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An algorithm for finding a common point of the solutions of fixed point and variational inequality problems in Banach spaces

Abstract Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth real Banach space E. Let $T : C \to C$ be relatively nonexpansive mapping and let $A_i : C \to E^*$ be L_i-Lipschitz monotone mappings, for $i = 1, 2$. In this paper, we introduce and study an iterative process for finding a common point of the fixed point set of a relatively nonexpansive mapping and the solution set of variational inequality problems for A_1 and A_2. Under some mild assumptions, we show that the proposed algorithm converges strongly to a point in $F(T) \cap VI(C, A_1) \cap VI(C, A_2)$. Our theorems improve and unify most of the results that have been proved for this important class of nonlinear operators.

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الملخص

لتكن C مجموعة جزئية غير خالية، مغلقة ومحدبة من فضاء باناخ حقيقي 2-محدب بشكل منتظم وأملس بشكل منتظم E. ليكن $C \to T:C$ راسماً غير متمدد نسبيا وليكن $A_i: C \to E^*$ راسم L_i-ليبشتز رتيبا لـ $i = 1,2$. في هذه الورقة، نقدم وندرس عملية تكرارية لإيجاد نقطة مشتركة بين مجموعة النقطة الثابتة لراسم غير متمدد نسبيا ومجموعة حل لمسائل متباينات تباينية لـ A_1 وَ A_2. تحت بعض الافتراضات الطفيفة، نثبت أن الخوارزمية المقترحة تتقارب إلى نقطة في $F(T) \cap VI(C, A_1) \cap VI(C, A_2)$. مبرهناتنا تحسن وتوحد معظم النتائج التي تم إثباتها لهذا الصف من المؤثرات غير الخطية.

1 Introduction

Let C be a nonempty subset of a real Banach space E. A mapping $T : C \to E$ is called *L-Lipschitz* if there exits $L \geq 0$ such that

$$||Tx - Ty| \leq L||x - y||, \quad \forall x, y \in C. \tag{1.1}$$

If $L < 1$, then T is called *a contraction* and if $L = 1$ then T is called *a nonexpansive* mapping. It follows that the class of Lipschitzian mappings includes the class of nonexpansive and hence the class of contraction mappings.

A mapping A of C into E^* is called *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \text{for all} \quad x, y \in C. \tag{1.2}$$

A is called *α-strongly monotone*, if there exists a positive real number α such that

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$$\langle Ax - Ay, x - y \rangle \geq \alpha ||x - y||^2, \quad \text{for all} \quad x, y \in C. \tag{1.3}$$

A is called γ-*inverse strongly monotone*, if there exists a positive real number γ such that

$$\langle Ax - Ay, x - y \rangle \geq \gamma ||Ax - Ay||^2, \quad \text{for all} \quad x, y \in C. \tag{1.4}$$

We observe that any γ-inverse strongly monotone mapping A is Lipschitz with $L = \frac{1}{\gamma}$.

Clearly, the class of monotone mappings includes the classes of α-strongly monotone and γ-inverse strongly monotone mappings.

Let C be a nonempty, closed and convex subset of E and let $A : C \rightarrow E^*$ be a nonlinear mapping. The variational inequality problem, denoted by VIP, is to find $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.5}$$

The solution set of the variational inequality problem is denoted by VI(C, A). The variational inequality problem is a fundamental problem in variational analysis. The theory of variational analysis has emerged as a very natural generalization of the theory of boundary value problems and allows us to consider new problems arising from many fields of applied mathematics, such as mechanics, physics, engineering, the theory of convex programming, and the theory of control: see, for instance [16,18,25,28–30,33].

There are several iterative methods for solving VIP (see, e.g., [4,5,7,11,18,30,33,35]). The basic idea consists of extending the projected gradient method for solving the problem of minimizing $f(x)$ subject to $x \in C$ given by

$$x_{n+1} = P_C[x_n - \alpha_n \nabla f(x_n)], \quad n \geq 0, \tag{1.6}$$

where $\{\alpha_n\}$ is a positive real sequence satisfying certain conditions and P_C is the metric projection onto C. For convergence properties of this method for the case in which $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is convex and differentiable function, one may see [2]. An immediate extension of method (1.6) to VIP is the projected gradient method for optimization problems, substituting the operator A for the gradient, so that we generate a sequence $\{x_n\}$ through:

$$x_{n+1} = P_C[x_n - \alpha_n A x_n], \quad n \geq 0. \tag{1.7}$$

Convergence results for this method require some monotonicity properties of A. This method converges under quite strong hypotheses. If A is Lipschitz continuous with Lipschitz constant L and α-strongly monotone, then the sequence generated by (1.7) converges to an element of VI(C, A), if $\{\alpha_n\} \subset (0, 2\alpha/L^2)$ provided that problem (1.5) has a solution.

We note that, in (1.7), there is no chance of relaxing the assumption on A to plain monotonicity. For example consider $C = \mathbb{R}^2$ and A, a rotation with a $\frac{\pi}{2}$ angle, which is certainly monotone and Lipschitz continuous with VI$(C, A) = \{0\}$. It is easy to check that $||x_{n+1}|| > ||x_n||$ for all $n \geq 0$ and for all $\alpha_n > 0$. Therefore, the sequence generated by (1.7) moves away from the solution, independent of the choice of the sequence α_n.

To deal with the weakness of the method defined by (1.7), Korpelevich [14] proposed a modification of the method, called the extragradient algorithm in the finite-dimensional Euclidean space $\mathbb{R}^n$ under the assumption that a set $C \subset \mathbb{R}^n$ is closed and convex and a mapping A of C into $\mathbb{R}^n$ is monotone and L-Lipschitz continuous,

$$\begin{cases} y_n = P_C[x_n - \lambda A x_n], \\ x_{n+1} = P_C[x_n - \lambda A y_n], \quad n \geq 0, \end{cases} \tag{1.8}$$

for all $n \geq 0$, where $\lambda \in (0, \frac{1}{L})$. He proved that the sequences $\{x_n\}$ and $\{y_n\}$, generated by (1.8), converge strongly to point $x^* \in$ VI(C, A) provided that VI(C, A) is nonempty.

The difference in (1.8) is that A is evaluated twice and the projection is computed twice at each iteration, but the benefit is significant, because the resulting algorithm is applicable to the whole class of variational inequalities for monotone mappings. The method of solving VIP by Korpelevichs has received great attention by many authors (see, e.g., [7,11,16,17,22,30,36,37] and the references therein).

In 2006, Nadezhkina and Takahashi [17] suggested the following modified Korpelevichs method for a solution of VIP for L-Lipschitz continuous monotone mapping A in infinite-dimensional Hilbert spaces. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in C$ by

$$\begin{cases} x_0 \in C, \\ y_n = P_C[x_n - \lambda_n A x_n], \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) P_C[x_n - \lambda_n A y_n], \quad n \geq 0, \end{cases} \tag{1.9}$$

where P_C is a metric projection from H onto C, $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/L)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then, they proved that the sequence $\{x_n\}$ converges *weakly* to an element of VI(C, A).

However, we realize that Korpelevichs modified method (1.9) has only weak convergence in the infinite-dimensional Hilbert spaces (you may also see, e.g., [6,7]). Therefore, several authors studied to obtain strong convergence by modifying the original method of Korpelevich. For example, in [4,9,31], it is proved that some very interesting Korpelevich-type algorithms strongly converge to a solution of VIP.

Recently, Yao et al. [32] investigated the problem of finding a solution of variational inequality problem for γ-inverse strongly monotone mapping A, by considering the following iterative algorithm:

$$\begin{cases} y_n = P_C[x_n - \lambda_n A x_n + \alpha_n(f(x_n) - x_n)], \\ x_{n+1} = P_C[x_n - \mu_n A y_n + \gamma_n(y_n - x_n)], \quad n \geq 0, \end{cases} \tag{1.10}$$

where $f : C \to H$ is a ρ-contractive mapping and $\{\alpha_n\}$, $\{\lambda_n\}$, $\{\mu_n\}$ and $\{\gamma_n\}$ are real sequences satisfying certain conditions. Then, they proved that the sequence $\{x_n\}$ generated by (1.10) converges strongly to $x^* \in$ VI(C, A).

More recently, Zegeye and Shahzad [38] investigated the problem of finding a common solution of a finite family of variational inequality problems for Lipschitz monotone mappings in Hilbert spaces. In fact, they proved the following theorem.

Theorem ZS *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A_i : C \to H$ be an L_i-Lipschitz monotone mappings with Lipschitz constants L_i, for $i = 1, 2$. Let $f : C \to C$ be a contraction mapping. Assume that $\mathcal{F} = VI(C, A_1) \cap VI(C, A_2)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in C$ by*

$$\begin{cases} z_n = P_C[x_n - \gamma_n A_2 x_n], \\ y_n = P_C[x_n - \gamma_n A_1 x_n], \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)(a_n x_n + b_n P_C[x_n - \gamma_n A_1 y_n] \\ \qquad\qquad + c_n P_C[x_n - \gamma_n A_2 z_n]), \end{cases} \tag{1.11}$$

where P_C is the metric projection from H onto C, and $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{\alpha_n\}$ are real sequences satisfying certain conditions. Then, $\{x_n\}$ converges strongly to a point $x^* \in \mathcal{F}$.

We observe that the strong convergence result in [38] is valid only in Hilbert spaces. This now leads to the following important question.

Question 1 *Can we obtain an iterative scheme which converges strongly to a point in $VI(C, A)$ in more general Banach spaces?*

It is well known that if C is a nonempty closed convex subset of a Hilbert space H the metric projection $P_C : H \to C$ is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. Recently, Alber [1] introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Let E be a smooth real Banach space with dual E^*. Let the Lyapunov functional $\phi : E \times E \to \mathbb{R}$, introduced by Alber [1], be defined by

$$\phi(y, x) = ||y||^2 - 2\langle y, Jx \rangle + ||x||^2, \quad \text{for} \quad x, y \in E, \tag{1.12}$$

where J is the normalized duality mapping from E into 2^{E^*} defined by $Jx := \{f^* \in E^* : \langle x, f^* \rangle = ||x||^2 = ||f^*||^2\}$, where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that E is smooth if and only if J is single valued and if E is uniformly smooth, then J is uniformly continuous on bounded subsets of E. In addition, if E is a reflexive and strictly convex Banach space with a strictly convex dual, then J^{-1} is duality mapping from E^* into E which is single valued, one-to-one, surjective and $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$ (see [24]). We note that in a Hilbert space, H, J is the identity mapping.

Let C be a subset of a smooth real Banach space E. The generalized projection $\Pi_C : E \to C$, introduced by Alber [1], is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the function $\phi(y, x)$, that is $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem: $\inf_{y \in C} \phi(y, x)$.

It is known that, if C is nonempty, closed and convex subset of a real reflexive, strictly convex, and smooth Banach space E and $x \in E$, then there exists a unique element $x_0 \in C$ such that $\phi(x_0, x) = \min\{\phi(z, x) : z \in C\}$ (see, e.g [1]).

From the definition of ϕ, we have that

$$(||x|| - ||y||)^2 \leq \phi(x, y) \leq (||x|| + ||y||)^2, \tag{1.13}$$

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2 \langle x - z, Jz - Jy \rangle. \tag{1.14}$$

If E is a Hilbert space, then $\phi(y, x) = ||y - x||^2$ and $\Pi_C = P_C$ is the metric projection of H onto C.

Let T be a mapping from C into itself. A point $x \in C$ is a *fixed point of T* if $Tx = x$ and we denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. A point p in C is said to be an *asymptotic fixed point of T* (see [19]) if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n \to \infty} ||x_n - Tx_n|| = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. A mapping T from C into itself is called *relatively nonexpansive* if (R1) $F(T) \neq \emptyset$; (R2) $\phi(p, Tx) \leq \phi(p, x)$ for $x \in C$ and (R3) $F(T) = \widehat{F}(T)$.

Clearly, if $E = H$, a real Hilbert space, then the class of relatively nonexpansive mappings includes the class of nonexpansive and hence contraction mappings with nonempty fixed point set.

Many authors have considered the problem of finding a common element of the fixed point set of a relatively nonexpansive mapping and the solution set of a variational inequality problem for $\gamma-$inverse strongly monotone mapping (see, e.g., [13,15,20,23,26,34]).

For finding an element of $F(T) \bigcap \hat{A}'VI(C, A)$ under the assumption that a set C is closed and convex subset of H, where H is a real Hilbert space, T is nonexpansive mapping of C into itself, and A is Lipschitz monotone mapping of C into H, Nadezhkina and Takahashi [17], introduced the following iterative process.

$$\begin{cases} x_0 \in C, \quad \text{chosen arbitrary,} \\ y_n = P_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)TP_C[x_n - \lambda_n Ay_n]. \end{cases}$$

They proved that the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to the same element of $F(T)\hat{A}' \bigcap VI(C, A)$.

In [10], Iiduka and Takahashi proposed the iteration scheme given by

$$\begin{cases} x_0 = x \in C, \quad \text{chosen arbitrary,} \\ x_{n+1} = \alpha_n x + (1 - \alpha_n)TP_C[x_n - \lambda_n Ax_n], \end{cases}$$

where A is γ-inverse strongly monotone, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\gamma)$ satisfying certain conditions. They showed that if $F(T) \bigcap VI(C, A)$ is nonempty, then the sequence $\{x_n\}$ converges strongly to some $z \in F(T) \bigcap VI(C, A)$.

Very recently, Zegeye et al. [33] introduced and studied the iterative scheme given below for a common point of the solution set of a variational inequality problem for γ-inverse strongly monotone mapping A and the fixed point set of a closed relatively quasi-nonexpansive mapping T in a 2-uniformly convex and uniformly smooth real Banach space E:

$$\begin{cases} x_0 \in C = C_1, \quad \text{chosen arbitrary,} \\ z_n = \Pi_C(J^{-1}(Jx_n - \lambda_n Ax_n)), \\ y_n = J^{-1}(\beta Jx_n - (1 - \beta)JTz_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), n \geq 0, \end{cases} \tag{1.15}$$

where C is a closed convex nonempty subset of E and $\{\lambda_n\}$ is a sequence satisfying certain conditions. They proved that the sequence $\{x_n\}$ converges strongly to an element of $F := F(T) \bigcap VI(C; A) \neq \emptyset$ provided that A satisfies $||Ax|| \leq ||Ax - Ap||$ for all $x \in C$ and $p \in F$.

This brings us to our second question.

Question 2 *Can we obtain an iterative scheme which converges strongly to a common point of the fixed point set of a relatively nonexpansive mapping T and the solution of a variational inequality problem for Lipschitz monotone mapping A in Banach spaces?*

It is our purpose in this paper to propose an extragradient-type method for finding a common point of the fixed point set of a relatively nonexpansive mapping T and the solution set of two variational inequality problems for Lipschitz monotone mappings in a 2-uniformly convex and uniformly smooth real Banach space E. As a consequence, we obtain a convergence theorem for approximating a common point of the solution of a finite family of variational inequality problems for Lipschitz monotone mappings. The results obtained in this paper improve and extend the results of Zegeye and Shahzad [38], Yao et al. [32] and some other results in this direction.

2 Preliminaries

Let E be a real Banach space. Let $S(E) = \{x \in E : ||x|| = 1\}$. Then the norm of E is said to be Gâteaux differentiable if

$$\lim_{t \to 0} \frac{||x + ty|| - ||x||}{t}, \tag{2.1}$$

exists for each $x, y \in S(E)$. In this case, E is called *smooth* (see e.g., [24]).

The *modulus of smoothness* of E is defined by

$$\rho_E(\tau) := \sup \left\{ \frac{||x + y|| + ||x - y||}{2} - 1 : ||x|| = 1, ||y|| = \tau \right\}.$$

A Banach space E is called *uniformly smooth* if $\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0$. It is well known that every uniformly smooth space (e.g., L_p and the Sobolev spaces W_p^m, $(1 < p < \infty)$) has Gâteaux differentiable norm (see e.g., [8]).

A normed space E is called *strictly convex* if for all $x, y \in E$ such that $||x|| = ||y|| = 1$, $x \neq y$, $||\frac{x+y}{2}|| < 1$.

The *modulus of convexity* of E is the function $\sigma : (0, 2] \to [0, 1]$ defined by $\sigma(\epsilon) = inf \left\{ 1 - ||\frac{x+y}{2}|| : ||x|| = 1 = ||y||; \epsilon = ||x - y|| \right\}$.

E is called *uniformly convex* if and only if $\sigma(\epsilon) > 0$, for every $\epsilon \in (0, 2]$. Clearly, every uniformly convex is strictly convex space.

Let $p > 0$. Then E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\sigma(\epsilon) \geq c\epsilon^p$, for all $\epsilon \in (0, 2]$. It is well known (see for example [27]) that

$$Ł_p \text{ or } ł_p = \begin{cases} p\text{-uniformly convex} & \text{if } p \geq 2, \\ 2\text{-uniformly convex} & \text{if } 1 < p \leq 2. \end{cases}$$

A monotone mapping $B : C \to 2^{E^*}$ is called *maximal monotone* if its graph $G(B)$ is not properly contained in the graph of any other monotone mapping. It is clear that a monotone mapping B is maximal if and only if for any $(x, u) \in E \times E^*$, if $\langle x - y, u - v \rangle \geq 0$, for every $(y, v) \in G(B)$, then it follows that $u \in Bx$. Let A be a monotone and L-Lipschitz mapping of C into E^* and let $N_C v$ be the normal cone to C at $v \in C$; i.e.,

$$N_C v = \{w \in E^* : \langle v - u, w \rangle \geq 0, \forall u \in C\}.$$

Define

$$Bv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases} \tag{2.2}$$

Then, B is maximal monotone and $0 \in Bv$ if and only if $v \in VI(C, A)$ (see, e.g., [21]).

In the sequel, we shall make use of the following lammas.

Lemma 2.1 ([35]) *Let C be a non-empty closed and convex subset of a real reflexive, strictly convex and smooth Banach space E. If $A : C \to E^*$ is continuous monotone mapping, then $VI(C, A)$ is closed and convex.*

Lemma 2.2 ([1]) *Let C be a non-empty closed and convex subset of a real reflexive, strictly convex and smooth Banach space E. Then for $\forall y \in C$,*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x).$$

We make use of the function $V : E \times E^* \to \mathbb{R}$ defined by $V(x, x^*) = ||x||^2 - 2 \langle x, x^* \rangle + ||x^*||^2$ for all $x \in E$ and $x^* \in E^*$, studied by Alber [1]. That is $V(x, x^*) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$.

Lemma 2.3 ([1]) *Let E be reflexive strictly convex and smooth Banach space with E^* as its dual. Then, $V(x, x^*) + 2 \langle J^{-1}x^* - x, y^* \rangle \leq V(x, x^* + y^*)$ for all $x \in E$ and $x^*, y^* \in E^*$.*

Lemma 2.4 ([1]) *Let C be a nonempty, closed and convex subset of a real smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle z - x_0, Jx - Jx_0 \rangle \leq 0, \quad \forall z \in C.$$

Lemma 2.5 ([3]) *Suppose that E is 2-uniformly convex Banach space. Then, there exists $\mu \geq 1$ such that $\frac{1}{\mu} ||x - y||^2 \leq \phi(x, y)$ for all $x, y \in E$.*

Lemma 2.6 ([33]) *Let E be a uniformly convex Banach space and $B_R(0)$ be a closed ball of E. Then, there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$||\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k||^2 \leq \sum_{i=0}^{k} \alpha_i ||x_i||^2 - \alpha_i \alpha_j g(||x_i - x_j||),$$

where $0 \leq i \leq j \leq k$, for each $\alpha_i \in (0, 1)$ and for $x_i \in B_R(0) := \{x \in E : ||x|| \leq R\}$, $i = 0, 1, 2, \ldots, k$ with $\sum_{i=0}^{k} \alpha_i = 1$.

Lemma 2.7 ([12]) *Let E be a real smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $x_n - y_n \to 0$ as $n \to \infty$.*

Remark 2.8 For any bounded sequences $\{x_n\}$ and $\{y_n\}$ in a 2-uniformly convex and uniformly smooth real Banach space E we have

$$x_n - y_n \to 0 \Leftrightarrow \phi(x_n, y_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{2.3}$$

Lemma 2.9 ([28]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \delta_n, n \geq n_0,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\delta_n\} \subset R$ satisfying the following conditions:
$\lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\limsup_{n \to \infty} \delta_n \leq 0$. *Then,* $\lim_{n \to \infty} a_n = 0$.

Lemma 2.10 ([16]) *Let $\{a_n\}$ be sequences of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$, for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

For the rest of this paper, Π_C is a generalized metric projection from E onto C, μ is a constant as in Lemma 2.5 and $\{\alpha_n\} \subset (0, c) \subset (0, 1)$ for all $n \geq 0$ satisfying $\lim_{n \to \infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$.

3 Main result

Theorem 3.1 *Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth real Banach space E. Let $A_i : C \to E^*$ be L_i-Lipschitz monotone mappings with Lipschitz constants L_i, for $i = 1, 2$. Let $T : C \to C$ be relatively nonexpansive mapping. Assume that $\mathcal{F} = VI(C, A_1) \cap VI(C, A_2) \cap F(T)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0, u \in C$ by*

$$\begin{cases} z_n = \Pi_C J^{-1}[Jx_n - \gamma_n A_2 x_n], \\ y_n = \Pi_C J^{-1}[Jx_n - \gamma_n A_1 x_n], \\ w_n = a_n Jx_n + b_n JTx_n + c_n J(u_n) + d_n J(v_n), \\ x_{n+1} = \Pi_C J^{-1}[\alpha_n Ju + (1 - \alpha_n)w_n], \end{cases} \tag{3.1}$$

where $u_n = \Pi_C J^{-1}[Jx_n - \gamma_n A_1 y_n]$, $v_n = \Pi_C J^{-1}[Jx_n - \gamma_n A_2 z_n]$, $\{\gamma_n\} \subset [a, b] \subset (0, \frac{1}{\mu L})$, for $L := \max\{L_1, L_2\}$ and $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\} \subset [e, 1) \subset (0, 1)$ such that $a_n + b_n + c_n + d_n = 1$. Then, $\{x_n\}$ converges strongly to a point x^* in $\mathcal{F}$ which is nearest to u with respect to the Lyapunov distance.

Proof Let $p \in \mathcal{F}$. Then, by (3.1), Lemma 2.2 and the monotonicity of A_1 we have

$$
\begin{aligned}
\phi(p, u_n) &= \phi(p, \Pi_C J^{-1}(Jx_n - \gamma_n A_1 y_n)) \\
&\leq \phi(p, J^{-1}(Jx_n - \gamma_n A_1 y_n)) - \phi(u_n, J^{-1}(Jx_n - \gamma_n A_1 y_n)) \\
&= ||p||^2 - 2 \langle p, Jx_n - \gamma_n A_1 y_n \rangle + ||Jx_n - \gamma_n A_1 y_n||^2 \\
&\quad - [||Jx_n - \gamma_n A_1 y_n||^2 - 2 \langle u_n, Jx_n - \gamma_n A_1 y_n \rangle + ||u_n||^2] \\
&= ||p||^2 - 2 \langle p, Jx_n \rangle + ||x_n||^2 - ||x_n||^2 + 2 \langle p, \gamma_n A_1 y_n \rangle \\
&\quad + 2 \langle u_n, Jx_n \rangle - ||u_n||^2 - 2 \langle u_n, \gamma_n A_1 y_n \rangle \\
&= \phi(p, x_n) - \phi(u_n, x_n) + 2 \langle p - u_n, \gamma_n A_1 y_n \rangle \\
&= \phi(p, x_n) - \phi(u_n, x_n) + 2 \langle p - y_n, \gamma_n A_1 y_n \rangle + 2 \langle y_n - u_n, \gamma_n A_1 y_n \rangle \\
&= \phi(p, x_n) - \phi(u_n, x_n) + 2\gamma_n \langle p - y_n, A_1 y_n - A_1 p \rangle + 2\gamma_n \langle p - y_n, A_1 p \rangle \\
&\quad + 2\gamma_n \langle y_n - u_n, A_1 y_n \rangle \\
&\leq \phi(p, x_n) - \phi(u_n, x_n) + 2\gamma_n \langle y_n - u_n, A_1 y_n \rangle,
\end{aligned}
\tag{3.2}
$$

and from (1.14), we obtain

$$
\phi(u_n, x_n) = \phi(u_n, y_n) + \phi(y_n, x_n) + 2 \langle u_n - y_n, Jy_n - Jx_n \rangle.
\tag{3.3}
$$

Thus, from (3.2) and (3.3) we get

$$
\begin{aligned}
\phi(p, u_n) &\leq \phi(p, x_n) - \phi(u_n, y_n) - \phi(y_n, x_n) \\
&\quad + 2 \langle y_n - u_n, \gamma_n A_1 y_n - Jx_n + Jy_n \rangle.
\end{aligned}
\tag{3.4}
$$

Moreover, by Lemma 2.4, we have that,

$$
\begin{aligned}
\langle y_n - u_n, \gamma_n A_1 y_n - Jx_n + Jy_n \rangle &= \langle u_n - y_n \gamma_n A_1 x_n - \gamma_n A_1 y_n \rangle \\
&\quad + \langle u_n - y_n, Jx_n - \gamma_n A_1 x_n - Jy_n \rangle \\
&\leq \gamma_n \langle u_n - y_n, A_1 x_n - A_1 y_n \rangle.
\end{aligned}
\tag{3.5}
$$

Then, since A_1 is Lipschitz, by (3.4) and (3.5), we have that,

$$
\begin{aligned}
\phi(p, u_n) &\leq \phi(p, x_n) - \phi(u_n, y_n) - \phi(y_n, x_n) \\
&\quad + 2\gamma_n \langle u_n - y_n, A_1 x_n - A_1 y_n \rangle \\
&\leq \phi(p, x_n) - \phi(u_n, y_n) - \phi(y_n, x_n) + 2\gamma_n ||u_n - y_n|| ||A_1 x_n - A_1 y_n|| \\
&\leq \phi(p, x_n) - \phi(u_n, y_n) - \phi(y_n, x_n) + 2L\gamma_n ||u_n - y_n|| ||x_n - y_n|| \\
&\leq \phi(p, x_n) - \phi(u_n, y_n) - \phi(y_n, x_n) + L\gamma_n[||u_n - y_n||^2 + ||x_n - y_n||^2].
\end{aligned}
\tag{3.6}
$$

Hence, from (3.6) and Lemma 2.5 we have,

$$
\begin{aligned}
\phi(p, u_n) &\leq \phi(p, x_n) - \frac{1}{\mu} ||y_n - x_n||^2 - \frac{1}{\mu} ||u_n - y_n||^2 \\
&\quad + L\gamma_n[||u_n - y_n||^2 + ||x_n - y_n||^2] \\
&\leq \phi(p, x_n) + \left(Lr_n - \frac{1}{\mu}\right)[||y_n - x_n||^2 + ||u_n - y_n||^2].
\end{aligned}
\tag{3.7}
$$

Likewise, we obtain that

$$
\phi(p, v_n) \leq \phi(p, x_n) + \left(Lr_n - \frac{1}{\mu}\right)[||z_n - x_n||^2 + ||v_n - z_n||^2].
\tag{3.8}
$$

Now, from (3.1), (3.7), (3.8), Lemma 2.2, Lemma 2.6 and the fact that $L\gamma_n - \frac{1}{\mu} < 0$, we have the following:

$$
\begin{aligned}
\phi(p, x_{n+1}) &= \phi(p, \Pi_C J^{-1}(\alpha_n Ju + (1-\alpha_n)w_n)) \\
&\leq \phi(p, J^{-1}(\alpha_n Ju + (1-\alpha_n)w_n)) \\
&= \phi(p, J^{-1}(\alpha_n Ju + (1-\alpha_n)[a_n Jx_n + b_n JTx_n + c_n Ju_n + d_n Jv_n])) \\
&\leq \alpha_n \phi(p, u) + (1-\alpha_n)[a_n \phi(p, x_n) + b_n \phi(p, x_n) + c_n \phi(p, u_n) \\
&\quad + d_n \phi(p, v_n) - a_n b_n g(\|JTx_n - Jx_n\|)] \\
&\leq \alpha_n \phi(p, u) + (1-\alpha_n)\phi(p, x_n) - (1-\alpha_n)a_n b_n g(\|JTx_n - Jx_n\|) \\
&\quad + (1-\alpha_n)\left(L\gamma_n - \frac{1}{\mu}\right)\left[c_n(\|u_n - y_n\|^2 + \|y_n - x_n\|^2)\right. \\
&\quad \left. + d_n(\|v_n - z_n\|^2 + \|z_n - x_n\|^2)\right] \\
&\leq \alpha_n \phi(p, u) + (1-\alpha_n)\phi(p, x_n).
\end{aligned}
\tag{3.9}
$$

Therefore, by induction we get that

$$
\phi(p, x_{n+1}) \leq \max\{\phi(p, u), \phi(p, x_0)\}, \quad \forall n \geq 0,
$$

which implies that $\{x_n\}, \{y_n\}, \{z_n\}, \{u_n\}$ and $\{v_n\}$ are bounded.

Let $x^* = \Pi_{\mathcal{F}} u$ and $t_n = J^{-1}(\alpha_n Ju + (1-\alpha_n)w_n)$. Then, using (3.1), Lemma 2.2, Lemma 2.3 and the fact that $L\gamma_n - \frac{1}{\mu} < 0$, we obtain

$$
\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi(x^*, \Pi_C J^{-1}(\alpha_n Ju + (1-\alpha_n)w_n)) \\
&\leq \phi(x^*, J^{-1}(\alpha_n Ju + (1-\alpha_n)[a_n Jx_n + b_n JTx_n + c_n Ju_n + d_n Jv_n])) \\
&= V(x^*, \alpha_n Ju + (1-\alpha_n)[a_n Jx_n + b_n JTx_n + c_n Ju_n + d_n Jv_n]) \\
&\leq V(x^*, (1-\alpha_n)[a_n Jx_n + b_n JTx_n + c_n Ju_n + d_n Jv_n] + \alpha_n Jx^*) \\
&\quad - 2\langle t_n - x^*, \alpha_n(Jx^* - Ju)\rangle,
\end{aligned}
$$

and

$$
\begin{aligned}
\phi(x^*, x_{n+1}) &\leq (1-\alpha_n)[a_n \phi(x^*, x_n) + b_n \phi(x^*, x_n) + c_n \phi(x^*, u_n) + d_n \phi(x^*, v_n)] \\
&\quad - (1-\alpha_n)a_n b_n g(\|JTx_n - Jx_n\|) + 2\alpha_n\langle t_n - x^*, Ju - Jx^*\rangle \\
&\leq (1-\alpha_n)\phi(x^*, x_n) + (1-\alpha_n)\left(L\gamma_n - \frac{1}{\mu}\right) \\
&\quad \times \left[c_n(\|u_n - y_n\|^2 + \|y_n - x_n\|^2) + d_n(\|v_n - z_n\|^2 + \|z_n - x_n\|^2)\right] \\
&\quad - (1-\alpha_n)a_n b_n g(\|JTx_n - Jx_n\|) + 2\alpha_n\langle t_n - x^*, Ju - Jx^*\rangle \\
&\leq (1-\alpha_n)\phi(x^*, x_n) + 2\alpha_n\langle t_n - x^*, Ju - Jx^*\rangle \\
&= (1-\alpha_n)\phi(x^*, x_n) + 2\alpha_n\langle t_n - x_n, Ju - Jx^*\rangle \\
&\quad + 2\alpha_n\langle x_n - x^*, Ju - Jx^*\rangle \\
&\leq (1-\alpha_n)\phi(x^*, x_n) + 2\alpha_n\|t_n - x_n\|\|Ju - Jx^*\| \\
&\quad + 2\alpha_n\langle x_n - x^*, Ju - Jx^*\rangle.
\end{aligned}
\tag{3.11}
$$

(3.10)

Now, we consider two cases.

Case 1 Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\phi(x^*, x_n)\}$ is decreasing for all $n \geq n_0$. Then, we get that $\{\phi(x^*, x_n)\}$ is convergent. Thus, from (3.10) and the fact that $a_n, b_n, c_n, d_n \geq e > 0$, $\gamma_n < b < \frac{1}{\mu L}$ for all $n \geq 0$ and $\alpha_n \to 0$ as $n \to \infty$, we have that

$$
\begin{aligned}
u_n - y_n &\to 0, \quad y_n - x_n \to 0, \quad z_n - v_n \to 0, \\
z_n - x_n &\to 0, \quad JTx_n - Jx_n \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}
\tag{3.12}
$$

As E^* is uniformly smooth , J^{-1} is uniformly continuous on bounded subsets of E^* and hence we obtain

$$J^{-1}(JTx_n) - J^{-1}(Jx_n) = Tx_n - x_n \to 0 \quad \text{as} \quad n \to \infty. \tag{3.13}$$

Moreover, by the property of ϕ, we have

$$\begin{aligned}
\phi(x_n, t_n) &= \phi\left(x_n, J^{-1}(\alpha_n Ju + (1-\alpha_n)[a_n Jx_n + b_n JTx_n + c_n Ju_n + d_n Jv_n])\right) \\
&\leq \alpha_n \phi(x_n, u) + (1-\alpha_n)[a_n \phi(x_n, x_n) + b_n \phi(x_n, Tx_n) \\
&\quad + c_n \phi(x_n, u_n) + d_n \phi(x_n, v_n)].
\end{aligned}$$

This together with (2.3), (3.12) and (3.13) implies that

$$\phi(x_n, t_n) \to 0 \quad \text{as} \quad n \to \infty, \tag{3.14}$$

which in turn implies that

$$x_n - t_n \to 0 \quad \text{as} \quad n \to \infty. \tag{3.15}$$

Furthermore, since $\{x_n\}$ is bounded subset of E which is reflexive, we can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z$ and $\limsup_{n\to\infty}\langle x_n - x^*, Ju - Jx^*\rangle = \lim_{j\to\infty}\langle x_{n_j} - x^*, Ju - Jx^*\rangle$. This implies from (3.12) that $z_{n_j} \rightharpoonup z$, $y_{n_j} \rightharpoonup z$, $u_{n_j} \rightharpoonup z$ and $v_{n_j} \rightharpoonup z$ as $j \to \infty$.

Note that relatively nonexpansiveness of T implies that $z \in F(T)$.

Now, we show that $z \in \cap_{i=1}^2 \mathrm{VI}(C, A_i)$. Since A_i, for each $i \in \{1, 2\}$, is Lipschitz continuous, we have

$$\|A_1 u_{n_j} - A_1 y_{n_j}\| \to 0 \quad \text{as} \quad j \to \infty.$$

Now, let

$$B_1 x = \begin{cases} A_1 x + N_C x, & \text{if} \quad x \in C, \\ \emptyset, & \text{if} \quad x \notin C, \end{cases} \tag{3.16}$$

where $N_C(x)$ is the normal cone to C at $x \in C$ given by $N_C(x) = \{w \in E^* : \langle x - u, w\rangle \geq 0 \text{ for all } u \in C\}$. Then, B_1 is maximal monotone and $0 \in B_1 x$ if and only if $x \in VI(C, A_1)$ (see, e.g. [21]). Let $(v, w) \in G(B_1)$. Then, we have $w \in B_1 v = A_1 v + N_C v$ and hence $w - A_1 v \in N_C v$. Thus, we get $\langle v - u, w - A_1 v\rangle \geq 0$, for all $u \in C$. On the other hand, since $u_{n_j} = \Pi_C J^{-1}(Jx_{n_j} - \gamma_{n_j} A_1 y_{n_j})$ and $v \in C$, we have $\langle Jx_{n_j} - \gamma_{n_j} A_1 y_{n_j} - Ju_{n_j}, u_{n_j} - v\rangle \geq 0$, and hence $\langle v - u_{n_j}, (Ju_{n_j} - Jx_{n_j})/\gamma_{n_j} + A_1 y_{n_j}\rangle \geq 0$. Thus, as $u_{n_j} \in C$, the above imply that

$$\begin{aligned}
\langle v - u_{n_j}, w\rangle &\geq \langle v - u_{n_j}, A_1 v\rangle \\
&\geq \langle v - u_{n_j}, A_1 v\rangle - \langle v - u_{n_j}, (Ju_{n_j} - Jx_{n_j})/\gamma_{n_j} + A_1 y_{n_j}\rangle \\
&= \langle v - u_{n_j}, A_1 v - A_1 u_{n_j}\rangle + \langle v - u_{n_j}, A_1 u_{n_j} - A_1 y_{n_j}\rangle \\
&\quad - \langle v - u_{n_j}, (Ju_{n_j} - Jx_{n_j})/\gamma_{n_j}\rangle \\
&\geq \langle v - u_{n_j}, A_1 u_{n_j} - A_1 y_{n_j}\rangle - \langle v - u_{n_j}, (Ju_{n_j} - Jx_{n_j})/\gamma_{n_j}\rangle. \tag{3.17}
\end{aligned}$$

Therefore, since E is uniformly smooth we have that J is uniformly continuous and hence $Ju_n - Jx_n \to 0$. Thus, from (3.17) we get that $\langle v - z, w\rangle \geq 0$ as $j \to \infty$. Then, maximality of B_1 gives that $z \in B_1^{-1}(0)$. Therefore, $z \in \mathrm{VI}(C, A_1)$. Similarly, with the use of $v_{n_j} = \Pi_C J^{-1}(Jx_{n_j} - \gamma_{n_j} A_2 z_{n_j})$ we get that $z \in \mathrm{VI}(C, A_2)$. Thus, we have that $z \in \mathrm{VI}(C, A_1) \cap \mathrm{VI}(C, A_2) \cap F(T)$. Therefore, by Lemma 2.4, we immediately obtain that

$$\begin{aligned}
\limsup_{n\to\infty}\langle x_n - x^*, Ju - Jx^*\rangle &= \lim_{j\to\infty}\langle x_{n_j} - x^*, Ju - Jx^*\rangle \\
&= \langle z - x^*, Ju - Jx^*\rangle \leq 0. \tag{3.18}
\end{aligned}$$

Hence, it follows from (3.11), (3.15), (3.18) and Lemma 2.9 that $\phi(x^*, x_n) \to 0$ as $n \to \infty$. Then, Lemma 2.7 implies that $x_n \to x^* = \Pi_{\mathcal{F}} u$.

Case 2 Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\phi(x^*, x_{n_i}) < \phi(x^*, x_{n_i+1}),$$

for all $i \in \mathbb{N}$. Then, by Lemma 2.10, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, and

$$\phi(x^*, x_{m_k}) \leq \phi(x^*, x_{m_{k+1}}) \quad \text{and} \quad \phi(x^*, x_k) \leq \phi(x^*, x_{m_{k+1}}), \tag{3.19}$$

for all $k \in \mathbb{N}$. Now, from (3.10) and the fact that $a_n, b_n, c_n, d_n \geq e > 0, \gamma_n < b < \frac{1}{\mu L}$ for all $n \geq 0$ and $\alpha_n \to 0$ as $n \to \infty$, we get that $u_{m_k} - y_{m_k} \to 0, y_{m_k} - x_{m_k} \to 0, z_{m_k} - v_{m_k} \to 0, z_{m_k} - x_{m_k} \to 0, JTx_{m_k} - Jx_{m_k} \to 0$ as $k \to \infty$. Thus, following the method in Case 1, we obtain

$$\limsup_{k \to \infty} \langle x_{m_k} - x^*, Ju - Jx^* \rangle \leq 0. \tag{3.20}$$

Now, from (3.11), we have that

$$\phi(x^*, x_{m_{k+1}}) \leq (1 - \alpha_{m_k})\phi(x^*, x_{m_k}) + 2\alpha_{m_k}\langle x_{m_k} - x^*, Ju - Jx^* \rangle$$
$$+ 2\alpha_{m_k}||t_{m_k} - x_{m_k}||.||Ju - Jx^*||, \tag{3.21}$$

and hence (3.19) and (3.21) imply that

$$\alpha_{m_k}\phi(x^*, x_{m_k}) \leq \phi(x^*, x_{mk}) - \phi(x^*, x_{m_{k+1}}) + 2\alpha_{m_k}\langle x_{m_k} - x^*, Ju - Jx^* \rangle$$
$$+ 2\alpha_{m_k}||t_{m_k} - x_{m_k}||.||Ju - Jx^*||$$
$$\leq 2\alpha_{m_k}\langle x_{m_k} - x^*, Ju - Jx^* \rangle$$
$$+ 2\alpha_{m_k}||t_{m_k} - x_{m_k}||.||Ju - Jx^*||. \tag{3.22}$$

But the fact that $\alpha_{m_k} > 0$ implies that

$$\phi(x^*, x_{m_k}) \leq 2\langle x_{m_k} - x^*, Ju - Jx^* \rangle + 2||t_{m_k} - x_{m_k}||.||Ju - Jx^*||.$$

Thus, using (3.15) and (3.20) we get that $\phi(x^*, x_{m_k}) \to 0$ as $k \to \infty$. This together with (3.21) implies that $\phi(x^*, x_{m_{k+1}}) \to 0$ as $k \to \infty$. But $\phi(x^*, x_k) \leq \phi(x^*, x_{m_{k+1}})$ for all $k \in \mathbb{N}$ gives that $x_k \to x^*$. Therefore, from the above two cases, we can conclude that $\{x_n\}$ converges strongly to a point $x^* = \Pi_{\mathcal{F}}u$. The proof is complete.

If, in Theorem 3.1, we assume that $T = I$, the identity mapping on C, then we obtain the following corollary.

Corollary 3.2 *Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth real Banach space E. Let $A_i : C \to E^*$ be L_i-Lipschitz monotone mappings with Lipschitz constants L_i, for $i = 1, 2$. Let u be any point in C. Assume that $\mathcal{F} = VI(C, A_1) \cap VI(C, A_2)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in C$ by*

$$\begin{cases} z_n = \Pi_C J^{-1}[Jx_n - \gamma_n A_2 x_n], \\ y_n = \Pi_C J^{-1}[Jx_n - \gamma_n A_1 x_n], \\ w_n = a_n Jx_n + b_n J(u_n) + c_n J(v_n), \\ x_{n+1} = \Pi_C J^{-1}[\alpha_n Ju + (1 - \alpha_n)w_n], \end{cases} \tag{3.23}$$

where $u_n = \Pi_C J^{-1}[Jx_n - \gamma_n A_1 y_n]), v_n = \Pi_C J^{-1}[Jx_n - \gamma_n A_2 z_n], \{\gamma_n\} \subset [a, b] \subset (0, \frac{1}{\mu L}),$ for $L := \max\{L_1, L_2\}$ and $\{a_n\}, \{b_n\}, \{c_n\} \subset [e, 1) \subset (0, 1)$ such that $a_n + b_n + c_n = 1$. Then, $\{x_n\}$ converges strongly to a point x^* in $\mathcal{F}$ which is nearest to u with respect to the Lyapunov distance.

If, in Corollary 3.2, we consider $A_1 = 0$ or $A_2 = 0$, then we obtain the following corollary for one variational inequality problem for a monotone mapping A.

Corollary 3.3 *Let C be a nonempty, closed and convex subset of a rm 2-uniformly convex and uniformly smooth real Banach space E. Let $A : C \to E^*$ be a Lipschitz monotone mapping with Lipschitz constant L and let u be any point in C. Assume that $VI(C, A)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in C$ by*

$$\begin{cases} y_n = \Pi_C J^{-1}[Jx_n - \gamma_n Ax_n], \\ x_{n+1} = \Pi_C J^{-1}[\alpha_n Ju + (1 - \alpha_n)(a_n Jx_n + (1 - a_n)J(u_n))], \end{cases}$$

where, $u_n = \Pi_C J^{-1}[Jx_n - \gamma_n Ay_n], \{\gamma_n\} \subset [a, b] \subset (0, \frac{1}{\mu L})$ and $\{a_n\} \subset [e, 1) \subset (0, 1)$. Then, $\{x_n\}$ converges strongly to a point x^* in $\mathcal{F}$ which is nearest to u with respect to the Lyapunov distance.

If, in Theorem 3.1, we assume that A_i, for $i = 1, 2$, are γ_i-inverse strongly monotone mappings, then both are L-Lipschitz with constant $L = \max\left\{\frac{1}{\gamma_1}, \frac{1}{\gamma_2}\right\}$ and hence we get the following corollary.

Corollary 3.4 *Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth real Banach space E. Let $A_i : C \to E^*$ be an γ_i-inverse strongly monotone mappings, for $i = 1, 2$. Let $T : C \to C$ be relatively nonexpansive mapping. Assume that $\mathcal{F} = VI(C, A_1) \cap VI(C, A_2) \cap F(T)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $u, x_0 \in C$ by*

$$\begin{cases} z_n = \Pi_C J^{-1}[Jx_n - \gamma_n A_2 x_n], \\ y_n = \Pi_C J^{-1}[Jx_n - \gamma_n A_1 x_n], \\ w_n = a_n Jx_n + b_n JTx_n + c_n J(u_n) + d_n J(v_n), \\ x_{n+1} = \Pi_C J^{-1}\big[\alpha_n Ju + (1 - \alpha_n)w_n\big], \end{cases} \tag{3.24}$$

where $u_n = \Pi_C J^{-1}[Jx_n - \gamma_n A_1 y_n])$, $v_n = \Pi_C J^{-1}[Jx_n - \gamma_n A_2 z_n]$, $\{\gamma_n\} \subset [a, b] \subset (0, \frac{1}{\mu L})$, for $L := \max\{\frac{1}{\gamma_1}, \frac{1}{\gamma_2}\}$, and $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\} \subset [e, 1) \subset (0, 1)$ such that $a_n + b_n + c_n + d_n = 1$. Then, $\{x_n\}$ converges strongly to a point x^* in $\mathcal{F}$ which is nearest to u with respect to the Lyapunov distance.

If, in Theorem 3.1, we assume that $C = E$, then the projection mapping Π_C is reduced to the identity mapping in E and $VI(C, A_1) = A_1^{-1}(0)$, $VI(C, A_2) = A_2^{-1}(0)$. Thus, we get the following corollary.

Corollary 3.5 *Let E be a 2-uniformly convex and uniformly smooth real Banach space E. Let $A_i : E \to E^*$ be L_i-Lipschitz monotone mappings with Lipschitz constants L_i, for $i = 1, 2$. Let $T : E \to E$ be relatively nonexpansive mapping. Assume that $\mathcal{F} = A_1^{-1}(0) \cap A_2^{-1}(0) \cap F(T)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $u, x_0 \in C$ by*

$$\begin{cases} z_n = J^{-1}[Jx_n - \gamma_n A_2 x_n], \\ y_n = J^{-1}[Jx_n - \gamma_n A_1 x_n], \\ w_n = a_n Jx_n + b_n JTx_n + c_n J(u_n) + d_n J(v_n), \\ x_{n+1} = J^{-1}\big[\alpha_n Ju + (1 - \alpha_n)w_n\big], \end{cases} \tag{3.25}$$

where $u_n = J^{-1}[Jx_n - \gamma_n A_1 y_n])$, $v_n = J^{-1}[Jx_n - \gamma_n A_2 z_n]$, $\{\gamma_n\} \subset [a, b] \subset (0, \frac{1}{\mu L})$, for $L := \max\{L_1, L_2\}$ and $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\} \subset [e, 1) \subset (0, 1)$ such that $a_n + b_n + c_n + d_n = 1$. Then, $\{x_n\}$ converges strongly to a point x^* in $\mathcal{F}$ which is nearest to u with respect to the Lyapunov distance.

We note that the method of proof of Theorem 3.1 provides a convergence theorem for a finite family of Lipschitzian monotone mappings. In fact, we have the following theorem.

Theorem 3.6 *Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth real Banach space E. Let $A_i : C \to E^*$ be L_i-Lipschitz monotone mappings with Lipschitz constants L_i, for $i = 2, 3, \ldots, N$. Let $T : C \to C$ be relatively nonexpansive mapping. Assume that $\mathcal{F} = \cap_{i=2}^N VI(C, A_i) \cap F(T)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $u, x_0 \in C$ by*

$$\begin{cases} y_{ni} = \Pi_C J^{-1}[Jx_n - \gamma_n A_i x_n], \\ x_{n+1} = \Pi_C J^{-1}\left[\alpha_n Ju + (1 - \alpha_n)\left(b_{n0} Jx_n + b_{n1} JTx_n + \sum_{i=2}^N b_{ni} J(u_{ni})\right)\right], \end{cases}$$

where $u_{ni} = \Pi_C J^{-1}[Jx_n - \gamma_n A_i y_{ni}]$, $\{\gamma_n\} \subset [a, b] \subset (0, \frac{1}{\mu L})$, for $L := \max\{L_i : i = 2, 3, \ldots, N\}$, $\{b_{ni}\} \subset [e, 1) \subset (0, 1)$ such that $\sum_{i=0}^N b_{ni} = 1$. Then, $\{x_n\}$ converges strongly to a point x^* in $\mathcal{F}$ which is nearest to u with respect to the Lyapunov distance.

We also note that the method of proof of Theorem 3.1 provides the following theorem for approximating the minimum-norm point of solution of two variational inequality problems.

Theorem 3.7 *Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth real Banach space E. Let $A_i : C \to E^*$ be an L_i-Lipschitz monotone mappings with Lipschitz constants L_i, for $i = 1, 2$. Assume that $\mathcal{F} = \cap_{i=1}^2 VI(C, A_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in E$ by*

$$\begin{cases} z_n = \Pi_C J^{-1}[Jx_n - \gamma_n A_2 x_n], \\ y_n = \Pi_C J^{-1}[Jx_n - \gamma_n A_1 x_n], \\ x_{n+1} = \Pi_C J^{-1}\big[(1 - \alpha_n)\big(a_n Jx_n + b_n JTx_n + c_n J(u_n) + d_n J(v_n)\big)\big], \end{cases} \quad (3.26)$$

where $u_n = \Pi_C J^{-1}[Jx_n - \gamma_n A_1 y_n]$, $v_n = \Pi_C J^{-1}[Jx_n - \gamma_n A_2 z_n]$, $\{\gamma_n\} \subset [a, b] \subset (0, \frac{1}{\mu L})$, for $L := \max\{L_1, L_2\}$, $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\} \subset [e, 1) \subset (0, 1)$ such that $a_n + b_n + c_n + d_n = 1$. Then, $\{x_n\}$ converges strongly to a minimum-norm point x^* of $\mathcal{F}$ with respect to the Lyapunov distance.

Remark 3.8 Theorem 3.1 extends Theorem 3.1 of Nadezhkina and Takahashi [17] in the sense that our scheme provides *strong convergence* to a common point of the fixed point set of a relatively nonexpansive mapping and solutions of two variational inequality problems for Lipschitz monotone mappings in spaces more general than Hilbert spaces.

Remark 3.9 Theorem 3.1 also extends Theorem 3.1 of Iiduka and Takahashi [10] and Theorem 3.2 of Zegeye et al. [33] in the sense that our scheme provides strong convergence to a common point of the fixed point set of a relatively nonexpansive mapping and solutions of two variational inequality problems for a more general class of Lipschitz monotone mappings in 2-uniformly convex and uniformly smooth real Banach spaces.

Remark 3.10 Corollary 3.2 extends Theorem 1 of Yao et al. [32] and Corollary 3.2 of Zegeye and Shahzad [38] in the sense that our scheme provides strong convergence to a common point of solutions of two variational inequality problems in spaces more general than Hilbert spaces.

4 Numerical example

Now, we give an example of a relatively nonexpansive mapping and two Lipschitzian monotone mappings satisfying Theorem 3.1 and some numerical experiment result to explain the conclusion of the theorem as follows:

Example 1 Let $H = \mathbb{R}$ with Euclidean norm. Let $C = [-1, 2]$ and $T : C \to C$ be defined by

$$Tx = \begin{cases} x^2, & x \in [-1, 0), \\ x, & x \in [0, 2], \end{cases} \quad (4.1)$$

and $A_1, A_2 : C \to \mathbb{R}$ be defined by

$$A_1 x = \begin{cases} 0, & x \in [-1, 1], \\ (x - 1)^2, & x \in (1, 2]. \end{cases} \quad (4.2)$$

$$A_2 x = \begin{cases} 2x, & x \in [-1, 0), \\ 0, & x \in [0, \frac{1}{2}), \\ x - \frac{1}{2}, & x \in [\frac{1}{2}, 2]. \end{cases} \quad (4.3)$$

Then, we observe that, T is continuous and hence $F(T) = [0, 2] = \widehat{F}(T)$.

Now, if $p \in F(T)$, then we have

$$\phi(p, Tx) = \begin{cases} (p - x^2)^2, & x \in [-1, 0), \\ (p - x)^2, & x \in [0, 2], \end{cases} \quad (4.4)$$

and $\phi(p, x) = (p - x)^2$ for all $x \in [-1, 2]$.

Thus, $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in [-1, 2]$ and hence T is relatively nonexpansive mapping which is not nonexpansive since for $x = -0.6$ and $y = -0.7$, we have $|Tx - Ty| = 0.13 > 0.1 = |x - y|$.

Moreover, we easily see that A_1 and A_2 are monotone mappings. Next, we show that A_1 is Lipschitz.

Case 1 Let $x, y \in [-1, 1]$. Then we have:

$$|A_1 x - A_1 y| = |0 - 0| = 0 \le |x - y|. \tag{4.5}$$

Case 2 Let $x \in [-1, 1]$ and $y \in [1, 2]$. Then we have:

$$|A_1 x - A_1 y| = |0 - (y - 1)^2| = (y - 1)^2 \le |y - x| = |x - y|. \tag{4.6}$$

Case 3 Let $x, y \in [1, 2]$. Then we have:

$$
\begin{aligned}
|A_1 x - A_1 y| = |(x - 1)^2 - (y - 1)^2| &\le |x^2 - y^2| + 2|x - y| \\
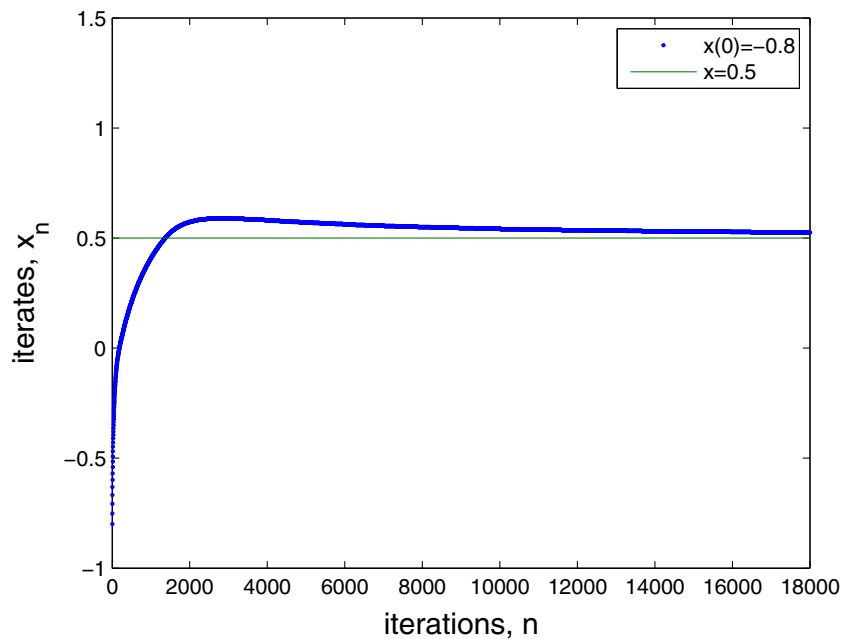&\le |x + y||x - y| + 2|x - y| \\
&\le 4|x - y| + 2|x - y| = 6|x - y|.
\end{aligned}
$$

From, Cases 1, 2 and 3 we conclude that A_1 is Lipschitz with Lipschitz constant $L_1 = 6$. Similarly, one can show that A_2 is Lipschitz mapping with Lipschitz constant $L_2 = 3$. It is also clear that $VI(C, A_1) \cap VI(C, A_2) \cap F(T) = \left[0, \frac{1}{2}\right]$.

Now, if we take, $\alpha_n = \frac{1}{n+100}$, $\gamma_n = \frac{1}{n+200} + 0.001$, $a_n = b_n = c_n = \frac{1}{n+100} + 0.001$, $d_n = 1 - \frac{3}{n+100} - 0.003$, and $u = 1.0$, we observe that the conditions of Theorem 3.1 are satisfied and Scheme (3.1) reduces to

$$
\begin{cases}
z_n = P_C[x_n - \gamma_n A_2 x_n], \\
y_n = P_C[x_n - \gamma_n A_1 x_n], \\
x_{n+1} = \alpha_n u + (1 - \alpha_n)(a_n x_n + b_n T x_n + c_n u_n + d_n v_n),
\end{cases}
\tag{4.7}
$$

where $u_n = P_C[x_n - \gamma_n A_1 y_n]$ and $v_n = P_C[x_n - \gamma_n A_2 z_n]$. Thus, for $x_0 = -0.8$, Scheme (4.7) converges strongly to $0.5 = P_{\mathcal{F}}(u)$. See the following table and figure.

n	0	184	1364	1365	3512	30,000	40,000	70,000	89,999
x_n	−0.8000	0.0010	0.4999	0.5001	0.5854	0.5156	0.5119	0.5070	0.5054

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