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Kato Classes for Lévy Processes

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Abstract We prove that the definitions of the Kato class through the semigroup and through the resolvent of the Lévy process in $\mathbb{R}^d$ coincide if and only if 0 is not regular for $\{0\}$. If 0 is regular for $\{0\}$ then we describe both classes in detail. We also give an analytic reformulation of these results by means of the characteristic (Lévy-Khintchine) exponent of the process. The result applies to the time-dependent (non-autonomous) Kato class. As one of the consequences we obtain a simultaneous time-space smallness condition equivalent to the Kato class condition given by the semigroup.

Keywords Kato class · Lévy process · Lévy-Khintchine exponent · Schrödinger perturbation · Unimodal isotropic Lévy process · Subordinator · Polarity of a one point set

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1 Introduction

The Kato class plays an important role in the theory of stochastic processes and in the theory of pseudo-differential operators that emerge as generators of stochastic processes.

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The definition of the Kato class may differ according to the underlying probabilistic or analytical problem. In the first case the primary definition of the Kato condition is

$$\lim_{t\to 0^+}\left[\sup_x \mathbb{E}^x\left(\int_0^t |q(X_u)|\,du\right)\right]=0\,. \tag{1}$$

Here q is a Borel function on the state space of the process $X=(X_t)_{t\geqslant 0}$. As shown in [13, section 3.2] through the Khas'minskii Lemma the condition yields sufficient local regularity of the corresponding Schrödinger (Feynman-Kac) semigroup

$$\widetilde{P}_t f(x)=\mathbb{E}^x\left[\exp\left(-\int_0^t q(X_u)\,du\right)f(X_t)\right].$$

In particular, the existence of a density, strong continuity or strong Feller property are inherited under (1) from properties of the original semigroup $P_t f(x)=\mathbb{E}^x f(X_t)$ (for details and further results see [13, Theorems 3.10–3.12]). Moreover, if we denote by L the generator of $(P_t)_{t\geqslant 0}$, we expect the semigroup $(\widetilde{P}_t)_{t\geqslant 0}$ to correspond to $L-q$ and to allow for the analysis of the Schrödinger operator $H=-L+q$ [14]. A fact that the Schrödinger operator is essentially self-adjoint and has bounded and continuous eigenfunctions is another consequence of Eq. 1, see [11, 32] and [18]. Applications of Eq. 1 to quadratic forms of Schrödinger operators are also known and we describe them shortly after Proposition 3.4.

The condition (1) can be understood as a smallness condition with respect to time. The alternative definition of the Kato condition is given by the following space smallness,

$$\lim_{r\to 0^+}\left[\sup_x \mathbb{E}^x\left(\int_0^\infty e^{-\lambda u}\mathbb{1}_{B(x,r)}(X_u)|q(X_u)|\,du\right)\right]=0, \tag{2}$$

for some $\lambda>0$ (equivalently for every $\lambda>0$; see Lemma 3.2).

In this paper we obtain a precise description of the equivalence of Eqs. 1 and 2 for Lévy processes in $\mathbb{R}^d$, $d\in\mathbb{N}$. In order to formulate the result we recall that a point $x\in\mathbb{R}^d$ is said to be *regular* for a Borel set $B\subseteq\mathbb{R}^d$ if

$$\mathbb{P}^x(T_B=0)=1,$$

where $T_B=\inf\{t>0: X_t\in B\}$ is the first hitting time of B.

Theorem 1.1 *Let X be a Lévy process in $\mathbb{R}^d$. The conditions* (1) *and* (2) *are NOT equivalent if and only if 0 is regular for* $\{0\}$.

Complete and direct descriptions of Eqs. 1 and 2 in the case of the *compound Poisson process* are given in Proposition 3.8. When X is not a compound Poisson process and *0 is regular for* $\{0\}$ we fully describe (1) and (2) in Theorems 4.6 and 4.12. To move right away to Section 4 we recommend to read Definition 2 and Section 2.2 first. In Section 2.2 the reader will also find analytic characterization of the situation when 0 is regular for $\{0\}$.

In [11, Theorem III.1] Carmona, Masters and Simon declare that Eq. 1 can be expressed by Eq. 2 under additional assumptions on the transition density of the Lévy process. However, the general equivalence of (i) and (iii) from [11, Theorem III.1] that is claimed therein does not hold. As we show in Theorem 4.6 it fails for the Brownian motion in $\mathbb{R}$ and for those one-dimensional unimodal Lévy processes for which $\{0\}$ is not polar. Recall that a Borel set $B\subseteq\mathbb{R}^d$ is called *polar* if

$$\mathbb{P}^x(T_B=\infty)=1 \qquad \text{for all} \quad x\in\mathbb{R}^d\,.$$

For example the function $q(x) = \sum_{k=1}^{\infty} 2^k \mathbb{1}_{(k,k+2^{-k})}(x)$ satisfies (i), but fails to satisfy (iii) in [11, Theorem III.1] for such processes. The paper [11] was very influential and the mistake reappears in the literature. For instance (1) and (3) of [17, Proposition 4.5] are *not* equivalent in general.

The special character of the one-dimensional case can also be seen in [25, Remark 3.1]. In [25, Definition 3.1 and 3.2] the authors discuss the Kato class of measures for symmetric Markov processes admitting upper and lower estimates of transition density with additional integrability assumptions, see [25, Theorem 3.2].

Theorem 1.1 allows also for results on the time-dependent Kato class for Lévy processes in $\mathbb{R}^d$. Such a class is used for instance in [5, 7, 9, 36, 37]. See [31] for a wider discussion of the Brownian motion case, c.f. [31, Theorem 2].

Corollary 1.2 *Let X be a Lévy process in $\mathbb{R}^d$. For $q\colon \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ we have*

$$\lim_{t\to 0^+}\left[\sup_{s,x}\mathbb{E}^x\left(\int_0^t |q(s+u,X_u)|\,du\right)\right]=0\,, \tag{3}$$

if and only if

$$\lim_{r\to 0^+}\left[\sup_{s,x}\mathbb{E}^x\left(\int_0^r \mathbb{1}_{B(x,r)}(X_u)|q(s+u,X_u)|\,du\right)\right]=0\,. \tag{4}$$

See Section 4 for the proof. If one uses Corollary 1.2 for time-independent q, i.e., let $q\colon \mathbb{R}^d \to \mathbb{R}$ and put $q(u,z)=q(z)$, then the quantity in Eq. 3 coincides with Eq. 1 and we obtain the following reinforcement of Eq. 1 to a time-space smallness condition.

Theorem 1.3 *Let X be a Lévy process in $\mathbb{R}^d$. Then* (1) *holds if and only if*

$$\lim_{r\to 0^+}\left[\sup_{x}\mathbb{E}^x\left(\int_0^r \mathbb{1}_{B(x,r)}(X_u)|q(X_u)|\,du\right)\right]=0\,. \tag{5}$$

In view of the equivalence of Eqs. 1 and 5 for every Lévy process (see Proposition 3.4 for other description of Eq. 1 true for Hunt processes) these conditions should be compared with Eq. 2 by its alternative form provided by Proposition 3.6 in a generality of a Hunt process, i.e.,

$$\lim_{r\to 0^+}\left[\sup_{x}\mathbb{E}^x\left(\int_0^t \mathbb{1}_{B(x,r)}(X_u)|q(X_u)|\,du\right)\right]=0\,, \tag{6}$$

for some (every) fixed $t>0$. The closeness or possible differences between Eqs. 1 and 2 are now more evident for Lévy processes through Eqs. 5 and 6.

The variety of conditions we point out is due to possible applications where one can choose a suitable version according to the knowledge about the process and derive a clear *analytic description* of the Kato condition (1). See also Theorems 4.14 and 4.15 for other conditions. For instance, in Example 1 we apply Theorem 1.1 and we make use of Eq. 6. On the other hand, by Theorem 1.1 and Eq. 2 we obtain that for a large class of subordinators (1) is equivalent to

$$\lim_{r\to 0^+}\sup_{x\in\mathbb{R}}\int_0^r |q(z+x)|\frac{\phi'(z^{-1})}{z^2\phi^2(z^{-1})}\,dz=0\,, \tag{7}$$

where ϕ is the Laplace exponent of the subordinator. See Section 5.2 for details. This is also usual that from Eqs. 2 and 6 one learns, like through Eq. 7, about acceptable singularities of q. Schrödinger perturbations of subordinators are interesting since they exhibit peculiar properties that indicate complexity of the matter. For instance, we easily see that if q is bounded, then $\widetilde{P}_t f(x) \leqslant c_N P_t f(x)$ for every $t \in (0, N]$, $x \in \mathbb{R}$, $f \geqslant 0$. On the other hand, if $-q \geqslant 0$ is time-independent and the above inequality holds for some $N > 0$ on the level of densities, then necessarily $q \in L^\infty(\mathbb{R})$ (see [5, Corollary 3.4]). Nevertheless, perturbation techniques yield an upper bound by means of an auxiliary density for (unbounded) q from the Kato class if an appropriate 4G inequality for the transition density of the subordinator holds (see [5, Proposition 2.4]). Generators of subordinators generalize fractional derivative operators that are used in statistical physics to model anomalous subdiffusive dynamics (see [16]).

A discussion of analytic counterparts of Eq. 1 should contain the fundamental example of the standard Brownian motion in $\mathbb{R}^d$, $d \in \mathbb{N}$. The famous result of Aizenman and Simon [1, Theorem 4.5] says that in this case (1) is equivalent to

$$\lim_{t\to 0^+} \left[\sup_x \int_{|z-x|<\sqrt{t}} \frac{|q(z)|}{|z-x|^{d-2}}\, dz \right] = 0\,, \quad \text{for} \quad d \geqslant 3\,, \tag{8}$$

$$\lim_{t\to 0^+} \left[\sup_x \int_{|z-x|<\sqrt{t}} |q(z)| \ln \frac{t}{|z-x|^2}\, dz \right] = 0\,, \quad \text{for} \quad d = 2\,, \tag{9}$$

$$\left[\sup_x \int_{|z-x|<1} |q(z)|\, dz \right] < \infty\,, \quad \text{for} \quad d = 1\,.$$

Here we also refer to Simon [32, Proposition A.2.6], Chung and Zhao [13, Theorem 3.6], Demuth and van Casteren [14, Theorem 1.27]. The above remains true if $\ln(t/|z-x|^2)$ is replaced by $\ln(1/|z-x|)$ for $d = 2$ and if $|q(z)|$ is multiplied by $|z-x|$ for $d = 1$. In fact, the expressions in square brackets of Eqs. 1 and 8 are comparable for $d \geqslant 3$, while for $d = 2$ and $d = 1$ similar but slightly different results hold (see Bogdan and Szczypkowski [9], Demuth and van Casteren [14, Theorem 1.28]). We emphasise that (8) was used by Kato [20] to prove by analytic methods that the operator $-\Delta + q$ is essentially self-adjoint (see [21] for extensions to second order elliptic operators). The equivalence of Eq. 1 with Eqs. 8 and 9 follows also from Theorem 1.1 (see [38]). The one-dimensional case is also covered by Theorem 4.6 of this paper.

In what follows we present and explain our main ideas in view of the literature. A major contribution to the understanding of the subject in a general probabilistic manner is made by Zhao [38]. Zhao considers a Hunt process $X = (\Omega, \mathscr{F}_t, X_t, \vartheta_t, \mathbb{P}^x)$ with state space (S, ρ) and life-time ζ, where S is a locally compact metric space with a metric ρ (see [4]). For a strong sub-additive functional A_t of X, $t \geqslant 0$, he discusses relations between the following three conditions

$$\lim_{r\to 0^+} \left\{ \sup_x \mathbb{E}^x \left[\int_0^\infty \mathbb{1}_{B(x,r)}(X_t)\, dA_t \right] \right\} = 0\,, \tag{C1}$$

$$\lim_{t\to 0^+} \left[\sup_x \mathbb{E}^x(A(t)) \right] = 0\,, \tag{C2}$$

$$\lim_{r\to 0^+} \left\{ \sup_x \mathbb{E}^x \left[A(\tau_{B(x,r)}) \right] \right\} = 0\,, \tag{C3}$$

in presence of three hypotheses on the process X,

$$h_1(X) \equiv \sup_{t>0} \inf_{r>0} \sup_{x\in S} \mathbb{P}^x\left(\tau_{B(x,r)} > t\right) < 1\,, \tag{H1}$$

$$h_2(X) \equiv \sup_{r>0} \inf_{t>0} \sup_{x\in S} \mathbb{P}^x\left(\tau_{B(x,r)} < t\right) < 1\,, \tag{H2}$$

$$h_3(X) \equiv \sup_{u>0} \inf_{r>0} \sup_{\substack{x,\, y\in S\\ \rho(x,y)\geqslant u}} \mathbb{P}^y\left(T_{B(x,r)} < \zeta\right) < 1\,. \tag{H3}$$

Here for any Borel set B in S, T_B is the first hitting time of B, $\tau_B = T_{S\setminus B}$ is the first exit time of B (we let $\inf\emptyset = \infty$) and $B(x,r) = \{y \in S : \rho(x,y) < r\}$, $x \in S$, $r > 0$. We present the main theorem of Zhao [38] on Fig. 1 below; for instance, under (H3), (C3) implies (C1).

In this paper we assume that A_t, $t \geqslant 0$, is the additive functional of the form

$$A_t = \int_0^t |q(X_u)|du\,, \tag{10}$$

and we note that any additive functional is a strong sub-additive functional; see [38, Lemma 1]. Then (C2) coincides with Eq. 1 and as such becomes the principal object of our considerations. We explain the origin and the choice of Eq. 2 using the concept of λ-subprocess X^λ, $\lambda > 0$, of the process X (see [4] for the definition). We first notice that (C2) holds for X if and only if it holds for X^λ (see Remark 9 and Definition 2). A similar statement is not true in general for (C1). For the standard Brownian motion in $\mathbb{R}^d$, $d \geqslant 3$, (C2) in fact coincides with (C1), which gives rise to Eq. 8, yet for $d = 2$ or $d = 1$ the expectation in (C1) is infinite for constant non-zero q, whereas that never happens for (C2). This shows that (C1) for X is too strong for a general equivalence result. Therefore we rely on the relations of Fig. 1 for X^λ, and then (C1) results in Eq. 2. We also observe that Eq. 2 holds for X if and only if it holds for $X^{\lambda'}$, $\lambda' > 0$ (see Remark 9). To ultimately clarify the choice of X^λ we note that $h_1(X^\lambda) = h_1(X)$, $h_2(X^\lambda) = h_2(X)$ and $h_3(X^\lambda) \leqslant h_3(X)$ (see Lemmas 2.10 and 2.11).

We now restrict ourselves to the case of the Lévy process in $\mathbb{R}^d$. Besides being a Hunt process in $\mathbb{R}^d$, X is also translation invariant. We point out that (H2) holds for every Lévy process and (H1) holds if and only if X is not a compound Poisson process (see Remark 8). The case of the compound Poisson process is entirely described in Proposition 3.8. Thus, in the remaining cases, (H3) for X^λ becomes decisive for understanding the confines of the applicability of Fig. 1 to X^λ. By Proposition 2.15 the study of $h_3(X^\lambda)$ reduces to the analysis of the first hitting time of a single point set by the original Lévy process X. Namely, we consider (see also Lemma 4.2)

$$h^\lambda(x) := \mathbb{E}^0 e^{-\lambda T_{\{x\}}}\,, \qquad x \in \mathbb{R}^d\,. \tag{11}$$

Eventually, by Corollary 2.16 and Remark 8 we obtain the following characterization.

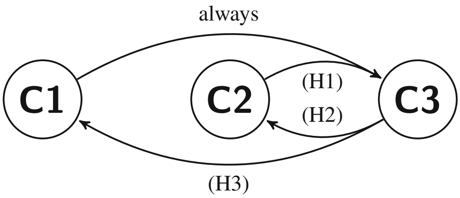

Fig. 1 Zhao [38] hypotheses and conditions

Proposition 1.4 *Let X be a Lévy process in $\mathbb{R}^d$ and $\lambda > 0$. All hypotheses* (H1), (H2) *and* (H3) *are satisfied for X^λ if and only if $\{0\}$ is polar.*

Therefore Theorem 1.1 goes much beyond the range of [38]. The reason is that in our work we also investigate all the cases that are not covered by Fig. 1. Our initial study effects in a list that classifies Lévy processes according to a non-degeneracy hypothesis (H0) and specific properties of h^λ, which is thoroughly examined by Bretagnolle [10] for one-dimensional non-Poisson Lévy processes. A full layout of our development is presented in Section 2.2. Theorem 1.1 results as a summary of Proposition 3.8 and 6 theorems of Section 4. We stress that the non-symmetric cases or those close to the compound Poisson process (without (H0)) are more delicate and require more precision.

In [38, Lemma 4] Zhao proposes sufficient conditions on X under which (H1)-(H3) are satisfied for X^λ. He uses them to re-prove the result of Aizenman and Simon [1] for $d \geqslant 2$. He also verifies hypotheses (H1)-(H3) directly for X in the case of Lévy processes admitting rotationally symmetric transition density with additional assumption on the behaviour of the density integrated in time [38, Lemma 5]. Finally he applies that to describe (1) for symmetric α-stable processes, $d > \alpha$, and the relativistic process. We generalize [38, Lemma 5] in Theorem 4.15.

The paper is organized as follows. In Section 2 we introduce the non-degeneracy hypothesis (H0) for a Lévy process. Next, we give a classification of Lévy processes that provides a detailed plan of our research. In the last part of Section 2 we prove results concerning hypotheses (H1)-(H3). In Section 3, for a Hunt process X, we define Kato classes $\mathbb{K}(X)$ and $\mathcal{K}(X)$ of functions q satisfying (1) and (2), respectively. We give other general descriptions of both of those classes and we establish their initial relations for Lévy processes. In Section 4 we prove the main description theorems for Lévy processes, separately under and without (H0). Section 4 ends with additional equivalence results involving the class $\mathcal{K}^0(X)$ (see (26)). In Section 5 we present a supplementary discussion on isotropic unimodal Lévy processes and subordinators. The paper finishes with examples.

2 Preliminaries

Our main focus in this paper is on a (general) Lévy process X in $\mathbb{R}^d$ (see [29]). The characteristic exponent ψ of X defined by $\mathbb{E}^0 e^{i\langle x, X_t\rangle} = e^{-t\psi(x)}$ equals

$$\psi(x) = -i\,\langle x, \gamma\rangle + \langle x, Ax\rangle - \int_{\mathbb{R}^d} \left(e^{i\langle x,z\rangle} - 1 - i\,\langle x, z\rangle\, \mathbb{1}_{|z|<1}\right) \nu(dz), \quad x \in \mathbb{R}^d,$$

where $\gamma \in \mathbb{R}^d$, A is a symmetric non-negative definite matrix and ν is a Lévy measure, i.e., $\nu(\{0\}) = 0$, $\int_{\mathbb{R}^d} \left(1 \wedge |z|^2\right) \nu(dz) < \infty$. If $\int_{\mathbb{R}^d} (1 \wedge |z|)\, \nu(dz) < \infty$, then the above representation simplifies to

$$\psi(x) = -i\,\langle x, \gamma_0\rangle + \langle x, Ax\rangle - \int_{\mathbb{R}^d} \left(e^{i\langle x,z\rangle} - 1\right) \nu(dz), \quad x \in \mathbb{R}^d,$$

where $\gamma_0 = \gamma - \int_{\mathbb{R}^d} z\mathbb{1}_{|z|<1}\nu(dz)$. Further, if $\gamma_0 = 0$, $A = 0$ and $\nu(\mathbb{R}^d) < \infty$, then X is called a compound Poisson process (see [29, Remark 27.3]). We say that X is non-Poisson if X is not a compound Poisson process. Recall that $\mathbb{E}^x F(X) = \mathbb{E}^0 F(X+x)$ for $x \in \mathbb{R}^d$ and Borel functions $F \geqslant 0$ on paths. In particular $h^\lambda(x) = \mathbb{E}^{(-x)} e^{-\lambda T_{\{0\}}}$, and thus the following holds.

Remark 1 $\{0\}$ is polar if and only if $h^\lambda(x) = 0$, $x \in \mathbb{R}^d$.

Remark 2 0 is regular for $\{0\}$ if and only if $h^\lambda(0) = 1$.

Remark 3 X is such that $A = 0$, $\gamma_0 \in \mathbb{R}^d$, $\int_{\mathbb{R}^d}(|x| \wedge 1)\nu(dx) < \infty$ if and only if X has finite variation on finite time intervals ([29, Theorem 21.9]). Then $\mathbb{P}^0(\lim_{s\to 0^+} s^{-1}X_s = \gamma_0) = 1$ ([33, Theorem 1]; see also [29, Theorem 43.20]).

Lemma 2.1 *Let X be non-Poisson. Then $\mathbb{P}^0(X_t = 0) = 0$ except for countably many $t > 0$.*

Proof By [29, Theorem 27.4] it suffices to consider compound Poisson process with non-zero drift. Let then ν and γ_0 be its Lévy measure and drift. According to the decomposition $\nu = \nu^d + \nu^c$ for discrete and continuous part (see [29, Chapter 5, Section 27]) we write $X_t = X^d_t + X^c_t + \gamma_0 t$. For $t > 0$, by [29, Remark 27.3] $\mathbb{P}^0(X^c_t \in dz)$ is continuous on $\mathbb{R}^d \setminus \{0\}$, therefore $\mathbb{P}^0(X^c_t \in C \setminus \{0\}) = 0$ for any countable set $C \subset \mathbb{R}^d$. By [29, Corollary 27.5 and Proposition 27.6] there is a countable set $C_{X^d} \subset \mathbb{R}^d$ such that $\mathbb{P}^0(X^d_t + \gamma_0 t = 0) > 0$ if and only if $(-\gamma_0 t) \in C_{X^d}$. Thus $\mathbb{P}^0(X^d_t + \gamma_0 t = 0) = 0$ except for countably many $t > 0$. Finally,

$$\begin{aligned}
\mathbb{P}^0(X^d_t + X^c_t + \gamma_0 t = 0) &= \mathbb{P}^0(X^c_t = 0,\ X^d_t + \gamma_0 t = 0) \\
&\quad + \mathbb{P}^0(X^c_t = -(X^d_t + \gamma_0 t),\ X^d_t + \gamma_0 t \neq 0) \\
&\leqslant \mathbb{P}^0(X^d_t + \gamma_0 t = 0) + \mathbb{P}^0(X^c_t \in -(C_{X^d} + \gamma_0 t) \setminus \{0\}) = 0\,,
\end{aligned}$$

except for countably many $t > 0$. □

We say that a Lévy process X is non-sticky if $\mathbb{P}^0(\tau_{\{0\}} > 0) = 0$, or equivalently that the hypothesis (H) from [10] holds. Lemma 2.1 reinforces remarks following [38, Lemma 3].

Remark 4 X is non-sticky if and only if X is non-Poisson.

If necessary we specify which Lévy process we have in mind by adding a superscript, for instance $h^{Z,\lambda}$ is the function given by Eq. 11 that corresponds to the process Z.

2.1 Non-Degeneracy Hypothesis (H0) for Lévy Processes

Before we introduce the main non-degeneracy hypothesis on a Lévy process X we recall the basic matrix notation. Let M be a matrix. We let M^* to be the transpose of M and $M(\mathbb{R}^d)$ the range of M. We call M a projection if it is symmetric and $M^2 = M$. For a subset V by $V^\perp$ we denote the orthogonal complement of V in $\mathbb{R}^d$. We use the following fact.

Lemma 2.2 *If A is symmetric non-negative definite and $M^*AM = 0$, then $A(\mathbb{R}^d) \subseteq M(\mathbb{R}^d)^\perp$.*

Remark 5 Let X be a Lévy process in a linear subspace V of $\mathbb{R}^d$ (see [29, Proposition 24.17]) and denote $d_0 = \dim(V)$. Then there exists a rotation given by a matrix $O \in \mathcal{M}_{d\times d}$ such that $Y = OX$ is a Lévy process in $\mathbb{R}^{d_0}$; the correspondence between X and Y is one-to-one.

Lemma 2.3 *Let X be a Lévy process in $\mathbb{R}^d$ and Π be a projection. If $\{0\}$ is polar for the process $Y = \Pi X$, then $\{0\}$ is polar for X.*

Proof If $X_t + x = 0$, then $Y_t + \Pi x = 0$, thus $\inf\{t > 0\colon X_t + x = 0\} \geqslant \inf\{t > 0\colon Y_t + \Pi x = 0\}$ and $\mathbb{P}^x(T_{\{0\}} < \infty) \leqslant \mathbb{P}^{\Pi x}(T^Y_{\{0\}} < \infty) = 0$. □

Definition 1 We say that (H0) holds for X if there is no linear subspace V of $\mathbb{R}^d$ such that

$$\dim(V) \leqslant \min\{1, d-1\},$$
$$A(\mathbb{R}^d) \subseteq V, \quad \nu(\mathbb{R}^d \setminus V) < \infty, \quad \text{and} \quad \gamma - \int_{\mathbb{R}^d \setminus V} z \mathbb{1}_{B(0,1)}(z)\nu(dz) \in V. \tag{12}$$

We give a precise probabilistic description of (H0).

Remark 6 For $d = 1$, (H0) holds if and only if X is non-Poisson. For $d > 1$, (H0) holds if and only if X is non-Poisson and is not of the form Eq. 13 below.

Proposition 2.4 *Let $d > 1$ and X be non-Poisson. Then (H0) does not hold if and only if*

$$X = Y + Z, \tag{13}$$

and there exist a linear subspace V of $\mathbb{R}^d$, $\dim(V) = 1$, such that

- *i) Y and Z are independent,*
- *ii) Y is either zero or a compound Poisson process with the Lévy measure vanishing on V,*
- *iii) Z is not a compound Poisson process,*
- *iv) Z is supported on V.*

Proof Since we assume that X is non-Poisson, if Eq. 12 holds and $\dim(V) \leqslant \min\{1, d-1\}$, then $\dim(V) = 1$. We let Y to be a compound Poisson process with the Lévy measure $\nu^Y = [\nu]_{\mathbb{R}^d \setminus V}$ and let Z to be a Lévy process with the Lévy triplet $(A, \gamma - \int_{\mathbb{R}^d \setminus V} z \mathbb{1}_{B(0,1)}(z)\nu(dz), [\nu]_V)$, where $[\nu]_B$ denotes the measure ν restricted to a set B. By definition $\psi = \psi^Y + \psi^Z$, hence $X = Y + Z$ and i), ii) and iii) are satisfied. The property iv) follows from [29, Proposition 24.17]. Conversely, if X is of the form (13), then its Lévy triplet is given by $A = A^Z$, $\gamma = \gamma^Z + \int_{\mathbb{R}^d \setminus V} z \mathbb{1}_{B(0,1)}(z)\nu^Y(dz)$ and $\nu = \nu^Y + \nu^Z$. Then Eq. 12 holds since $\nu = \nu^Y$ on $\mathbb{R}^d \setminus V$. □

The hypothesis (H0) agrees with the hypothesis (H) from [10] if $d = 1$. In particular, for $d = 1$ under (H0) we have that $\{0\}$ is essentially polar if and only if $\{0\}$ is polar. As known, in $d > 1$ $\{0\}$ is always essentially polar (see [3, Theorem 16 and Corollary 17]).

Proposition 2.5 *Let $d > 1$ and assume (H0). Then $\{0\}$ is polar.*

Proof Let V be the smallest in dimension linear subspace in $\mathbb{R}^d$ satisfying Eq. 12. Now, let Π_1 be the projection on V and define $Y = \Pi_1 X$. Observe that by (H0) we have $\dim(V) \geqslant 2$. We claim that there is no one-dimensional subspace $W \subset V$ such that the projection of Y on W is a compound Poisson process. For the proof assume that there is such W and let Π_2 be the projection on W. Then $Z = \Pi_2 Y = \Pi_2 X$ is a compound Poisson process. By

[29, Proposition 11.10] we have the following consequences. First, $\Pi_2 A \Pi_2 = 0$ and by Lemma 2.2 we obtain $A(\mathbb{R}^d) \subseteq V \cap W^{\perp}$. Next, $\nu(\mathbb{R}^d \setminus W^{\perp}) = \nu\Pi_2^{-1}(\mathbb{R}^d \setminus \{0\}) < \infty$ and then $\nu(\mathbb{R}^d \setminus (V \cap W^{\perp})) < \infty$. Further, since $\Pi_2 z = 0$ on $V \cap W^{\perp}$ we have

$$\begin{aligned} 0 &= \Pi_2\gamma - \int_{\mathbb{R}^d} \Pi_2 z \mathbb{1}_{B(0,1)}(z)\nu(dz) \\ &= \Pi_2\gamma - \int_{\mathbb{R}^d \setminus (V\cap W^{\perp})} \Pi_2 z \mathbb{1}_{B(0,1)}(z)\nu(dz) \\ &= \Pi_2 \left(\gamma - \int_{\mathbb{R}^d \setminus (V\cap W^{\perp})} z \mathbb{1}_{B(0,1)}(z)\nu(dz) \right) . \end{aligned}$$

Thus $\gamma_1 = \gamma - \int_{\mathbb{R}^d \setminus (V\cap W^{\perp})} z \mathbb{1}_{B(0,1)(z)}\nu(dz) \in W^{\perp}$. Finally, by $\mathbb{R}^d \setminus (V \cap W^{\perp}) = (\mathbb{R}^d \setminus V)\dot{\cup}(V \setminus W^{\perp})$ and by Eq. 12,

$$\gamma_1 = \left(\gamma - \int_{\mathbb{R}^d \setminus V} z \mathbb{1}_{B(0,1)}(z)\nu(dz) \right) - \int_{V \setminus W^{\perp}} z \mathbb{1}_{B(0,1)}(z)\nu(dz) \in V ,$$

which is a contradiction, because then Eq. 12 holds with $V \cap W^{\perp}$ in place of V and $\dim(V \cap W^{\perp}) < \dim(V)$. Now, by Remark 5 we can treat Y as a process in $\mathbb{R}^{d_0}$, $d_0 = \dim(V) \geqslant 2$, and then by [10, Theoreme 4] the set $\{0\}$ is a polar set for Y as well as for X by Lemma 2.3. $\square$

2.2 Classification of Lévy Processes

We outline our work-flow to analyze every Lévy process X.

Exclusively one of the following situations holds for a Lévy process in $\mathbb{R}^d$.

1. (H0) holds:
   - (a) $d > 1$ (then $h^{\lambda}(x) = 0$, $x \in \mathbb{R}^d$),
   - (b) $d = 1$
     - (A) $h^{\lambda}(x) = 0$, $x \in \mathbb{R}$,
     - (B) $h^{\lambda}(0) = \liminf_{x\to 0} h^{\lambda}(x) < \limsup_{x\to 0} h^{\lambda}(x) = 1$,
     - (C) $h^{\lambda}(0) = \lim_{x\to 0} h^{\lambda}(x) = 1$.
2. (H0) does not hold:
   - (a) a compound Poissson process ($d \geqslant 1$; then $h^{\lambda}(0) = 1$),
   - (b) given by (13) ($d > 1$)
     - (A') $h^{Z,\lambda}(v) = 0$, $v \in V$,
     - (B') $h^{Z,\lambda}(0) = \liminf_{v\in V,\, v\to 0} h^{Z,\lambda}(v) < \limsup_{v\in V,\, v\to 0} h^{Z,\lambda}(v) = 1$,
     - (C') $h^{Z,\lambda}(0) = \lim_{v\in V,\, v\to 0} h^{Z,\lambda}(v) = 1$.

The comment in the case case 1(a) is a consequence of Proposition 2.5 and Remark 1. The partition of the case 1(b) is due to Remarks 6, 4 and [10, Théorèmes 3 and 6]. The division of the case 2 results from Remark 6. The subcases of 2(b) follow from Remark 5 and [10].

The subcases of 1(b) translate equivalently into probabilistic properties of X, see [10, Théorèmes 6, 8] and Remark 3. We have

(A) $\{0\}$ is polar,
(B) X has finite variation and non-zero drift,
(C) 0 is regular for $\{0\}$.

The analytic counterpart by means of the characteristic exponent or the Lévy triplet is (see [10, Théorèmes 3, 7 and 8])

(A) $\int_{\mathbb{R}} \mathrm{Re}\left(\frac{1}{\lambda+\psi(z)}\right) dz = \infty$,
(B) $A = 0$, $\gamma_0 \neq 0$ and $\int_{\mathbb{R}}(|x| \wedge 1)\nu(dx) < \infty$,
(C) $A \neq 0$ or (A) does not hold and $\int_{\mathbb{R}}(|x| \wedge 1)\nu(dx) = \infty$.

We could similarly reformulate 2(b) for Z, but in proofs of Theorems 4.11 and 4.12 we use the following description.

(A') $\int_V \mathrm{Re}\left(\frac{1}{\lambda+\psi^Z(v)}\right) dv = \infty$ (dv is the one-dimensional Lebesgue measure on V),
(B') $A^Z = 0$, $\gamma_0^Z \neq 0$ and $\int_V(|x| \wedge 1)\nu^Z(dx) < \infty$,
(C') 0 is regular for $\{0\}$.

We translate (A'), (B') and (C') into X given by Eq. 13.

Lemma 2.6 $\{0\}$ *is polar for* X *if and only if* $\{0\}$ *is polar for* Z.

Proof If $\{0\}$ is polar for Z, then $\int_V \mathrm{Re}(1/[\lambda + \psi^Z(v)])dv = \infty$. By Lemma 2.3 to verify that $\{0\}$ is polar for X it suffices to show that it is polar for $\Pi X = \Pi(Y + Z) = \Pi Y + Z$, where Π is the projection on V. Since $\psi^{\Pi X} = \psi^{\Pi Y} + \psi^Z$ and $\psi^{\Pi Y}$ is bounded (ΠY is a compound Poisson process) we have by our assumption $\int_V \mathrm{Re}(1/[\lambda + \psi^{\Pi X}(v)])dv = \infty$. Thus Remark 5 and [10, Théorèmes 7, 3] end this part of the proof. If $\{0\}$ is not polar for Z, $\mathbb{P}^0(T^Z_{\{x\}} < \infty) > 0$ for some $x \in V$, we have for large $t > 0$

$$\mathbb{P}^0(T_{\{x\}} < \infty) \geqslant \mathbb{P}^0\left(Y_t = 0,\ T_{\{x\}} = T^Z_{\{x\}} < t\right) = \mathbb{P}^0(Y_t = 0)\mathbb{P}^0\left(T^Z_{\{x\}} < t\right) > 0\,.$$

□

Lemma 2.7 $\{0\}$ *is not polar for* X *if and only if* $\limsup_{x\to 0} h^\lambda(x) = 1$.

Proof If $\limsup_{x\to 0} h^\lambda(x) = 1$, then $h^\lambda(x) > 0$ for some $x \in \mathbb{R}^d$ and $\mathbb{P}^0(T_{\{x\}} < \infty) > 0$. Conversely, if $\{0\}$ is not polar for X then by Lemma 2.6 it is not polar for Z and $\limsup_{v\in V, v\to 0} h^{Z,\lambda}(v) = 1$. This implies $\limsup_{v\in V, v\to 0} \mathbb{P}^0(T^Z_{\{v\}} < t) = 1$ for every fixed $t > 0$. Thus we have for $t > 0$

$$\begin{aligned} h^\lambda(x) &\geqslant \mathbb{E}^0\left(Y_t = 0, T^Z_{\{x\}} < t; e^{-\lambda T_{\{x\}}}\right) = \mathbb{E}^0\left(Y_t = 0, T^Z_{\{x\}} < t; e^{-\lambda T^Z_{\{x\}}}\right) \\ &\geqslant \mathbb{P}^0(Y_t = 0)\mathbb{P}^0\left(T^Z_{\{x\}} < t\right) e^{-\lambda t}\,, \end{aligned}$$

which gives $\limsup_{x\to 0} h^\lambda(x) \geqslant \mathbb{P}^0(Y_t = 0)e^{-\lambda t}$. Finally, we let $t \to 0^+$. □

Lemma 2.8 *0 is regular for* $\{0\}$ *for* X *if and only if 0 is regular for* $\{0\}$ *for* Z.

Proof We observe that the set $\{Y_s = 0$ for all $s \in [0, \delta]$ for some $\delta > 0\}$ is of measure one with respect to $\mathbb{P}^0$. On that set $T_{\{0\}} = 0$ if and only if $T^Z_{\{0\}} = 0$. □

Corollary 2.9 *For the process* X *of the form* Eq. 13 *we have*

(A') $h^\lambda(x) = 0$, $x \in \mathbb{R}^d$,

(B') $h^\lambda(0) < \limsup_{x\to 0} h^\lambda(x) = 1$,
(C') $h^\lambda(0) = \limsup_{x\to 0} h^\lambda(x) = 1$,

and

(A') $\{0\}$ *is polar,*
(B') *X has finite variation and non-zero drift (see Remark 3),*
(C') *0 is regular for* $\{0\}$.

The last observation facilitates a discussion of (H3) in the next subsection.

Remark 7 For a non-Poisson Lévy process we have $\limsup_{x\to 0} h^\lambda(x) = 1$ or $h^\lambda(x) = 0$, $x \in \mathbb{R}^d$.

2.3 Hypotheses (H1)-(H3)

We start with a general case of a Hunt process X on S with life-time ζ. In the proofs of Lemmas 2.10 and 2.11 all objects corresponding to X^λ, the λ-subprocess of X, are indicated with a bar, e.g., $\overline{T}_B = \inf\{t > 0\colon X_t^\lambda \in B\}$.

Lemma 2.10 *Let* $\lambda > 0$. *We have* $h_1(X^\lambda) = h_1(X)$ *and* $h_2(X^\lambda) = h_2(X)$.

Proof Recall that $\inf\emptyset = \infty$. For any Borel set B in S and $t > 0$ we have $\{\overline{\tau}_B > t\} = \{\tau_B > t\} \times [0,\infty) \,\dot{\cup}\, \{\tau_B \leqslant t\} \times [0,\tau_B)$ and $\{\overline{\tau}_B < t\} = \{\tau_B < t\} \times (\tau_B,\infty)$. Thus,

$$\overline{\mathbb{P}}^x\left(\overline{\tau}_B > t\right) = \mathbb{P}^x\left(\tau_B > t\right) + \mathbb{E}^x\left(\tau_B \leqslant t; 1 - e^{-\lambda\tau_B}\right) \leqslant \mathbb{P}^x\left(\tau_B > t\right) + 1 - e^{-\lambda t},$$

and

$$\begin{aligned}\overline{\mathbb{P}}^x\left(\overline{\tau}_B < t\right) &= \mathbb{E}^x\left(\tau_B < t;\ e^{-\lambda\tau_B}\right) = \mathbb{P}^x\left(\tau_B < t\right) + \mathbb{E}^x\left(\tau_B < t;\ e^{-\lambda\tau_B} - 1\right)\\ &\geqslant \mathbb{P}^x\left(\tau_B < t\right) + e^{-\lambda t} - 1\,.\end{aligned}$$

Since we may change $\sup_{t>0}$ with $\limsup_{t\to 0^+}$, $h_1(X) \leqslant h_1(X^\lambda) \leqslant h_1(X) + \lim_{t\to 0^+}(1 - e^{-\lambda t})$ and since we may replace $\inf_{t>0}$ with $\liminf_{t\to 0^+}$, $h_2(X) \geqslant h_2(X^\lambda) \geqslant h_2(X) + \lim_{t\to 0^+}(e^{-\lambda t} - 1)$. This ends the proof. □

Lemma 2.11 *Let* $\lambda > 0$. *We have* $h_3(X^\lambda) \leqslant h_3(X)$, *more precisely*

$$h_3(X^\lambda) = \sup_{u>0}\inf_{r>0}\sup_{\substack{x,\,y\in S\\ \rho(x,y)\geqslant u}} \mathbb{E}^y\left(T_{B(x,r)} < \zeta;\, e^{-\lambda T_{B(x,r)}}\right).$$

Proof For any Borel set B in S we have $\{\overline{T}_B < \overline{\zeta}\} = \{T_B < \zeta\} \times (T_B,\infty)$. This results in $\overline{\mathbb{P}}^y(\overline{T}_B < \overline{\zeta}) = \mathbb{E}^y(T_B < \zeta; e^{-\lambda T_B})$. □

Now, let $S = \mathbb{R}^d$ be the Euclidean space and $\zeta = \infty$. The following lemmas and corollary address the question whether $h_3(X^\lambda) = \sup_{u>0}\inf_{r>0}\sup\limits_{|x-y|\geqslant u} \mathbb{E}^y e^{-\lambda T_{B(x,r)}} < 1$.

Lemma 2.12 *Let* $x \in \mathbb{R}^d$ *be fixed. Then*

$$\lim_{r\to 0^+} T_{\overline{B}(x,r)} = T_{\{x\}} \qquad \mathbb{P}^0\ a.s. \tag{14}$$

Proof Fix $x \in \mathbb{R}^d$. Define the stopping times $T_r = T_{\overline{B}(x,r)}$ and $T = \lim_{r\to 0^+} T_r$, $r > 0$. Obviously, $T_r \leqslant T \leqslant T_{\{x\}}$. It suffices to consider (14) on the set $\{T < \infty\}$, otherwise both sides of Eq. 14 are infinite. Since T_r is non-increasing in $r > 0$ we have by the quasi-left continuity $\lim_{r\to 0^+} X_{T_r} = X_T$ a.s. on $\{T < \infty\}$. On the other hand, by the right continuity we have $X_{T_r} \in \overline{B}(x,r)$ and thus $\lim_{r\to 0^+} X_{T_r} = x$ a.s. on $\{T < \infty\}$. Finally, $X_T = x$ and consequently $T \geqslant T_{\{x\}}$ a.s. on $\{T < \infty\}$. □

Lemma 2.13 *Let* $\tau_n = \tau_{B(0,n)}$. *Then* $\lim_{n\to\infty} \tau_n = \infty$ $\mathbb{P}^0$ *a.s.*

Proof Denote $\tau = \lim_{n\to\infty} \tau_n$. Since τ_n is non-decreasing, by the quasi-left continuity $X_{\tau_n} \xrightarrow{n\to\infty} X_\tau$ a.s. on $\{\tau < \infty\}$. On $\{\tau < \infty\}$ for $n \geqslant |X_\tau| + 1$ by the right continuity we have $|X_{\tau_n}| \geqslant |X_\tau|+1$, which is a contradiction; it shows that a.s $\tau < \infty$ does not occur. □

Lemma 2.14 *Let* $\lambda > 0$. *Then*

$$\sup_{u>0}\inf_{r>0}\sup_{|x|\geqslant u} \mathbb{E}^0 e^{-\lambda T_{\overline{B}(x,r)}} = \sup_{x\neq 0} \mathbb{E}^0 e^{-\lambda T_{\{x\}}}\,. \tag{15}$$

Proof Let $f_r(x) = \mathbb{E}^0 e^{-\lambda T_{\overline{B}(x,r)}}$, $r \geqslant 0$, $x \in \mathbb{R}^d$, where $\overline{B}(x,0) = \{x\}$. Notice that $f_r(x) \geqslant f_0(x)$. Therefore

$$a := \sup_{u>0}\inf_{r>0}\sup_{|x|\geqslant u} f_r(x) \geqslant \sup_{u>0}\inf_{r>0}\sup_{|x|\geqslant u} f_0(x) = \sup_{u>0,\,|x|\geqslant u} f_0(x) = \sup_{x\neq 0} f_0(x) \geqslant 0\,. \tag{16}$$

It suffices to prove the reverse inequality in the case $a \neq 0$, otherwise (15) holds by Eq. 16. Thus let $a \in (0,1]$. Then for $\varepsilon > 0$ there is $u > 0$ such that for all $r > 0$ we have $\sup_{|x|\geqslant u} f_r(x) > a - \varepsilon$. Hence, there is a sequence $\{x_n\}$ such that $f_{1/n}(x_n) > a - \varepsilon$ and $|x_n| \geqslant u$. We will show that $\{x_n\}$ is bounded. For $r \in (0,1]$, $m \in \mathbb{N}$ and $|x| \geqslant m+2$, we have $T_{\overline{B}(x,r)} \geqslant \tau_m$ thus by Lemma 2.13 and the dominated convergence theorem there is m_0 such that

$$\sup_{|x|\geqslant m_0+2} f_r(x) \leqslant \mathbb{E}^0 e^{-\lambda\tau_{m_0}} \leqslant a - \varepsilon\,.$$

This proves that $m_0 + 2 \geqslant |x_n| \geqslant u > 0$ for every n. We let $y \neq 0$ to be the limit point of $\{x_n\}$. Observe that for every $r > 0$ there is n such that $B(x_n, 1/n) \subseteq B(y,r)$, which implies $T_{\overline{B}(y,r)} \leqslant T_{\overline{B}(x_n,1/n)}$ and $f_r(y) \geqslant f_{1/n}(x_n) > a - \varepsilon$. Finally, by Lemma 2.12 and the dominated convergence theorem we obtain

$$\sup_{x\neq 0} \mathbb{E}^0 e^{-\lambda T_{\{x\}}} \geqslant \mathbb{E}^0 e^{-\lambda T_{\{y\}}} = \lim_{r\to 0} \mathbb{E}^0 e^{-\lambda T_{\overline{B}(y,r)}} = \lim_{r\to 0} f_r(y) \geqslant a - \varepsilon.$$

This ends the proof since $\varepsilon > 0$ was arbitrary. □

We continue discussing (H1)-(H3) for a Lévy process X in $\mathbb{R}^d$. Remark 4 and [38, Lemmas 2 and 3] ensure the following.

Remark 8 Clearly (H1) does not hold for any compound Poisson process.
(H1) holds for every non-Poisson Lévy process X with $h_1(X) = 0$.
(H2) holds for every Lévy process X with $h_2(X) = 0$.

Proposition 2.15 *Let X be a Lévy process in $\mathbb{R}^d$ and $\lambda > 0$. For h^λ defined in* Eq. 11 *we have*

$$h_3(X^\lambda) = \sup_{x \neq 0} h^\lambda(x)\,.$$

Proof By Lemma 2.11, $\overline{B}(x, r/2) \subseteq B(x,r) \subseteq \overline{B}(x,r)$ and Lemma 2.14

$$\begin{aligned} h_3(X^\lambda) &= \sup_{u>0} \inf_{r>0} \sup_{|x-y|\geqslant u} \mathbb{E}^y(T_{B(x,r)} < \infty; e^{-\lambda T_{B(x,r)}}) \\ &= \sup_{u>0} \inf_{r>0} \sup_{|x-y|\geqslant u} \mathbb{E}^0(e^{-\lambda T_{\overline{B}(x-y,r)}}) \\ &= \sup_{x\neq 0} \mathbb{E}^0 e^{-\lambda T_{\{x\}}}\,. \end{aligned}$$

□

By Proposition 2.15, Remarks 7 and 1 we obtain an improvement of [38, Lemma 4].

Corollary 2.16 *Let X be non-Poisson and $\lambda > 0$. Then* (H3) *holds for X^λ if and only if* $\{0\}$ *is polar for X. If this is the case, then we have $h_3(X^\lambda) = 0$.*

3 Kato Class

Let X be a Hunt process in $\mathbb{R}^d$. For $t \geqslant 0$ we define the *transition kernel* $P_t(x, dz)$ and the corresponding *transition operator* P_t by

$$P_t(x, B) = \mathbb{P}^x(X_t \in B)\,, \qquad P_t f(x) = \int_{\mathbb{R}^d} f(z) P_t(x, dz)\,.$$

Moreover, for $\lambda \geqslant 0$ and $t \in (0, \infty]$ we let

$$G_t^\lambda(x, B) = \int_0^t e^{-\lambda s} P_u(x, B) du\,, \quad G_t^\lambda f(x) = \int_{\mathbb{R}^d} f(z) G_t^\lambda(x, dz) = \int_0^t e^{-\lambda u} P_u f(x) du\,,$$

to be the (truncated) λ-*potential kernel* and the (truncated) λ-*potential operator* G_t^λ, respectively. We simplify the notation by putting $G^\lambda(x, dz) = G_\infty^\lambda(x, dz)$ and $G^\lambda = G_\infty^\lambda$.

Definition 2 Let $q : \mathbb{R}^d \to \mathbb{R}$. We write $q \in \mathbb{K}(X)$ if Eq. 1 holds, i.e.,

$$\lim_{t\to 0^+} \left[\sup_{x\in\mathbb{R}^d} G_t^0 |q|(x) \right] = 0. \tag{17}$$

We write $q \in \mathcal{K}(X)$ if Eq. 2 holds for some (every) $\lambda > 0$, i.e.,

$$\lim_{r\to 0^+} \left[\sup_{x\in\mathbb{R}^d} \int_{B(x,r)} |q(z)|\, G^\lambda(x, dz) \right] = 0. \tag{18}$$

If the process X is understood from the context we will write in short $\mathbb{K}$, $\mathcal{K}$ for $\mathbb{K}(X)$, $\mathcal{K}(X)$. In the next two lemmas we show that the definition of $\mathcal{K}$ is consistent. The first one is an apparent reinforcement of Eqs. 2 and 18.

Lemma 3.1 *For all* $\lambda \geqslant 0$, $t \in (0,\infty]$,

$$\left[\sup_{x,y\in\mathbb{R}^d}\int_{B(x,r)}|q(z)|\,G_t^\lambda(y,dz)\right]\leqslant\left[\sup_{x\in\mathbb{R}^d}\int_{B(x,2r)}|q(z)|\,G_t^\lambda(x,dz)\right],\qquad r>0\,.$$

Proof Let $T=T_{\overline{B}(x,r)}$. The strong Markov property leads to

$$\begin{aligned}&\mathbb{E}^y\left(\int_0^\infty e^{-\lambda s}\mathbb{1}_{(0,t](s)}\mathbb{1}_{B(x,r)}(X_s)|q(X_s)|\,ds\right)\\ &=\mathbb{E}^y\left(T<\infty;\int_T^\infty e^{-\lambda s}\mathbb{1}_{(0,t]}(s)\mathbb{1}_{B(x,r)}(X_s)|q(X_s)|\,ds\right)\\ &\leqslant\mathbb{E}^y\left(T<\infty;e^{-\lambda T}\int_0^\infty e^{-\lambda u}\mathbb{1}_{(0,t]}(u)\mathbb{1}_{B(x,r)}(X_u\theta_T)|q(X_u\theta_T)\,du\right)\\ &=\mathbb{E}^y\left(T<\infty;e^{-\lambda T}\mathbb{E}^{X_T}\left(\int_0^\infty e^{-\lambda u}\mathbb{1}_{(0,t]}(u)\mathbb{1}_{B(x,r)}(X_u)|q(X_u)|\,du\right)\right),\end{aligned}$$

where θ denotes the usual shift operator. By the right continuity $X_T\in\overline{B}(x,r)$ and $B(x,r)\subseteq B(X_T,2r)$ on $\{T<\infty\}$. Thus eventually

$$\begin{aligned}\int_{B(x,r)}|q(z)|\,G_t^\lambda(y,dz)&\leqslant\mathbb{E}^y\left(T<\infty;e^{-\lambda T}\mathbb{E}^{X_T}\left(\int_0^\infty e^{-\lambda u}\mathbb{1}_{(0,t]}(u)\mathbb{1}_{B(X_T,2r)}(X_u)|q(X_u)|\,du\right)\right)\\ &\leqslant\sup_{x\in\mathbb{R}^d}\mathbb{E}^x\left[\int_0^\infty e^{-\lambda u}\mathbb{1}_{(0,t](u)}\mathbb{1}_{B(x,2r)}(X_u)|q(X_u)|\,du\right]\\ &=\sup_{x\in\mathbb{R}^d}\int_{B(x,2r)}|q(z)|\,G_t^{\lambda_0}(x,dz)\,.\end{aligned}$$

□

Lemma 3.2 *If* Eqs. 2 *or* 18 *holds for some* $\lambda_0>0$, *then it holds for every* $\lambda>0$.

Proof Clearly, by the resolvent formula (see [4, Chapter 1, (8.10)]) it suffices to consider the measure $A\mapsto\int\mathbb{1}_A(z)\,G^{\lambda_0}G^\lambda(x,dz)=\iint\mathbb{1}_A(z)G^{\lambda_0}(y,dz)G^\lambda(x,dy)$. We have

$$\begin{aligned}\int_{B(x,r)}|q(z)|\,G^{\lambda_0}G^\lambda(x,dz)\ &=\textstyle\int_{\mathbb{R}^d}\left(\int_{B(x,r)}|q(z)|\,G^{\lambda_0}(y,dz)\right)G^\lambda(x,dy)\\ &\leqslant\lambda^{-1}\left[\textstyle\sup_{x,y\in\mathbb{R}^d}\int_{B(x,r)}|q(z)|\,G^{\lambda_0}(y,dz)\right].\end{aligned}$$

This ends the proof due to Lemma 3.1. □

Now, we give alternative characterisations of $\mathbb{K}(X)$ and $\mathcal{K}(X)$. We easily observe that

$$e^{-\lambda t}\,G_t^0(x,dz)\leqslant G_t^\lambda(x,dz)\leqslant G_t^0(x,dz)\,.\tag{19}$$

Lemma 3.3 *For* $\lambda>0$ *and* $t\in[1/\lambda,\infty]$ *we have*

$$(1-e^{-1})\sup_x\left[G_t^\lambda|q|(x)\right]\leqslant\sup_x\left[G_{1/\lambda}^0|q|(x)\right]\leqslant e\sup_x\left[G_t^\lambda|q|(x)\right].$$

Proof Actually, the upper bound holds pointwise as follows,

$$G^0_{1/\lambda}|q|(x) = \int_0^{1/\lambda} P_u|q|(x)du \leqslant e \int_0^{1/\lambda} e^{-\lambda u} P_u|q|(x)du \leqslant e\, G^\lambda_t|q|(x).$$

We prove the lower bound,

$$G^\lambda|q|(x) \leqslant \sum_{k=0}^{\infty} e^{-k} \int_{k/\lambda}^{(k+1)/\lambda} P_{k/\lambda}P_{u-k/\lambda}|q|(x)du = \sum_{k=0}^{\infty} e^{-k} P_{k/\lambda}\left(\int_0^{1/\lambda} P_u|q|(\cdot)du\right)(x)$$
$$\leqslant (1-e^{-1})^{-1} \sup_{z\in\mathbb{R}^d}\left[\int_0^{1/\lambda} P_u|q|(z)du\right].$$

□

Here is a conclusion from Eq. 19 and Lemma 3.3.

Proposition 3.4 *The following conditions are equivalent to* $q \in \mathbb{K}(X)$.

i) $\lim_{t\to 0^+}\left[\sup_{x\in\mathbb{R}^d} G^\lambda_t|q|(x)\right] = 0$ *for some (every)* $\lambda \geqslant 0$.
ii) $\lim_{\lambda\to\infty}\left[\sup_{x\in\mathbb{R}^d} G^\lambda_t|q|(x)\right] = 0$ *for some (every)* $t \in (0,\infty]$.

For resolvent operators R^λ, $\lambda > 0$, of a strongly continuous contraction semigroup on a Banach space we have $\lim_{\lambda\to\infty} \lambda R^\lambda\phi = \phi$. Thus $\lim_{\lambda\to\infty} R^\lambda\phi = 0$ in the norm for every element ϕ of the Banach space. For a Markov process the counterparts of the resolvent operators are the λ-potential operators G^λ_∞.

Proposition 3.4 extends the equivalence of (i) and (ii) of [11, Theorem III.1] from a subclass of Lévy processes to any Hunt process. Similar result is proved in [24, Lemma 3.1] where authors discuss the Kato class of measures for Markov processes possessing transition densities that satisfy the Nash type estimate (see [25] for the symmetric case). In Lemma 3.7 we also show that the uniform local integrability of V ([11, Theorem III.1]) is necessary for $V \in \mathbb{K}(X)$ for any Lévy process X in $\mathbb{R}^d$.

We briefly explain the role of Proposition 3.4. For the Brownian motion, as mentioned in [26] (see also [34]), by Stein's interpolation theorem the inequality $\sup_{x\in\mathbb{R}^d}[G^\lambda|q|(x)] \leqslant \gamma$ leads to $\||q|^{1/2}\phi\|_2^2 \leqslant \gamma\,(\|\nabla\phi\|_2^2 + \lambda\|\phi\|_2^2)$, $\phi \in C_c^\infty(\mathbb{R}^d)$ (a partial reverse result is proved in [1, Theorem 4.9]). For a counterpart of such implication for other processes see remarks preceding [17, Theorem 4.10]. The latter inequality with $\gamma < 1$ allows to define a self-adjoint Schrödinger operator in the sense of quadratic forms, cf. [27, Theorem 3.17], the analogue of Kato-Rellich theorem.

We use Lemma 3.1 to get a better insight into the result of Lemma 3.3.

Lemma 3.5 *For* $t \in (0,\infty)$ *we have* $G^0_t(x,dz) \leqslant e\,G^{1/t}(x,dz)$ *and*

$$(1-e^{-1}) \sup_{x\in\mathbb{R}^d}\left[\int_{B(x,r)} |q(z)|G^{1/t}(x,dz)\right] \leqslant \sup_{x\in\mathbb{R}^d}\left[\int_{B(x,2r)} |q(z)|G^0_t(x,dz)\right], \quad r>0.$$

Proof For a fixed $y \in \mathbb{R}^d$ by Lemma 3.3 with $\tilde{q}(z) = q(z)\mathbb{1}_{B(y,r)}(z)$ we have

$$(1-e^{-1})\int_{B(y,r)} |q(z)|G^{1/t}(y,dz) = (1-e^{-1})G^{1/t}|\tilde{q}|(y)$$
$$\leqslant \sup_{x\in\mathbb{R}^d}\int_0^t P_s|\tilde{q}|(x)ds = \sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|\tilde{q}(z)|G_t^0(x,dz) = \sup_{x\in\mathbb{R}^d}\int_{B(y,r)}|q(z)|G_t^0(x,dz).$$

Thus, by Lemma 3.1 we obtain

$$(1-e^{-1})\sup_{y\in\mathbb{R}^d}\int_{B(y,r)}|q(z)|G^{1/t}(y,dz) \leqslant \sup_{x\in\mathbb{R}^d}\int_{B(x,2r)}|q(z)|G_t^0(x,dz)\,.$$

□

The following is the aftermath of Eq. 19 and Lemma 3.5.

Proposition 3.6 *$q \in \mathcal{K}(X)$ if and only if*

$$\lim_{r\to 0^+}\left[\sup_{x\in\mathbb{R}^d}\int_{B(x,r)}|q(z)|G_t^{\lambda}(x,dz)\right] = 0\,,$$

for some (all) $t \in (0,\infty)$, $\lambda \geqslant 0$.

The above truncation in time is useful when the distribution $\mathbb{P}^x(X_s \in dz)$ is well estimated only for $s \in (0,t]$ near every $x \in \mathbb{R}^d$. See [19], [12, Theorems 2.4 and 3.1] for such estimates. In view of [25, (A2.3), Lemmas 4.1 and 4.3] Proposition 3.6 can also be regarded as an extension or counterpart of [25, Theorem 3.1]. We use Proposition 3.6 in Example 1 below.

Remark 9 Let $\lambda > 0$. Then $\mathbb{K}(X) = \mathbb{K}(X^{\lambda})$ and $\mathcal{K}(X) = \mathcal{K}(X^{\lambda})$.

Lemma 3.7 *Let X be a Lévy process in $\mathbb{R}^d$. Assume that there are $t > 0$ and $0 \leqslant M < \infty$ such that for all $x \in \mathbb{R}^d$,*

$$G_t^0|q|(x) = \int_0^t P_u|q|(x)\,du \leqslant M\,.$$

Then there is a constant $0 \leqslant M' < \infty$ independent of q such that

$$\sup_x \int_{B(x,1)}|q(z)|\,dz \leqslant M'\,. \tag{20}$$

Proof Let $\varphi \in C_0(\mathbb{R}^d)$ be such that $\varphi \geqslant 0$, $\varphi = 1$ on $B(0, 1)$ and $\int_{\mathbb{R}^d} \varphi(x)dx = N < \infty$. For $x_0 \in \mathbb{R}^d$ we have, for $h \leqslant t$,

$$\begin{aligned} MN &\geqslant \int_0^h \int_{\mathbb{R}^d} P_u|q|(x)\varphi(x_0 - x)\, dx du = \int_0^h \int_{\mathbb{R}^d} \mathbb{E}^0|q(X_u + x)|\varphi(x_0 - x)\, dx du \\ &= \int_0^h \mathbb{E}^0 \left[\int_{\mathbb{R}^d} |q(X_u + x)|\varphi(x_0 - x)\, dx \right] du = \int_0^h \mathbb{E}^0 \left[\int_{\mathbb{R}^d} |q(z)|\varphi(X_u + x_0 - z)\, dz \right] du \\ &= \int_0^h \int_{\mathbb{R}^d} |q(z)| P_u\varphi(x_0 - z)\, dz du \geqslant \int_0^h \int_{B(x_0,1)} |q(z)| P_u\varphi(x_0 - z)\, dz du \\ &\geqslant (\varepsilon/2) \int_{B(x_0,1)} |q(z)|\, dz\,, \end{aligned}$$

where $0 < \varepsilon \leqslant h$ is such that $\|P_u\varphi - \varphi\|_\infty \leqslant 1/2$ for $u \leqslant \varepsilon$ (see [29, Theorem 31.5]). $\square$

Here $C_0(\mathbb{R}^d)$ denotes the set of continuous functions $f\colon \mathbb{R}^d \to \mathbb{R}$ such that $\lim_{|x|\to\infty} f(x) = 0$. We write $q \in L^1_{unif}(\mathbb{R}^d)$ if Eq. 20 holds. By $B(\mathbb{R}^d)$ we denote the set of bounded (Borel) functions on $\mathbb{R}^d$. We collect basic properties of $\mathbb{K}(X)$ and $\mathcal{K}(X)$ for a Lévy process X in $\mathbb{R}^d$.

Proposition 3.8 *We have*

1. $\mathcal{K} \subseteq \mathbb{K} \subseteq L^1_{unif}(\mathbb{R}^d)$ *for every Lévy process,*
2. $B(\mathbb{R}^d) \subseteq \mathbb{K}$ *for every Lévy process,*
3. $B(\mathbb{R}^d) \subseteq \mathcal{K}$ *for every non-Poisson Lévy process,*
4. $\mathcal{K} = \{0\}$ *and* $\mathbb{K} = B(\mathbb{R}^d)$ *for every compound Poisson process.*

Proof The inclusion $\mathbb{K} \subseteq L^1_{unif}(\mathbb{R}^d)$ follows from Lemma 3.7. To complete 1. we let $q \in \mathcal{K}(X)$, which reads as (C1) for X^λ, $\lambda > 0$. By Remark 8 and Lemma 2.10, (H2) holds for X^λ and thus the result of Zhao on Fig. 1 implies that (C2) holds for X^λ, i.e., $q \in \mathbb{K}(X^\lambda) = \mathbb{K}(X)$ (see Remark 9). Plainly, 2. holds. Now, let X be non-Poisson. By Lemma 2.1 we get $P_t(\{0\}) = 0$ for almost all $t > 0$ and consequently $G^\lambda(\{0\}) = 0$. Further, since $G^\lambda(dx)$ is a finite measure, for $q \in B(\mathbb{R}^d)$ we have

$$\lim_{r\to\infty} \sup_{x\in\mathbb{R}^d} \int_{B_r} |q(x+z)|G^\lambda(dz) \leqslant \lim_{r\to 0^+} G^\lambda(B_r) \sup_{x\in\mathbb{R}^d} |q(x)| = G(\{0\}) \sup_{x\in\mathbb{R}^d} |q(x)| = 0\,,$$

and 3. holds. Finally, if X is a compound Poisson process, then $G^\lambda(\{0\}) \geqslant (\lambda + \nu(\mathbb{R}^d))^{-1} > 0$ and for every $r > 0$

$$\sup_{x\in\mathbb{R}^d} \int_{B_r} |q(x+z)|G^\lambda(dz) \geqslant \sup_{x\in\mathbb{R}^d} |q(x)|(\lambda + \nu(\mathbb{R}^d))^{-1}.$$

Hence $q \in \mathcal{K}$ if and only if $q \equiv 0$. Moreover,

$$\sup_{x\in\mathbb{R}^d} \int_0^t P_u|q|(x)du \geqslant \sup_{x\in\mathbb{R}^d} |q(x)| \int_0^t e^{-\nu(\mathbb{R}^d)u} du\,,$$

which proves 4. $\square$

4 Main Theorems

In this section we consider a Lévy process X in $\mathbb{R}^d$ and we pursue according to the cases of Section 2.2. Before that, we prove Corollary 1.2 directly from Theorem 1.1.

Proof of Corollary 1.2 Consider a Lévy process Y in $\mathbb{R}^{d+1} = \mathbb{R} \times \mathbb{R}^d$ defined by $Y_t = (t, X_t)$, $t \geqslant 0$, where X is an arbitrary Lévy process in $\mathbb{R}^d$, $d \geqslant 1$. Observe that for $(s, x) \in \mathbb{R}^{d+1}$ and a Borel set $B \subseteq \mathbb{R}^{d+1}$ we have $\mathbb{P}^{(s,x)}(Y_u \in B) = \mathbb{E}^x[\mathbb{1}_B(s+u, X_u)]$, $u \geqslant 0$. Since for Y 0 is not regular for $\{0\}$ Theorem 1.1 applies to Y. Finally, we use (2) taking into account that $\mathbb{1}_{B_{d+1}((s,x),r)}(s+u, X_u)$, where $B_{d+1}(x,r)$ denotes a ball in $\mathbb{R}^{d+1}$, can be replaced with $\mathbb{1}_{[0,r)}(u)\mathbb{1}_{B(x,r)}(X_u)$ and that $e^{-\lambda u}$ is comparable with one for $u \in [0, r)$. $\square$

4.1 Under (H0)

In this subsection we consider a Lévy process X satisfying (H0).

Theorem 4.1 *For $d > 1$ or $d = 1$ under (A) we have $\mathcal{K}(X) = \mathbb{K}(X)$.*

Proof By Proposition 3.8 we concentrate on $\mathbb{K}(X) \subseteq \mathcal{K}(X)$. Let $q \in \mathbb{K}(X) = \mathbb{K}(X^\lambda)$, $\lambda > 0$. This reads as (C2) for X^λ. Since X is non-Poisson, by Remark 8 and Lemma 2.10 the hypothesis (H1) holds for X^λ. To obtain (C1) for X^λ, that is to prove $q \in \mathcal{K}(X)$, it remains to verify (H3) for X^λ. In view of Corollary 2.16 it suffices to justify that $\{0\}$ is a polar set. For $d > 1$ this is assured by Proposition 2.5. For $d = 1$ it is our assumption. $\square$

From now on in this subsection we discuss the case of $d = 1$. For simplicity we recall from [10, Théorèmes 7, 1, 5, 6 and 8] the following facts.

Lemma 4.2 *Let $d = 1$ and $\int_{\mathbb{R}} \mathrm{Re}\left(\frac{1}{\lambda+\psi(z)}\right) dz < \infty$, $\lambda > 0$. Then $G^\lambda(dz)$ has a bounded density $G^\lambda(z) = k^\lambda\, h^\lambda(z)$, $z \in \mathbb{R}$, with respect to the Lebesgue measure which is continuous on $\mathbb{R}\backslash\{0\}$. Further, $G^\lambda(z)$ is continuous at 0 if and only if 0 is regular for $\{0\}$ (i.e. $h^\lambda(0) = 1$), and then $0 < h^\lambda(z) \leqslant 1$ for $z \in \mathbb{R}$.*

We investigate the properties of $G_t^\lambda(dz)$, $\lambda > 0$, $t \in (0, \infty)$.

Lemma 4.3 *Let $d = 1$ and $\int_{\mathbb{R}} \mathrm{Re}\left(\frac{1}{\lambda+\psi(z)}\right) dz < \infty$, $\lambda > 0$. Then $G_t^\lambda(dz)$ has a bounded density $G_t^\lambda(z)$ with respect to the Lebesgue measure which is lower semi-continuous on $\mathbb{R} \setminus \{0\}$.*

Proof According to Lemma 4.2 we define $F^\lambda(z) := G^\lambda(z)$ on $\mathbb{R} \setminus \{0\}$ and $F^\lambda(0) := \limsup_{z\to 0} F^\lambda(z)$. Then $F^\lambda(z)$ is a density of $G^\lambda(dz)$. Since $G_t^\lambda(B) \leqslant G^\lambda(B)$ and $G_t^\lambda(B) = G^\lambda(B) - e^{-\lambda t}\int_{\mathbb{R}} G^\lambda(B-z)P_t(dz)$, $G_t^\lambda(dx)$ is absolutely continuous and its density $G_t^\lambda(x)$ can be chosen as

$$G_t^\lambda(x) := F^\lambda(x) - e^{-\lambda t}\int_{\mathbb{R}} F^\lambda(x-z)P_t(dz). \tag{21}$$

To prove the lower semi-continuity of G_t^λ we observe that for $x_0 \in \mathbb{R} \setminus \{0\}$,

$$G_t^\lambda(x) = F^\lambda(x) - e^{-\lambda t}\left(\int_{\mathbb{R}\setminus\{x_0\}} F^\lambda(x-z)P_t(dz) + F^\lambda(x-x_0)P_t(\{x_0\})\right).$$

Then by continuity of F^λ on $\mathbb{R} \setminus \{0\}$ and the bounded convergence theorem

$$\begin{aligned}\liminf_{x\to x_0} G_t^\lambda(x) &= F^\lambda(x_0) - e^{-\lambda t}\left(\int_{\mathbb{R}\setminus\{x_0\}} \lim_{x\to x_0} F^\lambda(x-z)P_t(dz) + \limsup_{x\to x_0} F^\lambda(x-x_0)P_t(\{x_0\})\right)\\ &= G_t^\lambda(x_0).\end{aligned}$$

□

Theorem 4.4 *For $d = 1$ under (B) we have*

$$\mathcal{K}(X) = \mathbb{K}(X) = \left\{q\colon \lim_{r\to 0^+} \sup_{x\in\mathbb{R}} \int_{B(x,r)} |q(z)|dz = 0\right\}.$$

Proof Without loss of generality we may and do assume that $\gamma_0 > 0$. Due to Proposition 3.8 and Lemma 4.2 (boundedness of the function G^λ) it remains to prove $\mathbb{K}(X) \subseteq \{q\colon \lim_{r\to 0^+} \sup_{x\in\mathbb{R}} \int_{B(x,r)} |q(z)|dz = 0\}$. By Remark 3 we get $\mathbb{P}^0(\lim_{u\to 0^+} u^{-1}X_u = \gamma_0) = 1$. Hence, there is $\varepsilon > 0$ such that $\mathbb{P}^0(|X_u - \gamma_0 u| < \gamma_0 u) \geqslant 1/2$ for $u \leqslant \varepsilon$. This implies that for $t \leqslant \varepsilon$,

$$G_t^\lambda((0, 2\gamma_0 t]) = \int_0^t e^{-\lambda u}\mathbb{P}^0(X_u \in (0, 2\gamma_0 t])du \geqslant \int_0^t e^{-\lambda u}\mathbb{P}^0(|X_u - \gamma_0 u| < \gamma_0 u)du \geqslant \frac{1-e^{-\lambda t}}{2\lambda}.$$

Hence, $\sup_{z\in(0,2\gamma_0 t]} G_t^\lambda(z) \geqslant \frac{1-e^{-\lambda t}}{\lambda t}\frac{1}{4\gamma_0} \geqslant \frac{1-e^{-\lambda\varepsilon}}{\lambda\varepsilon}\frac{1}{4\gamma_0}$. Since $G_t^\lambda(z)$ is lower semi-continuous on $\mathbb{R} \setminus \{0\}$ there exist $0 < a_t < b_t \leqslant 2\gamma_0\varepsilon$ such that $G_t^\lambda(z) \geqslant \frac{1-e^{-\lambda\varepsilon}}{\lambda\varepsilon}\frac{1}{8\gamma_0}$ for $z \in (a_t, b_t)$. Now, let $q \in \mathbb{K}(X)$. We obtain for $t \leqslant \varepsilon$,

$$\int_{\mathbb{R}} |q(x+z)|G_t^\lambda(dz) \geqslant \frac{1-e^{-\lambda\varepsilon}}{8\lambda\varepsilon\gamma_0}\int_{a_t}^{b_t} |q(x+z)|dz.$$

Thus,

$$0 = \lim_{t\to 0^+} \sup_{x\in\mathbb{R}} \int_{a_t}^{b_t} |q(x+z)|dz \geqslant \lim_{r\to 0^+} \sup_{x\in\mathbb{R}} \int_{B(x,r)} |q(z)|dz.$$

□

Lemma 4.5 *Let 0 be regular for* $\{0\}$. *There is* $0 < M_{G^\lambda} < \infty$ *such that*

$$G^\lambda(x) \leqslant M_{G^\lambda}\, G^\lambda(y), \quad x,\ y \in \mathbb{R},\ \ |x-y| \leqslant 1. \tag{22}$$

Further, $G_t^\lambda(x)$ *given by* Eq. 21 *is continuous on* $\mathbb{R}$ *and*

$$G_t^\lambda(x) \leqslant G^\lambda(x)(\lambda t + ||P_t f - f||_\infty), \qquad f(x) = h^\lambda(-x) \in C_0(\mathbb{R}).$$

Proof Let F^λ be defined as in the proof of Lemma 4.3. By Lemma 4.2 the functions G^λ and F^λ are equal and continuous on $\mathbb{R}$. Further, Lemma 2.13 implies that the function

$h^\lambda(x) = G^\lambda(x)/k^\lambda = \mathbb{E}^0 e^{-\lambda T_{\{x\}}}$ is in $C_0(\mathbb{R})$. Since $h^\lambda(x+y) \geqslant h^\lambda(x)h^\lambda(y)$, $x, y \in \mathbb{R}$ (see remarks after [10, Lemma 2]), we get

$$\frac{G^\lambda(x-z)}{G^\lambda(x)} = \frac{h^\lambda(x-z)}{h^\lambda(x)} \geqslant h^\lambda(-z)\,.$$

By positivity and continuity of h^λ we obtain (22) with $M_{G^\lambda} = \sup_{|z|\le 1} 1/[h^\lambda(z)] < \infty$. Eventually, by Eq. 21,

$$\begin{aligned} G_t^\lambda(x) &= G^\lambda(x)\left(1 - e^{-\lambda t} + e^{-\lambda t}\int_{\mathbb{R}} \left(1 - \frac{G^\lambda(x-z)}{G^\lambda(x)}\right) P_t(dz)\right) \\ &\leqslant G^\lambda(x)\left(\lambda t + \int_{\mathbb{R}} \left(h^\lambda(0) - h^\lambda(-z)\right) P_t(dz)\right). \end{aligned}$$

□

Theorem 4.6 *For $d = 1$ under (C) we have $\mathcal{K}(X) \subsetneq \mathbb{K}(X)$,*

$$\mathcal{K}(X) = \left\{q\colon \lim_{r\to 0^+} \sup_{x\in\mathbb{R}} \int_{B(x,r)} |q(z)|\, dz = 0\right\},$$

and

$$\mathbb{K}(X) = L^1_{unif}(\mathbb{R}) = \left\{q\colon \sup_{x\in\mathbb{R}} \int_{B(x,1)} |q(z)| dz < \infty\right\}.$$

Proof For $\mathcal{K}(X)$ we just observe that $G^\lambda(z)$ is bounded and $G^\lambda(z) \geqslant \varepsilon > 0$ if $|z| \leqslant 1$. Now, we describe $\mathbb{K}(X)$. The condition $q \in L^1_{unif}(\mathbb{R})$ is necessary by Lemma 3.7. We show that it is sufficient. Let $\lambda > 0$ and denote $c_t = \lambda t + ||P_t f - f||_\infty$, where $f(x) = h^\lambda(-x) = \mathbb{E} e^{-\lambda T_{\{-x\}}}$. By Lemma 4.5

$$\begin{aligned} \int_{\mathbb{R}} |q(x+z)| G_t^\lambda(dz) &\leqslant c_t \int_{\mathbb{R}} |q(x+z)| G^\lambda(z) dz = c_t \sum_{k=-\infty}^{\infty} \int_{k-1/2}^{k+1/2} |q(x+z)| G^\lambda(z) dz \\ &\leqslant c_t\, M_{G^\lambda} \sum_{k=-\infty}^{\infty} G^\lambda(k) \int_{k-1/2}^{k+1/2} |q(x+z)| dz \\ &\leqslant c_t\, M_{G^\lambda} \sup_{x\in\mathbb{R}} \int_{B(x,1)} |q(z)| dz \sum_{k=-\infty}^{\infty} G^\lambda(k) \\ &\leqslant c_t\, (M_{G^\lambda})^2 \lambda^{-1} \sup_{x\in\mathbb{R}} \int_{B(x,1)} |q(z)| dz. \end{aligned} \tag{23}$$

Since $f \in C_0(\mathbb{R})$ we get $c_t \to 0$ as $t \to 0^+$ (see [29, Theorem 31.5]). □

4.2 Without (H0)

In this subsection we assume that (H0) does not hold. In view of Proposition 3.8 we assume that $d > 1$ and X is given by Eq. 13. We use results of Section 4.1 and analyze the cases (A'), (B') and (C').

Theorem 4.7 *Under (A') we have $\mathcal{K}(X) = \mathbb{K}(X)$.*

Proof Following the proof of Theorem 4.1 it remains to show that $\{0\}$ is polar for the process X. This is assured by Corollary 2.9. □

We proceed to the remaining cases. The transition kernel of X equals

$$P_t(dx) = P_t^Z * \sum_{n=0}^{\infty} e^{-t\nu^Y(\mathbb{R}^d)} \frac{t^n (\nu^Y)^{*n}}{n!}(dx)\,.$$

The characteristic exponent ψ of X can be written as $\psi = \psi^Y + \psi^Z$. We note that $\psi^Z(z) = \psi^Z(v)$ for $z = v + w \in \mathbb{R}^d$, $v \in V$, $w \in V^{\perp}$. For $\lambda > 0$, $t \in (0, \infty]$ and $n \in \mathbb{N}$ we define

$$G_t^{Z,\lambda,n}(dv) := \int_0^t u^n e^{-\lambda u} P_u^Z(dv)\, du\,.$$

We investigate n-moment λ-potentials $G^{Z,\lambda,n}(dv) := G_\infty^{Z,\lambda,n}(dv)$ and truncated λ-potentials $G_t^{Z,\lambda}(dv) := G_t^{Z,\lambda,0}(dv)$ of Z. We also write $G^{Z,\lambda}(dv) = G_\infty^{Z,\lambda,0}(dv)$ for λ-potentials of Z. The measures $G^{Z,\lambda}$, $G_t^{Z,\lambda}$, $G^{Z,\lambda,n}$ are concentrated on V. Observe that

$$G^\lambda(dx) = \sum_{n=0}^{\infty} \frac{1}{n!}\, G^{Z,\lambda+\nu^Y(\mathbb{R}^d),n} * (\nu^Y)^{*n}(dx)\,. \tag{24}$$

We reformulate Lemmas 4.3 and 4.5 in view of Remark 5. We write $C_0(V)$ for the set of continuous functions $f\colon V \to \mathbb{R}$ such that $\lim_{v\in V,\,|v|\to\infty} f(v) = 0$.

Lemma 4.8 *Let* $\int_V \mathrm{Re}\left(\frac{1}{\lambda+\psi^Z(v)}\right) dv < \infty$, $\lambda > 0$. *Then* $G_t^{Z,\lambda}(dv)$ *has a bounded density* $G_t^{Z,\lambda}(v)$ *with respect to the Lebesgue measure on* V *which is lower semi-continuous on* $V \setminus \{0\}$. *If 0 is regular for* $\{0\}$ *for Z then there is* $0 < M_{G^{Z,\lambda}} < \infty$ *such that*

$$G^{Z,\lambda}(v) \leqslant M_{G^{Z,\lambda}}\, G^{Z,\lambda}(v'), \qquad v,\ v' \in V,\quad |v - v'| \leqslant 1,$$

$G_t^{Z,\lambda}(v)$ *is continuous on* V *and*

$$G_t^{Z,\lambda}(v) \leqslant G^{Z,\lambda}(v)(\lambda t + ||P_t^Z f - f||_\infty)\,, \qquad f(v) \in C_0(V)\,.$$

Lemma 4.9 *Let* $\int_V \mathrm{Re}\left(\frac{1}{\lambda+\psi^Z(v)}\right) dv < \infty$, $\lambda > 0$. *Then* $G^{Z,\lambda,n}(dv)$ *has a density* $G^{Z,\lambda,n}(v)$ *with respect to the Lebesgue measure on* V, *and*

$$G^{Z,\lambda,n}(v) \leqslant \frac{n!}{\lambda^n} \int_V \mathrm{Re}\left(\frac{1}{\lambda + \psi^Z(u)}\right) du. \tag{25}$$

Proof By Remark 5 we assume that $V = \mathbb{R}$ and we observe that the Fourier transform of $G^{Z,\lambda,n}$ equals

$$\int_0^\infty t^n e^{-\lambda t} e^{-t\psi^Z(\xi)} dt = \frac{n!}{[\lambda + \psi^Z(\xi)]^{n+1}}\,, \qquad \xi \in \mathbb{R}\,.$$

Since $\mathrm{Re}(1/z) = \mathrm{Re}(\bar{z})/|z|^2$ and $\mathrm{Re}[\psi] \geqslant 0$ we obtain

$$\frac{1}{|\lambda + \psi^Z(\xi)|^{n+1}} \leqslant \lambda^{-n+1} \frac{1}{|\lambda + \psi^Z(\xi)|^2} \leqslant \lambda^{-n} \mathrm{Re}\left(\frac{1}{\lambda + \psi^Z(\xi)}\right).$$

This implies that the Fourier transform is integrable and (25) follows by the inversion formula. □

Lemma 4.10 *Let* $\int_V \mathrm{Re}\left(\frac{1}{\lambda+\psi^Z(v)}\right) dv < \infty$, $\lambda > 0$. *Then*

$$\sup_{x\in\mathbb{R}^d}\left(\int_{B(0,r)} |q(x+z)| G^\lambda(dz)\right) \leqslant \sup_{x\in\mathbb{R}^d}\left(\int_{B(0,r)\cap V} |q(x+v)|\, dv\right) C\left[1+\nu^Y(\mathbb{R}^d)/\lambda\right],$$

where dv *is the one-dimensional Lebesgue measure on* V *and* $C = \int_V \mathrm{Re}\left(1/[\lambda+\nu^Y(\mathbb{R}^d)+\psi^Z(u)]\right) du$.

Proof By Eqs. 24 and 25 we have

$$\begin{aligned}\int_{B(0,r)} |q(x+z)| G^\lambda(dz) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^d}\left(\int_V \mathbb{1}_{B(0,r)}(v+w)|q(x+v+w)| G^{Z,\lambda+\nu^Y(\mathbb{R}^d),n}(dv)\right)\\ &\quad\times(\nu^Y)^{*n}(dw)\\ &\leqslant \sup_{x,w\in\mathbb{R}^d}\left(\int_V \mathbb{1}_{B(0,r)}(v+w)|q(x+v+w)|\, dv\right)\sum_{n=0}^{\infty} C\left(\frac{\nu^Y(\mathbb{R}^d)}{\lambda+\nu^Y(\mathbb{R}^d)}\right)^n,\end{aligned}$$

and

$$\begin{aligned}\sup_{x,w\in\mathbb{R}^d}\left(\int_V \mathbb{1}_{B(0,r)}(v+w)|q(x+v+w)|\, dv\right) &= \sup_{x,w\in\mathbb{R}^d}\left(\int_{B(-w,r)\cap V} |q(x+v)|\, dv\right)\\ &= \sup_{x\in\mathbb{R}^d,\, w\in V}\left(\int_{B(-w,r)\cap V} |q(x+v)|\, dv\right)\\ &= \sup_{x\in\mathbb{R}^d}\left(\int_{B(0,r)\cap V} |q(x+v)|\, dv\right),\end{aligned}$$

where the last equality follows by the translation invariance of the Lebesgue measure on V. This ends the proof. □

Theorem 4.11 *Under (B') we have*

$$\mathcal{K}(X) = \mathbb{K}(X) = \left\{q\colon \lim_{r\to 0^+} \sup_{x\in\mathbb{R}^d} \int_{B(0,r)\cap V} |q(x+v)| dv = 0\right\},$$

where dv *is the one-dimensional Lebesgue measure on* V.

Proof Lemma 4.10 gives $\{q\colon \lim_{r\to 0^+}\sup_{x\in\mathbb{R}^d}\int_{B(0,r)\cap V}|q(x+v)|dv = 0\} \subseteq \mathcal{K}(X)$. By Proposition 3.8 it suffices to show $\mathbb{K}(X) \subseteq \{q\colon \lim_{r\to 0^+}\sup_{x\in\mathbb{R}^d}\int_{B(0,r)\cap V}|q(x+v)|dv = 0\}$. Since for $t > 0$ and $x \in \mathbb{R}^d$ we have

$$\int_0^t P_u|q|(x)\, du \geqslant \int_0^t \int_{\mathbb{R}^d} |q(x+z)|\, e^{-u\nu^Y(\mathbb{R}^d)} P_u^Z(dz)\, du = \int_{\mathbb{R}^d\cap V} |q(x+v)|\, G_t^{Z,\,\nu^Y(\mathbb{R}^d)}(dv),$$

the inclusion follows by adapting the proof of Theorem 4.4 to the one-dimensional process Z with the support of Lemma 4.8 and Remark 3. □

Theorem 4.12 *Under (C') we have* $\mathcal{K}(X) \subsetneq \mathbb{K}(X)$*,*

$$\mathcal{K}(X) = \left\{ q\colon \lim_{r \to 0^+} \sup_{x \in \mathbb{R}^d} \int_{B(0,r)\cap V} |q(x+v)|\, dv = 0 \right\},$$

and

$$\mathbb{K}(X) = \left\{ q\colon \sup_{x \in \mathbb{R}^d} \int_{B(0,1)\cap V} |q(x+v)|\, dv < \infty \right\},$$

where dv *is the one-dimensional Lebesgue measure on* V*.*

Proof The condition postulated for the description of $\mathcal{K}(X)$ is sufficient by Lemma 4.10. Next, by Remark 5 and Lemma 4.2 the λ-potential kernel of Z, that is $G^{Z,\lambda}(dv) = G^{Z,\lambda,0}(dv)$, has a density $G^{Z,\lambda}(v)$ with respect to the Lebesgue measure on V, such that $G^{Z,\lambda}(v) \geqslant \varepsilon > 0$ if $v \in B(0,1) \cap V$ (ε may depend on λ). Thus,

$$\int_{B(0,r)} |q(x+z)| G^{\lambda}(dz) \geqslant \int_{B(0,r)\cap V} |q(x+v)| G^{Z,\lambda+\nu^Y(\mathbb{R}^d)}(dv) \geqslant \varepsilon \int_{B(0,r)\cap V} |q(x+v)|\, dv\,,$$

which proves the necessity. Further, the necessity of the condition proposed to describe $\mathbb{K}(X)$ follows from Remark 5, Lemma 3.7 and

$$\begin{aligned}\int_0^t P_u|q|(x)du &\geqslant \int_0^t \int_{\mathbb{R}^d\cap V} |q(x+v)|\, e^{-u\nu^Y(\mathbb{R}^d)} P_u^Z(dv)du \\ &\geqslant e^{-t\nu^Y(\mathbb{R}^d)} \int_0^t \int_{\mathbb{R}^d\cap V} |q(x+v)|\, P_u^Z(dv)du.\end{aligned}$$

For the sufficiency we partially follow the proof of Theorem 4.6. Note that $\int_0^t u^n e^{-\lambda u} P_u^Z (dv)\, du \leqslant t^n G_t^{Z,\lambda}(dv)$ which gives

$$G_t^{\lambda}(dx) \leqslant \sum_{n=0}^{\infty} \frac{t^n}{n!} G_t^{Z,\lambda+\nu^Y(\mathbb{R}^d)} * (\nu^Y)^{*n}(dx)\,.$$

Thus by Lemma 4.8 and adaptation of Eq. 23 we have with $c_t = (\lambda + \nu^Y(\mathbb{R}^d))t + ||P_t^Z f - f||_\infty$,

$$\begin{aligned}\int_{\mathbb{R}^d} |q(x+z)| G_t^{\lambda}(dz) &\leqslant \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{\mathbb{R}^d} \left(\int_V |q(x+v+w)| G_t^{Z,\lambda+\nu^Y(\mathbb{R}^d)}(dv) \right) (\nu^Y)^{*n}(dw) \\ &\leqslant \left(c_t \left(M_{G^{Z,\lambda+\nu^Y(\mathbb{R}^d)}} \right)^2 (\lambda+\nu^Y(\mathbb{R}^d))^{-1} \sup_{x\in\mathbb{R}^d} \int_{B(0,1)\cap V} |q(x+v)| dv \right) \\ &\quad \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{\mathbb{R}^d} (\nu^Y)^{*n}(dw)\,,\end{aligned}$$

which ends the proof. □

4.3 Zero-Potential Kernel

In the previous sections and subsections we have already used measures G_t^{λ}, $\lambda \geqslant 0$, $t \in (0,\infty]$. Below we present additional sufficient assumptions on a Lévy process X under which $G^0 = G^0_\infty$ can be used to describe $\mathbb{K}(X)$. The condition we want to analyze now is $q \in \mathcal{K}^0(X)$ defined by

$$\lim_{r\to 0^+}\left[\sup_{x\in\mathbb{R}^d}\int_{B(0,r)}|q(z+x)|G^0(dz)\right]=0\,. \tag{26}$$

Since $G^\lambda(dz) \leqslant G^0(dz)$, Eq. 26 implies $q \in \mathcal{K}(X)$ and thus $\mathcal{K}^0(X) \subseteq \mathcal{K}(X) \subseteq \mathbb{K}(X)$ by Proposition 3.8. Our aim is to obtain the equivalence, i.e., the implication from $q \in \mathbb{K}(X)$ to Eq. 26, and this is the subcase of $\mathcal{K}(X) = \mathbb{K}(X)$. We will assume that X is transient and $\{0\}$ is polar (in Theorem 4.15 polarity follows implicitly from other assumptions). The transience is necessary, otherwise $G^0(dz)$ is locally unbounded (see [29, Theorem 35.4]) and non-zero constant functions do not belong to $\mathcal{K}^0(X)$, which shows $\mathcal{K}^0(X) \subsetneq \mathbb{K}(X)$. The polarity of $\{0\}$ assures $\mathcal{K}(X) = \mathbb{K}(X)$. Moreover, if $\{0\}$ is not polar, the class $\mathbb{K}(X)$ is explicitly described by our previous theorems. Both, transience and polarity of $\{0\}$ are to some extent encoded in the characteristic exponent ψ (see [29, Remark 37.7] and Section 2.2). Finally, we note that $q \in \mathcal{K}^0(X)$ is equivalent to (C1) and $q \in \mathbb{K}(X)$ to (C2). Thus according to Fig. 1 and Remark 8, we focus on showing (H3) for X.

Remark 10 If X is transient, then we have

$$\lim_{r\to 0^+}\mathbb{P}^0(T_{\overline{B}(x,r)}<\infty)=\mathbb{P}^0(T_{\{x\}}<\infty)\,,\quad x\in\mathbb{R}^d\,. \tag{27}$$

Such statement is not true in general, but here it follows from $\mathbb{P}^0(T_{\overline{B}(x,r)} < \infty) = \mathbb{P}^0(T_{\overline{B}(x,r)} < \infty, T_{\{x\}} < \infty) + \mathbb{P}^0(T_{\overline{B}(x,r)} < \infty, T_{\{x\}} = \infty)$, Lemma 2.12 and $\lim_{t\to\infty}|X_t| = \infty$ $\mathbb{P}^0$ a.s.

We say that a measure $G^0(dz)$ tends to zero at infinity if $\lim_{|x|\to\infty}\int_{\mathbb{R}^d} f(z+x)G^0(dz) = 0$ for all $f \in C_c(\mathbb{R}^d)$ (i.e., f is continuous with compact support). Under certain assumptions on the *group* of the Lévy process [29, Definition 24.21] $G^0(dz)$ tends to zero for every transient X if $d \geqslant 2$. The case $d = 1$ is more complicated. See [29, Exercise 39.14] and Remark 13.

Lemma 4.13 *Let X be transient. If $G^0(dz)$ tends to zero at infinity then*

$$h_3(X)=\sup_{x\neq 0}\mathbb{P}^0(T_{\{x\}}<\infty)\,.$$

Proof The statement follows by the same proof as for Proposition 2.15 but with $\lambda = 0$ and a version of Lemma 2.14 for $\lambda = 0$. To prove the latter one we also repeat its proof with functions f_r extended to $\lambda = 0$, i.e., $f_r(x) = \mathbb{P}^0(T_{\overline{B}(x,r)} < \infty)$ up to a moment when $a > 0$ and a sequence $\{x_n\}$ such that $f_{1/n}(x_n) > a - \varepsilon$ are chosen. The rest of the proof easily applies with Eq. 27 in place of Lemma 2.12 as soon as we can show that $\{x_n\}$ is bounded. To this end assume that the sequence is unbounded. Since $f_r(x) = \mathbb{P}^y(T_{\overline{B}(x+y,r)} < \infty)$, $r > 0$, $y \in \mathbb{R}^d$, for $r \in (0, 1]$ and $|x - x_n| < 1$ we have

$$a-\varepsilon<f_r(x_n)=\mathbb{P}^{-x}(T_{\overline{B}(x_n-x,r)}<\infty)\leqslant\mathbb{P}^{-x}(T_{\overline{B}(0,2)}<\infty)=f_2(x)\,, \tag{28}$$

Next, by [29, Theorem 42.8] there is a finite measure ρ supported on $\overline{B}(0,2)$ (see also [29, Definition 42.1]) such that for any $g \in C_c(\mathbb{R}^d)$ satisfying $\mathbb{1}_{B(0,1)} \leqslant g$ we get

$$\int_{\mathbb{R}^d} g(x_n - x) f_2(x)\,dx = \int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d} g(x_n + w - v) G^0(dv)\right]\rho(dw) \quad \xrightarrow{n\to\infty} 0\,,$$

since $G^0(dv)$ tends to zero at infinity. This contradicts (28) and ends the proof. □

Theorem 4.14 *Let X be transient, $\{0\}$ be polar and $G^0(dz)$ tend to zero at infinity. Then $q \in \mathbb{K}(X)$ if and only if* Eq. 26 *holds, i.e., $\mathcal{K}^0(X) = \mathcal{K}(X) = \mathbb{K}(X)$.*

In the next result we improve [38, Lemma 5] and we cover some cases when $G^0(dz)$ may not tend to zero at infinity.

Theorem 4.15 *Let X be transient and let $G^0(dz)$ have a density $G^0(z)$ with respect to the Lebesgue measure which is unbounded and bounded on $|z| \geqslant r$ for every $r > 0$. Then $\mathcal{K}^0(X) = \mathcal{K}(X) = \mathbb{K}(X)$.*

Proof We note that the polarity of $\{0\}$ follows by our assumptions (see [29, Theorems 41.15 and 43.3]). By [29, Proposition 42.13 and Definition 42.9] for $r > 0$ we have

$$\mathbb{P}^x(T_{B(0,r)} < \infty) = \int_{\overline{B}(0,r)} G^0(y - x)\, m_{B(0,r)}(dy), \quad x \in \mathbb{R}^d.$$

Next, for $u > 0$, $|x| \geqslant u$ and $0 < r < u/2$ we obtain,

$$\mathbb{P}^x(T_{B(0,r)} < \infty) \leqslant \Big[\sup_{|y|\geqslant u/2} G^0(y)\Big] C(B(0,r))\,,$$

where C($\cdot$) stands for 0-order capacity. By [29, Proposition 42.10 and (42.20)] and Remark 10 we have $\lim_{r\to 0^+} C(B(0,r)) = C(\{0\})$ (see also [28, Proposition 8.4]). This gives

$$\begin{aligned} h_3(X) = \sup_{u>0}\inf_{r>0}\sup_{|x|\geqslant u} \mathbb{P}^x(T_{B(0,r)} < \infty) &\leqslant \sup_{u>0}\Big[\sup_{|y|\geqslant u/2} G(y)\Big]\inf_{0<r<u/2} C(B(0,r)) \\ &= \sup_{u>0}\Big[\sup_{|y|\geqslant u/2} G(y)\Big] C(\{0\})\,. \end{aligned}$$

Finally, since $\{0\}$ is polar, by [29, Theorem 42.19] we have $C(\{0\}) = 0$ and so (H3) holds with $h_3(X) = 0$. □

5 Further Discussion and Applications

In this section we give additional results for isotropic unimodal Lévy processes concerning (the implication) $\mathcal{K}(X) \subseteq \mathbb{K}(X)$, we apply general results to a subclass of subordinators and we present examples.

We recall from [6] the definition of weak scaling. Let $\underline{\theta} \in [0,\infty)$ and ϕ be a non-negative non-zero function on $(0,\infty)$. We say that ϕ satisfies the *weak lower scaling condition* (at infinity) if there are numbers $\underline{\alpha} \in \mathbb{R}$ and $\underline{c} \in (0,1]$, such that

$$\phi(\eta\theta) \ge \underline{c}\,\eta^{\underline{\alpha}}\phi(\theta) \quad \text{for} \quad \eta \ge 1, \quad \theta > \underline{\theta}.$$

In short we say that ϕ satisfies WLSC($\underline{\alpha}$, $\underline{\theta}$, $\underline{c}$) and write $\phi \in \mathrm{WLSC}(\underline{\alpha}, \underline{\theta}, \underline{c})$. Similarly, we consider $\overline{\theta} \in [0,\infty)$. The *weak upper scaling condition* holds if there are numbers $\overline{\alpha} \in \mathbb{R}$ and $\overline{C} \in [1,\infty)$ such that

$$\phi(\eta\theta) \leq \overline{C}\eta^{\overline{\alpha}}\phi(\theta) \quad \text{for} \quad \eta \geq 1, \quad \theta > \overline{\theta}.$$

In short, $\phi \in \mathrm{WUSC}(\overline{\alpha}, \overline{\theta}, \overline{C})$.

5.1 Isotropic Unimodal Lévy Processes

A measure on $\mathbb{R}^d$ is called isotropic unimodal, in short, unimodal, if it is absolutely continuous on $\mathbb{R}^d \setminus \{0\}$ with a radial non-increasing density (such measures may have an atom at the origin). A Lévy process X is called (isotropic) unimodal if all of its one-dimensional distributions $P_t(dx)$ are unimodal. Unimodal pure-jump Lévy processes are characterized in [35] by isotropic unimodal Lévy measures $\nu(dx) = \nu(x)dx = \nu(|x|)dx$. The distribution of X_t has a radial non-increasing density $p(t,x)$ on $\mathbb{R}^d \setminus \{0\}$, and atom at the origin, with mass $\exp[-t\nu(\mathbb{R}^d)]$ (no atom if ψ is unbounded).

For a continuous non-decreasing function $\phi : [0,\infty) \to [0,\infty)$, such that $\phi(0) = 0$, we let $\phi(\infty) = \lim_{s\to\infty}\phi(s)$ and we define the generalized left inverse $\phi^- : [0,\infty) \to [0,\infty]$,

$$\phi^-(u) = \inf\{s \geqslant 0 \colon \phi(s) = u\} = \inf\{s \geqslant 0 \colon \phi(s) \geqslant u\}, \quad 0 \leqslant u < \infty,$$

with the convention that $\inf\emptyset = \infty$. The function is increasing and càglàd where finite. Notice that $\phi(\phi^-(u)) = u$ for $u \in [0,\phi(\infty)]$ and $\phi^-(\phi(s)) \leqslant s$ for $s \in [0,\infty)$. Moreover, by the continuity of ϕ we have $\phi^-(\phi(s)+\varepsilon) > s$ for $\varepsilon > 0$ and $s \in [0,\infty)$. We also define $f^*(u) = \sup_{|x|\leqslant u}|f(x)|$ for $f : \mathbb{R}^d \to \mathbb{R}$.

In view of general results for Schrödinger perturbations [8, Theorem 3] and the so-called 3G type inequalities [7, (40) and Corollary 11] it is desirable to have the following results which extend [14, Theorem 1.28] and [9, Proposition 4.3] (see also [8, Remark 2]).

Proposition 5.1 *Let X be unimodal. For $t_0 \in (0,\infty]$, $r > 0$ and $0 < t < t_0$,*

$$\sup_{x\in\mathbb{R}^d}\int_0^t P_u|q|(x)du \leqslant \left(1 + \frac{t}{|B(0,1/2)|r^d G^0_{t_0}(r)}\right)\left[\sup_{x\in\mathbb{R}^d}\int_{B(x,r)}|q(z)|G^0_{t_0}(z-x)dz\right],$$

where $G^0_{t_0}(z) = \int_0^{t_0} p(u,z)\,du$, $z \in \mathbb{R}^d$, and $G^0_{t_0}(r) = G^0_{t_0}(x)$, $|x| = r$.

Proof We use [9, Lemma 4.2] with $k(x) = \int_0^t p(u,x)du$ and $K(x) = G^0_{t_0}(x)$. □

In what follows we assume that $d \geqslant 3$ and that the Lévy-Khintchine exponent ψ is unbounded. Then since X is (isotropic) unimodal by [29, Theorem 37.8] it is transient and the measure $G^0(dz)$ has a radially non-increasing density $G^0(z)$. This density is unbounded (see [29, Theorems 43.9 and 43.3]). Thus Theorem 4.15 applies and $\mathcal{K}^0(X) = \mathcal{K}(X) = \mathbb{K}(X)$. Under additional assumptions we investigate this relations.

Remark 11 Below we use the result of [15, Theorem 3] which says that if X is unimodal and $d \geqslant 3$ we always have $G^0(x) \leqslant C/(|x|^d\psi^*(|x|^{-1}))$, $x \in \mathbb{R}^d$, for some $C > 0$. If additionally $\psi \in \mathrm{WLSC}(\underline{\alpha}, \underline{\theta}, \underline{c})$, $\underline{\alpha} > 0$, then $c/(|x|^d\psi^*(|x|^{-1})) \leqslant G^0(x)$ for $|x|$ small enough and some $c > 0$.

Corollary 5.2 *Let $d \geqslant 3$, X be unimodal with $\psi \in WLSC(\underline{\alpha}, \underline{\theta}, \underline{c})$, $\underline{\alpha} > 0$. There exist constants $C = C(d, \underline{\alpha}, \underline{c})$ and $b = (d, \underline{\alpha}, \underline{c})$ such that for any $0 < t < 1/\psi^*(\underline{\theta}/b)$ and $q : \mathbb{R}^d \to \mathbb{R}$,*

$$\sup_{x\in\mathbb{R}^d} \int_0^t P_u|q|(x)du \leqslant C \sup_{x\in\mathbb{R}^d} \int_{B(x,1/(\psi^*)^-(1/t))} |q(z)|G^0(z-x)dz.$$

Proof We let $t_0 = \infty$ in Proposition 5.1. For $0 < t < \infty$ we take $r = 1/(\psi^*)^-(1/t) > 0$. Since $\psi^*(r^{-1}) = 1/t$ by [15, Theorem 3] $r^d G^0(r) \geqslant c/\psi^*(r^{-1}) = ct$ if $1/(\psi^*)^-(1/t) \leqslant b/\underline{\theta}$ for some constant $c > 0$. The last holds if $t < 1/\psi^*(\underline{\theta}/b)$. □

Lemma 5.3 *Let $d \geqslant 3$, X be unimodal and $\psi \in WLSC(\underline{\alpha}, \theta, \underline{c}) \cap WUSC(\overline{\alpha}, \theta, \overline{C})$, $\underline{\alpha}, \overline{\alpha} \in (0, 2)$. Then there exist constants $c = c(d, \underline{\alpha}, \overline{\alpha}, \underline{c}, \overline{C})$ and $a = (d, \underline{\alpha}, \overline{\alpha}, \underline{c}, \overline{C})$ such that for any $0 < t < 1/\psi^*(\theta/a)$ and $q : \mathbb{R}^d \to \mathbb{R}$,*

$$\sup_{x\in\mathbb{R}^d} \int_0^t P_u|q|(x)du \geqslant c \sup_{x\in\mathbb{R}^d} \int_{B(x,1/(\psi^*)^-(1/t))} |q(z)|G^0(z-x)dz.$$

Proof Let $x \in \mathbb{R}^d$ be such that $|x| < 1/(\psi^*)^-(1/t)$, which gives $1/\psi^*(|x|^{-1}) \leqslant t$. Further, since $t < 1/\psi^*(\theta/a)$ implies $1/(\psi^*)^-(1/t) < a/\theta$ we get $|x| < a/\theta$ and also $u\,\psi^*(\theta/a) < 1$ if $u < 1/\psi^*(|x|^{-1})$. Then [6, Theorem 21 and Lemma 17] ($r_0 = a$) yield

$$\int_0^t p(u,x)du \geqslant \int_0^{1/\psi^*(|x|^{-1})} p(u,x)du \geqslant c^* \int_0^{1/\psi^*(|x|^{-1})} \frac{u\psi^*(|x|^{-1})}{|x|^d} du = \frac{c^*}{2|x|^d \psi^*(|x|^{-1})}\,.$$

Finally, we apply [15, Theorem 3] to obtain

$$\int_0^t p(u,x)du \geqslant c\, G^0(x)\,, \qquad \text{for} \quad |x| < 1/(\psi^*)^-(1/t)\,.$$

□

5.2 Subordinators

Let X be a subordinator (without killing) with the Laplace exponent ϕ. Then ϕ is a Bernstein function (in short BF) with zero killing term. Two important subclasses of BF are special Bernstein functions (SBF) and complete Bernstein functions (CBF). We refer the reader to [30] for definitions and an overview. Since the cases when ϕ is bounded (equivalently X is a compound Poisson process) or when X has a non-zero drift γ_0, are completely described by Theorems 3.8 and 4.4, we assume that

(S1) ϕ is unbounded (X is non-Poisson) and $\gamma_0 = 0$.

Note that for $d = 1$ if a Lévy process is non-Poisson and $A = 0$, $\gamma_0 = 0$, $\int_{\mathbb{R}}(|x| \wedge 1)\nu(dx) < \infty$, then we are in the case (A) of Section 2.2 (see Remark 6). Thus by Theorem 4.1 the following is true for subordinators.

Remark 12 If X satisfies (S1), then $\{0\}$ is polar and $\mathcal{K}(X) = \mathbb{K}(X)$.

We impose further assumptions on the exponent ϕ to study $G^\lambda(dz)$, $\lambda \geqslant 0$, and describe its behaviour near the origin:

(S2) $a+\phi \in \mathrm{SBF}$ for some $a \geqslant 0$ (see [30, Remark 11.21]),

(S3) $\dfrac{\phi'}{\phi^2} \in \mathrm{WUSC}(-\beta, \overline{\theta}, \overline{C})$, $\beta > 0$.

We shall mention that (S2) is always satisfied if $\phi \in \mathrm{CBF}$. Indeed, if $\phi \in \mathrm{CBF}$, then $a+\phi \in \mathrm{CBF}$, $a \geqslant 0$, and $\mathrm{CBF} \subset \mathrm{SBF}$.

Remark 13 Recall that X is a subordinator without killing, i.e., $\phi \in \mathrm{BF}$ with zero killing term. Note that $U(dz) = G^a(dz)$ is a potential kernel of (possibly killed) subordinator $S = X^a$, see [30, (5.2)]. The Laplace exponent of S equals $a+\phi$, thus by [30, Theorem 11.3, formulas (11.9) and Corollary 11.8] we have

(a) under (S2), the measure $G^a(dz)$ is absolutely continuous with respect to the Lebesgue measure if and only if $\nu(0,\infty) = \infty$ (X is non-Poisson) or $\gamma_0 > 0$,
(b) under (S1) and (S2), the density $G^a(z)$ of $G^a(dz)$ satisfies: $G^a(z) = 0$ on $(-\infty, 0]$, $G^a(z)$ is finite, positive and non-increasing on $(0,\infty)$, and $\lim_{z\to 0^+} G^a(z) = \infty$,
(c) under (S2) with $a = 0$, $G^0(dz)$ tends to zero if and only if $\int_1^\infty x\nu(dx) = \infty$.

We already know by Remark 12 that G^a, $a > 0$, describes $\mathbb{K}(X)$ by Eq. 18. We extend this observation to $a = 0$.

Proposition 5.4 *Assume (S1) and (S2) with* $a = 0$. *Then* $\mathcal{K}^0(X) = \mathcal{K}(X) = \mathbb{K}(X)$, *that is* $q \in \mathbb{K}(X)$ *if and only if*

$$\lim_{r\to 0^+} \left[\sup_{x\in\mathbb{R}} \int_0^r |q(z+x)| G^0(z) dz \right] = 0\,.$$

Proof Obviously X is transient and by Remark 13 the result of Theorem 4.15 applies. □

Lemma 5.5 *Assume (S1), (S2) and (S3) and let* $a \geqslant 0$ *be chosen according to (S2). Then the density* $G^a(z)$ *of* $G^a(dz)$ *satisfies*

$$G^a(z) \approx \frac{\phi'(z^{-1})}{z^2\phi^2(z^{-1})}, \quad 0 < z \leqslant 1.$$

Proof The Laplace transform of $G^a(z)$ is given by $\Phi = 1/[a+\phi]$. Note that

$$\Phi' = \frac{\phi'}{\phi^2} \left[\frac{\phi}{a+\phi} \right]^2 \approx \frac{\phi'}{\phi^2} \qquad \text{on} \qquad [1,\infty).$$

Thus by [6, Remark 3] $\Phi' \in \mathrm{WUSC}(-\beta, \overline{\theta} \vee 1, \overline{C}/c)$, $c = [\phi(1)/[a+\phi(1)]]^2$. Next, [6, Lemma 5] and a version of Lemma 13 from [6] imply $G^a(z) \approx z^{-2}\Phi'(z^{-1}) \approx z^{-2}\phi'(z^{-1})/\phi^2(z^{-1})$ as $z \to 0^+$ (see also [22, Proposition 3.4]). The result extends to $z \in (0,1]$ by the regularity of both sides of the estimate. □

Lemma 5.5, Remark 12 and Proposition 5.4 imply the following result.

Proposition 5.6 *Let* X *be a subordinator satisfying (S1), (S2) and (S3). Then* $q \in \mathbb{K}(X)$ *if and only if* Eq. 7 *holds.*

5.3 Examples

We refer the reader to [1, 11, 38] and [25] for basic examples of the Brownian motion, the relativistic process, symmetric α-stable processes and relativistic α-stable processes. We proceed towards our examples.

Example 1 Denote $A_1 = \{2^n : n \in \mathbb{Z}\}$ and

$$f(s) = \mathbb{1}_{(0,1]}(s)\, s^{-\alpha} + e^m \mathbb{1}_{(1,\infty)}(s)\, e^{-ms^\beta} s^{-\delta}, \quad s > 0,$$

where $m > 0$, $\beta \in (0, 1]$, $\delta > 0$ and $\alpha \in (0, 2)$. Define a Lévy measure in $\mathbb{R}$ as

$$\nu(dz) = \sum_{y \in A_1} f(|y|) \left(\delta_y(dz) + \delta_{-y}(dz)\right). \tag{29}$$

Let X be a Lévy process with $A = 0$, $\gamma = 0$ and (an infinite symmetric) ν given by Eq. 29. Then X is a recurrent process, $\psi(z)$ is a real valued function comparable with $|z|^2 \wedge |z|^\alpha$ (see [19, Example 4] and [29, Corollary 37.6]). Further, if $\alpha \in (1, 2)$ Theorem 4.6 applies and describes both $\mathcal{K}(X)$ and $\mathbb{K}(X)$. If now $\alpha \in (0, 1]$ by Theorem 4.1 we obtain $\mathcal{K}(X) = \mathbb{K}(X)$. By [23, Theorem 2.5] there are constants $c_1, c_2 \in (0, 1)$ such that $p(t, x) \geqslant c_1 t^{-1/\alpha}$ on $|x| \leqslant c_2\, t^{1/\alpha}$, $t \in (0, 1]$. Then for some $c > 0$

$$\int_0^1 p(u, x)\, du \geqslant c\, H(|x|), \qquad |x| \leqslant c_2/2.$$

where

$$H(r) = \begin{cases} r^{\alpha-1}, & 0 < \alpha < 1, \\ \ln(r^{-1}), & \alpha = 1. \end{cases}$$

Moreover, by [19, Example 4] there is $c_3 > 0$ so that $p(t, x) \leqslant c_3\, t^{-1/\alpha}(1 \wedge t\,|x|^{-\alpha})$ on $|x| \leqslant 1$, $t \in (0, 1]$. Thus, if $\alpha \in (1/2, 1]$, there exists a constant $c > 0$ such that

$$\int_0^1 p(u, x)\, du \leqslant c\, H(|x|), \qquad |x| \leqslant 1/2.$$

Finally, by Proposition 3.6 for $\alpha \in (1/2, 1]$ we have $q \in \mathcal{K}(X) = \mathbb{K}(X)$ if and only if

$$\lim_{r \to 0^+} \int_{B(x,r)} |q(z)| H(|z - x|) dz = 0.$$

We note that this considerations superficially resemble the results of [25] (see especially [25, Definition 3.2]). We explain why [25] cannot be applied in this example if $\alpha \leqslant 1$. Let $f(t, x)$ be a function that is non-increasing on $x \in (0, 1]$ for every fixed $t \in (0, 1]$. If $p(t, x) \leqslant f(t, x)$ by the lower bound for p and monotonicity of f we have $f(t, x) \geqslant c_4\, t^{-1/\alpha}(1 \wedge t\, 2^{\alpha k})$, $x \in (2^{-k-1}, 2^{-k}]$. Then for $n(t) = (1/\alpha) \log_2(1/t)$ we obtain

$$\int_0^1 f(t, x) dx \geqslant c_4\, t^{1-1/\alpha} \sum_{k=0}^{n(t)} 2^{(\alpha-1)k-1} \xrightarrow{t \to 0^+} \infty, \quad \text{if} \quad \alpha \in (0, 1].$$

Finally, if the upper bound assumption [25, (A2.3)] holds, i.e., $p(t, x) \leqslant t^{-1/\beta} \Phi_2(t^{-1/\beta}|x|) = f(t, x)$ for some $\beta > 0$, we have

$$\int_0^{t^{-1/\beta}} \Phi_2(z) dz = \int_0^1 f(t, x) dx \xrightarrow{t \to 0^+} \infty, \quad \text{if} \quad \alpha \in (0, 1],$$

which contradicts with the integrability assumption in [25, (A2.3)].

In fact, we have $p(s,x) \leqslant c_3 t^{-1/\alpha} \Phi_2(t^{-1/\alpha}|x|)$ for $|x| \leqslant 1$, $t \in (0,1]$ with $\Phi_2(r) = 1 \wedge r^{-\alpha}$, which is a precise estimate for $x \in A_1$ and $|x| \leqslant 1$, and the integrability condition for Φ_2 holds only if $\alpha \in (1,2)$.

Example 2 Let $\psi(x,y) = |x|^2 + iy$ that is $X_t = (B_t, t)$, where B_t is the standard Brownian motion in $\mathbb{R}^d$ (see [2, 10.4 and Example 13.30]). We note that in this case the transition kernel is not absolutely continuous but the potential kernel is. Then $q \in \mathbb{K}(X)$ reads as

$$\lim_{t\to 0^+} \sup_{x\in\mathbb{R}^d,\, y\in\mathbb{R}} \int_0^t \int_{\mathbb{R}^d} |q(z+x,u+y)|\, u^{-d/2} e^{-|z|^2/(4u)} dz du = 0\,,$$

and by Corollary 1.2 holds if and only if

$$\lim_{r\to 0^+} \sup_{x\in\mathbb{R}^d,\, y\in\mathbb{R}} \int_0^r \int_{B(0,r)} |q(z+x,u+y)|\, u^{-d/2} e^{-|z|^2/(4u)} dz du = 0\,.$$

Now we discuss in detail subordinators. Since functions ϕ presented below are unbounded CBF with zero drift term, see [30, Chapter 16: No 2 and 59, Proposition 7.1], they satisfy (S1) and (S2). The assumption (S3) can be easily checked. The first example covers the case of α-stable subordinator, $\alpha \in (0,1)$, and the inverse Gaussian subordinator.

Example 3 Let $\phi(u) = \delta[(u+m)^\alpha - m^\alpha]$, $\delta > 0$, $m \geqslant 0$, $\alpha \in (0,1)$. Then $q \in \mathbb{K}(X)$ if and only if

$$\lim_{r\to 0^+} \sup_{x\in\mathbb{R}} \int_x^{x+r} |q(z)|(z-x)^{\alpha-1}\, dz = 0\,.$$

Example 4 Let $\phi(u) = \ln(1+u^\alpha)$, where $\alpha \in (0,1]$. Then $q \in \mathbb{K}(X)$ if and only if

$$\lim_{r\to 0^+} \sup_{x\in\mathbb{R}} \int_x^{x+r} |q(z)| \frac{dz}{(z-x)\ln^2(z-x)} = 0\,.$$

Example 5 Let $\phi(u) = \dfrac{u}{\ln(1+u^\alpha)}$, where $\alpha \in (0,1)$. Then $q \in \mathbb{K}(X)$ if and only if

$$\lim_{r\to 0^+} \sup_{x\in\mathbb{R}} \int_x^{x+r} |q(z)||\ln(z-x)| dz = 0\,.$$

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