SERIEs (2011) 2:431–446
DOI 10.1007/s13209-011-0050-y

SERIEs Journal of the Spanish Economic Association

ORIGINAL ARTICLE

An individual manipulability of positional voting rules

Fuad Aleskerov · Daniel Karabekyan · M. Remzi Sanver · Vyacheslav Yakuba

Received: 17 February 2011 / Accepted: 28 February 2011 / Published online: 15 March 2011

Abstract We study a problem of individual manipulation in an impartial culture (IC) framework using computer modeling. We estimate the degree of manipulability of ten positional voting rules in the case of multiple choice for 3 and 4 alternatives.

Keywords Manipulability · Positional voting rules · Multiple choice · Extended preferences

JEL Classification D7

1 Introduction

Gibbard (1973) and Satterthwaite (1975) showed that for at least 3 alternatives and any single-valued choice rule every non-dictatorial voting rule is individually manipulable. Later Duggan and Schwartz (2000) generalized this result for the case of multiple choice (when more than 1 alternative can be socially chosen). But if we know that

F. Aleskerov (✉) · D. Karabekyan
National Research University Higher School of Economics, Moscow, Russia
e-mail: alesk@hse.ru

D. Karabekyan
e-mail: dkarabekyan@hse.ru

F. Aleskerov · V. Yakuba
Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia
e-mail: yakuba@ipu.ru

M. R. Sanver
Istanbul Bilgi University, Istanbul, Turkey
e-mail: sanver@bilgi.edu.tr

every social choice rule is manipulable, how can we find the least manipulable one? A non-exhaustive list of papers studying to which extent known social choice rules are manipulable includes Chamberlin (1985), Nitzan (1985), Kelly (1993), Aleskerov and Kurbanov (1999), Smith (1999), Favardin and Lepelley (2006), Pritchard and Wilson (2007) and Aleskerov et al. (2011a,b).

All those papers differ in main assumptions about profile probability distributions, a measure of manipulability, tie-breaking assumptions and sets of rules under study. There are several assumptions about individual preferences interdependence, but the most popular are impartial culture (IC) and impartial anonymous culture assumptions (IAC). Under the IC it is assumed that all individual orderings over alternatives are equally possible and individual preferences are independent. Thus, in this model one studies all profiles of preferences which are equally possible. Under IAC one looks only on those profiles which cannot be constructed one from another by changing the order of preferences in a given profile. Those profiles are called voting situations and under IAC it is assumed that these situations are equally possible. IAC is useful when one wants to find an exact formula for the number of manipulable voting situations (Gehrlein and Fishburn 1976). In this work we use Impartial Culture model in order to estimate the degree of manipulability of known voting rules. We use several measures of manipulability including most popular and most native one: the share of all manipulable profiles. This measure is used almost in all papers in this field.

The next important feature is the way to deal with the possibility of multiple choice. For all rules there are some profiles where these rules give a tie as the result of voting. Most of the papers use alphabetical tie-breaking rules: in the case of tie first alternative in alphabetical order is chosen (for example, see Nitzan 1985, Aleskerov and Kurbanov 1999, Favardin and Lepelley 2006). The deficiency of this method is that it breaks symmetry between the alternatives because first alternatives in alphabetical order have more chances to be selected as the final outcome. Pritchard and Wilson (2007) use the random tie-breaking rule where in the case of a tie the final outcome is choosing randomly. In this case one can compare some sets of alternatives using some stochastic order. Voting rules in a more general framework of multiple choice were studied in Aleskerov et al. (2011a,b). In our research we use the same model and estimate the degree of manipulability of ten positional voting rules in the case of multiple choice.

The structure of the paper is as follows. Section 2 introduces the basic notation and concepts. Section 3 presents the indices to measure the degree of manipulability of social choice rules and explains the computational scheme. Section 4 presents the social choice rules under study. Section 5 presents and discusses the results.

2 The framework

Here we use the same notations as in Aleskerov and Kurbanov (1999) and almost the same model as in Aleskerov et al. (2011b). We consider a finite set A consisting of m alternatives, $m = 3,\ 4$. Let $\mathsf{A} = 2^A \setminus \{\varnothing\}$ denote the set of all non-empty subsets of A. Each agent from a finite set $N = \{1, \ldots, n\},\ n > 1$, is assumed to have a *preference* $P_i \in \mathsf{L}$ over alternatives where L is the set of linear orders on A.

An ordered n-tuple of preferences P_i is called a *(preference) profile*, $\vec{P}$. A group decision is made by a social choice rule based on $\vec{P}$ and is considered to be an element of A. Thus we define a *social choice rule* as a mapping $C : \mathsf{L}^n \to \mathsf{A}$.

Every agent i is assumed to have an *extended preference* EP_i over A which is induced by her preference P_i over A.

There are many preference extension axioms. One can find them, for example, in Barbera (1977), Gärdenfors (1976) and Kelly (1977). The detailed survey can be found in Barbera et al. (2004). In this paper we use the concepts of weak and strong manipulation. In the weak manipulation case we assume that not all possible sets can be compared by an agent. In this paper to describe the weak manipulation case we use Kelly's Dominance Axiom (strong version) introduced in Kelly (1977) and presented according to Pattanaik (1978).

Kelly's Dominance Axiom (strong version) $\forall i \in N$ *and* $\forall \vec{P},\ \vec{P}' \in \mathrm{L}^n$, *if* $[(\forall x \in C(\vec{P})$ *and* $\forall y \in C(\overrightarrow{P'}) \Rightarrow\ xP_iy$ *or* $x = y)$ *and* $(\exists z \in C(\overrightarrow{P})$ *and* $\exists w \in C(\overrightarrow{P'})] \Rightarrow zP_iw)]$ *then* $C(\overrightarrow{P})EP_iC(\overrightarrow{P'})$.

For the strong manipulation case we use several concepts. First of all we consider two methods to obtain EP_i from P_i, both of which are based on lexicographic comparisons used by Pattanaik (1978). The methods we consider are the leximax and leximin extensions, as described by Ozyurt and Sanver (2009).

Under the leximax extension, two sets are compared according to their best elements. If they are the same, then the ordering is made according to the second best elements, etc. The elements according to which the sets are compared will disagree at some step—except possibly when one set is a subset of the other, in which case the smaller set is preferred. Formally, take any $P_i \in \mathsf{L}$ and any distinct $X, Y \in \mathsf{A}$. Write $X = \{x_1, \ldots, x_{|X|}\}$, $Y = \{y_1, \ldots, y_{|Y|}\}$ and let, without loss of generality, $\forall j \in \{1, \ldots, |X| - 1\}, x_{j+1}P_ix_j$ and $\forall j \in \{1, \ldots, |Y| - 1\}$ $y_{j+1}P_iy_j$. The leximax extended preference EP_i is defined as follows

1. If $|X| = |Y|$, then $X\ EP_i\ Y$ iff $x_hP_iy_h$ for the smallest $h \in \{1, \ldots, k\}$ for which $x_h \neq y_h$.
2. If $|X| \neq |Y|$ and $\exists h \in \{1, \ldots, \min\{|X|, |Y|\}\}$ for which $x_h \neq y_h$, then XEP_iY iff $x_hP_iy_h$ for the smallest $h \in \{1, \ldots, \min\{|X|, |Y|\}\}$ for which $x_h \neq y_h$.
3. If $|X| \neq |Y|$ and $\forall h \in \{1, \ldots, \min\{|X|, |Y|\}\}$ $x_h = y_h$ then XEP_iY iff $|X| < |Y|$.

The concept of the leximin extension is defined similarly so that it is based on the ordering of two sets according to a lexicographic comparison of their worst elements. Again the elements according to which the sets are compared will disagree at some step—except possibly when one set is a subset of the other, in which case the larger set is preferred. So, given any $P_i \in \mathsf{L}$ and any distinct $X, Y \in \mathsf{A}$, where $X = \{x_1, \ldots, x_{|X|}\}$ and $Y = \{y_1, \ldots, y_{|Y|}\}$ are such that $\forall j \in \{1, \ldots, |X| - 1\}$ $x_{j+1}P_ix_j$ and $\forall j \in \{1, \ldots, |Y| - 1\}$ $y_{j+1}P_iy_j$, the leximin extended preference EP_i is defined as follows

1. If $|X| = |Y|$, then $X\ EP_i\ Y$ iff $x_hP_iy_h$ for the greatest $h \in \{1, \ldots, k\}$ for which $x_h \neq y_h$.
2. If $|X| \neq |Y|$ and $\exists h \in \{1, \ldots, \min\{|X|, |Y|\}\}$ for which $x_h \neq y_h$, then $X\ EP_i\ Y$ iff $x_hP_iy_h$ for the smallest $h \in \{1, \ldots, \min\{|X|, |Y|\}\}$ for which $x_h \neq y_h$.

3. If $|X| \neq |Y|$ and $x_h = y_h \quad \forall h \in \{1, \ldots, \min\{|X|, |Y|\}\}$ then $X \; EP_i \; Y$ iff $|X| > |Y|$.

We also introduce two probabilistic methods of preference extension.

In contrast to lexicographic methods, these methods of preferences extension suggest that for a voter not only the presence of the alternative in a social choice is important, but the probability that this alternative would be the final outcome is important as well. Here two algorithms are considered: an ordering is constructed based on the probability of the best alternative and an ordering is constructed based on the probability of the worst alternative.

Ordering based on the probability of the best alternative is produced on the element-wise comparison of two social choices. If the best alternatives of two sets are the same, then the set, in which the probability that this alternative would be the final outcome is higher, is more preferable. In fact, it will be the smaller set. If the best alternatives are the same and have equal probability to be the final outcome, then next alternatives are compared in the same way.

Example In the set $\{a, b, c\}$ the probability that alternative a would be the final outcome equals $\frac{1}{3}$ (we assume that each alternative of the winning set has an equal probability to be chosen as the final outcome). In the set $\{a, c\}$ this probability equals $\frac{1}{2}$. In other words, if the preference over alternatives is $aP_i bP_i c$, in the extended preference based on the probability of the best alternative algorithm these sets are ordered as $\{a, c\}EP_i\{a, b, c\}$.

Let us describe this method formally. From the preferences $P_i \in L$ we can get extended preferences EP_i based on the probability of the best alternative by the following algorithm.

Two social choices $X, Y \in \mathsf{A}$ are compared. Let us sort alternatives from each social choice from the most preferred to the least one, i.e., let $X = \{x_1, \ldots, x_{|X|}\}$ and $Y = \{y_1, \ldots, y_{|Y|}\}$, where $\forall j \in \{1, \ldots, |X| - 1\} \; x_j P_i x_{j+1}$ and $\forall j \in \{1, \ldots, |Y| - 1\} \; y_j P_i y_{j+1}$. We put

- If $x_1 P_i y_1$, then $X \; EP_i \; Y$.
- If $x_1 = y_1$ and $|X| < |Y|$, then $X \; EP_i \; Y$.
- If $x_1 = y_1$ and $|X| = |Y| = k$, where $k \in \{2, \ldots, m - 1\}$, then $X \; EP_i \; Y$ if and only if $x_h P_i y_h$ for the least $h \in \{2, \ldots, k\}$ for which $x_h \neq y_h$.

For example, for three alternatives and the preference relation $aP_i bP_i c$ over them, the extended preferences EP_i based on the probability of the best alternative are

$$\{a\}EP_i\{a, b\}EP_i\{a, c\}EP_i\{a, b, c\}EP_i\{b\}EP_i\{b, c\}EP_i\{c\}$$

The ordering based on the probability of the worst alternative is similar to the previous one, but in this case the probability of the worst alternative is considered. The set in which this probability is higher is less preferable.

Let us give it formally. Two social choices $X, Y \in \mathsf{A}$ are compared. Let us sort alternatives from each social choice from the most preferred to the least one, i.e., $X = \{x_1, \ldots, x_{|X|}\}$ and $Y = \{y_1, \ldots, y_{|Y|}\}$, where $\forall j \in \{1, \ldots, |X| - 1\} \; x_j P_i x_{j+1}$ and $\forall j \in \{1, \ldots, |Y| - 1\} \; y_j P_i y_{j+1}$. We put

- If $x_{|X|} P_i y_{|Y|}$, then $X \ EP_i \ Y$.
- If $x_{|X|} = y_{|Y|}$ and $|X| > |Y|$, then $X \ EP_i \ Y$.
- If $x_{|X|} = y_{|Y|}$ and $|X| = |Y| = k$, where $k \in \{2, \ldots, m-1\}$, then $X \ EP_i \ Y$ if and only if $x_h P_i y_h$ for the least $h \in \{2, \ldots, k\}$ for which $x_h \neq y_h$.

For example, for 3 alternatives and preferences $aP_i bP_i c$ over them, extended preferences EP_i based on the probability of the worst alternative will be

$$\{a\}EP_i\{a, b\}EP_i\{b\}EP_i\{a, b, c\}EP_i\{a, c\}EP_i\{b, c\}EP_i\{c\}$$

3 Manipulability indices and computation scheme

Number of alternatives being m, the total number of possible linear orders is equal to $m!$, and the total number of profiles with n agents is equal to $(m!)^n$. Nitzan (1985) introduces the following index, which was also used by Kelly (1993). We call this index as Nitzan–Kelly's index and denote as *NK*, to measure the degree of manipulability of social choice rules

$$NK = \frac{d_0}{(m!)^n},$$

where d_0 is the number of profiles in which manipulation takes place.

Aleskerov and Kurbanov (1999) introduce an index to measure *the freedom of manipulation*. In Aleskerov et al. (2011a) we introduced two similar indices: *the degree of nonsensitivity to a preference change* and *the probability of getting worse*. Here we also introduce *the degree of an uncertain change*. This index is used here because we consider the case of weak manipulation, where not all outcomes of voting can be compared. Let us note that for an agent there are $(m! - 1)$ linear orders to use instead of her sincere preference. Denote as κ_{ij}^{+} $(i = 1, \ldots, n;\ 0 \leq \kappa_{ij}^{+} \leq m! - 1)$ the number of orderings in which voter i is better off in the jth profile. Similarly, κ_{ij}^{0} is the number of orderings in which the result of voting remains the same, κ_{ij}^{-} is the number of orderings in which the voter is worse off and $\kappa_{ij}^{?}$ is the number of orderings in which the result of voting changes to the outcome incomparable by the given extension axiom.[1] It is obvious that $\kappa_{ij}^{+} + \kappa_{ij}^{0} + \kappa_{ij}^{-} + \kappa_{ij}^{?} = (m! - 1)$. Dividing each κ_{ij} by $(m! - 1)$ one can find the share of each type of orderings for an agent i in the jth profile. Summing up each share over all agents and dividing it by n one can find the average share in the given profile. Summing the share over all profiles and dividing this sum to $(m!)^n$ we obtain four indices

$$I_1^{+} = \frac{\sum_{j=1}^{(m!)^n} \sum_{i=1}^{n} \kappa_{ij}^{+}}{(m!)^n \cdot n \cdot (m! - 1)}; \quad I_1^{0} = \frac{\sum_{j=1}^{(m!)^n} \sum_{i=1}^{n} \kappa_{ij}^{0}}{(m!)^n \cdot n \cdot (m! - 1)};$$
$$I_1^{-} = -\frac{\sum_{j=1}^{(m!)^n} \sum_{i=1}^{n} \kappa_{ij}^{-}}{(m!)^n \cdot n \cdot (m! - 1)}; \quad I_1^{?} = \frac{\sum_{j=1}^{(m!)^n} \sum_{i=1}^{n} \kappa_{ij}^{?}}{(m!)^n \cdot n \cdot (m! - 1)}.$$

[1] The last number is always equal to zero in the case of strong manipulation because all sets can be compared.

It is obvious that $I_1^+ + I_1^0 + I_1^- + I_1^? = 1$.

We performed the calculation of indices for 3 and 4 alternatives. For 3, 4 and 5 voters, the respective indices are computed exhaustively (i.e., all possible profiles are checked for the manipulability), and for larger number of voters the statistical scheme is used.

In both exhaustive and statistical schemes, for each profile under consideration, all $(m! - 1)$ manipulating orderings for each voter are generated and the respective choice sets of manipulating profiles are compared with the choice of the original profile.

All indices were calculated for the rules defined in the next session.

4 Voting rules

We consider the following ten social choice rules.

1. *Plurality Rule* Choose alternatives that are ranked first by the maximum number of agents, i.e.

$$a \in C(\vec{P}) \Leftrightarrow [\forall x \in A \; n^+(a, \vec{P}) \geq n^+(x, \vec{P})],$$

where $n^+(a, \vec{P}) = card\{i \in N | \forall y \in A \; aP_i y\}$

2. *q-Approval* Let us define

$$n^+(a, \vec{P}, q) = card\{i \in N | card\{D_i(a)\} \leq q - 1\},$$

where $D_i(a) = \{y \in A : yP_i a\}$ is the upper contour set of $a \in A$ in $P_i \in \mathsf{L}$. Let $n^+(a, \vec{P}, q)$ be the number of agents for which a is ranked among the first q alternatives in their preference ordering. The integer q can be called as the degree of the procedure. We define q-Approval as follows

$$a \in C(\vec{P}) \Leftrightarrow [\forall x \in A \; n^+(a, \vec{P}, q) \geq n^+(x, \vec{P}, q)],$$

i.e., the alternatives which are admitted to be among the q best by the highest number of agents are chosen. It can be easily seen that Plurality Rule is a special case of q-Approval where $q = 1$.

3. *Borda's Rule* Let $r_i(x, \vec{P})$ be the cardinality of the lower contour set of $x \in A$ in $P_i \in \vec{P}$, i.e. $r_i(x, \vec{P}) = |L_i(x)| = |\{b \in A : xP_i b\}|$. The sum of $r_i(x, \vec{P})$ over all $i \in N$ is called the Borda score of alternative a.

$$r(a, \vec{P}) = \sum_{i=1}^{n} r_i(a, P_i).$$

The alternatives with maximum Borda score are chosen., i.e.

$$a \in C(\vec{P}) \Leftrightarrow [\forall b \in A, \; r(a, \vec{P}) \geq r(b, \vec{P})].$$

4. *Black's Procedure* Let us define the majority relation μ for a given profile $\vec{P}$

$$x\mu y \Leftrightarrow card\{i \in N|xP_i y\} > card\{i \in N|yP_i x\}.$$

Condorcet winner $CW(\vec{P})$ in the profile $\vec{P}$ is an element undominated in the majority relation μ (constructed according to the profile), i.e.

$$CW(\vec{P}) = [a|\neg\exists x \in A,\ x\mu a]$$

Black's rule picks the unique Condorcet winner if it exists and the Borda winner(s) otherwise.

5. *Threshold rule* (Aleskerov et al. 2010) Let $v_1(x)$ be the number of agents for which the alternative x is the worst in their ordering, $v_2(x)$—is the number of agents placing x the second worst, and so on, $v_m(x)$—the number of agents considering the alternative x as their best one. Then we order the alternatives lexicographically. The alternative x is said to V-dominate the alternative y if $v_1(x) < v_1(y)$ or, if there exists k not more than m, s.t. $v_i(x) = v_i(y),\ i = 1, \ldots, k-1$, and $v_k(x) < v_k(y)$. In other words, first, the number of worst places are compared, if these numbers are equal then the number of second worst places are compared and so on. The alternatives which are not dominated by other alternatives via V are chosen.

6. *Hare's Procedure* First, if an alternative is chosen by a simple majority of voters, then this alternative is chosen, and the procedure stops. Otherwise, the alternative a with the minimum number of votes is omitted. Then the procedure is applied to the set $X = A\backslash\{a\}$ and to the profile $\vec{P}/X$ until the alternative ranked first by a simple majority is found.

7. *Antiplurality Rule* The alternative, which is regarded as the worst by the minimum number of agents, is chosen, i.e.,

$$a \in C(\vec{P}) \Leftrightarrow [\forall x \in A\ n^-(a, \vec{P}) \leq n^-(x, \vec{P})],$$

where $n^-(a, \vec{P}) = card\{i \in N|\forall y \in A\ yP_i a\}$.

8. *Inverse Borda's Procedure* For each alternative Borda's count is calculated. Then the alternative a with the minimum count is omitted. Borda's count are re-calculated for profile $\vec{P}/X$, $X = A\backslash\{a\}$, and procedure is repeated until choice is found.

9. *Nanson's Procedure (modified)*[2] For each alternative Borda's count is calculated. Then average count is calculated, $\bar{r} = (\sum_{a\in A} r(a, \vec{P}))/|A|$, and alternatives $c \in A$ are omitted for which $r(c, \vec{P}) < \bar{r}$. Then the set $X = \{a \in A : r(a, \vec{P}) \geq \bar{r}\}$ is considered, and the procedure is applied to the profile $\vec{P}/X$. Such procedure is repeated until choice set will not be empty.

10. *Coombs' Procedure* Alternative a which is the worst for the maximum number of agents is omitted. Then the profile is contracted to the $\vec{P}/X$, $X = A\backslash\{a\}$, and the procedure is repeated until the choice set will not be empty.

[2] As anonymous referee pointed out, in original Nanson's rule alternatives with the average Borda score are also eliminated.

5 Results

When we use all preferences extension methods defined above for three alternatives aP_ibP_ic we have four linear extended orderings

1. (**Leximin3**) $\{a\}EP_i\{a,b\}EP_i\{b\}EP_i\{a,c\}EP_i\{a,b,c\}EP_i\{b,c\}EP_i\{c\}$
2. (**Leximax3**) $\{a\}EP_i\{a,b\}EP_i\{a,b,c\}EP_i\{a,c\}EP_i\{b\}EP_i\{b,c\}EP_i\{c\}$
3. (**PWorst3**) $\{a\}EP_i\{a,b\}EP_i\{b\}EP_i\{a,b,c\}EP_i\{a,c\}EP_i\{b,c\}EP_i\{c\}$
4. (**PBest3**) $\{a\}EP_i\{a,b\}EP_i\{a,c\}EP_i\{a,b,c\}EP_i\{b\}EP_i\{b,c\}EP_i\{c\}$

For Kelly's Dominance Axiom we have only the following relations in the extended preferences

$$\textbf{(KellyDA3)}\ \begin{array}{l}\{a\}EP_i\{a,b\}EP_i\{b\}EP_i\{b,c\}EP_i\{c\}\\ \{a\}EP_i\{a,c\}EP_i\{c\}\\ \{a\}EP_i\{a,b,c\}EP_i\{c\}\end{array}$$

In Tables 1 and 2 the results of the NK index calculation for 3 alternatives and 3 and 4 voters are given. We also provide here the results from our previous papers. For comparison with the case of the single-valued choice, we provide in the TBR column, the results for alphabetical tie-breaking rule. For all rules except the Threshold rule the same results were obtained in Aleskerov and Kurbanov (1999).

As it is seen from the Tables the degree of manipulability of most social choice rules is underestimated in the case of the alphabetical tie-breaking rule. It is easy to show an example of profile which is manipulable for the extended preferences and not manipulable for tie-breaking framework. For the case of a random tie-breaking there are the results from Pritchard and Wilson (2007) for the first three rules. An interesting fact is that the results coincide with the results for KellyDA3. It can be explained by the fact that the algorithm used for the random tie-breaking mechanism gives the similar extended preferences as KellyDA for 3 alternatives.

Table 1 NK index for 3 alternatives and 3 voters

Rule	Extension					
	Leximin3	Leximax3	PWorst3	PBest3	KellyDA3	TBR
Plurality	0.2222	0	0.2222	0	0	0.1667
q-Approval q = 2	0.1111	0.6111	0.1111	0.6111	0.1111	0.2639
Borda	0.3056	0.4167	0.3056	0.4167	0.25	0.2361
Black	0.0556	0.1667	0.0556	0.1667	0	0.1111
Threshold	0.3056	0.4167	0.3056	0.4167	0.25	0.3611
Hare	0.2222	0	0.2222	0	0	0.1111
Inverse Borda	0.0556	0.1667	0.0556	0.1667	0	0.1111
Nanson	0.0556	0.1667	0.0556	0.1667	0	0.1111
Coombs	0.2222	0.5000	0.2222	0.5000	0.1667	0.2222

Table 2 NK index for 3 alternatives and 4 voters

Rule	Extension					
	Leximin3	Leximax3	PWorst3	PBest3	KellyDA3	TBR
Plurality	0.3333	0.3333	0.3333	0.3333	0.3333	0.1852
q-Approval q = 2	0.2963	0.2963	0.2963	0.2963	0.2963	0.2755
Borda	0.3611	0.4028	0.3611	0.4028	0.2917	0.3102
Black	0.2361	0.2778	0.2778	0.2361	0.1667	0.1435
Threshold	0.4028	0.4028	0.4028	0.4028	0.4028	0.3380
Hare	0.3333	0.3333	0.3333	0.3333	0.3333	0.0926
Inverse Borda	0.2361	0.2778	0.2778	0.2361	0.1667	0.1435
Nanson	0.2361	0.2778	0.2778	0.2361	0.1667	0.1435
Coombs	0.2778	0.2778	0.2778	0.2778	0.2778	0.2222

Also an interesting result is that NK index for lexicographic methods is equal to the same index for probabilistic methods. To be precise, for most rules and for 3 voters case the index for Leximax3 is equal to PBest3, and the index for Leximin3 is equal to PWorst3. One can see that Leximax3 and PBest3, as well as Leximin3 and PWorst3 differ only on the pairs $\{a, c\}$ and $\{a, b, c\}$. These results imply that a manipulation between these sets is not recognized for the rules, for which the NK indices are equal for Leximax3 and PBest3 (or Leximin3 and PWorst3).

The results for KellyDA3 and for the strong manipulation case are almost the same or differ a bit for most rules. This means that relations added by stronger methods do not strongly influence the results because manipulation between incomparable sets is not often possible. We will show this by introducing the results for $I_1^?$ index.

In Figs. 1 and 2 NK index for PWorst3 and PBest3, respectively, is shown. On X-axis the logarithm of the number of voters is given. We calculate all indices for 3 to 25 voters and then 29, 30, 39, 40 and so on up to 100. That explains such strange behavior of the index. For easy presentation we provide only figures for five rules: rules 6, 8–10 from the list given above and Black's procedure. We do not give the results for Antiplurality rule because for 3 alternatives it is the same as q-Approval voting for q = 2. We already presented the results for rules 1–5 and Leximax3 and Leximin3 (which are similar to the probabilistic methods as we show above) in Aleskerov et al. (2011a) and Black's procedure was the least manipulable rule in almost all cases.

As it is seen from Figs. 1 and 2, the behavior of the index depends on the rule considered. For rules 6–10 there is a period of 2 in the index changes for Inverse Borda and Nanson rule, and a period of 6 for the Hare and Coombs rules. Antiplurality rule (not shown) as mentioned above coincides with q-Approval q = 2 and has the period of 3 (the number of alternatives). An interesting result is that the length of the period for the Hare and Coombs rules remains unchanged when we consider 4 alternatives. This very result gives an insight to algebraic properties of the rules and it might help in an analytical study of manipulability of voting rules. In Fig. 3 NK index for PWorst4 is given.

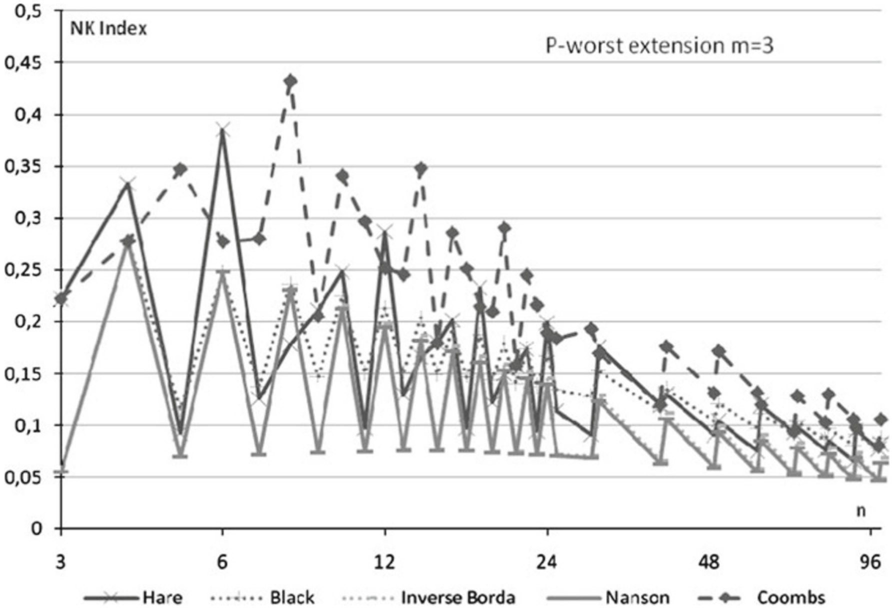

Fig. 1 NK index for PWorst3

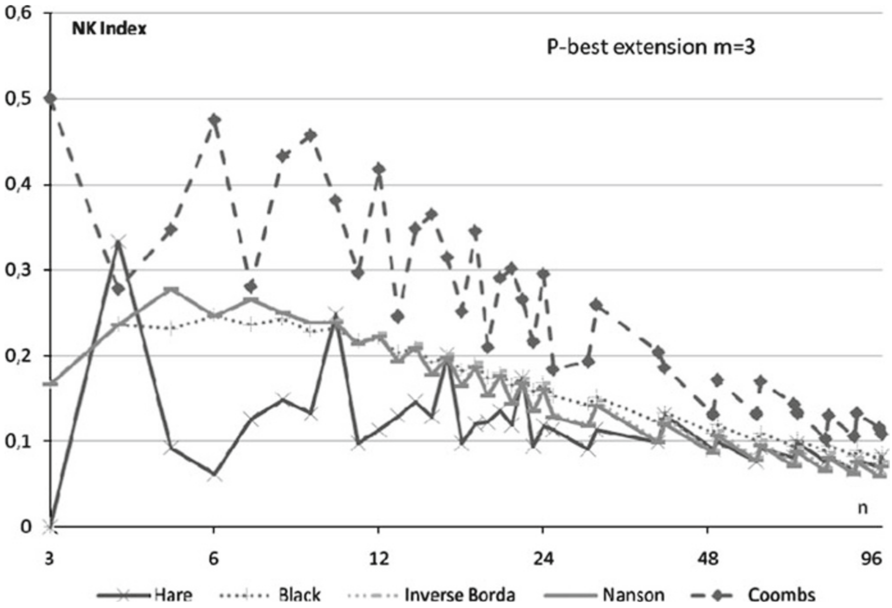

Fig. 2 NK index for PBest3

From these figures we observe that the relative position of the rules from the NK index point of view is quite similar for PWorst3. The periods in index changes remain the same for all shown rules except Antiplurality. Now the period of this rule is 4.

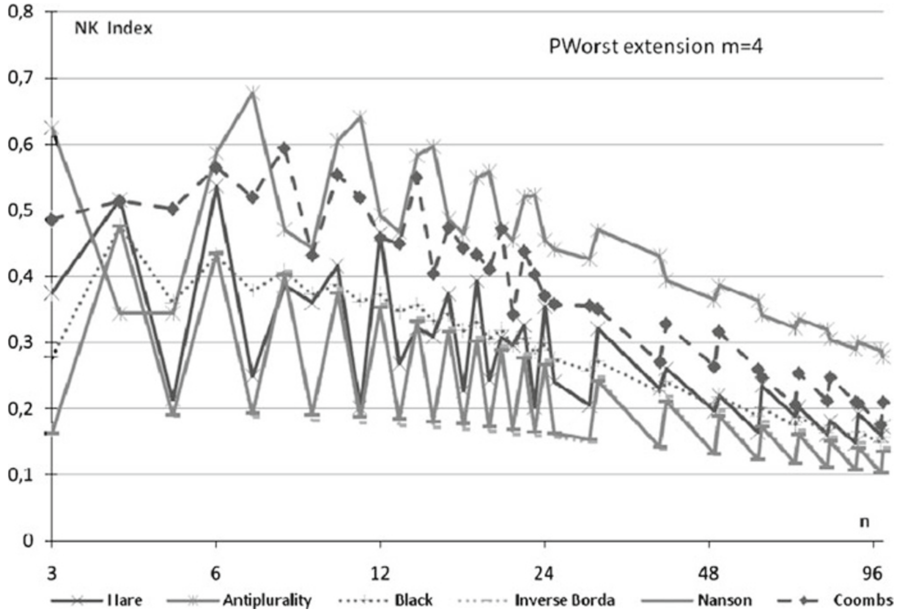

Fig. 3 NK index for PWorst4

An interesting result is that the Inverse Borda rule becomes less manipulable than Nanson's rule for odd number of voters.

Let us return to the case of 3 alternatives to compare the results of strong and weak manipulation. In Fig. 4 NK index for KellyDA3 is given.

As one can see the rules have the same behavior (specially the index periodicity). The only main difference is the value of index. We can interpret NK index for KellyDA3 as the degree of minimal manipulation (for all methods of strong manipulation NK index is higher or at least the same).

We can summarize the results in the following tables. Tables 3 and 4 show the least manipulable rules in the Nitzan–Kelly sense for 3 and 4 alternatives, respectively.

We can outline two main results from the tables and figures. First of all, the least manipulable rule in most cases depends on extension axiom used. But in some cases we can find the least manipulable rule. For example, for 3 alternatives and 8, 14 or 20 voters the best rule for every method is Hare's rule. Nanson's rule is the best one for 16 voters.

In Aleskerov et al. (2011a) five rules were studied—Plurality rule, Approval with q = 2, Borda rule, Threshold rule and Black's procedure. It turned out that for all extension methods and almost all number of voters the least manipulable rule was Black's procedure. However, now it is the least manipulable only in few cases. Nanson's and Hare's rules are the best rules in Nitzan–Kelly sense.

As it is seen from Table 4 Nanson's and Inverse Borda rules appear in a switching manner in the table. The difference between them is defined on the fourth decimal of the value of NK index. We believe that more detailed study is needed.

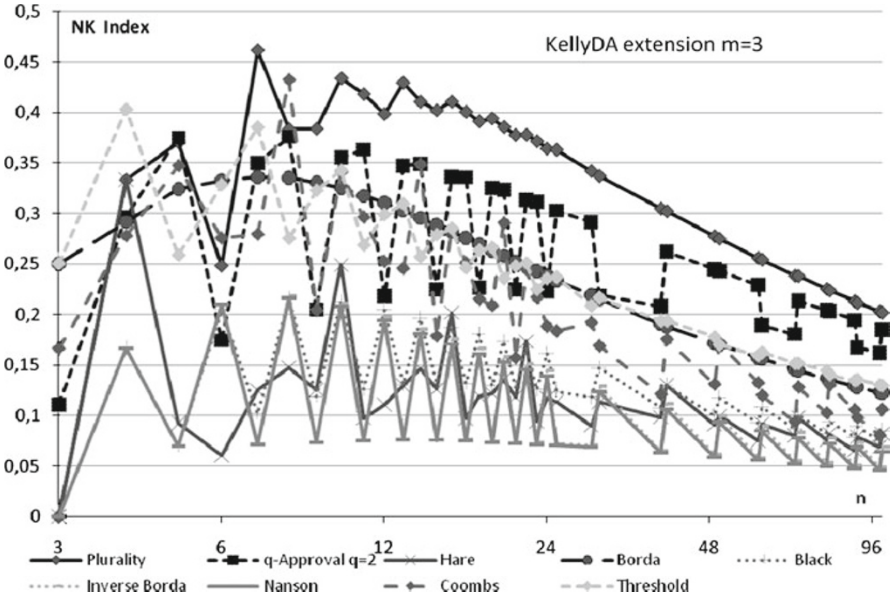

Fig. 4 NK index for KellyDA3

Table 3 The least manipulable rules according to NK index and 3 alternatives

Method	Number of voters																	
	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Leximin3	Bl IB N	Bl IB N	IB N	H	IB N	H	IB N	N	N	H	N	H	N	N	N	N	N	H
Leximax3	P H	Bl IB N C	H	Bl	H	H	H	Bl	H	Bl	H	H	H	N	H	N	H	H
PWorst3	Bl IB N	Bl IB N C	IB N	2-A	IB N	H	IB N	N	N	N	N	H	N	N	N	N	N	H
PBest3	P H	Bl IB N	H	H	H	H	H	Bl	H	H	H	H	H	N	H	H	H	H
KellyDA3	P H Bl IB N	Bl IB N	Bl IB N	H	IB N	H	IB N	N	N	H	N	H	N	N	N	H	N	H

Table 4 The least manipulable rules according to NK index and 4 alternatives

Method	Number of voters																	
	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Leximin4	IB N	A	IB	H	IB	H	IB	N	IB	H	IB	H	IB	N	IB	N	IB	N
Leximax4	P H	P H	H	Bl	H	H	H	N	H	N	H	H	H	N	H	N	H	H
PWorst4	IB N	A	IB	Bl	IB	H	IB	N	IB	N	IB	H	IB	N	IB	N	IB	N
PBest4	P H	Bl IB N	H	H	H	H	H	N	H	H	H	H	H	N	H	H	H	H
KellyDA4	P H	Bl IB N	IB	H	IB	H	IB	N	IB	H	IB	H	IB	N	IB	H	IB	H

P Plurality, *B1* Black, *IB* Inverse Borda, *2-A* q-Approval q = 2, *A* Antiplurality, *N* Nanson, *H* Hare, *C* Coombs

In order to find the most suitable rule to implement in most cases we should use additional criteria. One of them can be another way to compare rules from the freedom of manipulation point of view. The results for I_1^+ index and PWorst3 are presented in Fig. 5.

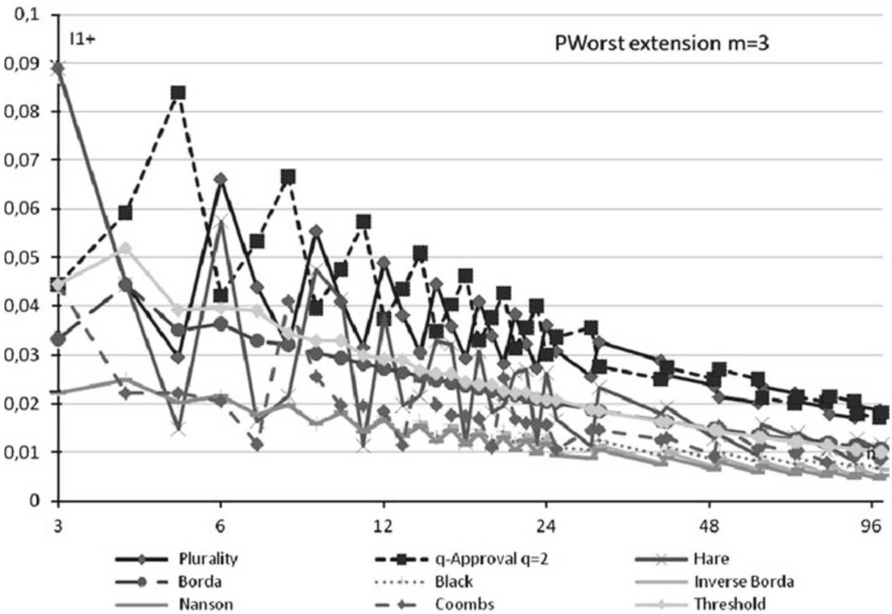

Fig. 5 I_1^+ index for PWorst3

It is important to note that this index in some sense addresses the problem to what extent a manipulation is hard to implement. The lesser is the freedom of manipulation the harder it is to find the way to manipulate. As one can see from the figures Coombs' rule has smaller freedom of manipulation than Hare's rule in most cases. The results here also depend on the extension axiom used. The results are summarized in the following table.

An interesting result here is that Nanson's rule has the least freedom of manipulation when the number of voters is at least 14. Although there is no dominating rule for small number of voters, for 4, 7, 9–11, 13–20 the results do not depend on the extension method used for the case of strong manipulation.

Another useful index is $I_1^?$. As we already mentioned above, using $I_1^?$ index we can explain in some way why the degree of manipulability for the weak and strong manipulation does not differ a lot. In Table 6 the calculation of all I_1 indices for Nanson's rule, 3 alternatives, 4 voters and all extension methods is given. For all four methods of the strong manipulation we have the same results for this number of voters, so they are grouped together.

Table 5 The least manipulable rules according to I_1^+ index and 3 alternatives

Method	Number of voters																	
	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Leximin3	Bl IB N	C	H	IB N	C	N	IB N	N	H	N	C	N	N	N	N	N	N	N
Leximax3	P H	C	H Bl IB N	P	C	H	IB N	N	H	N	C	N	N	N	N	N	N	N
PWorst3	Bl IB N	C	H	C	C	N	IB N	N	H	N	C	N	N	N	N	N	N	N
PBest3	P H	C	H Bl IB N	H	C	H	IB N	N	H	H	C	N	N	N	N	N	N	N

Table 6 I_1 index for Nanson's rule

Type	Index			
	I_1^+ (%)	I_1^0 (%)	$I_1^?$	I_1^- (%)
Weak manipulation	1.11	33.89	10.28%	54.72
Strong manipulation	2.50	33.89	–	63.61

One can see that the value of $I_1^?$ in KellyDA3 is mainly added to the I_1^- value when we use stronger axioms. Moreover, $I_1^?$ is rather small, that is why the results of the strong and weak manipulability do not differ a lot.

In this paper we have compared ten different positional rules from their vulnerability to manipulation point of view using different measures and extension axioms. We show that there is no rule which dominates the others for all extension methods, but from several points of view Nanson's and Hare's rules are the least manipulable. It is important to note that if we add additional rules to our analysis they can outperform these rules in terms of manipulabilty.

Acknowledgements The work of Fuad Aleskerov and Daniel Karabekyan is partially supported by the Scientific Foundation of the Higher School of Economics (grants # 08-04-0008 and # 10-04-0030), Russian Foundation for Basic Research (grant # 08-01-00039a) and Laboratory DECAN of Higher School of Economics. Remzi Sanver acknowledges the support of the Turkish Academy of Sciences Distinguished Young Scientist Award Program (TUBA-GEBIP) and the Scientific and Technological Research Council of Turkey (TUBITAK) through the project # 107K560. The work of Vyacheslav Yakuba is also supported by Russian Foundation for Basic Research (grant # 08-01-00039a). William Zwicker took part at the very beginning of this project. His ideas and comments were very useful. We also thank an anonymous referee, whose comments allow us to improve the text.

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