable and there exists $\delta > 0$ such that $\mathbb{E}[\|X_h^0\|_H^2] < \delta$ implies $\lim_{j \rightarrow \infty} \mathbb{E}[\|X_h^j\|_H^2] = 0$. Furthermore, it is called *asymptotically mean-square unstable* if it is not asymptotically mean-square stable.

For convenience, the abbreviation (*asymptotic*) *mean-square stability* is used for the (asymptotic) mean-square stability of the zero solution of (2.1) or (1.1) if it is clear from the context.

When applied to $Y_j = X_h^j$, the following lemma provides an equivalent condition for mean-square stability in terms of the tensor-product-space-valued process $X_h^j \otimes X_h^j \in V_h^{(2)}$. Here, for a general Hilbert space H , the abbreviation $H^{(2)} = H \otimes H$ is used and $H^{(2)}$ is defined as the completion of the algebraic tensor product with respect to the norm induced by

$$\langle v, w \rangle_{H \otimes H} = \sum_{i=1}^N \sum_{j=1}^M \langle v_{1,i}, w_{1,j} \rangle_H \langle v_{2,i}, w_{2,j} \rangle_H,$$

where $v = \sum_{i=1}^N v_{1,i} \otimes v_{2,i}$ and $w = \sum_{j=1}^M w_{1,j} \otimes w_{2,j}$ are representations of elements v and w in the algebraic tensor product.

Lemma 2.1 *Let V_h be a finite-dimensional subspace of H . Then, for any sequence $(Y_j, j \in \mathbb{N}_0)$ of V_h -valued, square-integrable random variables, $\lim_{j \rightarrow \infty} \mathbb{E}[Y_j \otimes Y_j] = 0$ if and only if $\lim_{j \rightarrow \infty} \mathbb{E}[\|Y_j\|_H^2] = 0$.*

Proof By Parseval’s identity, for an orthonormal basis $(\psi_1, \dots, \psi_{N_h})$ of V_h , we have

$$\begin{aligned} \|\mathbb{E}[Y_j \otimes Y_j]\|_{H^{(2)}}^2 &= \sum_{k,\ell=1}^{N_h} |\mathbb{E}[\langle Y_j \otimes Y_j, \psi_k \otimes \psi_\ell \rangle_{H^{(2)}}]|^2 \\ &= \sum_{k,\ell=1}^{N_h} |\mathbb{E}[\langle Y_j, \psi_k \rangle_H \langle Y_j, \psi_\ell \rangle_H]|^2 \end{aligned}$$

and similarly

$$\mathbb{E}[\|Y_j\|_H^2] = \sum_{k=1}^{N_h} \mathbb{E}[\langle Y_j, \psi_k \rangle_H^2].$$

Therefore, one implication is immediately obtained, while the other follows from the fact that

$$\|\mathbb{E}[Y_j \otimes Y_j]\|_{H^{(2)}} \leq \mathbb{E}[\|Y_j \otimes Y_j\|_{H^{(2)}}] = \mathbb{E}[\|Y_j\|_H^2].$$

This finishes the proof. □

This lemma enables us to show the following sufficient condition for asymptotic mean-square stability.

Theorem 2.1 Let $X_h = (X_h^j, j \in \mathbb{N}_0)$ given by (2.1) satisfy Assumption 2.1 and set

$$\mathcal{S}_j = D_{\Delta t, h}^{\det} \otimes D_{\Delta t, h}^{\det} + \mathbb{E} \left[D_{\Delta t, h}^{\text{stoch}, j} \otimes D_{\Delta t, h}^{\text{stoch}, j} \right].$$

Then the zero solution of (2.1) is asymptotically mean-square stable, if

$$\lim_{j \rightarrow \infty} \|\mathcal{S}_j \cdots \mathcal{S}_0\|_{L(V_h^{(2)})} = 0.$$

Proof Let us first remark that $\mathcal{S}_j \in L(V_h^{(2)})$ for all $j \in \mathbb{N}_0$ by the properties of $D_{\Delta t, h}^{\det}$ and $D_{\Delta t, h}^{\text{stoch}, j}$ and of the Hilbert tensor product. In order to show asymptotic mean-square stability, it suffices to show $\mathbb{E}[X_h^j \otimes X_h^j] \rightarrow 0$ as $j \rightarrow \infty$ by Lemma 2.1. For this, consider

$$\begin{aligned} \mathbb{E}[X_h^{j+1} \otimes X_h^{j+1}] &= \mathbb{E} \left[\left(D_{\Delta t, h}^{\det} + D_{\Delta t, h}^{\text{stoch}, j} \right) X_h^j \otimes \left(D_{\Delta t, h}^{\det} + D_{\Delta t, h}^{\text{stoch}, j} \right) X_h^j \right] \\ &= \mathbb{E} \left[\left(D_{\Delta t, h}^{\det} \otimes D_{\Delta t, h}^{\det} \right) \left(X_h^j \otimes X_h^j \right) \right] \\ &\quad + \mathbb{E} \left[\left(D_{\Delta t, h}^{\text{stoch}, j} \otimes D_{\Delta t, h}^{\text{stoch}, j} \right) \left(X_h^j \otimes X_h^j \right) \right] \\ &\quad + \mathbb{E} \left[\left(D_{\Delta t, h}^{\det} \otimes D_{\Delta t, h}^{\text{stoch}, j} \right) \left(X_h^j \otimes X_h^j \right) \right] \\ &\quad + \mathbb{E} \left[\left(D_{\Delta t, h}^{\text{stoch}, j} \otimes D_{\Delta t, h}^{\det} \right) \left(X_h^j \otimes X_h^j \right) \right]. \end{aligned}$$

The mixed terms vanish by the observation that

$$\begin{aligned} &\mathbb{E} \left[\left(D_{\Delta t, h}^{\det} \otimes D_{\Delta t, h}^{\text{stoch}, j} \right) \left(X_h^j \otimes X_h^j \right) \right] \\ &= \mathbb{E} \left[\left(D_{\Delta t, h}^{\det} \otimes \mathbb{E}[D_{\Delta t, h}^{\text{stoch}, j} | \mathcal{F}_{t_j}] \right) \left(X_h^j \otimes X_h^j \right) \right] = 0, \end{aligned}$$

since X_h^j and $D_{\Delta t, h}^{\det}$ are \mathcal{F}_{t_j} -measurable and $\mathbb{E}[D_{\Delta t, h}^{\text{stoch}, j} | \mathcal{F}_{t_j}] = 0$ by Assumption 2.1.

Applying Assumption 2.1 once more, we therefore conclude

$$\begin{aligned} &\mathbb{E}[X_h^{j+1} \otimes X_h^{j+1}] \\ &= \mathbb{E} \left[\left(D_{\Delta t, h}^{\det} \otimes D_{\Delta t, h}^{\det} + D_{\Delta t, h}^{\text{stoch}, j} \otimes D_{\Delta t, h}^{\text{stoch}, j} \right) \left(X_h^j \otimes X_h^j \right) \right] \\ &= \mathbb{E} \left[\left(D_{\Delta t, h}^{\det} \otimes D_{\Delta t, h}^{\det} + \mathbb{E} \left[D_{\Delta t, h}^{\text{stoch}, j} \otimes D_{\Delta t, h}^{\text{stoch}, j} | \mathcal{F}_{t_j} \right] \right) \left(X_h^j \otimes X_h^j \right) \right] \\ &= \left(D_{\Delta t, h}^{\det} \otimes D_{\Delta t, h}^{\det} + \mathbb{E} \left[D_{\Delta t, h}^{\text{stoch}, j} \otimes D_{\Delta t, h}^{\text{stoch}, j} \right] \right) \mathbb{E} \left[X_h^j \otimes X_h^j \right]. \end{aligned}$$

and obtain

$$\mathbb{E} \left[X_h^{j+1} \otimes X_h^{j+1} \right] = \mathcal{S}_j \mathbb{E} \left[X_h^j \otimes X_h^j \right] = (\mathcal{S}_j \cdots \mathcal{S}_0) \mathbb{E} \left[X_h^0 \otimes X_h^0 \right].$$

Since $\lim_{j \rightarrow \infty} \|\mathcal{S}_j \cdots \mathcal{S}_0\|_{L(V_h^{(2)})} = 0$, mean-square stability is shown with the computation

$$\begin{aligned} & \mathbb{E} \left[\|X_h^{j+1}\|_H^2 \right]^2 \\ &= \left(\sum_{k=1}^{N_h} \mathbb{E} \left[\langle X_h^{j+1}, \psi_k \rangle_H^2 \right] \right)^2 \leq N_h \sum_{k=1}^{N_h} \mathbb{E} \left[\langle X_h^{j+1}, \psi_k \rangle_H^2 \right]^2 \\ &\leq N_h \left\| \mathbb{E} \left[X_h^{j+1} \otimes X_h^{j+1} \right] \right\|_{H^{(2)}}^2 \leq N_h \|\mathcal{S}_j \cdots \mathcal{S}_0\|_{L(V_h^{(2)})}^2 \mathbb{E} \left[\|X_h^0\|_H^2 \right]^2. \end{aligned}$$

For asymptotic mean-square stability, note that for any \mathcal{F}_0 -measurable initial value $X_h^0 \in L^2(\Omega; V_h)$ it holds that $\lim_{j \rightarrow \infty} \mathbb{E}[X_h^j \otimes X_h^j] = 0$ if and only if

$$\lim_{j \rightarrow \infty} \left\| (\mathcal{S}_j \cdots \mathcal{S}_0) \mathbb{E} \left[X_h^0 \otimes X_h^0 \right] \right\|_{V_h^{(2)}} = 0,$$

for which a sufficient condition is given by $\lim_{j \rightarrow \infty} \|\mathcal{S}_j \cdots \mathcal{S}_0\|_{L(V_h^{(2)})} = 0$. Therefore, the proof is finished. □

In many examples the operators $(D_{\Delta t, h}^{\text{stoch}, j}, j \in \mathbb{N}_0)$ have a constant covariance, i.e., they satisfy for all $j \in \mathbb{N}_0$

$$\mathbb{E} \left[D_{\Delta t, h}^{\text{stoch}, j} \otimes D_{\Delta t, h}^{\text{stoch}, j} \right] = \mathbb{E} \left[D_{\Delta t, h}^{\text{stoch}, 0} \otimes D_{\Delta t, h}^{\text{stoch}, 0} \right]. \tag{2.2}$$

Often as in the following example they are even independent and identically distributed, which implies (2.2).

Example 2.1 Consider the one-dimensional geometric Brownian motion driven by an adapted, real-valued Brownian motion $(\beta(t), t \geq 0)$

$$dX(t) = \lambda X(t)dt + \sigma X(t)d\beta(t), \quad t \geq 0,$$

with initial condition $X(0) = x_0 \in \mathbb{R}$ and $\lambda, \sigma \in \mathbb{R}$. The solution can be approximated by the explicit Euler–Maruyama scheme

$$X_{j+1} = X_j + \lambda \Delta t X_j + \sigma \Delta \beta^j X_j,$$

for $j \in \mathbb{N}_0$, where $\Delta \beta^j = \beta(t_{j+1}) - \beta(t_j)$, or by the Milstein scheme

$$X_{j+1} = X_j + \lambda \Delta t X_j + \sigma \Delta \beta^j X_j + 2^{-1} \sigma^2 \left((\Delta \beta^j)^2 - \Delta t \right) X_j.$$

Then the deterministic operators in (2.1)

$$D_{\Delta t, \text{EM}}^{\text{det}} = D_{\Delta t, \text{Mil}}^{\text{det}} = 1 + \lambda \Delta t$$

are equal for both schemes, and the corresponding approximations of the stochastic integrals are given by

$$D_{\Delta t,EM}^{\text{stoch},j} = \sigma \Delta \beta^j, \quad D_{\Delta t,Mil}^{\text{stoch},j} = \sigma \Delta \beta^j + 2^{-1} \sigma^2 \left((\Delta \beta^j)^2 - \Delta t \right)$$

for $j \in \mathbb{N}_0$. Both families of stochastic approximation operators satisfy Assumption 2.1 and

$$\mathbb{E} \left[D_{\Delta t,EM}^{\text{stoch},j} \otimes D_{\Delta t,EM}^{\text{stoch},j} \right] = \sigma^2 \Delta t, \quad \mathbb{E} \left[D_{\Delta t,Mil}^{\text{stoch},j} \otimes D_{\Delta t,Mil}^{\text{stoch},j} \right] = \sigma^2 \Delta t \left(1 + 2^{-1} \sigma^2 \Delta t \right)$$

do not depend on j . We observe that the equidistant time step Δt is essential here.

Having this example in mind, we are able to give a necessary and sufficient condition for asymptotic mean-square stability when assuming (2.2) and therefore to specify Theorem 2.1. The condition relies on the spectrum of a single linear operator $\mathcal{S} \in L(V_h^{(2)})$.

Corollary 2.1 *Let $X_h = (X_h^j, j \in \mathbb{N}_0)$ given by (2.1) satisfy Assumption 2.1 and (2.2). Then the zero solution of (2.1) is asymptotically mean-square stable if and only if*

$$\mathcal{S} = D_{\Delta t,h}^{\text{det}} \otimes D_{\Delta t,h}^{\text{det}} + \mathbb{E} \left[D_{\Delta t,h}^{\text{stoch},0} \otimes D_{\Delta t,h}^{\text{stoch},0} \right] \in L(V_h^{(2)})$$

satisfies $\rho(\mathcal{S}) = \max_{i=1,\dots,N_h^2} |\lambda_i| < 1$, where $\lambda_1, \dots, \lambda_{N_h^2}$ are the eigenvalues of \mathcal{S} .

Furthermore, it is asymptotically mean-square stable if $\|\mathcal{S}\|_{L(V_h^{(2)})} < 1$.

Proof Setting $\mathcal{S}_j = \mathcal{S}$ for all $j \in \mathbb{N}_0$ in Theorem 2.1, we obtain by the same arguments

$$\mathbb{E} \left[X_h^{j+1} \otimes X_h^{j+1} \right] = (\mathcal{S}_j \cdots \mathcal{S}_0) \mathbb{E} \left[X_h^0 \otimes X_h^0 \right] = \mathcal{S}^{j+1} \mathbb{E} \left[X_h^0 \otimes X_h^0 \right].$$

As a consequence, $\lim_{j \rightarrow \infty} \mathbb{E} [X_h^j \otimes X_h^j] = 0$ if and only if $\lim_{j \rightarrow \infty} \mathcal{S}^j = 0$ which is equivalent to $\rho(\mathcal{S}) < 1$ by the same arguments as, e.g., in [8, 11, 17]. This completes the proof of the first statement. Since $\rho(\mathcal{S}) \leq \|\mathcal{S}\|_{L(V_h^{(2)})}$, a sufficient condition for asymptotic mean-square stability is given by $\|\mathcal{S}\|_{L(V_h^{(2)})} < 1$. \square

In the framework of SDE approximations, note that this corollary is an SPDE version formulated with operators of the results for finite-dimensional linear systems in [11]. There, the proposed method relies on a matrix eigenvalue problem. For SPDE approximations, this approach is not suitable, since the dimension of the considered eigenvalue problem increases heavily with space refinement. More precisely, for $h > 0$, the spectral radius of an $(N_h^2 \times N_h^2)$ -matrix has to be computed. To overcome this problem, we perform, in what follows, mean-square stability analysis of SPDE approximations based on operators as introduced above.

3 Application to Galerkin methods

We continue by applying the previous results to the analysis of some classical numerical approximations of (1.1) which admits by results in [27, Chapter 9] an up to modification unique mild càdlàg solution and is for $t \geq 0$ given by

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(X(s)) ds + \int_0^t S(t-s)G(X(s)) dL(s). \tag{3.1}$$

We assume further that the operator $-A : \mathcal{D}(-A) \subset H \rightarrow H$ of (1.1) is densely defined, self-adjoint, and positive definite with compact inverse. This implies that $-A$ has a non-decreasing sequence of positive eigenvalues $(\lambda_i, i \in \mathbb{N})$ for an orthonormal basis of eigenfunctions $(e_i, i \in \mathbb{N})$ in H and fractional powers of $-A$ are provided by

$$(-A)^{r/2}e_i = \lambda_i^{r/2}e_i$$

for all $i \in \mathbb{N}$ and $r > 0$. For each $r > 0$, $\dot{H}^r = \mathcal{D}((-A)^{r/2})$ with inner product $\langle \cdot, \cdot \rangle_r = \langle (-A)^{r/2} \cdot, (-A)^{r/2} \cdot \rangle_H$ defines a separable Hilbert space (see, e.g., [20, Appendix B]).

We assume that the sequence $(V_h, h \in (0, 1])$ of finite-dimensional subspaces fulfil $V_h \subset \dot{H}^1 \subset H$ and define the discrete operator $-A_h : V_h \rightarrow V_h$ by

$$\langle -A_h v_h, w_h \rangle_H = \langle v_h, w_h \rangle_1 = \left\langle (-A)^{1/2}v_h, (-A)^{1/2}w_h \right\rangle_H$$

for all $v_h, w_h \in V_h$. This definition implies that $-A_h$ is self-adjoint and positive definite on V_h and therefore has a sequence of orthonormal eigenfunctions $(e_{h,i}, i = 1, \dots, N_h)$ and positive non-decreasing eigenvalues $(\lambda_{h,i}, i = 1, \dots, N_h)$ (see e.g., [20, Chapter 3]). By using basic properties of the Rayleigh quotient, we bound the smallest eigenvalue $\lambda_{h,1}$ of $-A_h$ from below by the smallest eigenvalue λ_1 of $-A$ through

$$\lambda_{h,1} = \min_{v_h \in V_h \setminus \{0\}} \frac{\langle v_h, v_h \rangle_1}{\|v_h\|_H^2} \geq \min_{v \in H \setminus \{0\}} \frac{\langle v, v \rangle_1}{\|v\|_H^2} = \lambda_1, \tag{3.2}$$

since $V_h \subset H$, cf. [9]. This estimate turns out to be useful when comparing asymptotic mean-square stability of (1.1) and its approximation later in this section.

Let the covariance of the Lévy process L be self-adjoint, positive semidefinite, and of trace class. Then results in [27, Chapter 4] imply the existence of an orthonormal basis $(f_i, i \in \mathbb{N})$ of U and a non-increasing sequence of non-negative real numbers $(\mu_i, i \in \mathbb{N})$ such that for all $i \in \mathbb{N}$, $Qf_i = \mu_i f_i$ with $\text{Tr}(Q) = \sum_{i=1}^\infty \mu_i < \infty$ and L admits a Karhunen–Loève expansion

$$L(t) = \sum_{i=1}^\infty \sqrt{\mu_i} L_i(t) f_i, \tag{3.3}$$

where $(L_i, i \in \mathbb{N})$ is a family of real-valued, square-integrable, uncorrelated Lévy processes satisfying $\mathbb{E}[(L_i(t))^2] = t$ for all $t \geq 0$. Note that due to the martingale property of L , the real-valued Lévy processes satisfy $\mathbb{E}[L_i(t)] = 0$ for all $t \geq 0$ and $i \in \mathbb{N}$. This implies, together with the stationarity of the Lévy increments $\Delta L_i^j = L_i(t_{j+1}) - L_i(t_j)$, that for all $i \in \mathbb{N}$ and $j \in \mathbb{N}_0$,

$$\mathbb{E}[\Delta L_i^j] = \mathbb{E}[\Delta L_1^0] = \mathbb{E}[L_1(\Delta t)] = 0.$$

Since the series representation of L can be infinite, an approximation of L might be required to implement a fully discrete approximation scheme, which is typically done by truncation of the Karhunen–Loève expansion, i.e., for $\kappa \in \mathbb{N}$, set the truncated process $L^\kappa(t) = \sum_{i=1}^\kappa \sqrt{\mu_i} L_i(t) f_i$. Note that the choice of κ is essential and should be coupled with the overall convergence of the numerical scheme as is discussed in, e.g., [3, 5, 25]. Within this work, we consider numerical methods based on the original Karhunen–Loève expansion (3.3) of L . However, this does not restrict the applicability of the results since L^κ fits in the framework by setting $\mu_i = 0$ for all $i > \kappa$.

As standard example in this context we consider the stochastic heat equation which is used for simulations in Sect. 4.

Example 3.1 (Stochastic heat equation) Let $H = L^2([0, 1])$ be the separable Hilbert space of square-integrable functions on $[0, 1]$. On this space we consider the operator $A = \nu \Delta$, where $\nu > 0$ and Δ denotes the Laplace operator with homogeneous zero Dirichlet boundary conditions which is the generator of a C_0 -semigroup, cf. [20, Example 2.21]. The equation

$$dX(t) = \nu \Delta X(t) dt + G(X(t)) dL(t)$$

is referred to as the (homogeneous) stochastic heat equation.

It is known (see, e.g., [20, Chapter 6]) that the eigenvalues and eigenfunctions of the operator $-A$ are given by

$$\lambda_i = \nu i^2 \pi^2, \quad e_i(y) = \sqrt{2} \sin(i\pi y), \quad i \in \mathbb{N}, y \in [0, 1].$$

We first assume, for simplicity, that $U = H = L^2([0, 1])$ and that the operator Q diagonalizes with respect to the eigenbasis of $-A$, i.e., $f_i = e_i$ for all $i \in \mathbb{N}$. For this choice, we consider the operator $G = G_1$ that gives rise to a *geometric Brownian motion in infinite dimensions*, cf. [20, Sect. 6.4]. It is for all $u, v \in H$ defined by the equation

$$G_1(v)u = \sum_{i=1}^\infty \langle v, e_i \rangle_H \langle u, e_i \rangle_H e_i.$$

As a second example, we let $U = \dot{H}^1$ with the same diagonalization assumption as before, i.e., $f_i = \lambda_i^{1/2} e_i$ for all $i \in \mathbb{N}$. Here, we let the operator $G = G_2$ be a *Nemytskii operator* which is defined pointwise for $x \in [0, 1]$, $u \in \dot{H}^1$ and $v \in H$ by

$$(G_2(v)u)[x] = v(x)u(x).$$

To see that $G \in L(H; L(U; H))$ note that for $u, v \in H$, by the triangle inequality and Cauchy–Schwarz we have for G_1

$$\|G_1(v)u\|_H \leq \sum_{i=1}^{\infty} |\langle v, e_i \rangle_H| |\langle u, e_i \rangle_H| \leq \left(\sum_{i=1}^{\infty} \langle v, e_i \rangle_H^2 \right)^{1/2} \left(\sum_{i=1}^{\infty} \langle u, e_i \rangle_H^2 \right)^{1/2} = \|v\|_H \|u\|_H.$$

Next, for G_2 with $v \in H$ and $u \in \dot{H}^1$, it holds that

$$\begin{aligned} \|G_2(v)u\|_H^2 &= \int_0^1 u(x)^2 v(x)^2 \, dx = \int_0^1 \left(\sum_{i=1}^{\infty} \lambda_i^{1/2} \langle u, e_i \rangle_H \lambda_i^{-1/2} e_i(x) \right)^2 v(x)^2 \, dx \\ &\leq \left(\sum_{i=1}^{\infty} \lambda_i |\langle u, e_i \rangle_H|^2 \right) \int_0^1 \left(\sum_{i=1}^{\infty} \lambda_i^{-1} e_i(x)^2 \right) v(x)^2 \, dx \\ &\leq \|u\|_{\dot{H}^1}^2 \left(2 \sum_{i=1}^{\infty} \lambda_i^{-1} \right) \int_0^1 v(x)^2 \, dx = \left(2 \sum_{i=1}^{\infty} \lambda_i^{-1} \right) \|u\|_{\dot{H}^1}^2 \|v\|_H^2. \end{aligned}$$

Here, the first inequality is an application of the Cauchy–Schwarz inequality, while the second follows from the fact that the sequence $(|e_i(x)|, i \in \mathbb{N})$ is bounded by $\sqrt{2}$ for all $x \in [0, 1]$. Therefore, we obtain

$$\|G_1\|_{L(H;L(H))} \leq 1, \quad \|G_2\|_{L(H;L(\dot{H}^1,H))} \leq \left(2 \sum_{i=1}^{\infty} \lambda_i^{-1} \right)^{1/2}.$$

3.1 Time discretization with rational approximations

Let us first recall that a *rational approximation of order p* of the exponential function is a rational function $R : \mathbb{C} \rightarrow \mathbb{C}$ satisfying that there exist constants $C, \delta > 0$ such that for all $z \in \mathbb{C}$ with $|z| < \delta$

$$|R(z) - \exp(z)| \leq C|z|^{p+1}.$$

Since R is rational there exist polynomials r_n and r_d such that $R = r_d^{-1}r_n$. We want to consider rational approximations of the semigroup S generated by the operator $-A$ and of its approximations $-A_h$ as they were considered in [30]. With the introduced notation, the linear operator $R(\Delta t A_h)$ is given for all $v_h \in V_h$ by

$$R(\Delta t A_h)v_h = r_d^{-1}(\Delta t A_h)r_n(\Delta t A_h)v_h = \sum_{k=1}^{N_h} \frac{r_n(-\Delta t \lambda_{h,k})}{r_d(-\Delta t \lambda_{h,k})} \langle v_h, e_{h,k} \rangle_H e_{h,k}. \tag{3.4}$$

To start, let us consider the mean-square stability properties of a Galerkin Euler–Maruyama method, which is given by the recursion

$$X_h^{j+1} = \left(D_{\Delta t, h}^{\det} + D_{\Delta t, h}^{\text{EM}, j} \right) X_h^j \tag{3.5}$$

for $j \in \mathbb{N}_0$ with initial condition $X_h^0 = P_h X_0$, where

$$\begin{aligned} D_{\Delta t, h}^{\det} &= R(\Delta t A_h) + r_d^{-1}(\Delta t A_h) \Delta t P_h F, \\ D_{\Delta t, h}^{\text{stoch}, j} &= D_{\Delta t, h}^{\text{EM}, j} = r_d^{-1}(\Delta t A_h) P_h G(\cdot) \Delta L^j \end{aligned} \tag{3.6}$$

with $\Delta L^j = L(t_{j+1}) - L(t_j)$. Note that the linear operators $(D_{\Delta t, h}^{\text{stoch}, j}, j \in \mathbb{N}_0)$ satisfy all assumptions of Corollary 2.1 since they only depend on the Lévy increments $(\Delta L^j, j \in \mathbb{N}_0)$. For this type of numerical approximation, the result from Corollary 2.1 can be specified:

Proposition 3.1 *The zero solution of the numerical method (3.5) is asymptotically mean-square stable if and only if*

$$\mathcal{S} = D_{\Delta t, h}^{\det} \otimes D_{\Delta t, h}^{\det} + \Delta t (C \otimes C) q \in L(V_h^{(2)})$$

satisfies that $\rho(\mathcal{S}) < 1$, where $q = \sum_{k=1}^{\infty} \mu_k f_k \otimes f_k \in U^{(2)}$ and $C \in L(U; L(V_h))$ with

$$Cu = r_d^{-1}(\Delta t A_h) P_h G(\cdot) u.$$

Proof Note that since V_h is finite-dimensional, $L(V_h) = L_{\text{HS}}(V_h)$ so $(C \otimes C)$ is well-defined as an element of $L(U^{(2)}, L_{\text{HS}}^{(2)}(V_h)) \subset L(U^{(2)}, L(V_h^{(2)}))$ by [18, Lemma 3.1(ii)], which yields for $j \in \mathbb{N}$

$$\mathbb{E} \left[D_{\Delta t, h}^{\text{EM}, j} \otimes D_{\Delta t, h}^{\text{EM}, j} \right] = \mathbb{E} [C \Delta L^j \otimes C \Delta L^j] = (C \otimes C) \mathbb{E} [\Delta L^j \otimes \Delta L^j].$$

Since $\mathbb{E} [\Delta L^j \otimes \Delta L^j] = \Delta t q$ by Lemma 5.1, the proof follows from Corollary 2.1. □

The still rather abstract condition can be specified to an explicit sufficient condition.

Corollary 3.1 *A sufficient condition for asymptotic mean-square stability of (3.5) is*

$$\begin{aligned} & \left(\max_{k=1, \dots, N_h} |R(-\Delta t \lambda_{h, k})| + \max_{k=1, \dots, N_h} |r_d^{-1}(-\Delta t \lambda_{h, k})| \Delta t \|F\|_{L(H)} \right)^2 \\ & + \max_{k=1, \dots, N_h} |r_d^{-1}(-\Delta t \lambda_{h, k})|^2 \Delta t \text{Tr}(Q) \|G\|_{L(H; L(U; H))}^2 < 1. \end{aligned}$$

Proof We first note that by the triangle inequality and the properties of the linear operator induced by the rational approximation R defined in Eq. (3.4) we obtain that

$$\begin{aligned} \|D_{\Delta t, h}^{\det}\|_{L(V_h)} &= \|R(\Delta t A_h) + r_d^{-1}(\Delta t A_h)\Delta t P_h F\|_{L(V_h)} \\ &\leq \max_{k=1, \dots, N_h} |R(-\Delta t \lambda_{h,k})| + \max_{k=1, \dots, N_h} |r_d^{-1}(-\Delta t \lambda_{h,k})| \|\Delta t F\|_{L(H)} \end{aligned}$$

and, similarly

$$\|C\|_{L(U; L(V_h))} \leq \max_{k=1, \dots, N_h} |r_d^{-1}(-\Delta t \lambda_{h,k})| \|G\|_{L(H; L(U; H))}.$$

Since

$$\|(C \otimes C)q\|_{L(V_h^{(2)})} \leq \sum_{k=1}^{\infty} \mu_k \|C f_k\|_{L(V_h)}^2 \leq \text{Tr}(Q) \|C\|_{L(U; L(V_h))}^2$$

and $\|D_{\Delta t, h}^{\det} \otimes D_{\Delta t, h}^{\det}\|_{L(V_h^{(2)})} = \|D_{\Delta t, h}^{\det}\|_{L(V_h)}^2$, we obtain the claimed condition, which is sufficient by Corollary 2.1. \square

We continue with the higher order Milstein scheme. Applying [3] in our context reads

$$X_h^{j+1} = (D_{\Delta t, h}^{\det} + D_{\Delta t, h}^{\text{EM}, j} + D_{\Delta t, h}^{\text{M}, j})X_h^j, \tag{3.7}$$

where $D_{\Delta t, h}^{\det}$ and $D_{\Delta t, h}^{\text{EM}, j}$ are as in (3.6) and

$$D_{\Delta t, h}^{\text{M}, j} = \sum_{k, \ell=1}^{\infty} r_d^{-1}(\Delta t A_h) \sqrt{\mu_k \mu_\ell} P_h G(G(\cdot) f_k) f_\ell \int_{t_j}^{t_{j+1}} \int_{t_j}^s dL_k(r) dL_\ell(s).$$

Remark 3.1 In order to compute the iterated integrals of $D_{\Delta t, h}^{\text{M}, j}$, one may assume (cf. [3, 16]) that for all H -valued, adapted stochastic processes $\chi = (\chi(t), t \geq 0)$ and all $i, j \in \mathbb{N}$, the diffusion operator G satisfies the commutativity condition

$$G(G(\chi) f_j) f_i = G(G(\chi) f_i) f_j.$$

Under this assumption satisfied in Example 3.1, $D_{\Delta t, h}^{\text{M}, j}$ simplifies to

$$D_{\Delta t, h}^{\text{M}, j} = \frac{1}{2} \sum_{k, \ell=1}^{\infty} \sqrt{\mu_k \mu_\ell} r_d^{-1}(\Delta t A_h) P_h G(G(X_h^j) f_k) f_\ell (\Delta L_k^j \Delta L_\ell^j - \Delta[L_k, L_\ell]^j),$$

where $\Delta[L_k, L_\ell]^j = [L_k, L_\ell]_{t_{j+1}} - [L_k, L_\ell]_{t_j}$. Here, $[L_k, L_\ell]_t$ denotes the quadratic covariation of L_k and L_ℓ evaluated at $t \geq 0$, which is straightforward to compute when

L_k, L_ℓ are jump-diffusion processes (cf. [3]). For the simulation of more general Lévy processes in the context of SPDE approximation, we refer to [7, 13].

As for the Euler–Maruyama scheme, Corollary 2.1 can be specified for this Milstein scheme.

Proposition 3.2 *Assume that the mapping $C'(u_1, u_2) = r_d^{-1}(\Delta t A_h) P_h G(G(\cdot)u_1)u_2$ for $u_1, u_2 \in U$ can be uniquely extended to a mapping $C' \in L(U^{(2)}, L(V_h))$. Then the zero solution of (3.7) is asymptotically mean-square stable if and only if*

$$\mathcal{S} = D_{\Delta t, h}^{\det} \otimes D_{\Delta t, h}^{\det} + \Delta t (C \otimes C)q + \frac{\Delta t^2}{2} (C' \otimes C')q'$$

satisfies that $\rho(\mathcal{S}) < 1$. Here, $q' = \sum_{k, \ell=1}^\infty \mu_k \mu_\ell (f_k \otimes f_\ell) \otimes (f_k \otimes f_\ell) \in U^{(4)}$ and C and q as in Proposition 3.1.

Proof Note that $C' \otimes C' : U^{(4)} \rightarrow L(V_h^{(2)})$ and $C' \otimes C : U^{(2)} \otimes U \rightarrow L(V_h^{(2)})$ are well-defined by the same arguments as in Proposition 3.1. Since $D_{\Delta t, h}^{\text{stoch}, j} = D_{\Delta t, h}^{\text{EM}, j} + D_{\Delta t, h}^{\text{M}, j}$, we obtain for $j \in \mathbb{N}_0$

$$\begin{aligned} \mathbb{E} \left[D_{\Delta t, h}^{\text{stoch}, j} \otimes D_{\Delta t, h}^{\text{stoch}, j} \right] &= \mathbb{E} \left[D_{\Delta t, h}^{\text{EM}, j} \otimes D_{\Delta t, h}^{\text{EM}, j} \right] + \mathbb{E} \left[D_{\Delta t, h}^{\text{M}, j} \otimes D_{\Delta t, h}^{\text{EM}, j} \right] \\ &\quad + \mathbb{E} \left[D_{\Delta t, h}^{\text{EM}, j} \otimes D_{\Delta t, h}^{\text{M}, j} \right] + \mathbb{E} \left[D_{\Delta t, h}^{\text{M}, j} \otimes D_{\Delta t, h}^{\text{M}, j} \right]. \end{aligned}$$

The first term and $D_{\Delta t, h}^{\det} \otimes D_{\Delta t, h}^{\det}$ are given in Proposition 3.1. For the second term, writing $\Delta^{(2)}L = \sum_{k, \ell=1}^\infty \sqrt{\mu_k \mu_\ell} \left(\int_{t_j}^{t_j+1} \int_{t_j}^s dL_k(r) dL_\ell(s) \right) f_k \otimes f_\ell$, Lemma 5.2 yields

$$\mathbb{E} \left[D_{\Delta t, h}^{\text{M}, j} \otimes D_{\Delta t, h}^{\text{EM}, j} \right] = \mathbb{E} \left[C' \Delta^{(2)} L^j \otimes C \Delta L^j \right] = (C' \otimes C) \mathbb{E} \left[\Delta^{(2)} L^j \otimes \Delta L^j \right] = 0$$

and analogously the same for the third term. Finally, Lemma 5.2 also implies

$$\mathbb{E} \left[D_{\Delta t, h}^{\text{M}, j} \otimes D_{\Delta t, h}^{\text{M}, j} \right] = \mathbb{E} \left[C' \Delta^{(2)} L^j \otimes C' \Delta^{(2)} L^j \right] = (C' \otimes C') \mathbb{E} \left[\Delta^{(2)} L^j \otimes \Delta^{(2)} L^j \right]$$

and the statement follows directly from Corollary 2.1. □

Remark 3.2 The assumption on C' in Proposition 3.2 holds for the operators G_1 and G_2 in the setting of Example 3.1. One can get rid of this assumption by using that the bound on $G \in L(H; L(U; H))$ allows for an extension of the bilinear mapping to the projective tensor product space $U \otimes_\pi U$, cf. [18]. One would then have to assume additional regularity on L to ensure that $\Delta^{(2)}L^j$ in the proof of Proposition 3.2 is in the space $L^2(\Omega; U \otimes_\pi U)$. Alternatively, one considers finite-dimensional truncated noise, which leads to equivalent norms.

3.2 Examples of rational approximations

Let us next consider specific choices of rational approximations R and investigate their influence on mean-square stability. First, we derive sufficient conditions based on Corollary 3.1 for Euler–Maruyama schemes with standard rational approximations. More specifically, we consider the backward Euler, the Crank–Nicolson, and the forward Euler scheme.

Theorem 3.1 *Consider the approximation scheme (3.5).*

1. (Backward Euler scheme) *Let R be given by $R(z) = (1 - z)^{-1}$. Then (3.5) is asymptotically mean-square stable if*

$$\frac{(1 + \Delta t \|F\|_{L(H)})^2 + \Delta t \operatorname{Tr}(Q) \|G\|_{L(H;L(U;H))}^2}{(1 + \Delta t \lambda_{h,1})^2} < 1.$$

2. (Crank–Nicolson scheme) *Let R be given by $R(z) = (1 + z/2)/(1 - z/2)$. Then (3.5) is asymptotically mean-square stable if*

$$\left(\max_{k \in \{1, N_h\}} \left| \frac{1 - \Delta t \lambda_{h,k}/2}{1 + \Delta t \lambda_{h,k}/2} \right| + \Delta t \frac{\|F\|_{L(H)}}{(1 + \Delta t \lambda_{h,1}/2)} \right)^2 + \Delta t \frac{\operatorname{Tr}(Q) \|G\|_{L(H;L(U;H))}^2}{(1 + \Delta t \lambda_{h,1}/2)^2} < 1.$$

3. (Forward Euler scheme) *Let R be given by $R(z) = 1 + z$. Then (3.5) is asymptotically mean-square stable if*

$$\left(\max_{\ell \in \{1, N_h\}} |1 - \Delta t \lambda_{h,\ell}| + \Delta t \|F\|_{L(H)} \right)^2 + \Delta t \operatorname{Tr}(Q) \|G\|_{L(H;L(U;H))}^2 < 1.$$

Proof Let us start with the backward Euler scheme. Since the functions $r_d^{-1}(z)$ and $R(z)$ are equal and it holds for all $k = 1, \dots, N_h$ that $|R(-\Delta t \lambda_{h,k})| \leq |R(-\Delta t \lambda_{h,1})|$, we obtain by Corollary 3.1 asymptotic mean-square stability if

$$(1 + \Delta t \lambda_{h,1})^{-2} \left((1 + \Delta t \|F\|_{L(H)})^2 + \Delta t \operatorname{Tr}(Q) \|G\|_{L(H;L(U;H))}^2 \right) < 1.$$

For the Crank–Nicolson scheme, note that the function R is decreasing on \mathbb{R}^- and that $R(z) \in [-1, 1]$ for all $z \in \mathbb{R}^-$. Thus, the maximizing eigenvalue is either the largest, λ_{h,N_h} , or the smallest, $\lambda_{h,1}$, and therefore,

$$|R(-\Delta t \lambda_{h,k})| \leq \max_{\ell \in \{1, N_h\}} |R(-\Delta t \lambda_{h,\ell})|.$$

Since $|r_d^{-1}(-\Delta t \lambda_{h,k})| \leq |r_d^{-1}(-\Delta t \lambda_{h,1})|$ for all $k = 1, \dots, N_h$, the claim follows with Corollary 3.1.

By the same arguments, we obtain for the forward Euler scheme that $|R(-\Delta t \lambda_{h,i})|$ is maximized either at $z = -\Delta t \lambda_{h,1}$ or $z = -\Delta t \lambda_{h,N_h}$. Therefore, since $r_d^{-1}(z) = 1$, the claim follows again with Corollary 3.1, which finishes the proof. \square

For the Milstein scheme, Proposition 3.2 yields the following sufficient condition.

Proposition 3.3 *Under the assumptions of Proposition 3.2, the Milstein scheme (3.7) with $R(z) = (1 - z)^{-1}$ is asymptotically mean-square stable if*

$$(1 + \Delta t \|F\|_{L(H)})^2 + \Delta t \operatorname{Tr}(Q) \|G\|_{L(H;L(U;H))}^2 + \frac{\Delta t^2}{2} \operatorname{Tr}(Q)^2 \|G\|_{L(H;L(U;H))}^4 < (1 + \Delta t \lambda_{h,1})^2.$$

Proof In the same way as in the proof of Corollary 3.1, we bound

$$\begin{aligned} \|(C' \otimes C')q'\|_{L(V_h^{(2)})} &\leq \|C'\|_{L(U^{(2)};L(V_h))}^2 \operatorname{Tr}(Q)^2 \\ &\leq (1 + \Delta t \lambda_{h,1})^{-2} \|G\|_{L(H;L(U;H))}^4 \operatorname{Tr}(Q)^2. \end{aligned}$$

Hence, our assumption ensures that $\|\mathcal{S}\|_{L(V_h^{(2)})} < 1$, which by Corollary 2.1 is sufficient for asymptotic mean-square stability. □

Note that the sufficient condition for the Milstein scheme is more restrictive than the sufficient condition presented in Theorem 3.1(1) for the backward Euler–Maruyama method due to the additional positive term in the estimate in Proposition 3.3.

3.3 Relation to the mild solution

To connect existing results on asymptotic mean-square stability of (1.1) to the results for discrete schemes in Sect. 3.2, we have to restrict ourselves to Q -Wiener processes $W = (W(t), t \geq 0)$ due to the framework for analytical solutions in [24]. Specifically, we consider

$$dX(t) = (AX(t) + FX(t)) dt + G(X(t)) dW(t). \tag{3.8}$$

The following special case of [24, Proposition 3.1.1] provides a sufficient condition for the asymptotic mean-square stability of (1.1) by a Lyapunov functional approach.

Theorem 3.2 *Assume that $X_0 = x_0 \in \dot{H}^1$ is deterministic and there exists $c > 0$ such that*

$$2\langle v, Av + F(v) \rangle_H + \operatorname{Tr}[G(v)Q(G(v))^*] \leq -c\|v\|_H^2$$

for all $v \in \dot{H}^2$. Then the zero solution of (3.8) is asymptotically mean-square stable.

We use this theorem to derive simultaneous sufficient mean-square stability conditions for (3.8) and the corresponding backward Euler scheme (3.5).

Corollary 3.2 *Assume that $X_0 = x_0 \in \dot{H}^1$ is deterministic. Then the zero solutions of (3.8) and its approximation (3.5) with $R(z) = (1 - z)^{-1}$ are asymptotically mean-square stable for all h and Δt if*

$$2(\|F\|_{L(H)} - \lambda_1) + \text{Tr}(Q)\|G\|_{L(H;L(U;H))}^2 < 0. \tag{3.9}$$

Proof We show first that (3.9) yields asymptotic mean-square stability of (3.8) by reducing it to the assumption in Theorem 3.2. For the second term there, note that for any $v \in \dot{H}^2$,

$$\begin{aligned} & \text{Tr}[G(v)Q(G(v))^*] \\ &= \text{Tr}[(G(v))^*G(v)Q] = \sum_{k=1}^{\infty} \langle G(v)Qf_k, G(v)f_k \rangle \\ &\leq \sum_{k=1}^{\infty} \mu_k \|G\|_{L(H;L(U;H))}^2 \|v\|_H^2 \|f_k\|_U^2 = \text{Tr}(Q)\|G\|_{L(H;L(U;H))}^2 \|v\|_H^2, \end{aligned}$$

where the first equality follows from the properties of the trace. The first term satisfies

$$\begin{aligned} \langle v, Av + F(v) \rangle &= \langle v, F(v) \rangle + \langle v, Av \rangle \leq \|F\|_{L(H)} \|v\|_H^2 - \|v\|_1^2 \\ &\leq (\|F\|_{L(H)} - \lambda_1) \|v\|_H^2 \end{aligned}$$

using the definition of $\|\cdot\|_1$. Altogether, we therefore obtain

$$\begin{aligned} & 2\langle v, Av + F(v) \rangle_H + \text{Tr}[G(v)Q(G(v))^*] \\ &\leq (2(\|F\|_{L(H)} - \lambda_1) + \text{Tr}(Q)\|G\|_{L(H;L(U;H))}^2) \|v\|_H^2, \end{aligned}$$

i.e., with (3.9) asymptotic mean-square stability of (3.8) by setting

$$c = -(2(\|F\|_{L(H)} - \lambda_1) + \text{Tr}(Q)\|G\|_{L(H;L(U;H))}^2).$$

We continue with (3.5) observing first that $\lambda_{h,1} \geq \lambda_1$ by (3.2). Therefore, the condition in Theorem 3.1(1) yields

$$\Delta t(2(\|F\|_{L(H)} - \lambda_1) + \text{Tr}(Q)\|G\|_{L(H;L(U;H))}^2) + \Delta t^2(\|F\|_{L(H)}^2 - \lambda_1^2) < 0.$$

This is satisfied and finishes the proof since the first term is negative by assumption and so is the second using (3.9) and

$$\begin{aligned} & \|F\|_{L(H)}^2 - \lambda_1^2 \\ &= (\|F\|_{L(H)} + \lambda_1)(\|F\|_{L(H)} - \lambda_1) \\ &\leq (\|F\|_{L(H)} + \lambda_1)((\|F\|_{L(H)} - \lambda_1) + 2^{-1} \text{Tr}(Q)\|G\|_{L(H;L(U;H))}^2) < 0. \quad \square \end{aligned}$$

Note that under (3.9) in Corollary 3.2, the backward Euler–Maruyama scheme preserves the qualitative behaviour of the analytical solution without any restriction on h

and Δt . Hence, it can be applied to numerical methods requiring different refinement parameters in parallel such as multilevel Monte Carlo, which efficiently approximates quantities $\mathbb{E}[\varphi(X(T))]$ (see, e.g., [4,6] for details). Here, it is essential that the behaviour is preserved on all refinement levels [1].

On the other hand, note that in the homogeneous case, i.e., $F = 0$, the stability condition in Theorem 3.1(1) reduces to

$$\text{Tr}(Q)\|G\|_{L(H;L(U;H))}^2 < \lambda_{h,1}(2 + \Delta t\lambda_{h,1})$$

so that even if (1.1) is asymptotically mean-square unstable, its approximation (3.5) can always be rendered stable by letting Δt be large enough. In that case the analytical solution and its approximation have a different qualitative behaviour for large times.

Remark 3.3 Based on Theorem 3.1, it is also possible to examine the relation between asymptotic mean-square stability of (3.8) and its approximation by the other rational approximations. However, due to the nature of the sufficient conditions in Theorem 3.1, analogous results to Corollary 3.2 include restrictions on h and Δt .

For the Milstein scheme considered in Proposition 3.3 we can also derive a sufficient condition for the simultaneous mean-square stability not relying on h and Δt . However, due to the additional term in Proposition 3.3, the condition becomes slightly more restrictive than in Corollary 3.2. More precisely we obtain the following:

Corollary 3.3 *Assume that $X_0 = x_0 \in \dot{H}^1$ is deterministic and $F = 0$. Then the zero solutions of (3.8) and its Milstein approximation (3.7) with $R(z) = (1 - z)^{-1}$ are asymptotically mean-square stable for all h and Δt if*

$$-\sqrt{2}\lambda_1 + \text{Tr}(Q)\|G\|_{L(H;L(U;H))}^2 < 0.$$

Proof The asymptotic mean-square stability of (3.8) follows by Corollary 3.2 since

$$-2\lambda_1 + \text{Tr}(Q)\|G\|_{L(H;L(U;H))}^2 < -\sqrt{2}\lambda_1 + \text{Tr}(Q)\|G\|_{L(H;L(U;H))}^2 < 0.$$

The sufficient condition for (3.7) in Proposition 3.3 can be rewritten as

$$\Delta t(-2\lambda_{h,1} + \text{Tr}(Q)\|G\|_{L(H;L(U;H))}^2) + \Delta t^2(-2\lambda_{h,1}^2 + \text{Tr}(Q)^2\|G\|_{L(H;L(U;H))}^4) < 0.$$

The first summand is negative since

$$-2\lambda_{h,1} + \text{Tr}(Q)\|G\|_{L(H;L(U;H))}^2 < -\sqrt{2}\lambda_1 + \text{Tr}(Q)\|G\|_{L(H;L(U;H))}^2 < 0.$$

The assumption $\sqrt{2}\lambda_1 > \text{Tr}(Q)\|G\|_{L(H;L(U;H))}^2$ implies for the second summand that

$$-2\lambda_{h,1}^2 + \text{Tr}(Q)^2\|G\|_{L(H;L(U;H))}^4 < 0.$$

Thus, asymptotic mean-square stability of (3.7) follows. □

4 Simulations

In this section we adopt the setting of Example 3.1 and use numerical simulations to illustrate our theoretical results. More specifically, we consider the stochastic heat equation

$$dX(t) = \nu \Delta X(t) dt + G(X(t)) dW(t). \tag{4.1}$$

with $X_0(x) = \sqrt{30}x(1 - x)$, then $\mathbb{E}[\|X_0\|_H^2] = 1$. We consider a Q -Wiener process $W(t) = \sum_{i=1}^\infty \sqrt{\mu_i} \beta_i(t) e_i$, where $(\beta_i, i \in \mathbb{N})$ is a sequence of independent, real-valued Brownian motions, and assume $\mu_i = C_\mu i^{-\alpha}$ with $C_\mu > 0$ and $\alpha > 1$. Here, C_μ scales the noise intensity and α controls the space regularity of W , see, e.g., [23, 25].

4.1 Spectral Galerkin methods

For $G = G_1$ in Example 3.1, we obtain with the approach presented in [20, Sect. 6.4] the infinite-dimensional counterpart of the geometric Brownian motion

$$X(t) = \sum_{i=1}^\infty \langle X(t), e_i \rangle_H e_i = \sum_{i=1}^\infty x_i(t) e_i,$$

where each of the coefficients $x_i(t)$ is the solution to the one-dimensional geometric Brownian motion

$$dx_i(t) = -\lambda_i x_i(t) dt + \sqrt{\mu_i} x_i(t) d\beta_i(t).$$

Furthermore, the second moment is given by

$$\mathbb{E} \left[\|X(T)\|_H^2 \right] = \sum_{i=1}^\infty \mathbb{E} \left[|x_i(T)|^2 \right] = \sum_{i=1}^\infty \langle X_0, e_i \rangle_H^2 \exp((-2\lambda_i + \mu_i)T).$$

As a consequence, the asymptotic mean-square stability of (4.1) holds if and only if $-2\lambda_i + \mu_i < 0$ for all $i \in \mathbb{N}$. By using the explicit representation of the eigenvalues λ_i and μ_i , this corresponds to $-2\nu i^2 \pi^2 + C_\mu i^{-\alpha} < 0$ which is equivalent to the condition $-2\lambda_1 + \mu_1 = -2\nu\pi^2 + C_\mu < 0$, i.e., (4.1) is asymptotically mean-square unstable if and only if $C_\mu > 2\nu\pi^2$.

For the spectral Galerkin approximation, we choose $V_h = \text{span}(e_1, \dots, e_{N_h})$, $N_h < \infty$. Thus, we consider $X_h(t) = \sum_{k=1}^{N_h} x_k(t) e_k$. To obtain a fully discrete scheme, we approximate the one-dimensional geometric Brownian motions in time by the three considered rational approximations in Theorem 3.1 and Proposition 3.3. Propositions 3.1 and 3.2 yield asymptotic mean-square stability if and only if the corresponding linear operators \mathcal{S} satisfy $\rho(\mathcal{S}) < 1$. The operator is in the first case for $k, \ell = 1, \dots, N_h$ given by

$$\begin{aligned} \mathcal{S}(e_k \otimes e_\ell) &= (D_{\Delta t, h}^{\det} \otimes D_{\Delta t, h}^{\det})(e_k \otimes e_\ell) + \Delta t((C \otimes C)q)(e_k \otimes e_\ell) \\ &= (D_{\Delta t, h}^{\det} e_k \otimes D_{\Delta t, h}^{\det} e_\ell) + \Delta t \sum_{m=1}^{\infty} \mu_m(((C e_m) e_k) \otimes ((C e_m) e_\ell)). \end{aligned}$$

Since

$$D_{\Delta t, h}^{\det} e_k = R(\Delta t A_h) e_k = \sum_{r=1}^{N_h} R(-\Delta t \lambda_r) \langle e_k, e_r \rangle_H e_r = R(-\Delta t \lambda_k) e_k$$

and

$$\begin{aligned} (C e_m) e_k &= r_d^{-1}(\Delta t A_h) P_h G_1(e_k) e_m = r_d^{-1}(\Delta t A_h) P_h \left(\sum_{n=1}^{\infty} \langle e_k, e_n \rangle_H \langle e_m, e_n \rangle_H e_n \right) \\ &= \delta_{k, m} r_d^{-1}(\Delta t A_h) e_k = \delta_{k, m} r_d^{-1}(-\Delta t \lambda_k) e_k, \end{aligned}$$

the corresponding eigenvalues $\Lambda_{k, \ell}$ are

$$\Lambda_{k, \ell} = R(-\Delta t \lambda_k) R(-\Delta t \lambda_\ell) + \delta_{k, \ell} \Delta t \mu_k r_d^{-1}(-\Delta t \lambda_k) r_d^{-1}(-\Delta t \lambda_\ell).$$

Using a Milstein scheme instead, we obtain for \mathcal{S} in Proposition 3.2 with similar computations as before and the observations that the commutativity condition in Remark 3.1 is fulfilled and that $\Delta[\beta_k, \beta_\ell]^j = \delta_{k, \ell} \Delta t$

$$\Lambda_{k, \ell} = R(-\Delta t \lambda_k) R(-\Delta t \lambda_\ell) + \delta_{k, \ell} r_d^{-1}(-\Delta t \lambda_k) r_d^{-1}(-\Delta t \lambda_\ell) \left(\Delta t \mu_k + \Delta t^2 \mu_k^2 / 2 \right).$$

Note that for both operators \mathcal{S} , the eigenvalues $\Lambda_{k, \ell}$ with $k \neq \ell$ satisfy

$$|\Lambda_{k, \ell}| = |R(-\Delta t \lambda_k) R(-\Delta t \lambda_\ell)| \leq R(-\Delta t \lambda_s)^2 \leq A_{s, s},$$

where $|R(-\Delta t \lambda_s)| = \max_{j=1, \dots, N_h} |R(-\Delta t \lambda_j)|$. Hence, $\rho(\mathcal{S}) < 1$ is equivalent to $|\Lambda_{k, k}| < 1$ for all $k = 1, \dots, N_h$. In Table 1 the eigenvalues $\Lambda_{k, k}$ and sufficient and necessary conditions for asymptotic mean-square stability are collected.

As it is noted above, (4.1) is asymptotically mean-square stable if and only if the condition $-2\lambda_1 + \mu_1 < 0$ holds. As can be seen from Table 1 and the choice of the eigenvalues, the Euler–Maruyama scheme (3.5) with backward Euler and Crank–Nicolson rational approximation shares this property without any restriction on V_h and Δt . In Fig. 1a the qualitative behaviour of the Euler–Maruyama method with the three rational approximations in Theorem 3.1 is compared. We choose our parameters to be $\nu = 1$, $N_h = 15$, and $\mu_i = i^{-3}$ for $i \in \mathbb{N}$, i.e., $C_\mu = 1$ and $\alpha = 3$. Since $-2\lambda_1 + C_\mu = -2\pi^2 + 1 < 0$, the analytical solution to (4.1) is asymptotically mean-square stable.

Table 1 Spectral Galerkin methods with corresponding eigenvalues $\Lambda_{k,k}$ and asymptotic mean-square stability conditions

Rational approximation/stochastic approximation	$\Lambda_{k,k}$	$\rho(\mathcal{S}) < 1 \Leftrightarrow$ for all $k = 1, \dots, N_h$:
Backward Euler/EM	$\frac{1+\Delta t\mu_k}{(1+\Delta t\lambda_k)^2}$	$-2\lambda_k + \mu_k - \Delta t\lambda_k^2 < 0$
Backward Euler/Milstein	$\frac{1+\Delta t\mu_k+\Delta t^2\mu_k^2/2}{(1+\Delta t\lambda_k)^2}$	$-2\lambda_k + \mu_k + \Delta t(-\lambda_k^2 + \mu_k^2/2) < 0$
Crank–Nicolson/EM	$\frac{(1-\Delta t\lambda_k/2)^2+\mu_k\Delta t}{(1+\Delta t\lambda_k/2)^2}$	$-2\lambda_k + \mu_k < 0$
Forward Euler/EM	$(1 - \Delta t\lambda_k)^2 + \mu_k\Delta t$	$-2\lambda_k + \mu_k + \Delta t\lambda_k^2 < 0$

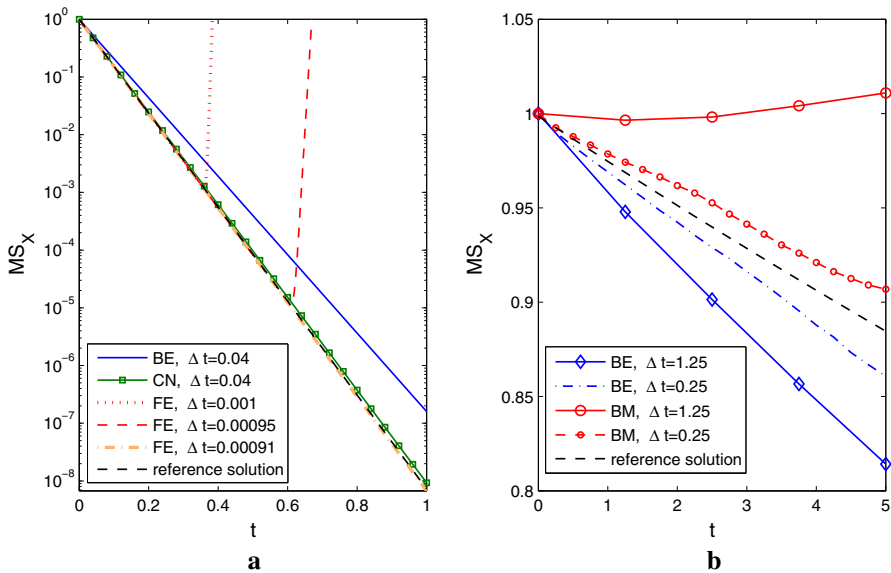


Fig. 1 Spectral Galerkin approximate of (4.1) with G_1 , $N_h = 15$, and different Δt . **a** Crank–Nicolson (CN), backward (BE) and forward (FE) Euler. **b** Euler–Maruyama (BE) and Milstein (BM) based on backward Euler

For the approximation of $\mathbb{E}[\|X_h^j\|_H^2]$ we use $M = 10^6$ samples in a Monte Carlo simulation, i.e., we approximate

$$\mathbb{E} \left[\|X_h^j\|_H^2 \right] \approx \text{MS}_X(t_j) = \frac{1}{M} \sum_{i=1}^M \sum_{k=1}^{N_h} |\widehat{x}_k^{j,(i)}|^2,$$

where $(\widehat{x}_k^{j,(i)}, i = 1, \dots, M)$ consists of independent samples of numerical approximations of $x_k(t_j)$ with different schemes. The reference solution is

$$\mathbb{E} \left[\|X_h(t)\|_H^2 \right] = \sum_{k=1}^{N_h} \mathbb{E} \left[|x_k(t)|^2 \right] = \sum_{k=1}^{N_h} \langle X_0, e_k \rangle_H^2 \exp((-2\lambda_k + \mu_k)t).$$

As it can be seen in Fig. 1a, the backward Euler and the Crank–Nicolson scheme reproduce the mean-square stability of (4.1) already for large time step sizes ($\Delta t = 1/25$), but the forward Euler scheme requires a 44 times smaller Δt . Here, the finest time step size is given by $\Delta t = 1/1100$ which satisfies the restrictive bound in Table 1 such that $\rho(\mathcal{S}) < 1$. Due to a rapid amplification of oscillations caused by negative values of X_h^j for coarser time step sizes outside the stability region (i.e. $\Delta t = 1/1000$ and $1/1050$), the mean-square process deviates rapidly from the reference solution at a certain time point.

In Fig. 1b the qualitative behaviour of the Euler–Maruyama and Milstein schemes with a backward Euler rational approximation on the time interval $[0, 5]$ is compared. The parameters $\nu = 8/(5\pi^4)$ and $\mu_i = 3/10 i^{-3}$ are chosen such that the Milstein scheme is asymptotically mean-square unstable for $\Delta t = 1.25$ and asymptotically mean-square stable for $\Delta t = 0.25$ while the Euler–Maruyama scheme is asymptotically mean-square stable for both choices. These theoretical results are reproduced in the simulation.

4.2 Galerkin finite element methods

Let us continue with $G = G_2$ in Example 3.1 and a Galerkin finite element setting, similar to that of [22]. This is to say, we let V_h be the span of piecewise linear functions on an equidistant grid of $[0, 1]$ with N_h interior nodes so that V_h is an N_h -dimensional subspace of \dot{H}^1 with refinement parameter $h = (N_h + 1)^{-1}$. With the exception that $U = \dot{H}^1$, all other parameters are as in Fig. 1a of Sect. 4.1.

In contrast to the setting in Sect. 4.1, the solution and its approximation are no longer sums of one-dimensional geometric Brownian motions and thus, analytical necessary and sufficient conditions for $\rho(\mathcal{S}) < 1$ are not available. We therefore consider the results of Theorem 3.1 instead. With the setting of this section,

$$\lambda_{h,i} = 12\nu h^{-2} (2 + \cos(i\pi h))^{-1} (\sin(i\pi h/2))^2$$

for $i \in \mathbb{N}$, which was derived in [20, Sect. 6.1]. For the convenience of the reader, the sufficient conditions of Theorem 3.1 for the considered approximation schemes are collected in simplified form in Table 2, expressed in terms of stability parameters ρ_{BE} , ρ_{CN} and ρ_{FE} . By setting $\hat{g} = \left(2 \sum_{i=1}^{\infty} \lambda_i^{-1}\right)^{1/2}$, we replace $\|G_2\|_{L(H;L(U;H))}$ in these conditions with the upper bound derived in Example 3.1. Note that Corollary 3.2 with these parameters implies simultaneous asymptotic mean-square stability of (4.1) and the finite element backward Euler scheme (3.5).

As in Sect. 4.1 we compare the mean-square behaviour of the backward Euler and the forward Euler scheme in Fig. 2a but now for the finite element discretization up to $T = 2.5$. We observe that the increase of the time step size by a very small amount, i.e., from $\Delta t = 0.00066$ to $\Delta t = 0.00067$, causes the forward Euler system to switch

Table 2 Finite element methods with sufficient conditions for $\rho(\mathcal{S}) < 1$

Rational approximation	$\rho(\mathcal{S}) < 1 \Leftarrow$:
Backward Euler	$\rho_{BE} = \Delta t \operatorname{Tr}(Q)\hat{g}^2 - 2\Delta t\lambda_{h,1} - \Delta t^2\lambda_{h,1}^2 < 0$
Crank–Nicolson	$\rho_{CN} = \max_{k \in \{1, N_h\}} \left \frac{1 - \Delta t\lambda_{h,k}/2}{1 + \Delta t\lambda_{h,k}/2} \right ^2 + \frac{\Delta t \operatorname{Tr}(Q)\hat{g}^2}{(1 + \Delta t\lambda_{h,1}/2)^2} - 1 < 0$
Forward Euler	$\rho_{FE} = \max_{k \in \{1, N_h\}} (1 - \Delta t\lambda_{h,k})^2 + \Delta t \operatorname{Tr}(Q)\hat{g}^2 - 1 < 0$

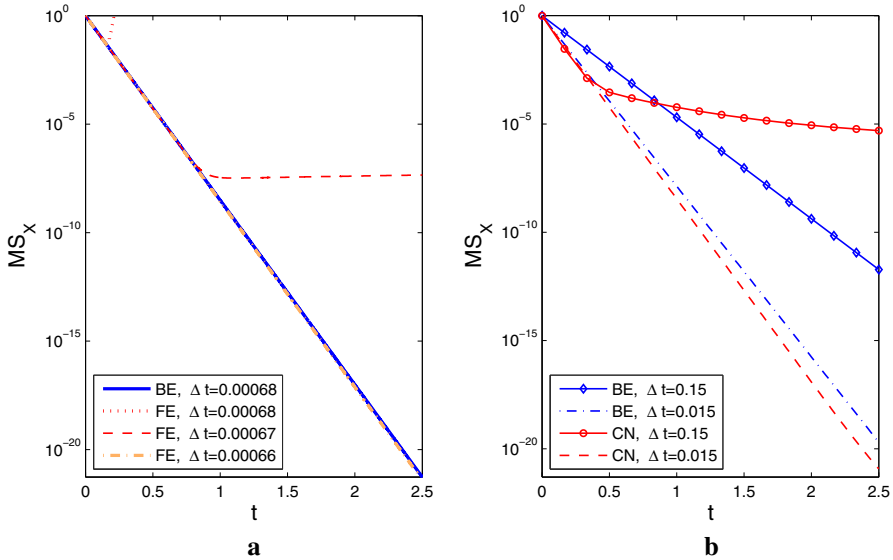


Fig. 2 Finite Element approximation of (4.1) with $G = G_2, N_h = 15$, and different Δt . **a** Backward (BE) and forward (FE) Euler. **b** Backward Euler (BE) and Crank–Nicolson (CN).

from a stable to an unstable behaviour. This agrees with the theory in Table 3, as ρ_{FE} changes sign in that interval, i.e., stability is only guaranteed for the smaller time step. Therefore we conclude that the sufficient condition is sharp in our model problem.

For the approximation of $\mathbb{E}[\|X_h^j\|_H^2]$, we use the same method as before but take $M = 10^4$ samples in the Monte Carlo approximation. For the computation of the norm in H , we use the fact that for given representation $X_h^j = \sum_{m=1}^{N_h} x_m \phi_m$ with respect to the hat functions $\{\phi_m, m = 1 \dots, N_h\}$ that span V_h

$$\|X_h^j\|_H^2 = \sum_{m=1}^{N_h} \sum_{n=1}^{N_h} x_m x_n \langle \phi_m, \phi_n \rangle_H .$$

In Fig. 2a the mean-square behaviour of the the backward Euler scheme and the Crank–Nicolson scheme for $\Delta t = 0.015$ to $\Delta t = 0.15$ is compared. We see from Table 3 that ρ_{CN} changes sign when the time step size is increased, which occurs

Table 3 Specific values of Table 2 for varying Δt

Δt	ρ_{BE}	ρ_{CN}	ρ_{FE}
0.15	-5.11613e+00	2.08460e-03	1.99602e+05
0.015	-3.13089e-01	-1.58504e-01	1.91542e+03
0.00068	-1.32387e-02	-1.31050e-02	6.09395e-02
0.00067	-1.30434e-02	-1.29135e-02	3.39709e-04
0.00066	-1.28480e-02	-1.27221e-02	-1.27626e-02

for significantly larger time steps than for the forward Euler scheme. The simulation results show a substantial change in the decay behaviour of $\mathbb{E}[\|X_h^j\|_H^2]$ for the Crank–Nicolson scheme with time step size $\Delta t = 0.15$ compared to $\Delta t = 0.015$, which is no longer convincing to be mean-square stable. Since the sufficient condition $\rho_{CN} < 0$ from Table 2 is not fulfilled for $\Delta t = 0.15$, it is unclear from the theory if asymptotic mean-square stability holds in that case.

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5 Appendix: Properties of Lévy increments

In this appendix we derive properties of the U -valued, square-integrable Lévy process that are used in the proofs of Propositions 3.1 and 3.2. We apply the same setting and notation as in Sect. 3.

Lemma 5.1 *Let L be a U -valued Lévy process and let, for $0 \leq a < b$, $\Delta L = L(b) - L(a)$ and $\Delta t = b - a$. Then*

$$\mathbb{E}[\Delta L \otimes \Delta L] = \Delta t \sum_{k=1}^{\infty} \mu_k f_k \otimes f_k.$$

Proof We first note that $\Delta L \otimes \Delta L$ is well-defined as a member of $L^1(\Omega; U^{(2)})$ since

$$\mathbb{E}[\|\Delta L \otimes \Delta L\|_{U^{(2)}}] = \mathbb{E}[\|\Delta L\|_U^2] = \text{Tr}(Q)\Delta t < \infty.$$

The increments $\Delta L_k = L_k(b) - L_k(a)$ of $\Delta L = \sum_{k=1}^\infty \sqrt{\mu_k} \Delta L_k f_k$ fulfil $\mathbb{E}[\Delta L_k \Delta L_\ell] = \delta_{k,\ell} \Delta t$ for $k, \ell \in \mathbb{N}$. Thus, we obtain

$$\mathbb{E}[\Delta L \otimes \Delta L] = \sum_{k,\ell=1}^\infty \sqrt{\mu_k \mu_\ell} \mathbb{E}[\Delta L_k \Delta L_\ell] f_k \otimes f_\ell = \Delta t \sum_{k=1}^\infty \mu_k f_k \otimes f_k$$

which finishes the proof. □

Lemma 5.2 *Let L be a U -valued, square-integrable Lévy process and set for $0 \leq a < b$ with $\Delta t = b - a$,*

$$\Delta^{(2)}L = \sum_{k,\ell=1}^\infty \sqrt{\mu_k \mu_\ell} \left(\int_a^b \int_a^s dL_k(r) dL_\ell(s) \right) f_k \otimes f_\ell \in L^2(\Omega; U^{(2)}).$$

Then

1. $\mathbb{E}[\Delta^{(2)}L \otimes \Delta L] = 0,$
2. $\mathbb{E}[\Delta^{(2)}L \otimes \Delta^{(2)}L] = \frac{\Delta t^2}{2} \sum_{k,\ell=1}^\infty \mu_k \mu_\ell ((f_k \otimes f_\ell) \otimes (f_k \otimes f_\ell)).$

Proof Since L is stationary, we may assume without loss of generality that $a = 0$ and $b = t > 0$. We first note that

$$\begin{aligned} \mathbb{E} \left[\left(\int_0^t \int_0^s dL_i(r) dL_j(s) \right) \left(\int_0^t \int_0^s dL_k(r) dL_\ell(s) \right) \right] \\ = \mathbb{E} \left[\left(\int_0^t L_i(s-) dL_j(s) \right) \left(\int_0^t L_k(s-) dL_\ell(s) \right) \right]. \end{aligned}$$

To simplify this expression, we use the angle bracket process $(\langle X, Y \rangle_t, t \geq 0)$, which for two real-valued semimartingales X and Y with (locally) integrable quadratic covariation $[X, Y]$ is defined as the unique compensator which makes $([X, Y]_t - \langle X, Y \rangle_t, t \geq 0)$ a local martingale. For this, we have the *polarization identity*,

$$\langle X, Y \rangle_t = \frac{1}{4} (\langle X + Y, X + Y \rangle_t - \langle X - Y, X - Y \rangle_t),$$

which can be found, along with an introduction to this process, e.g., in [28, Sect. III.5].

For square-integrable martingales M , it holds (see, e.g., [19, Sect. 8.9]) that $\mathbb{E}[\langle M, M \rangle_t] = \mathbb{E}[M(t)^2]$ and therefore, by the polarization identity, if N is another square-integrable martingale, then,

$$\mathbb{E}[\langle M, N \rangle_t] = \frac{1}{4} \left(\mathbb{E}[(M(t) + N(t))^2] - \mathbb{E}[(M(t) - N(t))^2] \right) = \mathbb{E}[M(t)N(t)].$$

Applying this to the Lévy integral, which is a martingale, we obtain

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^t L_i(s-) \, dL_j(s) \right) \left(\int_0^t L_k(s-) \, dL_\ell(s) \right) \right] \\ &= \mathbb{E} \left[\left\langle \int_0^\cdot L_i(s-) \, dL_j(s), \int_0^\cdot L_k(s-) \, dL_\ell(s) \right\rangle_t \right] \\ &= \mathbb{E} \left[\int_0^t L_i(s-) L_k(s-) \, d\langle L_j, L_\ell \rangle_s \right], \end{aligned}$$

where the last equality is a property of the angle bracket process and the stochastic integral, see [19, Sect. 8.9]. Now, when $j = \ell$, we have, since L_j is a Lévy process and $\mathbb{E}[L_j^2(s)] = s$, that $\langle L_j, L_\ell \rangle_s = \langle L_j, L_j \rangle_s = s$ by [27, Chapter 8]. When $j \neq \ell$ on the other hand, $L_j L_\ell$ is a square-integrable martingale by [27, Theorem 4.49(ii)]. Integration by parts yields

$$[L_j, L_\ell]_s = L_j(s)L_\ell(s) - \int_0^s L_j(r-) \, dL_\ell(r) - \int_0^s L_\ell(r-) \, dL_j(r).$$

Therefore, $[L_j, L_\ell]$ is also a square-integrable martingale (with zero mean), because the right hand side is a square-integrable martingale. Since $(\langle L_j, L_\ell \rangle_s, s \geq 0)$ is the unique compensator of $[L_j, L_\ell]$ it must follow that $\langle L_j, L_\ell \rangle_s = 0$ for all $s \geq 0$. Thus, $\mathbb{E}[\int_0^t L_i(s-) L_k(s-) \, d\langle L_j, L_\ell \rangle_s]$ is non-zero only if $j = \ell$, and in that case

$$\mathbb{E} \left[\int_0^t L_i(s-) L_k(s-) \, d\langle L_j, L_j \rangle_s \right] = \int_0^t \mathbb{E} [L_i(s-) L_k(s-)] \, ds.$$

In conclusion we have obtained

$$\mathbb{E} \left[\left(\int_0^t \int_0^s \, dL_i(r) \, dL_j(s) \right) \left(\int_0^t \int_0^s \, dL_k(r) \, dL_\ell(s) \right) \right] = \begin{cases} t^2/2 & \text{for } j = \ell \text{ and } i = k, \\ 0 & \text{otherwise,} \end{cases} \tag{5.1}$$

which yields by the monotone convergence theorem that $\Delta^{(2)}L \in L^2(\Omega; U^{(2)})$ with

$$\begin{aligned} \mathbb{E} \left[\|\Delta^{(2)}L\|_{U^{(2)}}^2 \right] &= \sum_{k,\ell=1}^\infty \mu_k \mu_\ell \mathbb{E} \left[\left(\int_0^t \int_0^s \, dL_k(r) \, dL_\ell(s) \right)^2 \right] \\ &= \frac{t^2}{2} \sum_{k,\ell=1}^\infty \mu_k \mu_\ell = \frac{t^2}{2} \text{Tr}(Q)^2 < \infty. \end{aligned}$$

This entails that $\Delta^{(2)}L \otimes \Delta L \in L^1(\Omega; U^{(2)} \otimes U)$, since

$$\begin{aligned} \left(\mathbb{E} \left[\|\Delta^{(2)}L \otimes \Delta L\|_{U^{(2)} \otimes U} \right] \right)^2 &= \left(\mathbb{E} \left[\|\Delta^{(2)}L\|_{U^{(2)}} \|\Delta L\|_U \right] \right)^2 \\ &\leq \mathbb{E} \left[\|\Delta^{(2)}L\|_{U^{(2)}}^2 \right] \mathbb{E} \left[\|\Delta L\|_U^2 \right] = \frac{t^3}{2} \text{Tr}(Q)^3 < \infty. \end{aligned}$$

by the Cauchy–Schwarz inequality. Similarly, it holds that $\Delta^{(2)}L \otimes \Delta^{(2)}L \in L^1(\Omega; U^{(2)} \otimes U^{(2)})$. Therefore, we obtain

$$\mathbb{E} \left[\Delta^{(2)}L \otimes \Delta L \right] = \sum_{k,\ell,m=1}^{\infty} \sqrt{\mu_k \mu_\ell \mu_m} \mathbb{E} \left[\Delta L_m \left(\int_0^t \int_0^s dL_k(r) dL_\ell(s) \right) \right] (f_k \otimes f_\ell) \otimes f_m,$$

and, in the same way as the first observation of this proof,

$$\begin{aligned} \mathbb{E} \left[\Delta L_m \left(\int_0^t \int_0^s dL_k(r) dL_\ell(s) \right) \right] &= \mathbb{E} \left[\left\langle \int dL_m(s), \int L_k(s-) dL_\ell(s) \right\rangle_t \right] \\ &= \mathbb{E} \left[\int_0^t L_k(s-) d \langle L_m, L_\ell \rangle_s \right] = 0. \end{aligned}$$

This is justified since $\langle L_m, L_\ell \rangle_s \neq 0$ only if $m = \ell$ and that in this case the expectation of the integral is still zero since L_k has zero expectation.

We note that by (5.1)

$$\begin{aligned} &\mathbb{E} \left[\Delta^{(2)}L \otimes \Delta^{(2)}L \right] \\ &= \sum_{i,j,k,\ell=1}^{\infty} \sqrt{\mu_i \mu_j \mu_k \mu_\ell} ((f_i \otimes f_j) \otimes (f_k \otimes f_\ell)) \\ &\quad \cdot \mathbb{E} \left[\left(\int_0^t \int_0^s dL_i(r) dL_j(s) \right) \left(\int_0^t \int_0^s dL_k(r) dL_\ell(s) \right) \right] \\ &= \frac{t^2}{2} \sum_{k,\ell=1}^{\infty} \mu_k \mu_\ell ((f_k \otimes f_\ell) \otimes (f_k \otimes f_\ell)), \end{aligned}$$

which shows the second claim. □

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