



# Critical Theorising from Studies of Undergraduate Mathematics Teaching for Students' Meaning Making in Mathematics

Barbara Jaworski<sup>1</sup> · Angeliki Mali<sup>1</sup> ·  
Georgia Petropoulou<sup>2</sup>

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**Abstract** This paper focuses directly on mathematics teaching practice at university level, exploring “what teachers do and think daily, in class and out, as they perform their teaching work” (Speer et al. in *The Journal of Mathematical Behavior*, 29, 99–114, 2010, p. 99). It is based on a small corpus of research conducted from sociocultural perspectives with qualitative and interpretative studies into mathematics teaching and the meanings it promotes for students learning mathematics. Two settings of teaching are explored: lectures and small group tutorials in which teachers are research mathematicians or mathematics educators. Analyses make use of a range of theoretical perspectives to interpret the complexities of teaching based on observations of teaching activity and interviews with teachers. Critical Theorising is seen through critical use of existing theory and construction of new theory: through the analytical process it provides insights into teaching and ways in which teaching does or can develop to promote students' meaning making in mathematics. The paper draws attention to the finer details of teaching practice and teachers' thinking that lie behind judgments such as ‘traditional’ or ‘transmissionist’.

**Keywords** University mathematics teaching · Mathematics teaching practice · Students' meaning making in mathematics · Critical theorising

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✉ Barbara Jaworski  
[b.jaworski@lboro.ac.uk](mailto:b.jaworski@lboro.ac.uk)

Angeliki Mali  
[a.mali@lboro.ac.uk](mailto:a.mali@lboro.ac.uk)

Georgia Petropoulou  
[enaprosena@yahoo.gr](mailto:enaprosena@yahoo.gr)

<sup>1</sup> Loughborough University, Loughborough, UK

<sup>2</sup> University of Athens, Athens, Greece

## Research Focus, Aims and Questions

A university (or college) mathematics teacher is given the task of teaching a particular course, or module, in a given mathematical topic (for example, calculus, or linear algebra), to a given cohort of undergraduate students. The teacher might be a mathematician (actively engaged in research in mathematics) or a mathematics educator (actively engaged in research in mathematics education).<sup>1</sup> We ask how he or she conceptualises the teaching—how aspects of mathematics, didactics and pedagogy figure in his or her thinking; how the teacher thinks about the students; how he or she uses the resource environment to facilitate teaching. We are also interested in what the teaching looks like: this includes the interactive setting, the roles taken by its participants, and the ways in which abstract topics in mathematics are approached and worked on. We ask particularly how the teacher addresses the meaning-making of the students and works with them to develop established mathematical meanings.

In order to address these questions, as mathematics education researchers, we collaborate with mathematicians or mathematics educators, teaching mathematics at university level, to gain insights into the processes and practices of teaching as they relate to creating students' meaning-making in mathematics. Our principal research question derives from Speer et al. 2010, who refer to “what teachers do and think daily, in class and out, as they perform their teaching work” (p. 99). Thus our research addresses:

What is it that mathematics teachers do and think as they perform their teaching work in a university setting, and how does this relate to the mathematical meaning making of their students?

## Mathematics Teaching at the University Level

Traditionally, at university level, students are taught in large groups, often with hundreds of students. Most of the teaching is delivered through a uni-vocal lecture format often described as *transmission teaching*: the teacher is a lecturer who exposes the mathematics for the students who listen, copy from the board and go away to make their own meanings from the experience (e.g., Alsina 2001; Pritchard 2010; Wagner et al. 2007; Weber 2004). It has been observed that many students have difficulties with the mathematics presented, not being prepared from their school experience for the degrees of abstraction and formalism experienced through such lectures (e.g., Dorier and Sierpiska 2001; Hawkes and Savage 2000; Nardi 2008; Stewart and Thomas 2010). These observations are significant because the cohorts of students who experience such teaching and learning go on to key roles in society, becoming future engineers, scientists, financial analysts, and, importantly, *mathematics teachers* at school level. The mathematical understandings of such teachers, the meanings they make of mathematics in their university setting, form the basis of their future teaching, both in terms of their concepts of mathematics and also their philosophy in developing

<sup>1</sup> We are NOT talking about graduate students teaching mathematics.

their students' understandings. If they have been enculturated into seeing teaching as a process of transmitting knowledge, then their own teaching is most likely also to take this form (Lortie 1975; Romberg and Carpenter 1986; Stigler and Hiebert 1999).

## Teaching for Promoting Students' Mathematical Meaning-Making

We are interested in what it looks like to “provide meaning” for students in ways that students can make their own sense; how teachers communicate their own meanings to students and how they foster students' meanings. Artique et al. (2007) suggest that learning at university level is seen as *enculturation* in advanced mathematical practices: if we see advanced mathematics to involve ways of being and thinking which embrace abstraction and formalism, then learners have to grow into these ways of thinking. Thus teaching can be seen to take an enculturative dimension. Similarly, Ben-Zvi and Arcavi (2001) write that making meaning in mathematics is a process of “socialisation” into the culture and values of “doing mathematics” (p. 36). We need to know more about these processes of socialization and enculturation, how they are manifested in the teaching that students experience and how teachers think about drawing their students into the mathematical cultures of which they themselves are a part.

We bring a sociocultural and ethical/moral perspective to the questions above. Thus, we take a Vygotskian approach to learning and teaching, seeing learning as occurring through social participation mediated by tools, both psychological and material, with the teacher having a significant role and responsibility in the learning process. We take account of as much of the *social context* of the teaching-learning environment as we can while attending to the need for anonymity of its participants. We consider a study of teaching to make the same demands as Clarke (2002) observes for a study of learning:

[A] study of learning in classroom settings would be incomplete without the simultaneous documentation of the social and cultural practices in which the learner participated, the instructional materials, and...the teacher's actions that preceded and followed the learning under investigation” (Clarke 2002; Cited by Weber 2004, p 131).

Such factors should also be central to analyses of teaching.

So, starting from the questions above, which are about teaching, we seek to conceptualise teaching activity in relation to the learning with which it is associated—the learning of mathematics in a university environment. Within these assumptions, we believe that a teacher has goals for the learning of his/her students, beliefs or values about what constitutes learning and how to achieve it and ways of acting in relation to the environment in which action takes place. These factors will be addressed through the specific studies we discuss below.

## Methodology

We report from four studies in which we have been involved and which seek to characterise teaching processes related to the learning of the students who are taught

and the setting in which the teaching takes place. These studies start to form a corpus of research “focused directly on teaching practice [at university level]—what teachers do and think daily, in class and out, as they perform their teaching work” (Speer et al. 2010, p. 99). They arise from an in-depth focus on specific examples of teaching practice with qualitative analyses which reveal aspects of teaching, and capture the intentions and reflective thinking of the teacher. They expand on existing theory and introduce new theory, illuminate the nature of teaching at this level and its relations to students’ meaning-making, raise and elaborate on issues that arise in the practice of teaching and offer insights that can be of relevance and significance more generally. Their commonality is that they have studied, or are studying, university teaching in either lectures or small group tutorials, focusing on the activity of particular teachers. The teachers concerned are largely chosen through circumstances which are convenient to the nature and location of the study. There is no attempt to claim that the teachers are representative of teachers more generally; however, they are also not untypical of teachers who work in the contexts studied.

The teachers we have studied teach in *both* lecturing and small group tutorial (SGT) settings; however, in our studies we view them in only one of these settings. Thus we see them as lecturers or as small group tutors. The lecturer typically works with a large cohort of students (50–200+) while the tutor works with a small group of students (2–8) focusing on problems set by the lecturers in the mathematics courses or modules the students take in their first year studies. Our purpose in conducting these studies has been to try to get inside the thinking of teachers as they plan to teach and as they act in teaching practice. We take for granted that teaching implies something to be learned (here, mathematics) and someone who is supposed to learn it (here, the university students). We therefore explore the ways in which teachers think about their students and act to promote students’ meaning-making in mathematics. Thus we focus on dimensions of mathematics, didactics and pedagogy and relations between them. We see didactics in the ways in which teachers work with the mathematics and construct their approach to offering it to students, and pedagogy in the ways they interact with the students in the medium of the teaching activity. Because these dimensions are deeply inter-related, we do not always specify them separately.

Our methodological approach therefore uses qualitative methods to gather data through which to address research questions. These methods largely involve observation of teaching-learning events and interviews or conversations with the teachers. Data consist of notes from, or recordings (audio or video) of lectures or tutorials, including words uttered by teachers and students in monologue or dialogue and use of a variety of media or tools, material or psychological (including projectors, visualizers, computers, software, VLE, boards, chalk or pens, printed materials, etc. and mathematical symbols, representations and examples). Analyses are grounded in the data: they include close and repeated scrutiny of the contents of written, audio, or video materials (possibly with transcription), with related coding and categorization (with a suitable grain size). They take into account research questions and methodological assumptions leading to interpretations grounded in the data from which critical theorising has taken place. We will say more about critical theorising below. Such analyses do not provide ‘answers’ but nevertheless suggest experiences and principles which might be seen to be germane in a wider variety of contexts.

In all of these studies theory has either emerged from the grounded approach or been used to make sense of findings emerging from the grounded analysis. We cannot go far in such exploration without the use of a terminology to describe or explain what we are looking for and what we see. Expanding on the words above we try to express teachers' *intentions* for their teaching and their *practice* in teaching events. For example, in Study C, terminology of *tools* and *strategies* is used to categorise teaching. These terms build on the Vygotskian idea of 'tool' as mediating between subject and object, such as between the tutor and his intentions for students' learning. In Study A, intentions and practice have been discussed in terms of *goals* and *actions*, terminology which is taken from Leont'ev's writing on Activity Theory (e.g., Leont'ev 1979). Since we operate within a sociocultural context, Activity Theory is theoretically commensurable, and has been used in several related studies (Jaworski and Goodchild 2006; Jaworski and Potari 2009) to offer frameworks for making sense of the complexities of teaching. Other frameworks we have used also contribute to this aim, i.e., making sense of the complexities of teaching—the Teaching Triad (e.g., Jaworski 1994; Potari and Jaworski 2002) and the Spectrum of Pedagogical Awareness (e.g., Nardi et al. 2005, p. 289); these are elaborated further below. Thus, the theoretical constructs in these frameworks and the associated terminology have developed precisely to try to make sense of the complexity of the teaching process within a sociocultural frame.

As the methodological literature has emphasized over almost half a century, there are many issues in applying theory within a grounded approach to analysis (e.g. Charmaz 2006; Glaser 1978, 1992, 1998; Strauss and Corbin 1990). The very methods used for collecting data beg questions about the nature of the data and interpretations from the data. Here, we recognize that talking with teachers about what they do and why they do it results in teachers reflecting more consciously on their practice in an attempt to try to explain something that has been intuitive and grounded in practice (their own and that of others around them). In these respects the research itself prompts the data it collects. It has therefore been important to analyse, side by side, the observations we have made in teaching events and what teachers say about what they do in those events; taking the view that teachers' expressions of their teaching derive from *Sensible Systems* in which any inconsistencies derive from problems in articulation of belief or of researcher interpretations (Leatham 2006). We therefore see ourselves as conducting *developmental research* in partnership with those teaching mathematics, with the concomitant development of knowledge and understanding of teaching processes for the teachers concerned.

## Associated Literature

Before embarking on the four studies, we refer to some of the literature relating to mathematics teaching at university level which offers a backdrop to this work. A literature review relating to undergraduate mathematics teaching (Treffert-Thomas and Jaworski 2015) made a distinction between three sorts of literature in this area: *research*, *professional*, and *pedagogical* literature. The *professional* literature offers accounts from the personal experiences of mathematicians or mathematics educators reporting from and reflecting on their teaching practice. The *pedagogic* literature goes further to draw out and distil aspects of pedagogy on which a teaching approach is

based. While not underestimating the value of such accounts in providing insights to practices and their use of pedagogy, our main focus here is on the *research* literature which *accounts for* what is observed, offering theoretical perspectives to explain and justify outcomes. We have cited the paper by Speer et al. (2010) from which we quoted above: “very little research has focused directly on teaching practice [at university level]—what teachers do and think daily, in class and out, as they perform their teaching work” (p. 99). Since this is precisely our area of research focus it seemed important to seek out the ‘little’ that has been done before 2010 and relevant research since then. Our focus is on research which explores teaching-learning situations through observing teaching in practice, so we exclude papers in which data reflects only the reports of teachers about teaching, survey results reflecting teachers’ perspectives, or research which focuses mainly on the learning experiences of students; all of these can be extremely valuable in the wider teaching-learning scene, but are not what we are addressing here.

One of the earliest papers focusing on university teaching through observations of practice is from Weber (2004) who studied the traditional teaching of one mathematician in a proof-oriented university mathematics course taught in a “definition-theorem-proof” (DTP) format. The resulting case study, based on data-grounded analyses, includes a description of the professor’s actions in the classroom which, it is claimed, are the result of his beliefs about mathematics, students, and education, as well as his knowledge of the material being covered, and which influenced the way that his students attempted to learn the material. Observations support the claim that DTP instruction is not a single teaching paradigm, but rather a diverse collection of pedagogical techniques sharing some core features. Weber discusses how such a study impacts on mathematics education research and suggests the importance of teachers and researchers engaging in a mutual negotiation about goals for advanced mathematics courses and beliefs about mathematics education.

From a cognitive analytical perspective, Wagner et al. (2007) and Speer and Wagner (2009), in the US, studied the teaching practices of professional mathematicians teaching an undergraduate course in differential equations. For the first time, the mathematicians used an inquiry-oriented curriculum which posed considerable challenges in formulating and addressing particular instructional goals, previously unfamiliar to the mathematicians. Of one mathematician the authors write: “His perceptions [i.e. reports of the challenges he faced in implementing such a non-traditional instructional approach] are corroborated by observations of his teaching and post-class interviews carried out by a mathematics education researcher” (2007, p. 248). The authors claim that by studying the teaching practices of professional mathematicians, it is possible to identify forms of knowledge (other than mathematical knowledge) that are essential to inquiry-oriented teaching.

Another study focusing on the teaching of one professional mathematician was conducted in Greece by Petropoulou et al. (2011) who investigated the mathematics teaching of one of their team in the context of first year Calculus lectures in a programme of study leading to a mathematics degree. The focus was on the lecturer’s teaching decisions, actions and reflections and on the way that these were linked to his different sources of experience. Different teaching practices that the lecturer adopted to challenge the students to develop high level mathematical reasoning, being at the same time sensitive to their needs, were identified. It was found that he brought experiences

from his practice as a research mathematician; his involvement in mathematics education research; his participation in the group where the focus was to inquire into teaching at the university level; and his actual teaching practice. The researchers used self-reflection (the lecturer), critical questioning (the other two researchers), and coordinated different interpretations of specific teaching actions and decisions developing a deeper understanding of what characterizes university mathematics teaching.

Another study looking into the teaching (of abstract algebra) of one mathematician was conducted by Fukawa-Connelly (2012) who investigated the mathematician's presentation of proofs. In particular, proof writing and the mathematician's intentions to provide her students, through questioning, with modes of thought that develop proof-writing abilities were explored. Despite the mathematician's good intentions for teaching, the author's analysis of the nature of questions indicated a funnelling pattern from "questions that a mathematician should ask while writing proofs, such as, "What does that mean?", "What comes next?", and "What do I still need to do?" (p. 343) to factual questions, such as, asking for definitions, restatements, and calculations. Fukawa-Connelly concluded that the reduced cognitive demand of the factual questions, which were left to students, did not engage them significantly. He also highlighted the dearth of research in observational studies of lectures at university level, to which he considered his study to be a contribution.

In a longitudinal study in the UK, Pampaka et al. (2012) conducted interviews and observations of pedagogy with nine teachers from five colleges, reporting different pedagogic practices (described e.g., as transmissionist or connectionist) from their observations. In particular, they stress that most teachers' teaching was transmissionist and teacher-centered; however, observations and interviews indicated that some teachers' practice was connectionist and student-centered with students involved in conceptually-focused discussions. They report on the association between teaching and a range of learning outcomes (e.g. dispositions).

Focusing on theory related to teachers' *listening*, Johnson and Larsen (2012) report on their study of teaching of three teachers of abstract algebra in an inquiry-based curriculum development project in a US university. Researchers videotaped all regular class sessions, took notes during each class session and held regular videotaped debriefings with teachers. Although the three teachers were generally successful at implementing the curriculum, analysis revealed a number of instances in which students expressed confusion or difficulties understanding the mathematics, and their teachers were not able to make sense of them. Further analysis of data from one teacher led to a categorisation of interactions involving student difficulties. While the teacher was particularly sensitive to dealing with students' difficulties, it was seen that at times she was unable to enter into the students' conceptions of the particular mathematics involved.

From a sociocultural perspective, Jaworski et al. (2012) studied teaching in the implementation of an inquiry-based innovation into the teaching of mathematics to first year engineering students in a UK university. The teacher was one of a teaching-research team (of 4); data consisted of observations of teaching practice, oral and written reflections of the teacher, surveys of student perceptions, and focus group interviews with students. Findings showed the development of teaching throughout the course as the teacher reflected on and modified her teaching in discussion with the team. The study revealed tensions between the perspectives of the team in creating and



teaching the course and those of students taking the course, which were analysed through an Activity Theory frame.

In a study of teaching in a community college in the US, Mesa et al. (2014), used interviews with teachers and observations of teaching to address the relationship between the teaching approaches that teachers described during interviews and the teaching approaches that they enacted in their classrooms. They considered what happens in the classroom when teachers interact with students and the mathematical content, and how that interaction might be influenced by specific classroom and institutional environments. They categorised teaching as content centred, or *traditional*, or student-centred, involving *meaning-making* or *student-support*, and classified *framing talk* and *mathematical questioning* in the classroom. Findings suggest that teachers' descriptions of their practice were well aligned with their enactment in the classroom.

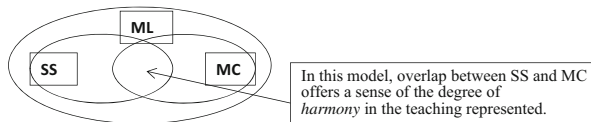
In his doctoral study in Sweden, Viirman (2014a) observed seven mathematicians' lectures and problem solving sessions in first-semester calculus and algebra at three universities. The focus was on the teaching of functions. In his discursive analysis with the *commognitive* framework of Sfard, Viirman studied the mathematician's written and oral mathematical discourse (2014b) and pedagogical discourse (2015) in "chalk talk" teaching. For instance, within the mathematical discourse, he recognized formal language in writing and informal language orally; and within the pedagogical discourse, he identified control questions (e.g. Does it make sense?), rhetorical questions, questions asking for facts (e.g. terminology, calculations) and enquiries for reflection on mathematics. Viirman (2014b) also stressed the underexploration of the teaching practices actually used by university mathematics teachers when giving lectures.

All of the above studies focused on the teaching of one or a very small number of teachers, using qualitative approaches to study teaching in depth. We notice that while three of these studies focus on teachers' interpretation of inquiry-based teaching (Jaworski et al.; Johnson & Larsen; Wagner, et al.), other studies observe teaching in its 'normal' state, i.e. teachers not trying to innovate or explore new approaches. Within these studies, some seek to categorise the teaching observed (e.g., Mesa; Pampaka; Viirman) using terms such as chalk-talk, traditional, meaning-making, and student-support, connectionist and transmissionist. Such introduction of terminology seems important in starting to develop a vocabulary for making sense of teaching at this level. Some papers noted the forms of *knowledge* (e.g., Johnson; Wagner; Weber) that teachers demonstrate in relation to their teaching, in some cases relating this to more extensive research on teacher knowledge at school level. Mesa et al. particularly pointed out an alignment between ways teachers talked about teaching and their enactment in practice. Johnson and Larsen focused on teachers *listening* to students and their responses to student difficulties. Jaworski et al., in a sociocultural frame, noticed *tensions* between student perceptions of teaching and those of the teaching team. Several papers pointed to the value for practitioners of research into their practice promoting deeper reflections and potential teaching development. All make clear the value of such in-depth studies of teaching, for mathematics education research into higher education practices, with several recognising their scarcity to date.

In the studies we discuss below, all involve observation of teaching practice as well as interviews with the teachers; teaching practice in all cases is 'normal' practice, with no intention of the researchers to study or to promote innovative (e.g., inquiry-based) teaching. Researchers categorise the observed teaching seeking to make sense of it in







**Fig. 2** One diagrammatic representation of the Teaching Triad

Theoretical perspectives used or developed in this research (viz: respectively the *Teaching Triad* (TT) and the *Spectrum of Pedagogical Awareness* (SPA)) are used further in the four studies reported below.

### A Corpus of Research Seen Through Four Studies

The four studies discussed below characterise the observed teaching focusing on “what teachers do and think daily, in class and out, as they perform their teaching work” (Speer et al. 2010). In Study A the focus was on just one teacher in a lecture module in linear algebra. Here a theoretical model emerged from a grounded analytical process. Study B is also a study of lecturing while Studies C and D have a focus on teaching in small group tutorials. Studies B and C are ongoing. Study D focuses on just one SGT and the reflective learning of the tutor in collaboration with a researcher. While the scale of these studies is small, it allows a detailed insight into the nature of the observed teaching and the thinking behind it through characterisations of teaching from which broader theories can emerge.

Research Studies	Lecturing	Small Group Tutorials
Single teacher	A	D
Multiple teachers	B	C

#### Characterising One Lecturer’s Teaching of Linear Algebra (A)

A UK study of the teaching and learning of linear algebra involved a small community of inquiry of two mathematics educators and one mathematician (the lecturer) in which *teaching in lectures* was explored in depth and characterised. The focus was centrally on the thinking and actions of the lecturer with the three members engaging deeply with the mathematics of linear algebra, approaches to teaching and the engagement/meaning-making of students. The teacher talked extensively about the mathematics and the ways in which he would (and did) work with students on this mathematics, reflecting particularly after a lecture in which he had given time to students to work on a task and observed their work. For example, after a lecture on subspaces, he reflected:

At some points I realised I need to find different ways of phrasing the questions in order to make them more accessible. One example of that was the introductory example on subspaces, where I had asked the students to find solutions to a homogeneous equation system with unknown coefficient matrix, given that they know a couple of solutions that I’ve given them. That was one question where I

saw quite clearly that some of the students found it very easy, and some of the students didn't have the slightest idea even if they tried. (Thomas 2012, p. 117)

Central to analysis was the lecturer's articulation of teaching goals and his realization of the goals in his day to day teaching. The study showed episodes of teaching and talking about teaching in the SPA realms of 'intuitive and questioning' and 'reflective and analytic' as the lecturer expressed his goals for teaching for the benefit of his co-researchers. We noted particularly two modes of reflection which we called 'expository' (in which the lecturer expressed the mathematical meanings he wanted students to make) and 'didactic' (in which he articulated his goals for teaching and the associated actions he would, and did, take in his lectures) (Jaworski et al. 2009). The study provides important insights into the teaching of linear algebra (Thomas 2012; Treffert-Thomas 2015). One researcher (Treffert-Thomas) brought Activity Theory into her findings to conceptualise the teachers' actions and goals in lectures, using the model attributed to Leont'ev, mentioned above. Treffert-Thomas (2015) writes

With each goal I identified and associated a number of actions. In addition, determining how goals related to one another opened up a hierarchical structure of intermediate goals in pursuit of a more over-arching goal (p.69).

The model, reproduced in Fig. 3 below, shows a hierarchy of goals, with teaching actions relating to each goal. The author writes further,

[The] first two goals and their associated actions were crucial in developing students' learning and understanding so that, in turn, the higher goals could be attained (p.70).

While at one level the teaching here might be seen as largely transmissive, this analysis goes more deeply into the nature of teaching and shows the very careful

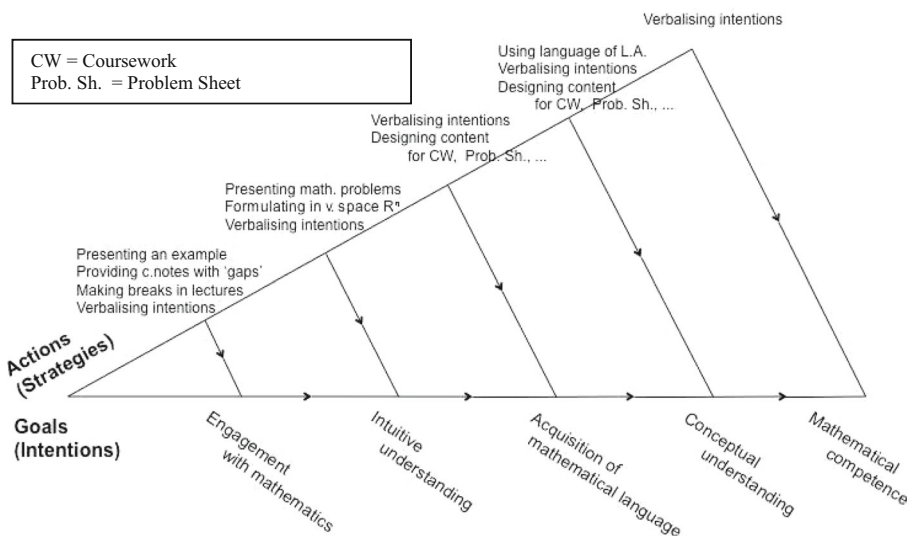


Fig. 3 “Modelling the teaching process” The Actions-Goals Model (Treffert-Thomas 2015)

analysis of this teacher of the mathematics of the module and the ways of presenting it to students. It was also clear to the researchers that the teacher learned pedagogically from his reflections prompted by the researchers. The theoretical model emerging from analyses reflects the researcher's critical theorising through interpretations of the lecturer's teaching activity. For details of the links between particular actions and goals we refer the reader to Treffert-Thomas (2015).

### Characterising Teaching in Mathematics Lectures (B)

An ongoing study in Greece, of teaching in mathematics lectures in two University Mathematics Departments, has involved observation of lectures and interviews with six lecturers who are research mathematicians. The study derives from a desire to enrich mathematics students' learning experiences at university level taking into account

- a) how university teachers conceptualize students' learning needs;
- b) what they think is important for the students to make meaning of (besides the particular content they teach) and
- c) how they communicate all these in their teaching.

Data from **three** of these teaching cases have been analysed in depth in order to characterize the observed teaching. In addition to individual interviews with the teachers, data from group meetings between teachers and researchers, discussing issues from teaching, are analysed. Three layers of analysis were conducted:

- 1) Grounded coding of lecturers' typical teaching actions (mostly observations from the lectures) as well as use of codes from the literature used to characterize teaching practices (e.g. Anghileri 2006; Drageset 2014).
- 2) Teaching actions were further analyzed using the Teaching Triad (TT) (Jaworski 1994; 2002) and grouped into characteristics according to the rationale behind these actions (the rationale and the teaching goals were identified from data of interviews and group discussions).
- 3) A cross-cases analysis using the Teaching Triad allowed insights into the nature of university mathematics lecturing (in these three cases) and consideration of the potential meanings the Teaching Triad's elements gain in this level.

The following episode illustrates the analytical process (Transcript 1: from a lecture focusing on series and convergence in the beginning of which, after defining a convergent series, the lecturer gave the geometrical series  $\sum_{k=0}^{+\infty} x^k = \frac{1}{1-x}$ , when  $|x| < 1$  as an example of a convergent series (a familiar example from high school) and subsequently, the divergent series  $\sum_{k=1}^{+\infty} (-1)^k$ ). The three columns hold (1) the dialogue as recorded and translated<sup>2</sup>; (2) descriptive codes interpreting the teaching actions; (3) Elements of the Teaching Triad that correspond to the perceived teaching actions.

<sup>2</sup> All dialogue and quotations are translated from the Greek by the third author.

In the episode, one lecturer aims to stimulate students affectively by drawing on students' experiences and encouraging their engagement in the lecture through interaction which is for him an important element of the learning process. He challenges students mathematically, encouraging mathematical investigation by posing open questions and making connections by exemplifying basic mathematical ideas in dialogue with students.

The TT comprises three domains of teacher's activity: a) Management of Learning (ML) which describes teacher's organizing role in the learning process; Sensitivity to Students—Cognitive (SSC) which is related to teacher's knowledge of students' thinking or Affective (SSA) which is related to the students' confidence; and Mathematical Challenge (MC) which describes the challenges offered to students. Teaching actions such as directing discussion or exposition are judged to be managerial because they suggest to the students particular ways of thinking.

*Transcript 1: Study B—Teacher-Student Interactions with Series and Convergence*

The students in this lecture are in their first year studies in mathematics in a Mathematics Department (first semester). About 80 students attended the particular lecture. The mathematical background of these students was not so strong according to the lecturer.

An episode (The lecturer facilitates students to arrive at a conjecture)	Teaching actions	TT
L: Now, can you hypothesize, when a series may converge?	Challenging students to engage in mathematical thinking/exploration	MC
You can base [your thinking] on the series S1 and S2: $S1 = \sum_{k=0}^{+\infty} x^k = \frac{1}{1-x},  x  < 1 \text{ converges}$ $S2 = \sum_{k=1}^{+\infty} (-1)^k \text{ does not converge}$ (No response from students)	Prompting students to focus on specific examples	SSC
L: Can you see any differentiation in these two cases [S1 and S2]? They are two specific cases of course but... is there any difference between them?	Asking a more focused question	SSC
St 1: Yes. The base [of the $x^k$ ] in the first case (S1) is a positive number.	Demonstrating insight from a student's response and	ML
L: Not necessarily. If $x$ is a negative number the sign [of $x^k$ ] changes.	Offering a counterexample	MC SSC
[he moves his hand left and right showing the alternation of the signs over an imaginary number line]		SSA ML
St 2: Counting starts from different numbers	Demonstrating insight from another student's response	ML
L: No, it has nothing to do with... We could start from the same number. [He changed (in S1) the counting from $k=0$ to $k=1$ on the board. Now on the board the two series were: $\sum_{k=1}^{+\infty} x^k \text{ (S1a) and } \sum_{k=1}^{+\infty} (-1)^k \text{ (S2)].}$	Reducing the example to a simpler form	SSC
L2: Here you are! Now the series ((S1a) and (S2)) seem comparable. Finite number of terms at the beginning of	Offering his own explanation	ML

a series does not have any influence on its behaviour [related to its convergence].		
St 3: Is the difference that the 1st series (S1a) has a variable?	Demonstrating insight from a third student's response	ML
L: It's not a matter of variable. OK. If I set here... I don't want to confuse you. Let me take a specific geometrical series. [He writes the series $\sum_{k=1}^{+\infty} (-1)^k \frac{1}{10^k}$ (S1b)]. I took units everywhere. This number... I made it to look about the same!	Specifying the example	SSA SSC
St 1: Is the difference that the 1st case (S1a) has the condition $ x  < 1$ ?	Relevant responses	ML
St 4: Is the difference that in the 2nd case (S2) the absolute value is 1 ( $  -1  ^k = 1$ )?		SSA
L: This observation (of St 4) may imply something. It's a correct observation.	Confirming	SSC/SSA
Your fellow student said we should take these terms (the sequence in S2) and we'd consider their absolute value. This absolute value will always be 1. This might be a difference. Whilst in this case (S1b) the absolute value of the sequence in the series is $1/10^k$ . And?	Building on students' ideas	MC/SSC MC
St 5: ... and this absolute value is less than 1. L: Yes. And? (No response)		MC
L: If $k$ tends to infinity, where does $1/10^k$ tend to? St 1: To zero. Then the lecturer summarizes the result of the investigation and formulates the conjecture.	Directing discussion	ML

Teaching actions such as confirming students' contributions and building on students' ideas are judged to be *sensitive* to students *affectively* because they are supportive and reassuring to students and they make them feel that their contributions are valued. Teaching actions such as focusing, specifying or reducing an example to a simpler form when students have difficulties are judged to be *sensitive* to students *cognitively* because they take into account students' thinking at the particular stage of their reasoning. Finally, the action of posing an open question in the beginning of the episode offered students the challenge to be involved in mathematical exploration and thus it was judged to be challenging mathematically.

Dialogue allows the lecturer access to students' meanings, to which he can then respond. Analyses based on the lecturer's own reflections as well as on his interactions with students are showing "An uneasy balance" between Sensitivity to Students and Mathematical Challenge in this teaching, reflected in a tension between the lecturer's intentions to include students' thinking in the activity of a lecture while at the same time presenting mathematical meanings in a rigorous form (Petropoulou et al. 2013). Moreover, we see that the lecturer builds on responses and comments from a small number of students. This is as an expression of the lecturer's sensitivity to the particular students, but raises questions about the many other students. For example, are the students' inputs comprehensible to their classmates? How can the lecturer draw on these inputs and synthesize them to make them meaningful to the rest? In this episode we see some of his attempts such as summarizing the main ideas, offering familiar



examples, interpreting students' ideas and clarifying points that he considers difficult for the students. The ways in which the lecturer addresses such questions in teaching episodes can be seen to fit with differing positions on the Spectrum of Pedagogical Awareness.

In interviews and group discussions the three lecturers, whose teaching is analysed in the depth illustrated above, emphasized their beliefs as in the following quotations:

“I believe that in order to be engaged in the learning process someone must have the appropriate psychology. I think that the 50 % of learning is the learner's psychology. I have to tone up students' psychology so as to learn. I work for this and I pay attention to this. Because, in my view, this might be the most important element” (L1)

“Here we talk about masses of students and how to achieve practical results for masses. This is the issue! (L2)

“Students should learn what it means [to have] rigorous proof. These are the foundations to become mathematicians” (L3)

Analysis shows that:

- a) Affective, social and cognitive aspects are involved in the way the lecturers L1, L2 and L3 respectively take students into account. Confidence in students' engagement with the advanced mathematics (L1); success with the final examination to go on with their studies (L2) and guidance in their encounter with the new academic learning style (L3) are what the lecturers consider as students' learning needs.
- b) Linking the lecturers' goals with the characteristics of their teaching practice, it seems that elements of mathematical research practice and mathematical production (to L1); routinisation of advanced mathematical processes and methods so students will be able to “get used to it” (to L2) and initiation into mathematical structure and mathematical language (to L3), are of great importance to the three lecturers for their students.
- c) All these issues are communicated to students through different teaching approaches. The lecturer L1 tries to involve students in the lecture and develop their intuition. L2 tries to prepare students for the final exam in many ways, having also in mind “all these students that cannot attend the course because they have to work, for example”. L3 tries to bring students into contact with the formal mathematical language.

Some researchers (e.g., Solomon 2007) discussing learning in undergraduate mathematics suggest that students' identities are formed as part of the way undergraduate mathematics is taught in the lectures which thus impacts on students' mathematical meaning-making. For example, we might conjecture, in the case of L1, that students are introduced to advanced mathematical thinking and mathematical production (e.g., writing proofs, solving problems) thus developing identities about mathematics and its learning which are aligned with the research practices of mathematics community (e.g., dealing with intuition

and formalism); in the case of L2, students are introduced to advanced methods and processes towards the final assessment, thus developing more instrumental identities about mathematics and its learning which are aligned with the institutional values of the university community; in the case of L3, students are introduced to the structure of mathematics, thus developing identities aligned with the epistemic values of the mathematics community.

Preliminary conclusions deriving from analysis suggest:

*First*, we see that the lecture is not a unified teaching approach and, as Weber (2004) notes, under the lecture format general umbrella, different teaching practices may take place.

*Second*, we can speak about a teacher's sensitivity to a large group of students so that sensitivity to students (SS) gains a visible social dimension.

*Third*, we reassess what mathematical challenge (MC) can mean at this level as it seems that this element of Teaching Triad is related with the processes of mathematical discovery and production as well as the formal consideration of advanced mathematics.

Our observations show that many lectures at the university level are still taught in a transmissive way in the sense that the lecturer conveys information and the students listen and passively take notes. However, this study is giving evidence that there are alternative ways for teaching in the instructional context of the lecture and identification of these might be used as a lecturer's 'self-awareness springboard' towards improving teaching. As pointed out in the secondary school teaching literature (e.g., Jaworski 1998, 2008), Mathematics Education researchers contribute towards this end. Lecturer L3 commented:

You pose questions at any rate and you make me think ...I had never this alternative view of teaching. I had teaching in my mind in a scientific way. I had never thought if students were interested. I took it for granted that students are interested. The only thing I had to do was to explain the content better (L3)

### **Characterizing Teaching in Mathematics Small-Group Tutorials (C)**

In this research, university mathematics teaching is studied in the context of small group tutorials (SGTs). The research focuses on a characterization of the teaching practice of tutors (mathematicians or mathematics educators): 26 tutors have been observed and interviewed, with three extended cases of teaching studied in depth. The study sheds light on the link between:

- a) the teaching that is observed by the researcher, and
- b) the thinking about teaching on the part of the tutor.

In the context of SGTs, observational data allows an emerging plethora of instances for discerning the nature of students' difficulties, and for analyzing the teaching which addresses difficulties. In consideration with the existing

literature on students' difficulties (e.g. Nardi 1996; Nardi et al. 2005), the study affords this possibility since the students' responsibility is to bring to the SGT, work with which they face difficulties. The work usually covers mathematical topics or tasks from the material of the lectures, such as problem sheets and coursework. The tutor, not having planned the teaching, should support and facilitate the students in the process of making meaning of the mathematics. On these occasions, the tutor makes decisions about his teaching and the work in the tutorial. The observer largely records the small group without participation in the action of the tutorial.

In this study, analysis of the tutor's teaching is based on the identification of teaching episodes, which are extracts in transcriptions of observations. Episodes concern a whole mathematical topic or work on a task. Interview data, collected after each observation, includes the tutors' thinking about the observed teaching. The researcher discerns the students' meaning through the students' responses in observations or the tutors' evaluation of students' meaning in interviews.

A grounded analysis of the observational and interview data led to a characterization of teaching practice in terms of: tutor's *tools* for teaching (e.g. graphical representations) and tutor's *strategies* for teaching (*modes of action* with tools e.g. explaining). Through the descriptive codes "graphical representations" and "explaining", we exemplify how *tools* and *strategies* characterize Phanes<sup>3</sup>' teaching practice in Transcript 2. "Explaining" is one of Phanes' *strategies* for teaching which were identified repeatedly in the data. The *tools* associated with "explaining", discerned in teaching episodes, are kinds of representations, such as graphical or symbolic representations. We italicize here the *tool* and the *strategy* to denote two dimensions of teaching which emerged as common in all three extended cases. Thus, the grounded analysis of teaching reveals the *tool* as the unit of analysis for practice; relating it to the tool in Vygotskian theory. The study also links its characterizations of tutors' teaching for students' meaning making of mathematics to the *Spectrum of Pedagogical Awareness*, in order to situate it with respect to the earlier research.

The episode in Transcript 2 is from Phanes' teaching with regard to the *absolute value* function. Students have suggested working with the problem-sheet set by their lecturer on tasks with which they found difficulty. The episode concerns the following task:

Rewrite  $||x| - 1|$  without modulus signs, using several cases where necessary.

Phanes gave the students 3'20" to work on the task and then demonstrated at the board.

#### *Transcript 2: Study C—Use of Graphs as Tools in Explaining Absolute Value*

The students in the tutorial were in their first year of a mathematics degree. Their performance as observed and according to the tutor was at a low level.

<sup>3</sup> Tutors were invited to choose their own pseudonyms: this tutor chose Aristophanes, of which Phanes is an abbreviation.

<p>An episode (Phanes explains the cases for <math>  x -1 </math>.) (adapted from Mali (2015))</p>	<p>Descriptive codes: “Explaining”; “graphical representations”.</p>
<p>Phanes: So, how do I solve this problem? I’ll show you. [...] So, what am I going to do? I will do it step-by-step. First, I will construct <math> x </math>, right? <math> x </math> is this. [Phanes sketched the graph of Fig. 4.] Ok? Then, we do <math> x -1</math>. <math> x -1</math> means that you take <math> x </math> and you shift it down by 1. This means <math>-1</math>, right? So, it gives you this [g in Fig. 4]. These points are 1 and <math>-1</math>. And this point is <math>-1</math>. This is the expression under the modulus sign. And then, you take the modulus of this function and it means that you reflect this negative bit about the <math>x</math> axis, right? And you get this function. Ok? This is the graph of the function. Now, we have to write down the equations for this. You can see that it’s given by different functions on different intervals. For instance, this expression is what [f in Fig. 4]? This was <math>y=x</math> [e in Fig. 4], and then, you shift it by 1, so this is <math>x-1</math> [f in Fig. 4]. Is this clear? Please stop me if something is unclear. So, this is <math>x-1</math> [f in Fig. 4]. So, what is this [c in Fig. 4]? What is this <math>-</math> this bit [c in Fig. 4]? It has the same slope as <math>x-1</math> but it’s shifted up.</p>	<p>Graph of <math> x </math> (assumed to be recognised by students)                  Graph of <math> x -1</math> (construction was explained)                  Graph of <math>  x -1 </math> (construction was explained)                  Encouraging students’ visualization of functions                  Graphs of <math>y=x</math>, <math>  x -1 =x-1</math> (construction was explained)                  Encouraging students’ feedback</p>
<p>St2: It’s <math>x+1</math>.</p>	<p>Graph of <math>  x -1 =x+1</math> (construction was probed)</p>
<p>Phanes: It’s <math>x+1</math>. So, this bit is <math>x+1</math> [c in Fig. 4]. Now, what is this [a in Fig. 4]? This graph is <math>y=-x</math> [b in Fig. 1], and we shift it down, so it’s <math>-x-1</math>. So, this thing is <math>-x-1</math>. And what is this [d in Fig. 4]?</p>	<p>St2’s correct input</p>
<p>St4: <math>-x+1</math>.</p>	<p>Graphs of <math>y=-x</math>, <math>  x -1 =-x-1</math> (construction was explained)</p>
<p>Phanes: <math>-x+1</math>. <math>-x+1</math>. So, what can we now say about this function <math>  x -1 </math>? It equals. Now, it depends on where <math>x</math> is, right? So, we know for this function that on this [Phanes points to interval <math>[1, +\infty)</math>], it’s <math>x-1</math> if <math>x</math> is greater than or equal to 1. Agree? It is <math>-x+1</math>, <math>-x+1</math> if <math>x</math> belongs to <math>(0,1)</math>. It is <math>x+1</math>, <math>x+1</math> if <math>x</math> belongs to <math>(-1,0)</math>. And finally, it’s <math>-x-1</math> if <math>x</math> belongs to <math>(-\infty, -1)</math>.</p>	<p>Graph of <math>  x -1 =-x+1</math>                  St4’s correct input                  Encouraging students’ visualization of intervals</p>

In this episode, the approach of teaching is to a great extent transmissive. By building the graph, stage by stage, Phanes explains or probes the construction of graphs for the cases of  $||x|-1|$ ; thereby encouraging students’ visualization of functions

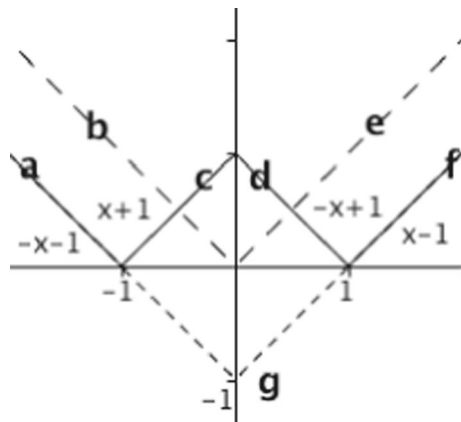


Fig. 4 Graph on board, episode 1

$(x - 1, x + 1, -x - 1, -x + 1)$  and intervals. In contrast, the students' inputs occupy two lines in which two different students correctly identify two graphs, the  $x + 1$  and the  $-x + 1$ . The students' visualization does not however provide insight into the endpoints of intervals. After the episode, Phanes stresses to students that  $||x| - 1|$  is continuous so "it doesn't matter" if the endpoint is included in the interval.

In this analysis, Phanes' *tools* for "explaining" the cases for  $||x| - 1|$  are the graphical representations of:

- a)  $|x|, |x| - 1, ||x| - 1|$  ( $|x|$  and  $|x| - 1$  were used for the construction of  $||x| - 1|$ )
- b)  $-x, x - 1, x + 1$ , ( $x$  was used for the construction of  $x - 1$ ;  $x - 1$  was used for probing the construction of  $x + 1$ )
- c)  $-x, -x - 1, -x + 1$ . ( $-x$  was used for the construction of  $-x - 1$ ;  $-x - 1$  was used for probing the construction of  $-x + 1$ )

Recognition of *tools* is related to Vygotskian theory, in particular psychological *tools* and material *tools* (Wertsch 1991), although Cole (1994, p. 94) argued a *tool* carries its material and psychological nature inherently; in other words, it exhibits "a dual nature". In this study, the concept *tool* carries the dual material and psychological nature. For example, the graphical representations of a), b) and c) above are Phanes' *tools* to stimulate the two students' (and potentially the remaining students') graphical imagery so that they have insight into the nature of  $||x| - 1|$ ; and thus share some meaning for reasoning by cases (psychological nature). The graphical representations, however, are curves drawn on some material, whiteboard or page (material nature).

In the grounded analysis, tutor's actions in teaching are considered to be *modes of action with tools* (referred to as *strategies*) for enabling the students' meaning making of mathematics. *Modes of action with tools* address the observed teaching and the tutor's associated thinking. For example, a researcher's code for Phanes' action (referred to as *strategy*) is "explaining" with graphical representations. Here, the graphical representations are the *tools* so that students have insight into the nature of  $||x| - 1|$ . "Explaining" is the *mode of action* using the *graphical representations*. Thus, Phanes's *strategy*, "explaining", refers to the observed teaching (constructing a sequence of graphical representations) and the associated thinking (for students' meaning making of rewriting  $||x| - 1|$  without modulus signs).

The tutors' awareness of students' difficulties is revealed in interviews with the tutors, and provides insight into the pedagogic characterizations of the teaching with regard to SPA. Phanes' small group tutorial includes almost exclusively low performance students. In spite of his 15-year teaching experience, Phanes expresses recognition that he is a learner for teaching such a group of students, and questions his approach. In the case of episodes of his teaching, the characterization towards *Intuitive and Questioning* is highly dependent on the students' engagement with the mathematics, in the tutorial and more generally.

We think that findings about Phanes' teaching go beyond the SGT setting. In tutorials, each time, he had to respond to students' suggestions without having planned the teaching. His responses to students, usually in analysis and linear algebra, revealed that his practice was related to a great extent to *tools* and *strategies* for mathematical thinking. So, the openness of the SGT enabled him to teach students to think mathematically, and allowed the researcher to explore what mathematical thinking was for

him. “Explaining” with graphical representations is a *strategy* for mathematical thinking. In its implementation, Phanes dissected the mathematical task to make its aspects more visible to students and used the graphs to make fundamentally mathematical ways of thinking transparent to students.

The characterization of tutors’ practice is an outcome which can be used in qualitative research studying teaching from a sociocultural perspective. Practice is understood through *tools*, and characterizations of teaching are based on tutor’s *acting with tools*. This characterization of practice offers various *strategies* of a tutor’s teaching (e.g. selecting tasks, evaluating students’ sense making of mathematics); ultimately, gaining in-depth insight into the tutor’s doing and thinking for the students’ mathematical meaning making. Analysing Phanes’ practice, we interpret that a *strategy* of his teaching is “explaining”. This *strategy* was identified in the other two extended cases of this study with other tutors, although implemented with different *tools*, and different views on and values of mathematics and mathematics teaching. A cross-case analysis of practice is currently showing possibilities for a clearer articulation of what “explaining” and other teaching strategies mean in an open SGT setting.

### **Reflecting on Teaching in One Small-Group Tutorial (D)**

In this fourth study the focus was on understanding and developing teaching in order to promote students’ mathematical meaning making in one SGT. The study involved one teacher-researcher and one researcher collaborating to design, study and develop the teaching, which dealt with aspects of calculus and linear algebra from first year modules. The teacher-researcher, a mathematics educator experienced in both teaching and researching teaching, had the roles of designing teaching, teaching (or *tutoring*), reflecting on teaching and its outcomes, analysing data and feeding back to design and planning. The researcher had the roles of observing teaching, collecting data from observation, promoting teacher reflection and analysing data. The five students were in their first year, following a programme of *Mathematics and Sport Science*; they were low-performing students (in mathematics) as revealed by their coursework submissions, marked by the tutor. Data from recordings of classroom dialogue and oral and written reflections of the tutor were collected and analysed from SGTs over 12 weeks of one semester with a tutorial of 1 hour each week. Analysis was undertaken by both researchers, firstly through a grounded coding and categorising of the data and then applying the Teaching Triad to the categorisations.

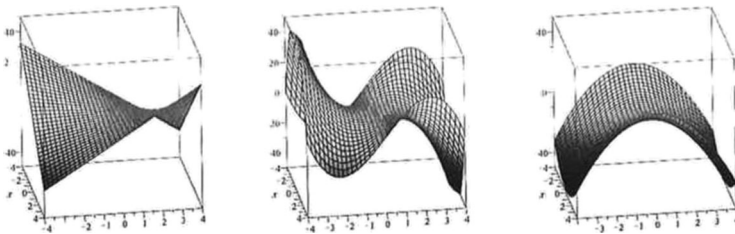
The teacher-researcher, or *tutor* henceforth, was an experienced pedagogue who planned teaching taking into account the mathematical topic, ways of working with the mathematics with the students, and the students’ meaning-making as tutorials progressed. Getting access to students’ meaning-making proved to be exceptionally challenging, and teaching approaches were modified over time to improve this access. Tutorials generally focused on problems set in the weekly problem sheets of the module lecturers in calculus and linear algebra. The tutor asked students to let her know, either before or during tutorials, which problems they had found difficult. However, it became clear rapidly that students often had little idea of how to even start with a problem and often had not looked at the problems before coming to the tutorial. The tutor showed considerable flexibility in changing what she had planned when her attempts to access meaning making revealed a poverty of mathematical meanings.



One example arose from the introduction in a calculus lecture of the definition of ‘limit’ and its use in finding limits of sequences. When students seemed to struggle with a problem requiring them to find limits, the tutor referred them to their notes for the definition and asked one student to read the definition. It became clear that, not only did the student not *understand* the definition, he could not *read* its symbolisation and punctuation; nor could any of the other students. From that point, problems were put on one side and the tutorial proceeded with a study of the definition, its symbols and their meanings, and the reading of the definition as a complex whole. The tutor addressed questions to each of the students and waited for student participation. Quantifiers were recognised and explained, the tutor making an input where students appeared ignorant of meanings. Gradually the students, one by one, were able to read, part by part, the definition and to give some indication of understanding the punctuation and overall meaning. At the end of this activity, the tutor asked students whether they valued what had been achieved; although there was some agreement, one student said “but we have not tackled any of the problems”. It became clear that help from the tutor with solutions to the problem sheet was a main reason for students attending a tutorial.

This raised the issue of what kind of help with problems could be most effective. The tutor wanted to avoid a transmission approach of tutor presenting and explaining the solution. In general a questioning approach was adopted as indicated in the following example. The topic was differentiation of two-variable functions. From the problem sheet that had been set in the lecture the tutor selected problems a) with which, knowing the students, she expected students to have difficulty, and b) which she expected to reveal important concepts within the topic. One such problem was presented as follows.

The three graphs below show a function  $f$  and its partial derivatives  $f_x$  and  $f_y$ . Which is which and why.



The following transcript reveals some of the dialogue during discussion of this problem. The dialogue is typified by the tutor’s *questioning* of the students; in analysis, several types of questions are recognised. Codes shown here are only those relating to questioning.

**Meaning** (Qm) or (Qmw)—seeking students’ expression/articulation of meaning, often in response to the question “why?”

**Inviting** (Qi)—asking students to respond; (Qig)—offering the question *generally* (to all students) or (Qid) *directly* to one student (named). The question can be a specific question (Qigs or Qids) or non-specific question (Qig or Qid).

Examples of Qm, Qmw, Qig, Qid, Qigs and Qids can be seen respectively at lines 1, 12, 6, 10, 18 and 32.

#	Who	Words and actions	Codes
1	T	So, first of all what are these things $f_x$ and $f_y$ ? Alun. What is, what do you mean, if you write $f_x$ and $f_y$ ?	Qm
2	A	Dee-f-dee-x ...	
3	T	And how would you write it?	Qid
4		[Alun indicates with his hand the partial derivative symbol, $\partial$ ]	
5	T	Yes <i>partial</i> $df/dx$ and similarly $f_y$ is <i>partial</i> $df/dy$ . When you say dee-f-dee-x, it's not clear, so you want to be clear, we would say here <i>partial</i> dee-f-dee-x and <i>partial</i> dee-f-dee-y [She writes on the board $\partial f/\partial x$ and $\partial f/\partial y$ ]	
6	T	So in the question then, we have three graphs; one of them is a function $f$ and the other two are the <i>partial</i> derivatives $\partial f/\partial x$ and $\partial f/\partial y$ [She points at the board.]. Now, which is which?	Qig
7		[silence]	
8	T	Anybody have a stab at that? What do you say Brian? [He pulls a face and people laugh]	Qids
9		[Response unclear]	
10	T	No? OK, how about you Erik?	Qid
11	E	... not really sure but I guess that, er, $f$ will be the middle one	
12	T	OK, <i>why</i> do you think that?	Qmw
13	E	... because it is got the, er, the slants of the first one, and the...	
14	T	so you're seeing a relationship between the one of the middle and the other two. What do you mean by the <i>slants</i> ?	Qm
15	E	er, I don't know, just the, the <i>gradient</i> there.	
16	T	<i>if</i> you're right and the function is middle one, erm, before we go any further, Alun, do you think the function is the middle one or would you say one of the others?	Qids
17	A	... it looks like the more complex	
18	T	aah.. "It looks like the more complex". So would you expect the function graph look more complex than its two ...?	Qigs
19	S	I would.	
20	T	you would. <i>Why</i> ?	Qmw
21	S	[pause] I don't know	
22	T	do you agree with him Carol?	Qid
23	C	yeah	
24	T	can you say why?	Qmw
25	C	erm, because it has in this $x$ <i>and</i> $y$ , functions of both $x$ and $y$ .	
26	T	well, don't they all	Qig
27	C	more functions	
28	T	more functions?	Qig
29	C	er, I don't know!	
30	T	Come on we're getting there. Brian?	Qid
31	B	Well, I guess when you differentiate, you're almost simplifying it to your next.[inaudible]	
32	T	OK, so if what we have got is, in some sense a polynomial, then when we differentiate a polynomial we get a lower degree ... So is that what you meant by 'simplifying'? So is everybody agreed then that the middle one is the function? OK. It is!! It is.	Qids

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- So look to the one on the right, Erik, and tell me how the one on the right fits with what you see in the middle. Is that going to be the partial derivative  $f_x$  or is it going to be the partial derivative  $f_y$ ?
- 33 E em, derivative of x ...
- 34 T Can you say why? Qmw
- 35 E aah.. because, I dunno, it looks as if ... along x and the other as if it kind of moving up and down y....
- 36 T so you are changing your mind? [they laugh] How about you Alun? Qids
- 37 A em ...[long pause] I dunno.  
5'50"

*The discussion continues in a similar style to the above (up to turn 66) with tutor asking questions and students responding.*

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In the 10 min of the episode there are 66 turns and 36 of these are tutor turns. This means that there are 30 responses from students. The pattern is entirely one of ‘tutor-student-tutor- ...’. An interpretation is that the tutor is controlling the mathematical focus through her questions. However, the nature of the questions is that students are expected to offer a response and to explain their response. Over time, they may be more ready to respond, expecting that this is required; perhaps recognizing that responding helps them to articulate what they think or mean. The tutor draws on their use of informal language to infer meaning—for example the word “slant” (13) seems to point to some perception of the significance of gradient and when she asks for the meaning, the tutor is rewarded by the student becoming more precise with the word “gradient” (15).

Recorded reflection of the tutor shows that the tutor was aware that her questions led the conversation, and that dialogue was largely one-to-one between her and individual students. She would have liked students to become more involved in discussing mathematics. On other occasions she had used other strategies to get students talking with each other, such as periods within the tutorial when she asked them to work in a pair or three on a problem and share their working with the group. However, using questions in this way indicated to students that they were expected to think and contribute, and moreover to listen to each other’s contributions from which their own conceptions could develop. Contrasting this use of questions with the cases discussed by Viirman (2015, p. 1176–8), we see similarity with what Viirman calls “genuine” or “inquiry” questions, seeking the students’ perspectives on what is asked. However, here, the questions are largely not “rhetorical” questions: they mainly prompt student responses rather than seeking to emphasise key aspects of the mathematics in focus.

Further analysis using the Teaching Triad suggested that the questioning approach formed the tutor’s management of the learning environment (ML). The responses given by students enabled her to become more aware of their dispositions and cognitive needs (SS) which then led to the degree of challenge (MC) she could offer through further questioning. Regarding MC, her questions kept the focus on key aspects of the mathematical concepts involved and probed/prompted students to think and respond.

## Discussion

Where university teaching of mathematics is concerned, our studies in two teaching settings, lectures and small group tutorials (SGT), have tried to reveal thinking, about teaching, that goes beyond what may be seen as implicit in any one setting. The two settings impose different demands on the teacher: what is needed or acceptable in a lecture is very different from an SGT. However, in both cases the teacher mediates between students and mathematics, with the objective of students' sense-making of the mathematics in focus. We therefore gain insights into what teachers are able to achieve in the two settings (or not) and ways in which the setting supports or constrains teaching potential.

Thus, we have analysed data from what we have seen and from what teachers have said to us, either in their own reflecting aloud on practice, in the presence of a researcher, or in response to researchers' questions. In critical theorizing from our analyses we have chosen terminology that enables us to make sense of our university mathematics teaching, how it is constructed, what it looks like in practice and how it seeks meaning making for students—how what happens in the teaching situation results in sense making for students (or not). We are imputing meaning from what we see and hear in order to try to say more about our collection of interpretations. So, critical theorizing involves the critical drawing of theory from interpretative analysis, or the use of theory to make sense of the interpretative analysis. This brings us to the particular studies and our use of theory to characterize teaching.

We start from the Spectrum of Pedagogical Awareness (SPA), applied in Studies A, B and C. The SPA focuses on *pedagogy*, which is on how the teachers (together with their students) construct the learning environment and foster learning in mathematics.

In the seminal study, reported by Nardi et al. (2005), the episodes of teaching (not the teachers) were categorized according to one of the positions on the spectrum. Where what was said and done was interpreted as paying little attention to pedagogy, perhaps focusing almost totally on the mathematics, sometimes with very unhelpful references to students' attitudes, the positioning was towards the left end of SPA. As more attention was paid to designing teaching to suit the students and encourage meaning-making, positions to the right became more appropriate. It is fair to say that the left→right hierarchy of positions provides some indication of the level of sophistication of pedagogic thinking. Nardi et al. (2005) provide examples of all the positions. These positions are of course not absolute, and the metaphor of a spectrum is more appropriate than one of specific positions. The spectrum becomes more useful as a (theoretical) tool for describing pedagogy as it is applied to more situations, and positions are compared. We say more about this below.

In Study A, the positioning of the teaching is often in the area of “intuitive and questioning”, although perhaps moving towards “reflective and analytic”. Following a grounded analysis, Leont'ev's *Activity Theory* model is applied to the relationship between teacher's intentions/goals and teaching practice/actions. In the theory *actions* are *goal related*, so the researcher has looked for evidence of such a relationship. The emerging model (Fig. 3) presents specific goals and actions and positions them in the triangular formation to indicate the relationship between goal and action and the increasing sophistication of the ways in which a particular intention/goal is interpreted in practice. We have to emphasise that this model is a result of the researcher's critical

theorizing, not ‘the way the teaching *is*’ (not an ontological representation) and not the lecturer’s articulation. The suggested positioning of this teaching on SPA increases our awareness of practices relating to the four SPA positions.

Returning to the UMPT study: after critical theorizing (largely from interviews with the teachers) which led to SPA, a grounded analysis was conducted of the observation data to which was applied the theory of the Teaching Triad (TT). While SPA brought insights into the degree of pedagogical awareness demonstrated in certain episodes of practice, the TT allowed a more detailed perspective on the practice itself and its didactical nature. In Jaworski (2002) we see an analysis of two teaching episodes (both relating to abstract algebra). In one the pedagogy is seen as being towards the left hand end of the spectrum, being almost entirely an explanation of the mathematical process from the tutor’s perspective and meanings, from which we gain little insight into how this contributed to students’ making of meaning. The second episode shows the tutor choosing a pedagogic device (bringing a student to the board to explain his thinking, with the tutor ‘chipping in’ to help the student along and to draw attention to important aspect of the concept under consideration). The SPA categorization here was much further to the right to indicate higher pedagogical sophistication. Analyses with the TT, suggested the first episode demonstrated the tutor’s management of learning (ML) but little sensitivity to students at the cognitive level (SSC). This judgment reflects the tutor’s desire to show his students how to determine the subgroups (perhaps SS in the affective domain—SSA), and thus to refute the converse of Lagrange’s theorem. His way of involving the students was to expound the mathematical process from which they were then left to make their own sense. In contrast, the second tutor challenged his student to present a mathematical argument (MC) and supported the student in doing so (SSA&C). These examples start to indicate how the TT offers insight into the teaching process for students’ meaning making that expands on SPA.

In Study B, into lecturing, the researcher uses the TT to provide insight into teaching episodes emerging from her grounded analysis. From this, we see an example from Lecturer L1, using the pedagogic approach of questioning his students during a lecture and modifying his approach as he gained from their responses some insight into their conceptualization. Analysis suggests that the lecturer had meanings into which he wanted to draw his students. His questioning revealed that a straightforward exposition of the mathematical concepts from his own perspective would not have been accessible to students’ own sense-making—i.e., the lecturer’s judgment was that students would not have been able to make sense according to established mathematical meanings of which the lecturer was aware. So, he reduced the cognitive level (a reduction of MC) of his exposition, to suit what he saw of students’ cognitive needs. This suggests a high degree of SSA, with intentions relating to SSC. Regarding MC, we might see the lecturer in this episode seeking the appropriate degree of challenge that his students could accept. This is not to say that it *was* appropriate (an ontological statement)—we can only relate our judgment to what it was possible to provide as evidence in the data. Critical theorizing suggests the positioning of this episode on SPA might be seen as towards right of centre in terms of the tutor’s awareness of the pedagogic issues he faced.

In Study C, focusing on teaching in SGTs, we see the tutor Phanes seeking to help his students make sense of the analysis of a composite function involving two levels of the absolute value function. Analysis here makes use of Vygotskian theory of *tools* and

*modes of action with tools* in the learning-teaching process. Analysis suggests that Phanes uses graphical tools to demonstrate meanings to his students who seem to make sense in accord with these meanings. If this analytical interpretation is trustworthy, then it appears that the tutor's action with tools, his teaching *strategy*, is effective for the students. In TT terms, we might say that his SSA, leading to SSC was appropriate, leading to MC at an appropriate level for students to make mathematical sense. We conjecture, for this episode, a positioning on SPA towards "intuitive and questioning" rather than "reflective and analytic", since the arguments for what was seen are not supported by the tutor's own articulation. However, we recognize that we have only the tutor's words, or lack of them, as evidence for such a statement. Current critical theorizing is relating tools and strategies to *knowledge in teaching*, thus developing theory which links teaching with the knowledge practitioners bring to the teaching process. While there is considerable research at school level on teachers' knowledge in practice (e.g., Ball and Bass 2003; Rowland et al. 2005), there is as yet little consideration of teachers knowledge at the university level (see however Johnson and Larsen 2012; Wagner et al. 2007).

Study D looked at the SGT teaching of a tutor familiar with such theorizing (Jaworski and Didis 2014). In such a situation, the reflections of the tutor on what was observed in the action can overtly relate goals to actions and make judgments about pedagogic approaches and positioning, and issues arising from it for the tutor. In such a case, analytical judgments can be grounded in the reflections of the tutor related to the theoretical perspectives with which the tutor is familiar. A difference here from studies B & C is that the tutor both teaches and analyses/theorises, with the danger of reduced objectivity. The presence of the second researcher acts as a critical device to question unrealistic interpretations. Ideally, as researchers it would be valuable to work with teachers who are willing and able to reflect in the knowledge of these various theoretical tools. We recognize that this will rarely be possible, and so we do the best we can in a critical theorising on the data which is available to us.

We have studied teaching in two settings, lecture and SGT. Both settings can be seen to offer the lecturer or tutor the possibility to teach using a transmissionist approach with little interaction with students. Such a description as "transmissionist" which we have found occurs often in the literature, is very 'broad brush'. It refers to the teachers engaging largely in the mode of exposition: that is, expounding or explaining the mathematical process to the students with students as relatively passive participants. However, the use of tools of various kinds or a variety of kinds of questions (including rhetorical questions) mediates between the exposition and the students to promote meaning making. Through the examples above we have started to show how this is or can be done. In our studies currently, we are seeing both differences and similarities between the two settings of teaching. In both studies in tutorial settings, researchers studied students' difficulties (or were led to study by the tutorial setting) and in both studies in lecture settings, researchers studied goals for teaching. It might be that the nature of the settings promotes some lines of inquiry. Some similarities lie in the effects on students of the approach used. For example, Phanes uses (as tools) graphs, chosen to engage students through mathematics that he knows they know, hence showing a sensitivity to students and helping them to develop established meanings of absolute value. Lecturer L1 uses examples whose complexity he gradually reduces, and questioning to gain insights to students' sense making, in order to reach a point at



which the students can engage with the minutiae of series convergence. This also shows considerable sensitivity to students' making of meaning. However, the conditions play an important role as well. In a lecture with a large number of students, only very few provide feedback to the lecturer. Potentially many others gain from this feedback, but it is not possible for the lecturer to engage directly with all and it is hard for the researchers to access meaning of the wider group. In the SGT, the tutor can engage with each student, either orally or in face to face visual engagement. However, if students are not happy with the ways the tutor engages with them, they are likely not to come to the tutorial.

We have no intention to evaluate the teaching we have observed; the teachers all worked to try to foster students' meaning-making in mathematics, and the extent to which they achieved this goal was of critical concern to both teachers and researchers. We continue to explore ways in which teachers use the setting to enable meaning-making for students. Further research will focus also more directly on the students' sense making in both settings.

## Conclusions

This paper has addressed the teaching of university-level mathematics to students in lectures and in small group tutorials. It has used a number of theoretical perspectives to reach into the thinking of teachers, their planning for teaching and its relation to teaching practice, and to the observations made of what occurs in the teaching-learning interface in lecture or SGT.

We have used the term “Critical Theorising” to emphasise the particular nature and importance of the theoretical work to which we have referred. Readers can see in the accounts above that theory has been used as a research tool in a number of ways and that theory has resulted from analyses. It is these two dimensions of theory which lead us to the term “Critical Theorising”. Some brief references follow.

In the seminal study (Nardi et al. 2005), the theoretical construct *Spectrum of Pedagogical Awareness* emerged from a grounded analysis of data—from a theorizing of this analysis. Subsequently, in Studies A, B and C, we see use of this theoretical construct (SPA) as a tool to capture essences of the teaching in the three studies, particularly with regard to their pedagogical sophistication. In the seminal study, we see use of another theoretical construct, the *Teaching Triad*, adapted as a tool from earlier research (e.g., Jaworski 1994; Potari and Jaworski 2002). This use of the TT as a tool, similarly in Studies B and D, was as an analytic framework enabling a characterization of the teaching data analysed. In Study A, we see a theoretical model emerging, relating actions and goals of teaching. To date, we are not aware of this model being used as a tool elsewhere. Studies B and C, both of which are still in progress, are each developing theory. In these cases the theory developed draws both on grounded analyses and on theoretical constructs such as SPA and TT. It seems fair to claim, based on these several examples, that Critical Theorising is a dynamic process which links research analyses with theory from previous research and itself leads to new theoretical constructs. The critical nature of the theorizing can be seen in the questioning of the data, the conceptualizing from patterns of activity and a close attention to issues, tensions and contradictions in the data.

We have talked here of “Theoretical Constructs” (e.g., SPA, TT), but at the same time located our research in broader theoretical perspectives such as *Sociocultural Theory* and, more specifically, *Activity Theory*. The theoretical constructs are more localised than these broader areas of theory. The broader theoretical areas, underpin the constructs. For example the TT was developed initially within a *Constructivist* frame (Jaworski 1994). We see it being used here within a Sociocultural frame. These two frames can be seen to differ considerably in the ways they conceptualise knowledge and learning and hence the ways in which analyses are conducted (e.g. Jaworski 2015). The broader frame determines the kinds of assumptions underpinning the research and its analyses, and consequently the use of the construct. Thus we see a linking of constructs with the research analyses to which they are applied within the broader philosophical frame. This kind of linking is yet in its infancy, but perhaps might develop a firmer basis if used (and developed) in further studies.

Critical theorizing is likely to remain in the province of the researchers and theoreticians. Theories of teaching can emerge and contribute to a more theoretical understanding of university mathematics teaching which can inform those who work with teachers to develop teaching and provide better learning environments for students.

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