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Circularly Symmetric Locally Univalent Functions

Leopold Koczan[1] · **Paweł Zaprawa**[1]

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Abstract Let $D \subset \mathbb{C}$ and $0 \in D$. A set D is circularly symmetric if, for each $\varrho \in \mathbb{R}^+$, a set $D \cap \{\zeta \in \mathbb{C} : |\zeta| = \varrho\}$ is one of three forms: an empty set, a whole circle, a curve symmetric with respect to the real axis containing ϱ. A function f analytic in the unit disk $\Delta \equiv \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and satisfying the normalization condition $f(0) = f'(0) - 1 = 0$ is circularly symmetric, if $f(\Delta)$ is a circularly symmetric set. The class of all such functions is denoted by X. In this paper, we focus on the subclass X' consisting of functions in X which are locally univalent. We obtain the results concerned with omitted values of $f \in X'$ and some covering and distortion theorems. For functions in X' we also find the upper estimate of the n-th coefficient, as well as the region of variability of the second and the third coefficients. Furthermore, we derive the radii of starlikeness, convexity and univalence for X'.

Keywords Locally univalent functions · Radius of univalence · Radius of starlikeness · Coefficients' estimates

Mathematics Subject Classification Primary 30C45; Secondary 30C75

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✉ Paweł Zaprawa
p.zaprawa@pollub.pl

Leopold Koczan
l.koczan@pollub.pl

1 Department of Mathematics, Lublin University of Technology, Nadbystrzycka 38D, 20-618 Lublin, Poland

1 Introduction

Let $\mathcal{A}$ denote the class of all functions analytic in the unit disk $\Delta \equiv \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ which satisfy the condition $f(0) = f'(0) - 1 = 0$. A function f is said to be typically real if the inequality $(\operatorname{Im} z)(\operatorname{Im} f(z)) \geq 0$ holds for all $z \in \Delta$. The class of functions which are typically real is denoted by $\tilde{T}$ and the class of typically real functions which belong to $\mathcal{A}$ is denoted by T. For a typically real function f, $z \in \Delta^+ \Leftrightarrow f(z) \in \mathbb{C}^+$ and $z \in \Delta^- \Leftrightarrow f(z) \in \mathbb{C}^-$. The symbols Δ^+, Δ^-, $\mathbb{C}^+$, $\mathbb{C}^-$ stand for the following sets: the upper and the lower halves of the disk Δ, the upper halfplane and the lower halfplane, respectively.

Jenkins [3] established the following definitions.

Definition 1 Let $D \subset \mathbb{C}$, $0 \in D$. A set D is circularly symmetric if, for each $\varrho \in \mathbb{R}^+$, a set $D \cap \{\zeta \in \mathbb{C} : |\zeta| = \varrho\}$ is one of three forms: an empty set, a whole circle, a curve symmetric with respect to the real axis containing ϱ.

Definition 2 A function $f \in \mathcal{A}$ is circularly symmetric if $f(\Delta)$ is a circularly symmetric set. The class of all such functions is denoted by X.

In fact Jenkins considered only those circularly symmetric functions which are univalent. This assumption is rather restrictive. A number of interesting problems appear while discussing non-univalent circularly symmetric functions. For these reasons, we decided to define a circularly symmetric function as in Definition 2. In this paper, we focus on the set X' consisting of locally univalent circularly symmetric functions.

According to Jenkins (see, [3]), if $f \in X$ is univalent then $\frac{zf'(z)}{f(z)}$ is a typically real function. Additionally, he observed that the property $\frac{zf'(z)}{f(z)} \in \tilde{T}$ does not guarantee the univalence of f. In fact, we have

$$f \in X \Leftrightarrow \frac{zf'(z)}{f(z)} \in \tilde{T}. \tag{1}$$

Jenkins also gave a nice geometric property of f in X. He proved that $f \in X$ if and only if, for a fixed $r \in (0, 1)$, a function $|f(re^{i\varphi})|$ is nonincreasing for $\varphi \in (0, \pi)$ and nondecreasing for $\varphi \in (\pi, 2\pi)$. From (1), all coefficients of the Taylor series expansion of $f \in X$ are real.

In [9] the following relation between X' and T was proved:

$$f \in X' \Leftrightarrow \frac{zf'(z)}{f(z)} = (1+z)^2 \frac{h(z)}{z}, \quad h \in T. \tag{2}$$

It is known that each function of the class T can be represented by the formula

$$h(z) = \int_{-1}^{1} \frac{z}{1 - 2tz + z^2} d\mu(t), \tag{3}$$

where μ is a probability measure on $[-1, 1]$ (see, [7]). Applying (3) in (2) and integrating it, one can write a function $f \in X'$ in the form

$$f(z) = z \exp\left(\int_0^z \int_{-1}^1 \frac{2(1+t)}{1-2t\zeta+\zeta^2}\, d\mu(t)\, d\zeta\right). \tag{4}$$

Putting $\cos\psi$ instead of t in (4) and integrating it with respect to ζ, we get the integral representation of a function in X':

$$f \in X' \Leftrightarrow f(z) = z \exp\left(\int_0^\pi i \cot\frac{\psi}{2} \log\frac{1-ze^{i\psi}}{1-ze^{-i\psi}}\, d\mu(\psi)\right), \tag{5}$$

where μ is a probability measure on $[0, \pi]$.

Applying the well-known equivalence

$$f \in T \Leftrightarrow \frac{1-z^2}{z} f(z) \in P_{\mathbb{R}}, \tag{6}$$

we obtain the relation between X' and the set $P_{\mathbb{R}}$ of functions with positive real part which have real coefficients:

$$f \in X' \Leftrightarrow \frac{zf'(z)}{f(z)} = \frac{1+z}{1-z} p(z), \quad p \in P_{\mathbb{R}}. \tag{7}$$

It is known (Robertson, [6]) that the set of extreme points for T has the form $\left\{\frac{z}{1-2zt+z^2} : t \in [-1, 1]\right\}$. Putting these functions into formula (2) as h, we get

$$\frac{zf'(z)}{f(z)} = \frac{(1+z)^2}{1-2zt+z^2}. \tag{8}$$

It is easy to check that the functions f, which satisfy (8), are of the form

$$f_t(z) = z \exp\left(i \cot\frac{\psi}{2} \log\frac{1-ze^{i\psi}}{1-ze^{-i\psi}}\right), \quad t \in [-1, 1), \tag{9}$$

where $t = \cos\psi$. Obviously, $t \in [-1, 1) \Leftrightarrow \psi \in (0, \pi]$. Furthermore,

$$\lim_{\psi\to 0^+}\left(i \cot\frac{\psi}{2} \log\frac{1-ze^{i\psi}}{1-ze^{-i\psi}}\right) = \frac{4z}{1-z},$$

so we can write

$$f_1(z) = z \exp\left(\frac{4z}{1-z}\right). \tag{10}$$

Besides f_t, we need another family of functions belonging to X'. Since the set T is convex, every linear combination of any two functions from T also belongs to T. Hence, taking $\frac{1+t}{2}\frac{z}{(1-z)^2} + \frac{1-t}{2}\frac{z}{(1+z)^2}$, $t \in [-1, 1]$ as h in (2), we obtain

$$\frac{zf'(z)}{f(z)} = \frac{1+2zt+z^2}{(1-z)^2}. \tag{11}$$

Let us denote by g_t the solutions of Eq. (11). From this equation

$$g_t(z) = z \exp\left(\frac{2(1+t)z}{1-z}\right), \quad t \in [-1, 1]. \tag{12}$$

In particular, we have

$$g_{-1}(z) = f_{-1}(z) = z \quad \text{and} \quad g_1(z) = f_1(z) = z \exp\left(\frac{4z}{1-z}\right).$$

2 Properties of f_t and g_t

Firstly, we shall describe the sets $f_t(\Delta)$ and $g_t(\Delta)$, where f_t, g_t are defined by (9) and (12), respectively.

For f_t, $t \in (-1, 1)$ (i.e. for $\psi \in (0, \pi)$), from (9) we obtain

$$\left|f_t(e^{i\varphi})\right| = \exp\left(\cot\frac{\psi}{2} \arg\frac{1-e^{i(\varphi-\psi)}}{1-e^{i(\varphi+\psi)}}\right).$$

We shall derive the argument which appears in the above expression. To do this, we need the following identity:

$$\arg\left(1+e^{i\alpha}\right) = \frac{\alpha}{2} - \pi \cdot \left\lfloor \frac{\alpha+\pi}{2\pi} \right\rfloor \quad \text{for} \quad \alpha \neq (1+2k)\pi, \quad k \in \mathbb{Z}.$$

Hence

$$\left|f_t(e^{i\varphi})\right| = \exp\left(-\psi \cot\frac{\psi}{2}\right) \quad \text{for} \quad \varphi \in (\psi, 2\pi - \psi) \tag{13}$$

and

$$\left|f_t(e^{i\varphi})\right| = \exp\left((\pi-\psi)\cot\frac{\psi}{2}\right) \quad \text{for} \quad \varphi \in [0, \psi) \cup (2\pi-\psi, 2\pi]. \tag{14}$$

From the above expressions, it follows that a function $|f_t(e^{i\varphi})|$, with fixed $t \in (-1, 1)$, does not depend on the variable φ. Moreover, $|f_t(e^{i\varphi})|$ in (13) is less than 1 and $|f_t(e^{i\varphi})|$ in (14) is greater than 1. Additionally,

$$\arg\left(f_t(e^{i\varphi})\right) = \varphi + \cot\frac{\psi}{2} \log\left|\frac{\sin\frac{\varphi+\psi}{2}}{\sin\frac{\varphi-\psi}{2}}\right|, \tag{15}$$

which means that the curves $\{f_t(e^{i\varphi}),\ \varphi \in [0, \phi)\}$ and $\{f_t(e^{i\varphi}),\ \varphi \in (\phi, \pi]\}$ wind around circles given by (13) and (14) infinitely many times. Hence, for $t \in (-1, 1)$,

$$f_t(\Delta) \subset \left\{w \in \mathbb{C} : |w| < \exp\left((\pi-\psi)\cot\frac{\psi}{2}\right)\right\}. \tag{16}$$

It is easily seen that $f_{-1}(\Delta) = \Delta$.

Now, we shall show that f_1 omits only one point on the real axis.

Theorem 1 *The condition $f_1(z) \neq -e^{-2}$ holds for all $z \in \Delta$.*

Proof On the contrary, suppose that there exists $z \in \Delta$ such that $f_1(z) = -e^{-2}$. In fact, we can assume that $\arg z \in [0, \pi]$ because the coefficients of f_1 are real. For this reason,

$$z \exp\left(\frac{2(1+z)}{1-z}\right) = -1, \tag{17}$$

or equivalently,

$$\frac{1+z}{1-z} = \frac{1}{2}\left(-\log|z| + i \arg\frac{-1}{z}\right). \tag{18}$$

Let $z = re^{i\varphi}$, $\varphi \in [0, \pi]$. Comparing the arguments of both sides of this equality, we get

$$\frac{2r \sin\varphi}{1-r^2} = \frac{\pi - \varphi}{-\log r}. \tag{19}$$

For a fixed $r \in (0, 1)$, let us consider a function

$$h(\varphi) = \frac{2r}{1-r^2}\sin\varphi + \frac{1}{\log r}(\pi - \varphi), \quad \varphi \in [0, \pi].$$

For $\varphi \in [0, \pi]$ the function $h'(\varphi)$ decreases and

$$h'(\varphi) \geq -\frac{2r}{(1-r^2)\log r} \cdot g(r), \tag{20}$$

where

$$g(r) = \log r + \frac{1-r^2}{2r}.$$

Since $g'(r) = -(1-r)^2/2r^2$, the function $g(r)$ decreases for $r \in (0, 1)$ and

$$g(r) \geq g(1) = 0.$$

Both factors on the right-hand side of (20) are positive. Consequently, $h(\varphi)$ increases for $\varphi \in (0, \pi)$, so $h(\varphi) \leq h(\pi) = 0$.

Taking into account the last inequality, we can see that $\varphi = \pi$ is the only solution of (19). It means that Eq. (18) is satisfied only for $z = re^{i\pi} = -r$; so

$$\frac{1-r}{1+r} + \frac{1}{2}\log r = 0. \tag{21}$$

Let us denote the left-hand side of (21) by $k(r)$. Since k is increasing for $r \in (0, 1)$,

$$\sup\{k(r) : r \in (0, 1)\} = k(1) = 0.$$

Therefore, (21) has no solutions in the open set (0, 1), which contradicts (18), and, consequently we obtain the desired result. □

Applying a similar argument, one can prove the following more general theorem.

Theorem 2 *For g_t, $t \in (-1, 1]$ given by (12),*

(i) $g_t(z) \neq -e^{-1-t}$ for $z \in \Delta$,
(ii) the equation $g_t(z) = -\varrho e^{-1-t}$ has a solution for any $\varrho > 1$,
(iii) the equation $g_t(z) = -e^{-1-t}e^{i\theta}$ has a solution for any $\theta \in (-\pi, \pi)$.

Proof **ad (i)** Suppose that there exists $z \in \Delta$ such that $g_t(z) = -e^{-1-t}$. Then

$$z \exp\left((1+t)\frac{1+z}{1-z}\right) = -1, \tag{22}$$

and putting $z = re^{i\varphi}$, we have

$$\frac{1+re^{i\varphi}}{1-re^{i\varphi}} = \frac{1}{1+t}\left(-\log r + i(\pi - \varphi)\right). \tag{23}$$

Comparing the arguments on both sides, we obtain the same function h as in the proof for Theorem 4; consequently $h(\varphi) \leq 0$. Therefore, Eq. (23) holds only if $z = -r$, but in this case

$$\frac{1-r}{1+r} + \frac{1}{1+t}\log r = 0. \tag{24}$$

Let the left-hand side of (24) be denoted by $k(r)$, which is an increasing and nonpositive function of $r \in (0, 1)$. It yields that (24) has no solutions in the open set (0, 1), which contradicts the assumption.
ad (ii) Consider an equation $g_t(z) = -\varrho e^{-1-t}$ with fixed $\varrho > 1$. It takes the following form

$$z \exp\left((1+t)\frac{1+z}{1-z}\right) = -\varrho. \tag{25}$$

Putting $z = re^{i\varphi}$ into (25), we have

$$\frac{1+re^{i\varphi}}{1-re^{i\varphi}} = \frac{1}{1+t}\left(\log\frac{\varrho}{r} + i(\pi - \varphi)\right). \tag{26}$$

Hence

$$\frac{2r\sin\varphi}{1-r^2} - \frac{\pi - \varphi}{\log\frac{\varrho}{r}} = 0. \tag{27}$$

Let $h(\varphi)$ denote the left-hand side of this equality. The function $h'(\varphi)$ is decreasing; $h'(0) = \frac{2r}{1-r^2} + \frac{1}{\log\frac{\varrho}{r}} > 0$ and $h'(\pi) = \frac{-2r}{1-r^2} + \frac{1}{\log\frac{\varrho}{r}}$. One can easily prove that there exists only one number $r_0 \in (0, 1)$ such that $h'(\pi) > 0$ for $r \in (0, r_0)$ and $h'(\pi) < 0$ for $r \in (r_0, 1)$.

Hence, for suitably taken r, the function h first increases and then decreases. Furthermore, $h(0) = -\pi/\log\frac{\varrho}{r} < 0$ and $h(\pi) = 0$. Consequently, there exists $\varphi_0 \in (0, \pi)$

such that $h(\varphi_0) = 0$. It means that (26) is satisfied by $z_0 = re^{i\varphi_0}$; so (25) holds for $z = z_0$.

ad (iii) The proof of this part is similar to the proof of (ii). □

Corollary 1 $g_t(\Delta) = \mathbb{C}\setminus\{-e^{-1-t}\}$ *for all* $t \in (-1, 1]$.

Proof Let $t \in (-1, 1]$ be fixed. For $r \in (0, 1)$, we have $g_t(-r) = -r\exp(-2(1 + t)r/(1 + r))$. Observe that $|g_t(-r)|$ is a continuous and increasing function of $r \in [0, 1)$. For this reason, $g_t(-r)$ achieves all values in $(-e^{-1-t}, 0]$. From the definition of circularly symmetric function it follows that if c belongs to the negative real axis and $c \in g_t(\Delta)$, then the whole circle with radius $|c|$ centered at the origin is also contained in this set. Hence, for each ϱ in $[0, 1)$ we have $\{w \in \mathbb{C} : |w| = \varrho e^{-1-t}\} \subset g_t(\Delta)$.

By Theorem 2, part (ii), $(-\infty, -e^{-1-t})$ is contained in $g_t(\Delta)$. Let $-\varrho e^{-1-t}$, $\varrho > 1$ be an arbitrary point of this ray. Applying the same argument as above, we conclude that for any $\varrho > 1$, $\{w \in \mathbb{C} : |w| = \varrho e^{-1-t}\} \subset g_t(\Delta)$.

Combining these facts with points (i) and (iii) of Theorem 2 completes the proof. □

Let us consider a function $b(\varphi) = \arg g_t(re^{i\varphi})$, $\varphi \in (0, \pi)$ where $t \in [-1, 1]$ and $r \in (0, 1)$ are fixed. Analyzing the derivative of this function it can be observed that if $t - 3 + 4\sigma^2 \le 0$, where $\sigma = \frac{2r}{1+r^2}$, then $b(\varphi)$ increases in $(0, \pi)$. On the other hand, if $t - 3 + 4\sigma^2 > 0$ then, for $\varphi \in (0, \pi)$, the function $b(\varphi)$ increases at the beginning, then it decreases, only to increase again at the end. From this observation we conclude that for small r, a set $g_t(\{z \in \mathbb{C} : |z| < r, \operatorname{Im} z \ge 0\})$ is contained in the upper halfplane. If r is greater than $r_t = \frac{\sqrt{3-t}}{2+\sqrt{1+t}}$, then this set is not contained in the upper halfplane; its boundary is wound around the origin.

If $r = 1$, then

$$g_t(e^{i\varphi}) = \exp\left(-1 - t + i\left(\varphi + (1 + t)\cot\frac{\varphi}{2}\right)\right). \tag{28}$$

Hence

$$\left|g_t(e^{i\varphi})\right| = \exp\left(-1 - t\right) \tag{29}$$

and

$$\arg\left(g_t(e^{i\varphi})\right) = \varphi + (1 + t)\cot\frac{\varphi}{2}. \tag{30}$$

Therefore, $g_t(\Delta)$ is wound around the origin infinitely many times, or, more precisely, it is wound around the circle with radius e^{-1-t}.

3 Coefficients of Functions in X'

To start with, let us look into the coefficients of f_1 given by (10). Although it is complicated to find an explicit formula for the n-th coefficient of this function, the formula of the logarithmic coefficients γ_n of f_1 can be easily derived. Indeed,

$$\frac{1}{2}\log\frac{f_1(z)}{z} = \sum_{k=1}^{\infty} 2z^k,$$

thus

$$\gamma_n = 2 \qquad \text{for all} \quad n \in \mathbb{N}.$$

The Taylor series expansion of f_1 is given by

$$f_1(z) = z + \sum_{k=1}^{\infty} \frac{4^k}{k!} z^{k+1}(1-z)^{-k} = z + \sum_{k=1}^{\infty} \frac{4^k}{k!} z^{k+1} \sum_{j=0}^{\infty} \binom{j+k-1}{k-1} z^j$$
$$= z + \sum_{j=0}^{\infty}\sum_{k=1}^{\infty} b_{j,k} z^{k+j+1},$$

where

$$b_{j,k} = \frac{4^k}{k!}\binom{j+k-1}{k-1}, \quad k \geq 1, \quad j \geq 0.$$

Denoting the n-th coefficient of f_1 by A_n, we can write

$$A_n = \sum_{s=0}^{n-2} b_{s,n-1-s} = \sum_{s=0}^{n-2} \frac{4^{n-1-s}}{(n-1-s)!}\binom{n-2}{n-2-s},$$

and consequently,

$$A_n = \sum_{j=1}^{n-1} \frac{4^j}{j!}\binom{n-2}{j-1}. \tag{31}$$

The first four values of A_n are

$$A_2 = 4, \quad A_3 = 12, \quad A_4 = \frac{92}{3}, \quad A_5 = \frac{212}{3}.$$

On the other hand, for A_n the formula

$$nA_{n+1} = (2n+2)A_n - (n-2)A_{n-1} \tag{32}$$

holds for $n \geq 2$. Indeed, expanding both sides of the equality

$$zf_1'(z) = f_1(z)\left(\frac{1+z}{1-z}\right)^2,$$

we get

$$\sum_{n=1}^{\infty} nA_n z^n = \sum_{n=1}^{\infty} A_n z^n \cdot \left(1 + \sum_{k=1}^{\infty} 2z^k\right)^2.$$

Comparing the coefficients at z^n, we obtain (32).

We shall now prove that the upper bound of n-th coefficient of a function $f \in X'$ is achieved when f is equal to f_1. To do this, we apply the relation (7).

Suppose that functions $f \in X'$ and $p \in P_{\mathbb{R}}$ are of the form $f(z) = \sum_{n=1}^{\infty} a_n z^n$ and $p(z) = \sum_{n=0}^{\infty} p_n z^n$ with $a_1 = 1$, $p_0 = 1$. Equation (7) yields

$$z + \sum_{n=2}^{\infty} n a_n z^n = \left(z + \sum_{n=2}^{\infty} a_n z^n\right) \cdot \left(1 + \sum_{n=1}^{\infty} c_n z^n\right),$$

where

$$c_n = p_n + 2 \sum_{k=0}^{n-1} p_k.$$

Comparing the coefficients at z^n, $n \geq 2$, we obtain

$$(n-1)a_n = \sum_{j=1}^{n-1} a_j c_{n-j} = \sum_{j=1}^{n-1} a_j \left(p_{n-j} + 2 \sum_{k=0}^{n-j-1} p_k\right). \tag{33}$$

Taking into account (33) and the coefficient estimates of a function in $P_{\mathbb{R}}$, we conclude

$$(n-1)a_n \leq 4 \sum_{j=1}^{n-1} |a_j|(n-j), \quad n \geq 2. \tag{34}$$

Equality in (34) holds only if all p_i in (33) are equal to 2, which means that $p(z) = \frac{1+z}{1-z}$.

From (34), when $n = 2$, there is $a_2 \leq 4|a_1| = 4$. Equality in this estimate holds for f_1 only. Now, it is sufficient to apply mathematical induction in order to prove that successive coefficients a_n of any $f \in X'$ are bounded by corresponding coefficients A_n of f_1. Hence

Theorem 3 *Let $f \in X'$ have the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and let A_n be given by (31). Then, for $n \geq 2$,*

$$a_n \leq A_n.$$

Our next problem is to find the set of variability of (a_2, a_3) for a function in X'. For a given class of analytic functions A, let $A_{i,j}(A)$ denote a set $\{(a_i(f), a_j(f)) : f \in A\}$.

For the class $P_{\mathbb{R}}$ of functions with positive real part and having real coefficients, the following result is known:

$$A_{1,2}(P_{\mathbb{R}}) = \{(x, y) : -2 \leq x \leq 2, x^2 - 2 \leq y \leq 2\}. \tag{35}$$

Based on this result, we can prove

Theorem 4

$$A_{2,3}(X') = \left\{(x, y) : 0 \le x \le 4, x^2 - x \le y \le \frac{1}{2}x^2 + x\right\}$$

and

Corollary 2 *Let $f \in X'$ have the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then $a_3 \ge -\frac{1}{4}$.*

Proof of Theorem 4 Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in X'$ and $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in P_{\mathbb{R}}$. It follows from (33) that

$$\begin{aligned} a_2 &= p_1 + 2, \\ 2a_3 &= p_2 + 2p_1 + 2 + a_2(p_1 + 2), \end{aligned}$$

or equivalently,

$$\begin{aligned} p_1 &= a_2 - 2, \\ p_2 &= 2a_3 - a_2{}^2 - 2a_2 + 2. \end{aligned}$$

Combining these relations with the estimates given in (35) completes the proof. □

The points of intersection of two parabolas described in Theorem 4, i.e.: (0, 0) and (4, 12) correspond to the functions $f_{-1}(z) = z$ and $f_1(z) = z \exp\left(\frac{4z}{1-z}\right) = z + 4z^2 + 12z^3 + \cdots$, respectively.

Observe that the class X' is not convex. Indeed, if X' is a convex set, then, for every fixed $\alpha \in (0, 1)$, a function $\alpha f_1(z) + (1 - \alpha) f_{-1}(z) = \alpha z \exp\left(\frac{4z}{1-z}\right) + (1 - \alpha)z = z + 4\alpha z^2 + 12\alpha z^3 + \cdots$, would be in X'. This will imply that $(4\alpha, 12\alpha) \in A_{2,3}(X')$, a contradiction with Theorem 4.

4 Distortion Theorems

Directly from the definition of a circularly symmetric function, it follows that

$$|f(-r)| \le |f(re^{i\varphi})| \le |f(r)| \tag{36}$$

for every function $f \in X'$ and for all $\varphi \in [0, 2\pi]$ and $r \in (0, 1)$.

From (4), for any function $f \in X'$ and any number $r \in (0, 1)$,

$$\begin{aligned} f(r) &= r \exp\left(\int_0^r \int_{-1}^1 \frac{2(1+t)}{1 - 2tx + x^2} \mathrm{d}\mu(t) \mathrm{d}x\right) \\ &\le r \exp\left(\int_0^r \frac{4}{(1-x)^2} \mathrm{d}x\right) = r \exp\left(\frac{4r}{1-r}\right) = f_1(r). \end{aligned} \tag{37}$$

Similarly,

$$
\begin{aligned}
|f(-r)| &= r\exp\left(\int_0^{-r}\int_{-1}^{1}\frac{2(1+t)}{1-2tx+x^2}\mathrm{d}\mu(t)\mathrm{d}x\right)\\
&= r\exp\left(-\int_0^{r}\int_{-1}^{1}\frac{2(1+t)}{1+2ty+y^2}\mathrm{d}\mu(t)\mathrm{d}y\right)\\
&\geq r\exp\left(-\int_0^{r}\frac{4}{(1+y)^2}\mathrm{d}y\right) = r\exp\left(\frac{-4r}{1+r}\right) = |f_1(-r)|. \qquad (38)
\end{aligned}
$$

Equalities in the above estimates hold only if μ is a measure concentrated in point 1; it means that $h(z) = \frac{z}{(1-z)^2}$. We have proved

Theorem 5 *For any $f \in X'$ and $r = |z| \in (0, 1)$,*

$$
r\exp\left(\frac{-4r}{1+r}\right) \leq |f(z)| \leq r\exp\left(\frac{4r}{1-r}\right), \qquad (39)
$$

Equalities in the above estimates hold only for f_1 and points $z = -r$ and $z = r$.

Corollary 3 *For any $f \in X'$, we have $f(\Delta) \supset \Delta_{e^{-2}}$.*

The estimates of $|f'(z)|$ for $f \in X'$ can be obtained from (2) and Theorem 5.

Theorem 6 *For any $f \in X'$ and $|z| = r \in (0, 1)$,*

$$
\left(\frac{1-r}{1+r}\right)^2\exp\left(\frac{-4r}{1+r}\right) \leq |f'(z)| \leq \left(\frac{1-r}{1+r}\right)^2\exp\left(\frac{4r}{1-r}\right). \qquad (40)
$$

Equalities in the above estimates hold only for f_1 and points $z = -r$ and $z = r$.

Proof Let $f \in X'$. From (2),

$$
f'(z) = (1+z)^2\frac{h(z)}{z}\frac{f(z)}{z},
$$

where $h \in T$. Therefore, if $|z| = r \in (0, 1)$ then

$$
|f'(z)| \leq (1+r)^2\frac{1}{(1-r)^2}\frac{f_1(r)}{r}
$$

and

$$
|f'(z)| \geq (1-r)^2\frac{1}{(1+r)^2}\frac{|f_1(-r)|}{r},
$$

which is equivalent to (40). Moreover, equalities in both estimates appear when h is equal to $\frac{z}{(1-z)^2}$ and z is equal to r and $-r$, respectively. It means that f_1 is the extremal function for (40). □

Finally, we shall prove two lemmas which will be useful in our research on the convexity of functions in X'.

Lemma 1 *For a fixed point $z \in \Delta^+$, the set $\Omega(z)$ of variability of the expression $\frac{zf'(z)}{f(z)}$, while f varies in X', is of the form*

$$\Omega(z) = \operatorname{conv} \gamma(z),$$

where $\gamma(z)$ is an upper halfplane located arc of a circle containing three nonlinear points: $z_0 = 0$, $z_1 = 1$, $z_2 = (\frac{1+z}{1-z})^2$, with endpoints z_1 and z_2.

Lemma 2 *For any $f \in X'$ and $z \in \Delta$,*

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geq \begin{cases} \operatorname{Re} \left(\frac{1+z}{1-z}\right)^2 & \text{for} \quad \operatorname{Re}(z + 1/z) \leq 2 \\ 1 & \text{for} \quad \operatorname{Re}(z + 1/z) \geq 2. \end{cases} \tag{41}$$

Proof of Lemma 1 Let $z \in \Delta^+$. Applying (2) and the representation formula for a function in T, we have

$$\frac{zf'(z)}{f(z)} = \int_{-1}^{1} \frac{(1+z)^2}{1 - 2zt + z^2} d\mu(t), \tag{42}$$

where μ is a probability measure on $[-1, 1]$.

With a fixed $z \in \Delta$, we denote by $q_z(t)$ an integrand in (42). The image set $\{q_z(t) : t \in \mathbb{R}\}$ coincides with a circle going through the origin. Furthermore, $q_z(-1) = 1$ and $q_z(1) = (\frac{1+z}{1-z})^2$.

For z such that $\operatorname{Im} z > 0$,

$$\operatorname{Im} \left(\frac{1+z}{1-z}\right)^2 = \operatorname{Im} \left(1 + \frac{4}{w-2}\right) = \frac{-4 \operatorname{Im} w}{|w-2|^2} = \frac{4(1/|z|^2 - 1) \operatorname{Im} z}{|w-2|^2} > 0,$$

where $w = z + 1/z$.

Hence, the set $\{q_z(t) : t \in [-1, 1]\}$ is an arc of the circle with endpoints $q_z(-1)$ and $q_z(1)$, which does not contain the origin. For this reason, this set coincides with $\gamma(z)$ and one endpoint of this arc is always 1, independent of z. Finally, $\Omega(z)$ is a section of the disk bounded by $\gamma(z)$ and the line segment with endpoints $q_z(-1)$ and $q_z(1)$. □

Proof of Lemma 2 Every function f in X' has real coefficients, so $f(\Delta)$ is symmetric with respect to the real axis. Hence, it is sufficient to prove (41) only for $z \in \Delta^+$. But Lemma 1 leads to

$$\inf \left\{ \operatorname{Re} \frac{zf'(z)}{f(z)} : f \in X' \right\} = \begin{cases} \operatorname{Re} q_z(1) & \text{for} \quad \operatorname{Re}(z + 1/z) \leq 2 \\ \operatorname{Re} q_z(-1) & \text{for} \quad \operatorname{Re}(z + 1/z) \geq 2. \end{cases} \tag{43}$$

□

It is easy to check that for $z \in \Delta$,

$$\operatorname{Re}\left(\frac{1+z}{1-z}\right)^2 \le 1 \quad \Leftrightarrow \quad \operatorname{Re}(z+1/z) \le 2. \tag{44}$$

Consequently, (41) can be written as follows:

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \ge \min\left\{\operatorname{Re}\left(\frac{1+z}{1-z}\right)^2,\ 1\right\}.$$

5 Starlikeness and Convexity

The relation (2) and the estimates of the argument for typically real functions imply that for $z \in \Delta^+$,

$$\arg\frac{zf'(z)}{f(z)} = 2\arg(1+z) + \arg\frac{g(z)}{z} \le 2\arg(1+z) + \arg\frac{1}{(1-z)^2} = 2\arg\frac{1+z}{1-z}. \tag{45}$$

Furthermore,

$$\left|\arg\frac{1+z}{1-z}\right| \le \arctan\frac{2r}{1-r^2}. \tag{46}$$

The condition for starlikeness $|\arg\frac{zf'(z)}{f(z)}| \le \frac{\pi}{2}$ and the bounds given above result in

$$r_{S^*}(X') = \sqrt{2} - 1. \tag{47}$$

Equality in (45) holds for $g(z) = \frac{z}{(1-z)^2}$, and, consequently, for $f = f_1$. This result will be generalized in two ways.

First, we estimate $\operatorname{Re}\frac{zf'(z)}{f(z)}$ for z in $H = \{z \in \Delta : |1+z^2| > 2|z|\}$. This set appears in the research on typically real functions. It is the domain of univalence and local univalence in T (see, [2]). The set H, called the Golusin lens, is the common part of two disks with radii $\sqrt{2}$ which have the centers in points i and $-i$. Moreover,

$$H = \left\{z \in \mathbb{C} : \operatorname{Re}\left(\frac{1+z}{1-z}\right)^2 > 0\right\}.$$

From Lemma 2, we obtain

Theorem 7 *For each $f \in X'$ and $z \in H$,*

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \ge 0.$$

It is worth noticing that this theorem is still true even if X' is replaced by T. This property is very interesting because the classes X' and T have a non-empty intersection, but one is not included in the other.

As a corollary, from Theorem 7 we obtain (47).

Another generalization of (47) refers to the radius of starlikeness of order α and the radius of strong starlikeness of order α (for definitions and other details the reader is referred to [1,5,8]).

Theorem 8 *The radius of starlikeness of order* α, $\alpha \in [0, 1)$, *in* X' *is equal to*

$$r_{S^*(\alpha)}(X') = \begin{cases} \sqrt{\frac{2}{1-2\alpha}} - \sqrt{\frac{1+2\alpha}{1-2\alpha}} & \text{for} \quad \alpha \in [0, 1/3], \\ \frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}} & \text{for} \quad \alpha \in [1/3, 1). \end{cases}$$

Corollary 4 $r_{S^*(1/2)}(X') = (\sqrt{2} - 1)^2 = 0.171\ldots.$

Theorem 9 *The radius of strong starlikeness of order* α, $\alpha \in (0, 1]$, *in* X' *is equal to*

$$r_{SS^*(\alpha)}(X') = \tan\left(\frac{\pi}{8}\alpha\right).$$

Corollary 5 $r_{SS^*(2/3)}(X') = 2 - \sqrt{3}$.

Proof of Theorem 8 By Lemma 2,

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geq \begin{cases} 1 + 4 \operatorname{Re} \frac{z}{(1-z)^2} & \text{for} \quad \operatorname{Re}(z + 1/z) \leq 2, \\ 1 & \text{for} \quad \operatorname{Re}(z + 1/z) \geq 2. \end{cases} \tag{48}$$

Let $r = |z|$ be a fixed number, $0 < r \leq 2 - \sqrt{3}$. It is known that $h(z) = \frac{z}{(1-z)^2}$ is convex for $|z| \leq 2 - \sqrt{3}$. Hence

$$\operatorname{Re} \frac{z}{(1-z)^2} \geq \frac{-r}{(1+r)^2} \tag{49}$$

with equality for $z = -r$.

From (48) and (49) it follows that

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geq \begin{cases} 1 - \frac{4r}{(1+r)^2} & \text{for} \quad \operatorname{Re}(z + 1/z) \leq 2, \\ 1 & \text{for} \quad \operatorname{Re}(z + 1/z) \geq 2, \end{cases}$$

and so

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geq \left(\frac{1-r}{1+r}\right)^2, \tag{50}$$

with equality for $z = -r$. For this z, the condition $\operatorname{Re}(z + 1/z) \leq 2$ is satisfied.

Now, suppose that $r \in (2-\sqrt{3}, 1)$. The real part of $\frac{z}{(1-z)^2}$ for $z = re^{i\varphi}$ can be written as a function $h(\cos\varphi)$, $\varphi \in [0, 2\pi]$, where $h(x) = \frac{r(1+r^2)x-2r^2}{(1-2rx+r^2)^2}$. If $r \in (2-\sqrt{3}, 1)$, one can check that

$$\min\{h(x) : x \in [-1, 1]\} = h(x_0) = -\frac{(1+r^2)^2}{8(1-r^2)^2},$$

where

$$x_0 = -\frac{1-6r^2+r^4}{2r(1+r^2)}.$$

Thus

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \geq \begin{cases} 1 - \frac{(1+r^2)^2}{2(1-r^2)^2} & \text{for} \quad \operatorname{Re}(z+1/z) \leq 2, \\ 1 & \text{for} \quad \operatorname{Re}(z+1/z) \geq 2. \end{cases}$$

Consequently

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \geq \frac{1-6r^2+r^4}{2(1-r^2)^2}, \tag{51}$$

with equalities for points $z_0 = re^{i\varphi_0}$ and $\overline{z_0}$, where $\varphi_0 = \arccos x_0$. Furthermore,

$$\operatorname{Re}(z_0 + 1/z_0) - 2 = (1/r + r)\cos\varphi_0 - 2 = (1/r + r)x_0 - 2 = -\frac{(1-r^2)^2}{2r^2}.$$

The condition $\operatorname{Re}(z_0 + 1/z_0) \leq 2$ is satisfied in this case also.

Combining (50) and (51), we get

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \geq \begin{cases} \left(\frac{1-r}{1+r}\right)^2 & \text{for} \quad r \in (0, 2-\sqrt{3}], \\ \frac{1-6r^2+r^4}{2(1-r^2)^2} & \text{for} \quad r \in [2-\sqrt{3}, 1). \end{cases} \tag{52}$$

In the first case, substituting $\left(\frac{1-r}{1+r}\right)^2$ by α, we obtain $r = \frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}$. The condition $r \in (0, 2-\sqrt{3}]$ is equivalent to $\alpha \in [1/3, 1)$.

While discussing the second possibility in (52), we should remember that the radius of starlikeness in X' is equal to $\sqrt{2}-1$. For this reason, we substitute $\frac{1-6r^2+r^4}{2(1-r^2)^2} = \alpha$ only for $r \in [2-\sqrt{3}, \sqrt{2}-1]$. This results in $r = \sqrt{\frac{2}{1-2\alpha}} - \sqrt{\frac{1+2\alpha}{1-2\alpha}}$ and $\alpha \in [0, 1/3]$.

The bound in (52) is sharp; equality holds for f satisfying $\frac{zf'(z)}{f(z)} = \frac{z}{(1-z)^2}$, so for f_1. □

The proof of Theorem 9 is easier. In fact, we need the condition for strong starlikeness and inequality (46). Thus we obtain

$$2\arctan\frac{2r}{1-r^2}\le\frac{\pi}{2}\alpha,$$

and hence

$$r^2+2r\cot\left(\frac{\pi}{4}\alpha\right)-1\le 0.$$

Solving this inequality with respect to r, the assertion of Theorem 9 follows.

The next theorem is concerned with the problem of convexity of a function in X'.

Theorem 10 *The radius of convexity for X' is equal to $r_{CV}(X')=r_0$, where $r_0=0.139\ldots$ is the only solution of equation $1-7r-r^2-r^3=0$. The extremal function is f_1.*

In the proof of this theorem, we need the following result of Todorov for $h\in T$ (see [10]):

$$\operatorname{Re}\frac{zh'(z)}{h(z)}\ge\begin{cases}\frac{1-r}{1+r}, & 0\le r\le 2-\sqrt{3},\\ \frac{1-6r^2+r^4}{1-r^4}, & 2-\sqrt{3}\le r<1.\end{cases}\tag{53}$$

Proof From (2), if $f\in X'$ then $\frac{zf'(z)}{f(z)}=(1+z)^2\frac{h(z)}{z}$, where $h\in T$. Hence

$$1+\frac{zf''(z)}{f'(z)}=\frac{zf'(z)}{f(z)}+\frac{zh'(z)}{h(z)}-\frac{1-z}{1+z}.\tag{54}$$

In further calculation, we shall apply Lemma 2.

Let $r\le 2-\sqrt{3}$. For z such that $\operatorname{Re}(z+1/z)\ge 2$,

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)\ge 1+\operatorname{Re}\left(\frac{zh'(z)}{h(z)}-\frac{1-z}{1+z}\right)=\operatorname{Re}\left(\frac{zh'(z)}{h(z)}+\frac{2z}{1+z}\right).$$

Estimate (53) and the inequality $\operatorname{Re}\frac{2z}{1+z}\ge-\frac{2r}{1-r}$ result in

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)\ge\frac{1-4r-r^2}{1-r^2}.$$

This estimate is not sharp because equalities in the two previous inequalities appear only if $z=-r$, but in this case $\operatorname{Re}(z+1/z)<2$. From the above, we conclude that if $\operatorname{Re}(z+1/z)\ge 2$ and $r\in[0,\sqrt{5}-2)$ then $\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>0$.

Assume now that $\mathrm{Re}(z+1/z) \le 2$. In this case

$$\begin{aligned}\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) &\ge \mathrm{Re}\left(\left(\frac{1+z}{1-z}\right)^2+\frac{zh'(z)}{h(z)}-\frac{1-z}{1+z}\right)\\ &= \mathrm{Re}\,\frac{zh'(z)}{h(z)}+\mathrm{Re}\,\frac{2z}{1+z}+\mathrm{Re}\,\frac{4z}{(1-z)^2}.\end{aligned} \tag{55}$$

The first two components can be estimated as above. Based on (49), the third one is greater than or equal to $\frac{-4r}{(1+r)^2}$. Since each estimate is sharp (with equality for $z=-r$), the estimate of the expression $\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)$ is also sharp. Consequently,

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) \ge \frac{1-7r-r^2-r^3}{(1+r)^2(1-r)}.$$

The function in the numerator of the right-hand side of this inequality is decreasing for $t \in \mathbb{R}$. For this reason, it has in $(0,1)$ the only solution r_0. We have proven that if $\mathrm{Re}(z+1/z) \le 2$ and $r \in [0, r_0]$ then $\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) \ge 0$. But $r_0 < \sqrt{5}-2$.

Taking into account both parts of the proof, we obtain the assertion. Equality in (53) holds for $h(z) = \frac{z}{(1-z)^2}$ and $z=-r$. It means that (55) is sharp, with equality for f_1 and $z=-r$. □

6 Univalence

The problems of the univalence of functions in X' are more complicated. Based on the already proved results, we know that the radius of univalence $r_S(X')$ is greater than or equal to $\sqrt{2}-1$. On the other hand, one can easily find the upper estimate of $r_S(X')$. Namely, discuss a function $F(z) = \frac{1}{r_*} f_1(r_* z)$, where f_1 is given by (10) and r_* is equal to $r_S(X')$ which we want to derive. The function F is univalent in Δ and it has normalization $F(0) = F'(0)-1 = 0$. From (31) it follows that $F(z) = z+4r_* z^2+\ldots$. The estimate of the second coefficient of functions in S results in $r_* \le 1/2$.

The main theorem of this section is as follows.

Theorem 11 *The radius of univalence in X' is equal to $r_S(X') = r_1$, where $r_1 = 0.454\ldots$ is the only solution of equation*

$$\arcsin\frac{1-r^2}{2r} + \frac{2(1-r^2)}{1+r^2-\sqrt{-1+6r^2-r^4}} = \pi \tag{56}$$

in $(\sqrt{2}-1, 1)$. The extremal function is f_1.

In the proof of this theorem we need two lemmas.

Lemma 3 *For each f_t, $t \in [-1,1]$ given by (9) and for each $r \in (0, \sqrt{3}/3)$ and $\varphi \in [0,\pi]$ the following inequality is true:*

$$\arg f_t(re^{i\varphi}) \le \arg f_1(re^{i\varphi}). \tag{57}$$

Lemma 4 *The function f_1 is univalent in the disk $|z| < r_1$, where r_1 is the only solution of* (56).

Proof of Lemma 3 Let $t \in [-1, 1]$ and $r \in (0, \sqrt{3}/3)$ be fixed. Let us denote by $g(\psi)$ the argument of $f_t(re^{i\varphi})$ with a fixed $\varphi \in [0, \pi]$, where ψ and t are connected by $t = \cos\psi$. Applying (9), g can be written as

$$g(\psi) = \varphi + \cot\frac{\psi}{2} k(\psi),$$

where

$$k(\psi) = \log\left|\frac{1 - re^{i(\varphi+\psi)}}{1 - re^{i(\varphi-\psi)}}\right|.$$

Since

$$\left|\frac{1 - re^{i(\varphi+\psi)}}{1 - re^{i(\varphi-\psi)}}\right| \ge 1,$$

for φ and ψ in $[0, \pi]$, we conclude that $k(\psi) \ge 0$ for all $\psi \in [0, \pi]$.

Now, we shall show that $g(\psi)$ is a decreasing function of the variable ψ. We have

$$g'(\psi) = \frac{-1}{2\sin^2\frac{\psi}{2}} \cdot h(\psi),$$

where

$$h(\psi) = k(\psi) - k'(\psi)\sin\psi.$$

A long and tedious calculation shows that

$$h'(\psi) = \frac{\sin\varphi(1 - \cos\psi)}{[(q\cos\psi - \cos\varphi)^2 + (q^2 - 1)\sin^2\psi]^2} \cdot W(\psi),$$

with $q = (1 + r^2)/2r$, $q > 1$ and

$$W(\psi) = [(1 - 2q^2)\cos\varphi - q]\cos^2\psi + 2[q\cos^2\varphi + \cos\varphi + q(q^2 - 1)]\cos\psi \\ - \cos^3\varphi - q\cos^2\varphi - (q^2 - 1)\cos\varphi + q(q^2 - 1).$$

Hence

$$W(0) = (q - \cos\varphi)(3q^2 - 4 + \cos^2\varphi) \quad \text{and} \quad W(\pi) = -(q + \cos\varphi)^3.$$

It is obvious that $W(\pi) < 0$. On the other hand, $W(0) > 0$, providing that $r \in (0, \sqrt{3}/3)$, or equivalently, $q^2 > 4/3$. From these observations, taking into account that W is a quadratic function of $\cos\psi$, we can see that $W(\psi)$ has exactly one solution in $[0, \pi]$; let us denote it by ψ_0. Hence, $h'(\psi)$ has only one solution ψ_0 in $(0, \pi)$. Thus, $h'(\psi)$ for $\psi \in (0, \psi_0)$ increases, and for $\psi \in (\psi_0, \pi)$ decreases. Combining it with $h(0) = h(\pi) = 0$, we obtain $h(\psi) \geq 0$ for $\psi \in [0, \pi]$. This implies $g'(\psi) \leq 0$ for $\psi \in (0, \pi)$; so $g(\psi)$ is decreasing in $(0, \pi)$.

Finally,

$$g(\psi) \leq g(0^+) \quad \text{for all} \quad \psi \in [0, \pi],$$

where

$$g(0^+) = \lim_{\psi\to 0^+} g(\psi) = \varphi + \lim_{\psi\to 0^+} \frac{k(\psi)}{\tan\frac{\psi}{2}} = \varphi + 2k'(0) = \varphi + \frac{2\sin\varphi}{q - \cos\varphi}.$$

Moreover, if $\varphi = 0$ then $g(\psi) = 0$, and, if $\varphi = \pi$ then $g(\psi) = \pi$ for all $\psi \in [0, \pi]$. Consequently, (57) holds also for $\varphi = 0$ and $\varphi = \pi$. □

Proof of Lemma 4 Consider a level curve $f_1(\{z \in \mathbb{C} : |z| = r\})$ with a fixed $r \in (0, 1)$. Since f_1 is a circularly symmetric function, $f_1(re^{i\varphi})$ decreases for φ from 0 to π. Hence, f_1 is univalent when the level curves has no self-intersection points. It happens at small r, ie. when $r < \sqrt{2} - 1$, because f_1 is starlike in this case. So it is enough to discuss for which $r \in [\sqrt{2} - 1, 1/2]$ the level curve $f_1(\{z \in \mathbb{C} : |z| = r\})$ is tangent to the real axis. Denoting the point of tangency by w_0, and denoting by $z_0 = re^{i\varphi_0}$ the corresponding point on circle $|z| = r$ for which $f(z_0) = w_0$, we obtain $\arg f(z_0) = \pi$.

The tangency of the level curve to the real axis in w_0 ensures that $\arg f_1(re^{i\varphi})$ is increasing for $\varphi \in (0, \varphi_0)$, decreasing for $\varphi \in (\varphi_0, \varphi_1)$, and once again increasing for $\varphi \in (\varphi_1, \pi)$, where φ_1 is a number from the interval (φ_0, π). Hence, $\mathrm{Re}\, \frac{z_0 f_1'(z_0)}{f_1(z_0)} = 0$.

For this reason, we need to solve the system

$$\begin{cases} \mathrm{Re}\, \frac{z_0 f_1'(z_0)}{f_1(z_0)} = 0, \\ \arg f_1(z_0) = \pi. \end{cases} \tag{58}$$

The first equation can be written, using (8), as $\mathrm{Re}\left(\frac{1+z_0}{1-z_0}\right)^2 = 0$. Since $z_0 = re^{i\varphi}$, we obtain

$$\varphi_0 = \arcsin\frac{1 - r^2}{2r}. \tag{59}$$

But

$$f_1(re^{i\varphi_0}) = re^{i\varphi_0} \exp\left(\frac{4re^{i\varphi_0} - 4r^2}{1 - 2r\cos\varphi_0 + r^2}\right),$$

so

$$\arg f_1(re^{i\varphi_0}) = \varphi_0 + \frac{4r\sin\varphi_0}{1-2r\cos\varphi_0 + r^2},$$

which, together with the second equation of (58), proves that r is a solution of (56).

Finally, it can be observed that the right-hand side of the equality given above, let us denote it by $b(r)$, satisfies

$$b'(r) = \frac{(1+r^2)(-1+6r^2-r^4) + 8r^2\sqrt{-1+6r^2-r^4} + 4r^2(1+r^2)}{r(1-r^2)^2\sqrt{-1+6r^2-r^4}}.$$

This means that $b(r)$ increases from $\pi/2+\sqrt{2}$ to infinity, while $r \in (\sqrt{2}-1, 1)$. For this reason, (56) has only one solution. □

Proof of Theorem 11 Let $L = \{\log \frac{f(z)}{z}, f \in X'\}$. In the paper [9], the authors showed that the extreme points of the class L are as follows:

$$l_\psi(z) = i\cot\frac{\psi}{2}\log\frac{1-ze^{i\psi}}{1-ze^{-i\psi}}, \quad \psi \in [0,\pi].$$

A functional $L \ni l \to \mathrm{Im}(l(z))$ is linear, so for a fixed $z \in \Delta$, there is

$$\max\{\mathrm{Im}\, l(z) : l \in L\} = \max\left\{\mathrm{Im}\, l_\psi(z) : \psi \in [0,\pi]\right\}. \tag{60}$$

But $\mathrm{Im}\, l(z) = \arg\frac{f(z)}{z}$ for $l \in L$. Therefore

$$\max\left\{\arg\frac{f(z)}{z} : f \in X'\right\} = \max\left\{\arg\frac{f_t(z)}{z} : t \in [-1,1]\right\}, \tag{61}$$

where f_t is given by (9). Hence, for $z \in \Delta\backslash\{0\}$,

$$\max\left\{\arg f(z) : f \in X'\right\} = \max\{\arg f_t(z) : t \in [-1,1]\}. \tag{62}$$

Applying Lemma 3, we conclude that for every $f \in X'$, $r \in (0, \sqrt{3}/3)$ and $\varphi \in [0,\pi]$, the following inequality holds:

$$\arg f(re^{i\varphi}) \le \arg f_1(re^{i\varphi}). \tag{63}$$

Consequently, for every function $f \in X'$, from $\left|\arg f_1(re^{i\varphi})\right| \le \pi$, it yields that $\left|\arg f(re^{i\varphi})\right| \le \pi$, which combined with Lemma 4 gives the assertion. □

In the paper [4], the class $\mathcal{T}$ of semi-typically real functions was defined. Namely, $f \in \mathcal{T}$ if

$$z \in (0,1) \quad \text{if and only if} \quad f(z) > 0.$$

This equivalence means that the values of f belonging to $\mathcal{T}$ are positive real numbers if and only if $z \in \Delta$ is positive and real. According to this definition, $T \subset \mathcal{T}$.

Based on the proof of Theorem 11, one can anticipate that functions $f \in X'$ are semi-typically real at most in the disk with radius $r_{\mathcal{T}}$. The number $r_{\mathcal{T}}$ is chosen such that the level curves $f(\{z \in \mathbb{C} : |z| = r\})$ for $r < r_{\mathcal{T}}$ and $f \in X'$ may wind around the origin, yet they do not touch the positive real halfaxis. Moreover, one can anticipate that the extremal function is still f_1.

Conjecture. The radius of semi-typical reality in X' is equal to $r_{\mathcal{T}}(X') = r_2$, where $r_2 = 0.718\ldots$ is the only solution of equation

$$\arcsin \frac{1-r^2}{2r} + \frac{2(1-r^2)}{1+r^2-\sqrt{-1+6r^2-r^4}} = 2\pi. \tag{64}$$

It is worth emphasizing that in the proof of Lemma 3 we did apply the assumption $r \in (0, \sqrt{3}/3)$, which is equivalent to $q^2 > 4/3$. The number $\sqrt{3}/3$ in this expression is not necesserily sharp. Hence, the argument given in the proof of Theorem 11 is not sufficient to prove this conjecture.

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