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ORIGINAL PAPER

Distance Magic Labeling and Two Products of Graphs

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Abstract Let $G = (V, E)$ be a graph of order n. A distance magic labeling of G is a bijection $\ell\colon V \to \{1, \ldots, n\}$ for which there exists a positive integer k such that $\sum_{x \in N(v)} \ell(x) = k$ for all $v \in V$, where $N(v)$ is the neighborhood of v. We introduce a natural subclass of distance magic graphs. For this class we show that it is closed for the direct product with regular graphs and closed as a second factor for lexicographic product with regular graphs. In addition, we characterize distance magic graphs among direct product of two cycles.

Keywords Distance magic graphs · Direct product · Lexicographic product

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1 Introduction and Preliminaries

All graphs considered in this paper are simple finite graphs. We use $V(G)$ for the vertex set and $E(G)$ for the edge set of a graph G. The *neighborhood* $N(x)$ (or more precisely $N_G(x)$, when needed) of a vertex x is the set of vertices adjacent to x, and the *degree* $d(x)$ of x is $|N(x)|$, i.e. the size of the neighborhood of x. By C_n we denote a cycle on n vertices.

Distance magic labeling (also called *sigma labeling*) of a graph $G = (V(G), E(G))$ of order n is a bijection $\ell \colon V \to \{1, \ldots, n\}$ with the property that there is a positive integer k (called *magic constant*) such that $w(x) = \sum_{y \in N_G(x)} \ell(y) = k$ for every $x \in V(G)$, where $w(x)$ is the *weight* of x. If a graph G admits a distance magic labeling, then we say that G is a *distance magic graph*.

The concept of distance magic labeling of a graph has been motivated by the construction of magic squares. However, finding an r-regular distance magic labeling is equivalent to finding equalized incomplete tournament EIT(n, r) [5]. In an *equalized incomplete tournament* EIT(n, r) of n teams with r rounds, every team plays with exactly r other teams and the total strength of the opponents of each team is k. For a survey, we refer the reader to [1].

The following observations were proved independently:

Observation 1.1 ([7,9,11,12]) *Let G be an r-regular distance magic graph on n vertices. Then $k = \frac{r(n+1)}{2}$.*

Observation 1.2 ([7,9,11,12]) *No r-regular graph with an odd r can be a distance magic graph.*

We recall three out of four standard graph products (see [6]). Let G and H be two graphs. All three, the *Cartesian product* $G \Box H$, the *lexicographic product* $G \circ H$, and the *direct product* $G \times H$ are graphs with vertex set $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent in:

- $G \Box H$ if and only if either $g = g'$ and h is adjacent with h' in H, or $h = h'$ and g is adjacent with g' in G;
- $G \circ H$ if and only if either g is adjacent with g' in G or $g = g'$ and h is adjacent with h' in H;
- $G \times H$ if and only if g is adjacent with g' in G and h is adjacent with h' in H.

For a fixed vertex g of G, the subgraph of any of the above products induced by the set $\{(g, h) : h \in V(H)\}$ is called an H*-layer* and is denoted ${}^gH$. Similarly, if $h \in H$ is fixed, then G^h, the subgraph induced by $\{(g, h) : g \in V(G)\}$, is a G*-layer*.

The main topic of this paper is the direct product (that is known also by many other names, see [6]). It is the most natural graph product in the sense that each edge of $G \times H$ projects to an edge in both factors G and H. This is also the reason that many times this product is the most difficult to handle among (standard) products. Even the distance formula is very complicated with respect to other products (see [8]), and

$G \times H$ does not need to be connected, even if both factors are. More precisely, $G \times H$ is connected if and only if both G and H are connected and at least one of them is non-bipartite [13].

The direct product is commutative, associative, and has attracted a lot of attention in the research community in last 50 years. Probably the biggest challenge (among all products) is the famous Hedetniemi's conjecture:

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}.$$

This conjecture suggests that the chromatic number of the direct product depends only on the properties of one factor and not both. This is not so rare and also in this work we show that it is enough for one factor to be a distance magic graph with one additional property and then the product with any regular graph will result in a distance magic graph. More details about the direct product and products in general can be found in the book [6].

Some graphs which are distance magic among (some) products can be seen in [2,3,9,10]. The following product cycle and product related results were proved by Miller, Rodger, and Simanjuntak.

Theorem 1.3 [9] *The cycle C_n of length n is a distance magic graph if and only if $n = 4$.*

Theorem 1.4 [9] *Let G be an r-regular graph and C_n the cycle of length n for $r \geq 1$ and $n \geq 3$. The lexicographic product $G \circ C_n$ admits a distance magic labeling if and only if $n = 4$.*

In particular we have:

Observation 1.5 *The lexicographic product $C_n \circ C_m$, $n, m \geq 3$ is a distance magic graph if and only if $m = 4$.*

Rao, Singh and Parameswaran characterized distance magic graphs among Cartesian products of cycles.

Theorem 1.6 [10] *The Cartesian product $C_n \square C_m$, $n, m \geq 3$, is a distance magic graph if and only if $n = m \equiv 2 \pmod 4$.*

In the next section we introduce a natural subclass of distance magic graphs. For this class of graphs we were able to generalize Theorem 1.4 and show that it is closed for the direct product with regular graphs. In the last section we characterize distance magic graphs among direct products of cycles. In particular, we prove that a graph $C_m \times C_n$ is distance magic if and only if $n = 4$ or $m = 4$ or $m, n \equiv 0 \pmod 4$).

2 Balanced Distance Magic Graphs

In order to obtain a large class of graphs for which their direct product is distance magic we introduce a natural subclass of distance magic graphs.

A distance magic graph G with an even number of vertices is called *balanced* if there exists a bijection $\ell : V(G) \to \{1, \dots, |V(G)|\}$ such that for every $w \in V(G)$ the following holds: if $u \in N(w)$ with $\ell(u) = i$, then there exists $v \in N(w)$, $v \neq u$, with $\ell(v) = |V(G)| + 1 - i$. We call u the *twin vertex* of v and vice versa (we will also say that u and v are *twin vertices*, or shortly *twins*) and ℓ is called a *balanced distance labeling*. Hence a distance magic graph G is balanced if for any $w \in V(G)$ either both or none of vertices u and v with labels $\ell(u) = i$ and $\ell(v) = |V(G)|+1-i$ are in the neighborhood of w. It also follows from the definition that twin vertices of a balanced distance magic graph cannot be adjacent and that $N_G(u) = N_G(v)$.

It is somewhat surprising that the condition $N_G(u) = N_G(v)$ plays an important role in finding the factorization of the direct product, see Chapter 8 of [6]. In particular, if a non-bipartite connected graph has no pairs of vertices with the property $N_G(u) = N_G(v)$, then it is easier to find the prime factor decomposition. Similarly, such pairs generate very simple automorphisms of G and have been called unworthy in [14]. However in both above mentioned cases not all vertices need to have a twin vertex as in our case.

It is easy to see that a balanced distance magic graph is r-regular for some even r. Recall that the magic constant is $\frac{r}{2}(|V(G)| + 1)$ by Observation 1.1. Trivial examples of balanced distance magic graphs are empty graphs on even number of vertices. Not all distance magic graphs are balanced distance magic graphs. The smallest example is P_3. More examples (regular graphs with an even number of vertices) will be presented in the next section.

The graph $K_{2n,2n}, n \geq 1$, is balanced distance magic. To verify this let $V(K_{2n,2n}) = \{v_1, \dots, v_{4n}\}$. Assume that the vertices are enumerated in such a way that the sets $U = \{v_i : i(\text{mod}4) \in \{0, 1\}\}$ and $W = V(K_{2n,2n}) - U$ form the bipartition of $V(K_{2n,2n})$. It is easy to see that the labeling

$$\ell(v_i) = i \quad \text{for } i \in \{1, \dots, 4n\}$$

is the desired balanced distance magic labeling for $n \geq 2$. In particular, for $n = 1$ note that $K_{2,2}$ is isomorphic to C_4 and consecutive vertices receive labels 1, 2, 4, 3.

Also $K_{2n} - M$ is balanced distance magic if M is a perfect matching of K_{2n}. Indeed, if u and v form the i-th edge of M, $i \in \{1, \dots, n\}$, then we set $\ell(u) = i$ and $\ell(v) = 2n + 1 - i$ which is a balanced distance magic labeling.

The distance magic graphs $G \circ C_4$ described in Theorem 1.4 are also balanced distance magic graphs. Let $V(G) = \{g_1, \dots, g_p\}$ be the vertex set of a regular graph G and $V(C_4) = \{h_1, h_2, h_3, h_4\}$ where indices of vertices in $V(C_4)$ correspond to labels of a distance magic labeling of C_4. It is not hard to verify that the labeling

$$\ell((g_i, h_j)) = \begin{cases} (j-1)p + i, & \text{if } j \in \{1, 2\}, \\ jp - i + 1, & \text{if } j \in \{3, 4\}, \end{cases}$$

is a balanced distance magic labeling of $G \circ C_4$. Using similar labeling we obtain a larger family of balanced distance magic graphs.

Next we give another description of balanced distance magic graphs, which was implicitly used in above examples and will help in forthcoming proofs.

Proposition 2.1 *Graph G is balanced distance magic if and only if G is regular, and $V(G)$ can be partitioned in pairs (u_i, v_i), $i \in \{1, \ldots, |V(G)|/2\}$, such that $N(u_i) = N(v_i)$ for all i.*

Proof Let G be a balanced distance magic graph. Twins generate a desired partition of $V(G)$ for which $N(u_i) = N(v_i)$ holds, since every vertex $w \in V(G)$ has either both or none of vertices u_i and v_i in its neighborhood. Also G is regular since every vertex must have the same amount of twins in its neighborhood to obtain the same weight equal to the magic constant.

Conversely let G be regular, and $V(G)$ can be partitioned in pairs (u_i, v_i), $i \in \{1, \ldots, |V(G)|/2\}$, such that $N(u_i) = N(v_i)$ for all i. By setting $\ell(u_i) = i$ and $\ell(v_i) = |V(G)| - i + 1$ we see that u_i and v_i are twins for every i. Moreover, $w(x) = \sum_{v \in N(x)} \ell(v) = \frac{d(x)}{2}(i + |V(G)| - i + 1) = \frac{d(x)(|V(G)|+1)}{2}$ for all $x \in V(G)$. Hence G is balanced distance magic graph. □

Theorem 2.2 *Let G be a regular graph and H be a graph not isomorphic to $\overline{K_n}$ where n is odd. Then $G \circ H$ is a balanced distance magic graph if and only if H is a balanced distance magic graph.*

Proof Let G be an r_G-regular graph and H be a graph not isomorphic to $\overline{K_n}$ for an odd n. Let first H be a balanced distance magic graph with the vertex set $V(H) = \{h_1, \ldots, h_t\}$ and let φ defined by $\varphi(h_j) = j$ be a balanced distance magic labeling of H (we can always enumerate the vertices in an appropriate way). Hence $(h_j, h_{|V(H)|-j+1})$ form a partition of $V(H)$ from Proposition 2.1. Clearly $((g_i, h_j), (g_i, h_{|V(H)|-j+1}))$, $i \in \{1, \ldots, |V(G)|\}$ and $j \in \{1, \ldots, |V(H)|/2\}$, form a partition of $V(G \circ H)$ with

$$N_{G \circ H}(g_i, h_j) = N_{G \circ H}(g_i, h_{|V(H)|-j+1}).$$

By Proposition 2.1 $G \circ H$ is balanced distance magic graph.

Conversely, let $G \circ H$ be a balanced distance magic (and hence regular) graph. If H is an empty graph on even number of vertices, then it is balanced distance magic graph.

If H is not empty, then for each pair of twins (g_1, h_1) and (g_2, h_2) we have

$$N(g_1, h_1) \cap {}^{g_1}H = N(g_2, h_2) \cap {}^{g_1}H$$

and

$$N(g_1, h_1) \cap {}^{g_2}H = N(g_2, h_2) \cap {}^{g_2}H,$$

what means that $g_1 = g_2$, i.e. (g_1, h_1) and (g_2, h_2) lie in the same H-layer ${}^{g_1}H$. In consequence

$$N_H(h_1) = N_H(h_2)$$

and by the Proposition 2.1 H is balanced distance magic. □

Note that in order to prove the equivalence in the above theorem we needed to exclude H as an empty graph with odd number of vertices. Namely, it is not hard to see that for positive integer k, $C_4 \circ \overline{K_{2k-1}}$ is a balanced distance magic graph, but $\overline{K_{2k-1}}$ is not (recall that by the definition an empty graph is balanced distance magic if it has an even order). As an example see the labeling of $C_4 \circ \overline{K_3}$ in the table below, where rows and columns represent labeling of vertices in C_4-layers and $\overline{K_3}$-layers, respectively (the latter ones refer to consecutive vertices of C_4).

3	6	10	7
2	5	11	8
1	4	12	9

The situation is even more challenging when we turn to the direct product. If one factor, say H, is an empty graph, also the product $G \times H$ is an empty graph. Hence for any graph G on even number of vertices $G \times \overline{K_{2k-1}}$ is a balanced distance magic graph, while $\overline{K_{2k-1}}$ is not. However, we can still obtain a result similar to Theorem 2.2. For this we need the following observations.

Lemma 2.3 *Let $G \times H$ be a balanced distance magic graph and let (g, h) and (g', h') with $g \neq g'$ and $h \neq h'$ be twin vertices for some balanced distance magic labeling. Then the labeling in which we exchange the labels of (g', h') and (g', h) is also balanced distance magic labeling with (g, h) and (g', h) being twin vertices.*

Proof Let $\ell : V(G \times H) \to \{1, \ldots, |V(G)||V(H)|\}$ be a balanced distance magic labeling where (g, h) and (g', h') are twin vertices with $g \neq g'$ and $h \neq h'$. Recall that $N_{G\times H}(a, b) = N_G(a) \times N_H(b)$ for every $(a, b) \in V(G \times H)$ and that twin vertices have the same neighborhood. Thus we derive

$$N_{G\times H}(g, h) = N_{G\times H}(g', h') = N_{G\times H}(g', h) = N_{G\times H}(g, h')$$

and we can exchange labelings of (g', h') and (g', h). By Proposition 2.1, ℓ is still a balanced distance magic labeling with twin vertices (g, h) and (g', h). □

This lemma has clearly a symmetric version if we exchange the labels of (g', h') and (g, h').

Lemma 2.4 *Let $G \times H$ be a balanced distance magic graph, and let (g, h) and (g', h) be twin vertices as well as (g, h_1) and (g, h_2) for some balanced distance magic labeling. The labeling in which we exchange the labels of (g, h_2) and (g', h_1) is balanced distance magic labeling with twins (g, h_1) and (g', h_1).*

Proof Let $\ell : V(G \times H) \to \{1, \ldots, |V(G)||V(H)|\}$ be a balanced distance magic labeling of $G \times H$ where $\{(g, h), (g', h)\}$ and $\{(g, h_1), (g, h_2)\}$ are pairs of twin vertices. Hence $N_{G\times H}(g, h) = N_{G\times H}(g', h)$ and therefore $N_G(g) = N_G(g')$. Now as in the proof of Lemma 2.3 we have

$$N_{G\times H}(g, h_1) = N_{G\times H}(g, h_2) = N_{G\times H}(g', h_1) = N_{G\times H}(g', h_2)$$

and we can exchange labels of (g, h_2) and (g', h_1). By Proposition 2.1, ℓ is still a balanced distance magic labeling with twin vertices (g, h_1) and (g', h_1). □

Lemma 2.5 *Let $G \times H$ be a balanced distance magic graph, and let (g, h) and (g', h) be twin vertices as well as (g, h') and (g'', h'), $g'' \neq g'$, for some balanced distance magic labeling. The labeling in which we exchange the labels of (g', h') and (g'', h') is a balanced distance magic labeling where (g, h') and (g', h') are twin vertices.*

Proof Let $\ell : V(G \times H) \to \{1, \ldots, |V(G)||V(H)|\}$ be a balanced distance magic labeling of $G \times H$ where $\{(g, h), (g', h)\}$ and $\{(g, h'), (g'', h')\}$ are pairs of twin vertices for $g'' \neq g'$. Hence $N_{G\times H}(g, h) = N_{G\times H}(g', h)$ and $N_{G\times H}(g, h') = N_{G\times H}(g'', h')$ and therefore $N_G(g) = N_G(g') = N_G(g'')$. One can observe that

$$N_{G\times H}(g, h') = N_{G\times H}(g', h') = N_{G\times H}(g'', h')$$

and we can exchange labels of (g', h') and (g'', h'). By Proposition 2.1, ℓ is still a balanced distance magic labeling with twin vertices (g, h') and (g', h'). □

Theorem 2.6 *The direct product $G \times H$ is balanced distance magic if and only if one of the graphs G and H is balanced distance magic and the other one is regular.*

Proof Assume first, without loss of generality, that G is regular and H is a balanced distance magic graph with $V(H) = \{h_1, \ldots, h_p\}$, where the suffix indicates the label of a balanced distance magic labeling of H. Thus for $j \leq \frac{p}{2}$, h_{p+1-j} is the twin vertex of h_j. Recall that H is r_H-regular for an even r_H. Let $V(G) = \{g_1, \ldots, g_t\}$. Pairs $((g_i, h_j), (g_i, h_{|V(H)|-j+1}))$, $i \in \{1, \ldots, t\}$ and $j \in \{1, \ldots, p/2\}$, form a partition of $V(G \times H)$ with $N_{G\times H}(g_i, h_j)) = N_{G\times H}(g_i, h_{p-j+1})$. By Proposition 2.1 $G \times H$ is a balanced distance magic graph.

Conversely, let $G \times H$ be a balanced distance magic graph (this implies that $G \times H$ is a regular graph and hence also both G and H are regular). There exists a balanced distance magic labeling $\ell : V(G \times H) \to \{1, \ldots, |V(G)||V(H)|\}$. First we show the following.

Claim *There exists a balanced distance labeling of $G \times H$ such that one of the following is true:*

1. *There exists an H-layer ${}^gH$, such that the twin vertex of any $(g, h) \in {}^gH$ lies in ${}^gH$.*
2. *There exists a G-layer G^h, such that the twin vertex of any $(g, h) \in G^h$ lies in G^h.*

If there exists an H-layer or a G-layer such that the twin vertex of any vertex in this layer also lies within this layer, then we are done. Hence assume that this is not the case, i.e. for every H-layer ${}^gH$ there exists a vertex (g, h) such that its twin vertex

(g', h') has the property $g' \neq g$ and for every G-layer G^h there exists a vertex (g, h) such that its twin vertex (g', h') has the property $h' \neq h$.

We use Algorithm 1 to rearrange the labels of vertices in $V(G \times H)$ in such a way that we either obtain an H-layer closed for twin vertices or we couple all H-layers, i.e. we find pairs of H-layers $\{{}^gH, {}^{g'}H\}$ with the property that the twin vertex of a vertex $(g, h) \in^g H$ lies in ${}^{g'}H$ and is of the form (g', h). The latter case implies that all G-layers (and in particular one of them, say G^h) are closed for twins, and the claim is proved.

Algorithm 1 Coupling H-layers

Step 1: Set $A = V(G)$. Go to step 2.

Step 2: If $A = \{g\}$ for some g, then STOP, ${}^gH$ is closed under twin vertices. If $A = \emptyset$ then STOP, all the H-layers are matched in such a way that in any pair $\{{}^gH, {}^{g'}H\}$ for every vertex (g, h) its twin vertex is of the form (g', h). If $|A| \geq 2$, then proceed to step 3.

Step 3: Choose any $g \in A$. If ${}^gH$ is closed for twin vertices, then STOP. Otherwise, there is a vertex $(g, h) \in {}^gH$ having the twin (g', h'), where $g' \in A$ and $g' \neq g$. If $h' \neq h$, then use Lemma 2.3 to obtain a new labeling with (g, h) and (g', h) being twins. If every vertex $(a, b) \in {}^gH \cup {}^{g'}H$ has its twin vertex (a', b') also in ${}^gH \cup {}^{g'}H$, then go to step 6. Otherwise go to step 4.

Step 4: While there exists a vertex $(g, h_1) \in {}^gH \cup G^h$ with the twin vertex (g'', h_2), where $g'' \notin \{g, g'\}$, $h_2 \neq h_1$, use Lemma 2.3 to obtain a new labeling where (g, h_1) and (g'', h_1) are twin vertices. Go to step 5.

Step 5: While there exists a vertex $(g, h_1) \in {}^gH \cup G^h$ with the twin vertex (g'', h_1), where $g'' \notin \{g, g'\}$, use Lemma 2.5 to obtain a new labeling with twin vertices (g, h_1) and (g', h_1). Proceed to step 6.

Step 6: While there exists a pair of twin vertices (g, h_1) and (g, h_2) , if the vertices (g', h_1) and (g', h_2) are not twins, then use Lemma 2.5 in order to make them twins.

Step 7: While there exists a pair of twin vertices $(g, h_1), (g, h_2) \in {}^gH$ with the property $h_2 \neq h_1$, use Lemma 2.4 to obtain a new labeling where (g, h_1) and (g', h_1) are twin vertices. Proceed to step 7.

Step 7: Set $A \leftarrow A - \{g, g'\}$ and go back to step 2.

Observe that after step 5, all the vertices in ${}^gH \cup G^h$ have their twins in ${}^gH \cup G^h$. Step 6 guarantees that each application of the Lemma 2.5 in step 7 in fact exchanges the twins of a vertex in ${}^gH$ and a vertex in G^h.

Assume that some H-layer, say ${}^gH$, is closed for twins. In particular for every pair of twins $(g, h_1), (g, h_2)$ we have

$$N_{G\times H}(g, h_1) = N_{G\times H}(g, h_2),$$

and in consequence

$$N_H(h_1) = N_H(h_2),$$

thus by the Proposition 2.1, H is balanced distance magic.

In the case when some G-layer G^h is closed for twins, we can prove in an analogous way that G is a balanced distance magic graph. □

3 Distance Magic Graphs $C_m \times C_n$

Let $V(C_m \times C_n) = \{v_{i,j} : 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$, where $N(v_{i,j}) = \{v_{i-1,j-1}, v_{i-1,j+1}, v_{i+1,j-1}, v_{i+1,j+1}\}$ and operation on the first suffix is taken modulo m and on the second suffix modulo n. We also refer to the set of all vertices $v_{i,j}$ with fixed i as i-th row and with fixed j as j-th column.

We start with direct products of cycles that are not distance magic.

Theorem 3.1 *If ($m \not\equiv 0 \pmod 4$ and $n \neq 4$) or ($n \not\equiv 0 \pmod 4$ and $m \neq 4$), then $C_m \times C_n$ is not distance magic.*

Proof By commutativity of the direct product we can assume that $m \not\equiv 0 \pmod 4$ and $n \neq 4$. Assume that $C_m \times C_n$ is distance magic with some magic constant k, which means there is a distance magic labeling ℓ. Let us consider sum of the labels of the neighbors of $v_{i+1,j+1}$ and $v_{i+3,j+1}$ for any $i \in \{0, \ldots, m-1\}$ and $j \in \{0, \ldots, n-1\}$, operations on indices taken modulo m and n, respectively:

$$w(v_{i+1,j+1}) = \ell(v_{i,j}) + \ell(v_{i,j+2}) + \ell(v_{i+2,j}) + \ell(v_{i+2,j+2}) = k,$$

$$w(v_{i+3,j+1}) = \ell(v_{i+2,j}) + \ell(v_{i+2,j+2}) + \ell(v_{i+4,j}) + \ell(v_{i+4,j+2}) = k.$$

It implies that

$$\ell(v_{i,j}) + \ell(v_{i,j+2}) = \ell(v_{i+4,j}) + \ell(v_{i+4,j+2}).$$

Repeating that procedure we obtain that

$$\ell(v_{i,j}) + \ell(v_{i,j+2}) = \ell(v_{i+4\alpha,j}) + \ell(v_{i+4\alpha,j+2})$$

for any natural number α.

It is well known that if $a, b \in \mathbb{Z}_m$ and $\gcd(a, m) = \gcd(b, m)$, then a and b generate the same subgroup of $\mathbb{Z}_m$, that is, $\langle a \rangle = \langle b \rangle$.

Since $m \not\equiv 0 \pmod 4$ we have $\gcd(2, m) = \gcd(4, m)$ and $2 \in \langle 4 \rangle$, which implies that there exists α' such that $4\alpha' \equiv 2 \pmod m$. We deduce that

$$\ell(v_{i,j}) + \ell(v_{i,j+2}) = \ell(v_{i+2,j}) + \ell(v_{i+2,j+2}) = \frac{k}{2}.$$

Substituting j with $j+2$ we obtain

$$\ell(v_{i,j+2}) + \ell(v_{i,j+4}) = \frac{k}{2}.$$

Thus for every i, j we have

$$\ell(v_{i,j}) = \ell(v_{i,j+4}),$$

which leads to a contradiction, since $n \neq 4$ and ℓ is not a bijection. □

Next we show that some of direct products of cycles are distance magic but not balanced distance magic. Used constructions are similar to those by Cichacz and Froncek in [4].

Theorem 3.2 *If $m, n \equiv 0 \pmod 4$ and $m, n > 4$, then $C_m \times C_n$ is distance magic but not balanced distance magic graph.*

Proof First we show that $C_m \times C_n$ is distance magic. We define the labeling ℓ by starting conditions (every second vertex of the row zero) followed by recursive rules that cover all the remaining vertices.

$$\ell(v_{0,4j+t}) = \begin{cases} 2j+1, & \text{if} \quad 0 \leq j \leq \lceil \frac{n}{8} \rceil - 1 \text{ and } \quad t=0, \\ \frac{n}{2} - 2j, & \text{if} \quad \lceil \frac{n}{8} \rceil \leq j \leq \frac{n}{4} - 1 \text{ and } \quad t=0, \\ mn - 2j - 1, & \text{if} \quad 0 \leq j \leq \lfloor \frac{n}{8} \rfloor - 1 \text{ and } \quad t=2, \\ mn - \frac{n}{2} + 2j + 2, & \text{if} \quad \lfloor \frac{n}{8} \rfloor \leq j \leq \frac{n}{4} - 1 \text{ and } \quad t=2. \end{cases}$$

Note that we have used every label between 1 and $\frac{n}{4}$ as well as between $mn - \frac{n}{4} + 1$ and mn exactly once for the starting conditions.

In the first recursive step we label every second vertex of row two in the order that is in a sense reverse to the one of row zero:

$$\ell(v_{2,j}) = \begin{cases} \ell(v_{0,n-2-j}) + \frac{n}{4}, & \text{if} \quad \ell(v_{0,n-2-j}) \leq \frac{mn}{2}, \\ \ell(v_{0,n-2-j}) - \frac{n}{4}, & \text{if} \quad \ell(v_{0,n-2-j}) > \frac{mn}{2}, \end{cases}$$

for $j \in \{0, 2, \ldots, n-2\}$. Clearly we use in this step every label between $\frac{n}{4} + 1$ and $\frac{n}{2}$ and between $mn - \frac{n}{2} + 1$ and $mn - \frac{n}{4}$ exactly once.

We continue with every second vertex in every even row. Hence for $2 \leq i \leq \frac{m}{2} - 1$ and for $j \in \{0, 2, \ldots, n-2\}$ let

$$\ell(v_{2i,j}) = \begin{cases} \ell(v_{2i-4,j}) + \frac{n}{2}, & \text{if} \quad \ell(v_{2i-4,j}) \leq \frac{mn}{2}, \\ \ell(v_{2i-4,j}) - \frac{n}{2}, & \text{if} \quad \ell(v_{2i-4,j}) > \frac{mn}{2}. \end{cases}$$

Again all the labels here are used exactly once and are between $\frac{n}{2} + 1$ and $\frac{mn}{8}$ and between $mn - \frac{mn}{8} + 1 = \frac{7mn}{8} + 1$ and $mn - \frac{n}{2}$.

Next we label every second vertex of every odd row and complete with this all even columns. For $0 \leq i \leq \frac{m}{2} - 1$ and for $j \in \{0, 2, \ldots, n-2\}$ we set:

$$\ell(v_{2i+1,j}) = \begin{cases} \ell(v_{2i,j}) + \frac{mn}{8} & \text{if} \quad \ell(v_{2i,j}) \leq \frac{mn}{2}, \\ \ell(v_{2i,j}) - \frac{mn}{8} & \text{if} \quad \ell(v_{2i,j}) > \frac{mn}{2}. \end{cases}$$

Labels used here are between $\frac{mn}{8} + 1$ and $\frac{mn}{4}$ and between $\frac{3mn}{4} + 1$ and $\frac{7mn}{8}$.

Finally we use all the remaining labels between $\frac{mn}{4} + 1$ and $\frac{3mn}{4}$ for all the vertices in every odd column. Thus for $0 \leq i \leq m-1$ and $j \in \{1, 3, \ldots, n-1\}$ let:

$$\ell(v_{i,2j+1}) = \begin{cases} \ell(v_{i,2j}) + \frac{mn}{4}, & \text{if } \ell(v_{i,2j}) \leq \frac{mn}{2}, \\ \ell(v_{i,2j}) - \frac{mn}{4}, & \text{if} \quad \ell(v_{i,2j}) > \frac{mn}{2}. \end{cases}$$

Obviously the labeling ℓ is a bijection from $V(C_m \times C_n)$ to $\{1, \dots, mn\}$. It is also straightforward to see that $k = 2mn + 2$ is the magic constant. Hence ℓ is distance magic labeling.

However, ℓ is not balanced distance magic, as none of the cycles C_m, C_n is (see Theorem 1.3) and thus their product cannot be balanced distance magic due to Theorem 2.6. □

The example of distance magic labeling of $C_{16} \times C_{16}$ is shown below, where $v_{0,0}$ starts in lower left corner and the first index is for the row and the second for the column:

196	132	62	126	194	130	64	128	193	129	63	127	195	131	61	125
228	164	30	94	226	162	32	96	225	161	31	95	227	163	29	93
57	121	199	135	59	123	197	133	60	124	198	134	58	122	200	136
25	89	231	167	27	91	229	165	28	92	230	166	26	90	232	168
204	140	54	118	202	138	56	120	201	137	55	119	203	139	53	117
236	172	22	86	234	170	24	88	233	169	23	87	235	171	21	85
49	113	207	143	51	115	205	141	52	116	206	142	50	114	208	144
17	81	239	175	19	83	237	173	20	84	238	174	18	82	240	176
212	148	46	110	210	146	48	112	209	145	47	111	211	147	45	109
244	180	14	78	242	178	16	80	241	177	15	79	243	179	13	77
41	105	215	151	43	107	213	149	44	108	214	150	42	106	216	152
9	73	247	183	11	75	245	181	12	76	246	182	10	74	248	184
220	156	38	102	218	154	40	104	217	153	39	103	219	155	37	101
252	188	6	70	250	186	8	72	249	185	7	71	251	187	5	69
33	97	223	159	35	99	221	157	36	100	222	158	34	98	224	160
1	65	255	191	3	67	253	189	4	68	254	190	2	66	256	192

Next theorem that completely describes distance magic graphs among direct product of cycles follows immediately by Theorems 3.1, 3.2, and 2.6.

Theorem 3.3 *A graph $C_m \times C_n$ is distance magic if and only if $n = 4$ or $m = 4$, or $m, n \equiv 0 \pmod 4$. Moreover, $C_m \times C_n$ is balanced distance magic if and only if $n = 4$ or $m = 4$.* □

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