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Delay in finite time capital accumulation

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Abstract This paper compares two classes of time-to-install/deliver and time-to build models in a finite time framework. We show that in case of no salvage value the two problems are equivalent. However, if a salvage value is added, this property is typically lost.

Keywords Capital accumulation · Delayed response · Time-to-build · Time-to-install/deliver · Optimal control

1 Introduction

This paper studies optimal firm behavior in accumulating capital, where the objective is to maximize the discounted profit stream. The profit rate equals the difference between the revenue and the costs of investment. Revenue is obtained by selling goods on the market, where a capital stock is needed to produce these goods. The higher the capital stock it owns, the more goods the firm produces, which in turn leads to higher revenue. The firm can increase capital stock by investing.

To study this problem one typically designs an optimal control model with capital stock as a state variable and investment as a control variable. The first contribution in

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this area is Eisner and Strotz (1963), in which the revenue function was assumed to be concave whereas the investment cost function was convex. Rothschild (1971) argued that arguments could be found in favor of a (partly) concave shape of the investment cost function (see also Davidson and Harris (1981), and Jorgensen and Kort (1993)). Also for the revenue function it can be reasonable to analyze other than strictly concave functions. Revenue functions that are partly convexly shaped typically lead to history dependent (Skiba!) solutions, as shown in, e.g., Dechert (1983), Davidson and Harris (1981), and Hartl and Kort (2004).

All the above mentioned contributions contain capital accumulation models without delays. This is striking inasmuch as there usually is a delay between the decision to launch a capital project and when that investment first bears fruit, whether the investment is in physical assets, such as building a factory, knowledge assets, such as inventing a new technology or product, or human capital, such as raising the educational level of one's workforce. An appropriate theory of control systems with delay is available; see e.g. Kolmanovskii (1992).

A paper on capital accumulation, that does take into account delays is Caulkins et al. (2010). There it was shown that some classes of problems of time-to-install/deliver and time-to-build are equivalent within an infinite time horizon problem. The present paper looks at this problem while the horizon date is finite. It will be shown that the delay equivalence still holds when no salvage value is considered, while the delay equivalence is typically lost when a salvage value is added to the objective. We illustrate the results with some examples.

2 Time-to-install/deliver

2.1 The model

In the first problem considered here, the firm has to spend money on investment a fixed time before the corresponding capital goods can be delivered or installed. This results in a capital accumulation model with control delay, in which I denotes investment, K is the capital stock. $R(K)$ is the revenue from producing with capital stock K, and $C(I)$ are the investment costs:

$$J = \int_0^T e^{-rt} \left(R\left(K\left(t\right)\right) - C\left(I\left(t\right)\right)\right) dt + e^{-rT} sK\left(T\right) \to \max_I, \tag{1}$$

$$\dot{K}\left(t\right) = I\left(t-\tau\right) - \delta K\left(t\right), \tag{2}$$

$$I\left(t\right) \geq 0, \tag{3}$$

with r being the discount rate and δ being the depreciation rate. The horizon time T is finite. The salvage value parameter s either equals 0 or 1. The initial conditions are:

$$K\left(0\right) = K_0, \tag{4}$$

$$I\left(t\right) = \bar{I}\left(t\right) \quad \text{for} \quad t \in [-\tau, 0]. \tag{5}$$

Note that, following Winkler (2010), we assume that investment goods have to be paid upon ordering while they are delivered τ units of time later.

2.2 Transformation to a standard control problem

Employing the state-time transformation

$$x(t) = K(t+\tau),$$

the problem can be rewritten as

$$\begin{aligned}
J &= \int_0^T e^{-rt} \left(R\left(x\left(t-\tau\right)\right) - C\left(I\left(t\right)\right)\right) dt + e^{-rT} sx\left(T-\tau\right) \\
&= \int_{-\tau}^{T-\tau} e^{-r(t+\tau)} R\left(x\left(t\right)\right) dt - \int_0^T e^{-rt} C\left(I\left(t\right)\right) dt + e^{-rT} sx\left(T-\tau\right) \to \max_I, \\
\dot{x}\left(t\right) &= I\left(t\right) - \delta x\left(t\right), \\
I\left(t\right) &\geq 0,
\end{aligned}$$

with initial condition

$$x\left(-\tau\right) = K_0.$$

Note that in $[-\tau, 0]$ the control problem is trivial, i.e. the control is given exogenously by $I\left(t\right) = \bar{I}\left(t\right)$ and the state is the solution of the linear ODE $\dot{x}\left(t\right) = \bar{I}\left(t\right) - \delta x\left(t\right)$. Hence, also $x_0 = x\left(0\right)$ is given exogenously. The (exogenously given) profit in this first interval is

$$\pi_{0c} = \int_{-\tau}^{0} e^{-r(t+\tau)} R\left(x\left(t\right)\right) dt = \int_0^{\tau} e^{-rt} R\left(x\left(t-\tau\right)\right) dt.$$

Now the final form of the optimization problem becomes

$$\begin{aligned}
J - \pi_{0c} = &\int_0^{T-\tau} e^{-rt} \left(e^{-r\tau} R\left(x\left(t\right)\right) - C\left(I\left(t\right)\right)\right) dt + e^{-rT} sx\left(T-\tau\right) \\
&- \int_{T-\tau}^{T} e^{-rt} C\left(I\left(t\right)\right) dt \quad \to \quad \max_I, \qquad (6)\\
\dot{x}\left(t\right) = &\; I\left(t\right) - \delta x\left(t\right), \qquad (7)
\end{aligned}$$

$$I(t) \geq 0, \tag{8}$$

$$x_0 = x(0). \tag{9}$$

Clearly, for $t \in (T-\tau, T]$ investing does not generate any benefit and hence $I = 0$ there. This implies that problem (6)–(9) can be rewritten into (with $z = T - \tau$):

$$J - \pi_{0c} = \int_0^z e^{-rt} \left(e^{-r\tau} R(x(t)) - C(I(t)) \right) dt + e^{-rz} e^{-r\tau} s x(z) \rightarrow \max_I, \tag{10}$$

$$\dot{x}(t) = I(t) - \delta x(t), \tag{11}$$

$$I(t) \geq 0, \tag{12}$$

$$x_0 = x(0). \tag{13}$$

Summing up, the problem (1)–(5) with the delay in the control (time-to-install/deliver) can be transformed into a standard (non-delayed) optimal control problem.

Remark 1 Note that also in case the firm has to pay upon delivery, i.e. the term $C(I(t))$ in the objective is replaced by $C(I(t-\tau))$, the problem can be transformed into a standard optimal control problem simply by employing a time shift transformation of the control, i.e. introducing a new control variable u as follows:

$$u(t) = I(t - \tau).$$

2.3 Example

Since the resulting problem is a standard (non-delayed) optimal control problem, it can be analyzed using phase plane analysis provided that the problem is concave in the control, the state is one-dimensional, and the problem is autonomous as assumed above. Clearly, the above transformations would also hold in the non-autonomous case. Rather than carrying out a general phase plane analysis, we focus on the linear quadratic case for simplicity. Hence, we assume that revenue and cost functions are quadratic:

$$R(K) = bK - \frac{a}{2}K^2, \tag{14}$$

$$C(I) = \frac{c}{2}I^2. \tag{15}$$

Then, from the Hamiltonian maximization condition we obtain, as usual, that control and shadow price are proportional:

$$cI = \lambda, \tag{16}$$

and the transversality condition $\lambda(T-\tau) = se^{-r\tau}$ becomes

$$I(T-\tau) = \frac{se^{-r\tau}}{c}. \tag{17}$$

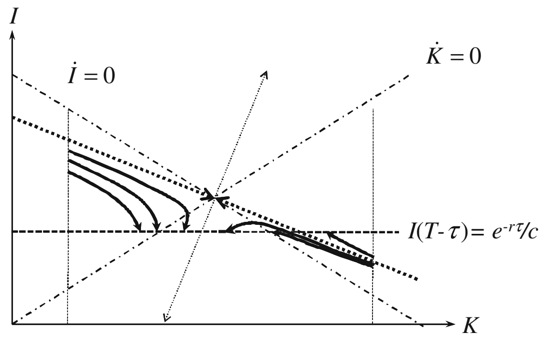


Fig. 1 The solutions in the time-to-install/deliver case with $s = 1$

Furthermore, isoclines and saddle point path (as well as the unstable path) are all straight lines. Hence, we arrive at the saddle point diagram depicted in Fig. 1. The isoclines are the dash-dotted lines; the stable path (saddle point path, optimal solution for infinite horizon) is depicted as the bold dotted line while the unstable path is the thin dotted line. The bold curves are the solutions for finite horizon T. We have sketched solutions for small and large initial capital stock for various different values of T. If the horizon T is short, the trajectory ends far away from the steady state. The larger T is, the more the trajectory approaches the steady state. In Fig. 1, we have plotted the case with *salvage value* $s = 1$. Without salvage value, $s = 0$, the (bold) finite horizon trajectories would end at the K-axis.

Clearly, the monotonicity of the state trajectory, as observed in the infinite horizon case (cf. Caulkins et al. 2010; Hartl 1987) may be lost here.

3 Time-to-build

3.1 The model

Another type of delay, named time-to-build, appeared in Asea and Zak (1999). Here, capital equipment acquired at time t only generates revenue from time $t + \tau$ onwards. In other words, it takes τ units of time to build the marketable product, and the capital stock at time t generates revenue $R(t)$ at time $t + \tau$. Combining this feature with our framework results in the following capital accumulation model with state delay:

$$J = \int_0^T e^{-rt} \left(R\left(K\left(t-\tau\right)\right) - C\left(I\left(t\right)\right)\right) dt + e^{-rT} sK\left(T\right) \to \max_I, \tag{18}$$

$$\dot{K}(t) = I(t) - \delta K(t), \tag{19}$$

$$I(t) \geq 0, \tag{20}$$

$$K(t) = \bar{K}(t) \quad \text{for} \quad t \in [-\tau, 0]. \tag{21}$$

Depreciation starts right after the capital good is ordered. This in particular makes sense when depreciation is driven by technological obsolescence, as with some software products. Alternatively, we could have assumed that depreciation only starts after the capital good becomes active in the production process. This would be the case when depreciation is caused by use, as with most physical production assets. Note that the work in process (WIP) at the horizon date T, that consists of production started in the interval $(T-\tau, T]$ and that has not led to finished goods by time T, does not generate any revenue. Hence, in the interval $(T-\tau, T]$ investment in the capital stock only makes sense in order to increase the salvage value of the terminal capital stock $K(T)$.

Since $K(t)$ cannot be influenced for $t < 0$, the term

$$\pi_{0s} = \int_0^{\tau} e^{-rt} R\left(K\left(t-\tau\right)\right) dt = \int_{-\tau}^{0} e^{-r(t+\tau)} R\left(\bar{K}\left(t\right)\right) dt$$

is a given constant. Hence, the objective can be reformulated as

$$\begin{aligned} J &= \int_0^T e^{-rt}\left(R\left(K\left(t-\tau\right)\right) - C\left(I\left(t\right)\right)\right) dt + e^{-rT} sK\left(T\right) \\ &= \int_{-\tau}^{T-\tau} e^{-r(t+\tau)} R\left(K\left(t\right)\right) dt - \int_0^T e^{-rt} C\left(I\left(t\right)\right) dt + e^{-rT} sK\left(T\right) \\ &= \pi_{0s} + \int_0^{T-\tau} e^{-rt}\left(e^{-r\tau} R\left(K\left(t\right)\right) - C\left(I\left(t\right)\right)\right) dt \\ &\quad - \int_{T-\tau}^{T} e^{-rt} C\left(I\left(t\right)\right) dt + e^{-rT} sK\left(T\right). \end{aligned}$$

The interesting aspect of the problem is that capital stock contributes to the objective in the interval $[0, T-\tau]$ via revenue $R(K)$ in the integral term and also at time T by the salvage value if $s = 1$. It does not, however, generate any positive benefits in the interval $(T-\tau, T)$.

On the one hand, if there is *no salvage value*, i.e. $s = 0$, it is easy to see that investment will be zero on the interval $(T-\tau, T)$. It would only incur costs and increase the capital stock that becomes productive only after the horizon date T. The resulting optimal control problem

$$J - \pi_{0s} = \int_0^{T-\tau} e^{-rt}\left(e^{-r\tau} R\left(K\left(t\right)\right) - C\left(I\left(t\right)\right)\right) dt \to \max_I, \qquad (22)$$

$$\dot{K}(t) = I(t) - \delta K(t),$$
$$I(t) \geq 0, \tag{23}$$
$$K(0) = \bar{K}(0),$$

is again a standard optimal control problem without delay. Analogous to Caulkins et al. (2010), it can be shown to be equivalent with problem (10)–(13) for $s = 0$ in the previous section.

On the other hand, if there is *a salvage value*, i.e. $s = 1$, the resulting standard optimal control problem is

$$J - \pi_{0s} = \int_0^T e^{-rt} \left(e^{-r\tau} \tilde{R}(K(t), t) - C(I(t)) \right) dt + e^{-rT} K(T) \to \max_I, \tag{24}$$

$$\dot{K}(t) = I(t) - \delta K(t), \tag{25}$$

$$I(t) \geq 0, \tag{26}$$

$$K(0) = \bar{K}(0), \tag{27}$$

where the new revenue function

$$\tilde{R}(K, t) = \begin{Bmatrix} R(K) \\ 0 \end{Bmatrix} \quad \text{for} \quad \begin{Bmatrix} 0 \leq t \leq T - \tau \\ T - \tau < t \leq T \end{Bmatrix}$$

is now discontinuous w.r.t. time. This situation is covered by standard optimal control theory.

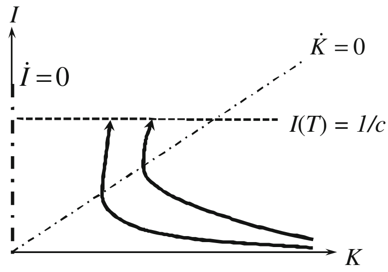
While $\tilde{R}(K, t)$ explicitly depends on time and therefore the problem is non-autonomous, it does not depend on time within both intervals $[0, T - \tau]$ and $(T - \tau, T]$. Hence some kind of phase plane analysis can be applied here. This is illustrated in the next subsection.

Alternatively, the problem can be considered as a 2-stage model with fixed switching time $T - \tau$.

3.2 Example

Since the resulting problem (24)–(27) is again a standard optimal control problem, it can be analyzed using phase plane analysis. For simplicity and for reasons of comparison, we again focus on the linear quadratic case, i.e., we assume that revenue and cost functions are quadratic as postulated in (14) and (15). Just like in the case of a control delay, control and shadow price are proportional (16). In the case of no-salvage value, $s = 0$, the problem is equivalent to the control delay problem of the previous section (see also the following section). Hence, we now only focus on the case with *salvage*

Fig. 2 The solutions in the time-to-build case with $s = 1$ close to the end time



value $s = 1$. Then, expression (24) leads to the transversality condition $\lambda (T) = 1$, which via (16) becomes

$$I (T) = \frac{1}{c}. \tag{28}$$

Observe that in the terminal interval, $(T - \tau, T)$, the revenue function vanishes, $\tilde{R} (K, t) = 0$. Hence, the adjoint equation becomes $\dot{\lambda} = (r + \delta) \lambda$ and the resulting ODE for the investment becomes

$$\dot{I} = (r + \delta) I. \tag{29}$$

Again, in this terminal interval isoclines and saddle point path (as well as the unstable path) are all straight lines. But because of (29), the isocline $\dot{I} = 0$ now coincides with the I-axis. This leads to the saddle point diagram depicted in Fig. 2.

Note however, that this figure and (29) only hold in the terminal interval $(T - \tau, T)$. Before this time interval, i.e. in $[0, T - \tau]$, the same canonical system as in Fig. 1 prevails. At the switching time $T - \tau$, the value of the investment rate can easily be computed from (29) and (28) as

$$I (T - \tau) = \frac{e^{-(r+\delta)\tau}}{c}.$$

Patching the two regimes before and after time τ together, we arrive at the final phase diagram depicted in Fig. 3. It is easy to see that some time before time $T - \tau$, which is the first point in time where investment is not productive anymore, investment typically decreases. Since at time T the remaining capital stock can be sold via the salvage value, investment then increases again.

From Fig. 3 we can immediately conclude that increasing the delay duration τ shifts the lower dashed line (switching curve) downwards. Hence, investment at the switching point $T - \tau$ decreases accordingly.

In case of no salvage value, $s = 0$, the terminal interval is not relevant anymore (since then $I = 0$ in this terminal interval) and all bold trajectories would end at the I-axis.

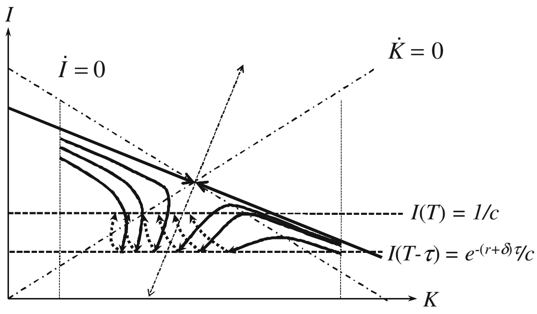


Fig. 3 The solutions in the time-to-build case with $s = 1$. The *bold dotted* trajectories refer to the terminal interval, while the *solid bold* ones correspond to the interval before time $T - \tau$

4 Compare time-to-install/deliver with time-to-build

From the previous two sections we conclude that without salvage value, the problems of the two delay cases are equivalent. Therefore the following theorem results.

Theorem 2 *Consider a control delay problem (1)–(5) with $s = 0$ with optimal solution (I_c, K_c) and a state delay problem (18)–(21) with $s = 0$ with optimal solution (I_s, K_s). Let furthermore the initial conditions be such that*

$$K_c(\tau) = K_s(0).$$

Then the two problems are equivalent in the following sense: if their optimal solutions are unique, they satisfy:

$$K_c(t+\tau) = K_s(t) \quad \textit{for} \quad t \geq 0,$$
$$I_c(t) = I_s(t) \quad \textit{for} \quad t \geq 0,$$

and the optimal objective function values J_c and J_s are connected by

$$J_c - \int_0^\tau e^{-rt} R(K_c(t))\, dt = J_s - \int_{-\tau}^0 e^{-r(t+\tau)} R\left(\bar{K}_s(t)\right) dt.$$

In case the optimal solutions are not unique there exists a pair of optimal solutions that satisfies the above equtions.

Proof Follows directly from the transformations in the previous sections. See (10)–(13) and (22). □

With salvage value, i.e. $s = 1$, the problems are fundamentally different. The difference is essentially that in the time-to-install/deliver problem the salvage value is brought back from time T to $T - \tau$, while in the time-to-build problem the salvage value is still collected at time T; see Fig. 4.

A way to make the problems equivalent also with positive salvage value would be the following. Note that in the time-to-install/deliver problem the salvage value only

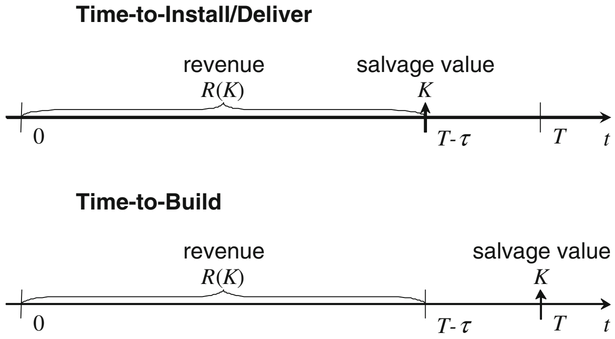


Fig. 4 Difference in salvage value timing between the two cases

depends on the productive capital stock at the horizon date. Alternatively, we could also have counted the capital stock that is not yet productive at time T, but becomes productive in the time interval $[T, T + \tau]$. Adding this not yet productive capital stock to the productive capital stock gives a salvage value that makes the time-to-install/deliver problem again equivalent to the time-to-build problem. Another possible interpretation of this alternative formulation would be that the value of the WIP at the horizon time is introduced in the salvage value function.

We conclude this paper by emphasizing that we do not claim that all kinds of delayed optimal control problems can be transformed into standard optimal control problems. Rather, we identified two classes of state delay and control delay problems where we were able to establish this eqivalence even in the finite horizon case, provided that there is no salvage value. As mentioned, in case of a salvage value the delay equivalence is typically lost.

Finally, we note that we did not use Pontryagin's maximum principle (cf. Feichtinger and Hartl 1986) in the above analysis, i.e. we did not need any assumptions on continuity or differentiability of the model functions. While the theory of control systems with delay (cf. Kolmanovskii 1992) was not needed here, it must be used if the above transformations do not work anymore.

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