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A characterization of ordinary abelian varieties by the Frobenius push-forward of the structure sheaf

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Abstract For an ordinary abelian variety X, $F^e_*\mathcal{O}_X$ is decomposed into line bundles for every positive integer e. Conversely, if a smooth projective variety X satisfies this property and the Kodaira dimension of X is non-negative, then X is an ordinary abelian variety.

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Contents

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1 Introduction

Let k be an algebraically closed field of characteristic $p > 0$. Let X be a smooth proper variety over k. When does X satisfy the following property $(*)$?

$(*)$ $F_*\mathcal{O}_X \simeq \bigoplus_j M_j$ where $F : X \to X$ is the absolute Frobenius morphism and each M_j is a line bundle.

For example, an arbitrary smooth proper toric variety satisfies this property $(*)$ (cf. [1,19]). Thus there are many varieties which satisfy $(*)$. But every toric variety has negative Kodaira dimension. On the other hand, we show that ordinary abelian varieties satisfy $(*)$. The main theorem of this paper is the following inverse result.

Theorem 1.1 (Theorem 4.7) *Let k be an algebraically closed field of characteristic $p > 0$. Let X be a smooth projective variety over k. Assume the following conditions.*

- *For infinitely many $e \in \mathbb{Z}_{>0}$, $F^e_*\mathcal{O}_X \simeq \bigoplus_j M_j^{(e)}$ where each $M_j^{(e)}$ is an invertible sheaf.*
- *K_X is pseudo-effective (e.g. the Kodaira dimension of X is non-negative).*

Then X is an ordinary abelian variety.

On the other hand, how about the opposite problem? More precisely, when does X satisfy the following property $(**)$?

$(**)$ $F_*\mathcal{O}_X$ is indecomposable, that is, if $F_*\mathcal{O}_X = E_1 \oplus E_2$ holds for some coherent sheaves E_1 and E_2, then $E_1 = 0$ or $E_2 = 0$.

We study this problem for abelian varieties and curves.

Theorem 1.2 (Theorem 5.3) *Let k be an algebraically closed field of characteristic $p > 0$. Let X be an abelian variety over k. Set r_X to be the p-rank of X. Then, for every $e \in \mathbb{Z}_{>0}$,*

$$F^e_*\mathcal{O}_X \simeq E_1 \oplus \cdots \oplus E_{p^{er_X}}$$

where each E_i is an indecomposable locally free sheaf of rank $p^{e(\dim X - r_X)}$. In particular, $F^e_*\mathcal{O}_X$ is indecomposable if and only if $r_X = 0$.

Theorem 1.3 (Theorem 5.5) *Let k be an algebraically closed field of characteristic $p > 0$. Let X be a smooth projective curve of genus g. Fix an arbitrary integer $e \in \mathbb{Z}_{>0}$. Then the following assertions hold.*

(0) *If $g = 0$, then $F^e_*\mathcal{O}_X \simeq \bigoplus L_j$ where every L_j is a line bundle.*
(1or) *If $g = 1$ and X is an ordinary elliptic curve, then $F^e_*\mathcal{O}_X \simeq \bigoplus L_j$ where every L_j is a line bundle.*
(1ss) *If $g = 1$ and X is a supersingular elliptic curve, then $F^e_*\mathcal{O}_X$ is indecomposable.*
(2) *If $g \geq 2$, then $F^e_*\mathcal{O}_X$ is indecomposable.*

By Theorem 1.3(2), it is natural to ask whether $F_*\mathcal{O}_X$ is indecomposable for every smooth projective variety of general type X. If we drop the assumption that X is smooth, then the following theorem gives a negative answer to this question.

Theorem 1.4 *Let k be an algebraically closed field of characteristic $p > 0$. Then, there exists a projective normal surface X over k which satisfies the following properties.*

(1) *The singularities of X are at worst canonical.*
(2) *K_X is ample.*
(3) *$F_*\mathcal{O}_X$ is not indecomposable.*

Remark 1.5 By [17], if X is a smooth projective curve of genus $g \geq 2$, then F_*E is a stable vector bundle whenever so is E. Theorem 1.4 shows that there exists a projective normal canonical surface of general type X such that $F_*\mathcal{O}_X$ is not a stable vector bundle with respect to an arbitrary ample invertible sheaf H on X.

Proof of Theorem 1.1: We overview the proof of Theorem 1.1. First of all, we can show that X is F-split, that is, $\mathcal{O}_X \to F_*\mathcal{O}_X$ splits as an $\mathcal{O}_X$-module homomorphism. This implies

$$H^0(X, -(p-1)K_X) \neq 0.$$

Since K_X is pseudo-effective, we obtain $(p-1)K_X \sim 0$. Then, by [6], we see that the Albanese map $\alpha : X \to \mathrm{Alb}(X)$ is surjective. There are two main difficulties as follows.

(1) To show that α is generically finite.
(2) To treat the case where α is a finite surjective inseparable morphism.

(1) Let us overview how to show that α is generically finite. Set r_X to be the p-rank of $\mathrm{Alb}(X)$. It suffices to show $\dim X = r_X$. Note that $\alpha : X \to \mathrm{Alb}(X)$ induces the following bijective group homomorphism:

$$\alpha^* : \mathrm{Pic}^0(\mathrm{Alb}(X)) \xrightarrow{\simeq} \mathrm{Pic}^0(X), \quad L \mapsto \alpha^* L.$$

Roughly speaking, since $\mathrm{Pic}^\tau(X)/\mathrm{Pic}^0(X)$ is a finite group, r_X can be calculated by the asymptotic behavior of the number of p^e-torsion line bundles on X. Thus, we count the number of p^e-torsion line bundles on X. More precisely, we prove that the number of p^e-torsion line bundles on X is $p^{e \dim X}$ for infinitely many e.

Now we have

$$F^e_*\mathcal{O}_X = \bigoplus_{1 \leq j \leq p^{e \dim X}} M_j$$

where each M_j is a line bundle. In our situation, we can show that every p^e-torsion line bundle L is isomorphic to some M_j (cf. Lemma 3.3). Therefore, it suffices to prove that each M_j is p^e-torsion. Tensor M_j^{-1} with the above equation and take H^0. Then, we obtain $H^0(X, M_j^{-p^e}) \neq 0$. If we have $H^0(X, M_j^{p^e}) \neq 0$, then we are done. For this, we take the duality, that is, apply $\mathcal{H}om_{\mathcal{O}_X}(-, \omega_X)$ to the above direct summand decomposition. Then we can also show $H^0(X, M^{p^e}) \neq 0$. For more details on this argument, see Lemma 4.4.

(2) We overview how to treat the inseparable case. To clarify the idea, we assume that α is a finite surjective purely inseparable morphism of degree p. Then, Frobenius map F_A of A factors through α:

$$F_A : A \to X \stackrel{\alpha}{\to} A.$$

By using the fact that X is F-split, we can show that

$$(F_A)_*\mathcal{O}_A \simeq \alpha_*\mathcal{O}_X \oplus E$$

for some coherent sheaf E. Since $(F_A)_*\mathcal{O}_A$ is the direct sum of the p-torsion line bundles, we obtain

$$\alpha_*\mathcal{O}_X \simeq \bigoplus_{j=1}^{p} M_j$$

where $M_1, \ldots, M_p$ are mutually distinct p-torsion line bundles. One of them, say M_1, satisfies $\alpha^*M_1 \not\simeq \mathcal{O}_X$. By tensoring M_1^{-1}, we obtain

$$\alpha_*(\alpha^*M_1^{-1}) \simeq \mathcal{O}_A \oplus \bigoplus_{j=2}^{p}(M_j \otimes_{\mathcal{O}_A} M_1^{-1})$$

which induces the following contradiction:

$$0 = H^0(X, \alpha^*M_1^{-1}) \simeq H^0(A, \mathcal{O}_A) \oplus H^0\left(A, \bigoplus_{j=2}^{p}\left(M_j \otimes_{\mathcal{O}_A} M_1^{-1}\right)\right) \neq 0.$$

In the proof of Theorem 1.1, there appear other technical difficulties. For more details on the inseparable case, see Step 5 of the proof of Theorem 4.7.

Related results:

(1) In [6], Hacon and Patakfalvi give a characterization of the varieties birational to ordinary abelian varieties.
(2) Achinger [1] gives a characterization of smooth projective toric varieties as follows. For a smooth projective variety X in positive characteristic, X is toric if and only if F_*L splits into line bundles for every line bundle L.

2 Preliminaries

2.1 Notation

We will not distinguish the notations line bundles, invertible sheaves and Cartier divisors. For example, we will write $L + M$ for line bundles L and M.

Throughout this paper, we work over an algebraically closed field k of characteristic $p > 0$. For example, a projective scheme means a scheme which is projective over k.

Let X be a noetherian scheme. For a coherent sheaf F on X and a line bundle L on X, we define $F(L) := F \otimes_{\mathcal{O}_X} L$.

In this paper, a *variety* means an integral scheme which is separated and of finite type over k. A *curve* or a *surface* means a variety whose dimension is one or two, respectively.

For a proper scheme X, let $\mathrm{Pic}(X)$ be the group of line bundles on X and let $\mathrm{Pic}^0(X)$ (resp. $\mathrm{Pic}^\tau(X)$) be the subgroup of $\mathrm{Pic}(X)$ of line bundles which are algebraically (resp. numerically) equivalent to zero:

$$\mathrm{Pic}^0(X) \subset \mathrm{Pic}^\tau(X) \subset \mathrm{Pic}(X).$$

For a normal variety X and a coherent sheaf M on X, we say M is *reflexive* if the natural map $M \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(M, \mathcal{O}_X), \mathcal{O}_X)$ is an isomorphism. We say M is *divisorial* if M is reflexive and $M|_{\mathcal{O}_{X,\xi}}$ is rank one where ξ is the generic point. It is well-known that a divisorial sheaf M is isomorphic to the sheaf $\mathcal{O}_X(D)$ associated to a Weil divisor D on X.

Let X be a scheme of finite type over k. We say X is *F-split* if the absolute Frobenius

$$\mathcal{O}_X \to F_*\mathcal{O}_X, \quad a \mapsto a^p$$

splits as an $\mathcal{O}_X$-module homomorphism.

We say a coherent sheaf F is *indecomposable* if, for every isomorphism $F \simeq F_1 \oplus F_2$ with coherent sheaves F_1 and F_2, we obtain $F_1 = 0$ or $F_2 = 0$.

We recall the definition of ordinary abelian varieties.

Definition-Proposition 2.1 Let X be an abelian variety. We say X is *ordinary* if one of the following conditions hold. Moreover, the following conditions are equivalent.

(1) For some $e \in \mathbb{Z}_{>0}$, the number of p^e-torsion points is $p^{e \cdot \dim X}$.
(2) For every $e \in \mathbb{Z}_{>0}$, the number of p^e-torsion points is $p^{e \cdot \dim X}$.
(3) $F : H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$ is bijective.
(4) $F : H^i(X, \mathcal{O}_X) \to H^i(X, \mathcal{O}_X)$ is bijective for every $i \geq 0$.
(5) X is F-split.

Proof (1) and (2) are equivalent by [13, Section 15, The$p$-rank]. (2) and (3) are equivalent by [13, Section 15, Theorem 3]. (Note that, in older editions of [13], there are two Theorem 2 in Section 15.) (3) and (4) are equivalent by [14, Example 5.4]. (4) and (5) are equivalent by [12, Lemma 1.1]. □

2.2 Albanese varieties

In this subsection, we recall the definition and fundamental properties of the Albanese varieties. For details, see [4, Section 9].

For a projective normal variety X and a closed point $x \in X$, there uniquely exists a morphism $\alpha_X : X \to \mathrm{Alb}(X)$ to an abelian variety $\mathrm{Alb}(X)$, called the *Albanese variety* of X, such that $\alpha_X(x) = 0$ and that every morphism to an abelian variety $g : X \to B$, with $g(x) = 0_B$, factors through α_X (cf. [4, Remark 9.5.25]). Note that $\mathrm{Alb}(X) \simeq \underline{\mathrm{Pic}}^0(\underline{\mathrm{Pic}}^0(X)_{\mathrm{red}})$, where $\underline{\mathrm{Pic}}(X) := \mathbf{Pic}_{X/k}$ in the sense of [4, Section 9].

The Albanese morphism $\alpha_X : X \to \mathrm{Alb}(X)$ induces a natural morphism

$$\alpha_X^* : \underline{\mathrm{Pic}}^0(\mathrm{Alb}(X)) \to \underline{\mathrm{Pic}}^0(X)_{\mathrm{red}}.$$

It is well-known that α_X^* is an isomorphism. In particular, the induced group homomorphism

$$\alpha_X^* : \mathrm{Pic}^0(\mathrm{Alb}(X)) \to \mathrm{Pic}^0(X)$$

is bijective.

2.3 The number of p^e-torsion line bundles

The asymptotic behavior of the number of p^e-torsion line bundles is determined by the p-rank of the Picard variety $\mathrm{Pic}^0(X)_{\mathrm{red}}$.

Proposition 2.2 *Let X be a projective normal variety. Then, the following assertions hold.*

(1) *There exists the following exact sequence*

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}^\tau(X) \to G(X) \to 0$$

where $G(X)$ is a finite group.

(2) *If r_X is the p-rank of $\underline{\mathrm{Pic}}^0(X)_{\mathrm{red}}$, then there exists $\xi \in \mathbb{Z}_{>0}$ such that*

$$p^{er_X} \leq |\mathrm{Pic}(X)[p^e]| \leq p^{er_X} \times \xi$$

for every $e \in \mathbb{Z}_{>0}$ where $\mathrm{Pic}(X)[p^e]$ is the group of p^e-torsion line bundles.

Proof The assertion (1) holds by [4, 9.6.17]. The assertion (2) follows from (1). □

As a consequence, we see that the p-rank of the Picard variety is stable under purely inseparable covers.

Proposition 2.3 *Let $f : X \to Y$ be a finite surjective purely inseparable morphism between projective normal varieties. Set r_X and r_Y to be the p-ranks of $\underline{\mathrm{Pic}}^0(X)_{\mathrm{red}}$ and $\underline{\mathrm{Pic}}^0(Y)_{\mathrm{red}}$, respectively. Then, $r_X = r_Y$.*

Proof We may assume that $[K(X) : K(Y)] = p$. Then, the absolute Frobenius morphism $F : Y \to Y$ factors through $f : X \to Y$:

$$F : Y \xrightarrow{g} X \xrightarrow{f} Y.$$

Thus, it suffices to show $r_Y \leq r_X$.

We show that the following inequality

$$p^{er_Y} \leq |\mathrm{Pic}(X)[p^{e+1}]|$$

holds for every $e \in \mathbb{Z}_{>0}$. Fix $e \in \mathbb{Z}_{>0}$. Let $L_1, \ldots, L_{p^{er_Y}}$ be mutually distinct p^e-torsion line bundles in $\mathrm{Pic}^0(Y)$. Then, since $\underline{\mathrm{Pic}}^0(Y)_{\mathrm{red}}$ is an abelian variety, we can find line bundles $M_1, \ldots, M_{p^{er_Y}}$ such that $M_j^p \simeq L_j$ for every $1 \leq j \leq p^{er_Y}$. We see that, for each j,

$$L_j \simeq M_j^p = F^* M_j \simeq g^* f^* M_j$$

and that $f^* M_1, \ldots, f^* M_{p^{er_Y}}$ are mutually distinct p^{e+1}-torsion line bundles on X. Thus, we obtain the required inequality $p^{er_Y} \leq |\mathrm{Pic}(X)[p^{e+1}]|$.

By Proposition 2.2(2), we can find $\xi \in \mathbb{Z}_{>0}$ such that the inequalities

$$p^{er_Y} \leq |\mathrm{Pic}(X)[p^{e+1}]| \leq p^{(e+1)r_X} \times \xi$$

hold for every $e \in \mathbb{Z}_{>0}$. By taking the limit $e \to \infty$, we obtain $r_Y \leq r_X$. □

3 Basic properties

In the main theorem (Theorem 1.1), we treat varieties such that $F_*^e \mathcal{O}_X$ is decomposed into line bundles. In this section, we summarize basic properties of such varieties. Since such varieties are F-split (Lemma 3.2), we also study F-split varieties. First, we give characterizations of F-split varieties.

Lemma 3.1 *Let X be a scheme of finite type over k. Then, the following assertions are equivalent.*

(1) *X is F-split.*
(2) *For every $e \in \mathbb{Z}_{>0}$, there exists a coherent sheaf E such that $F_*^e \mathcal{O}_X \simeq \mathcal{O}_X \oplus E$.*
(3) *$F_*^e \mathcal{O}_X \simeq \mathcal{O}_X \oplus E$ for some $e \in \mathbb{Z}_{>0}$ and coherent sheaf E.*
(4) *$F_*^e \mathcal{O}_X \simeq L \oplus E$ for some $e \in \mathbb{Z}_{>0}$, p^e-torsion line bundle L and coherent sheaf E.*

Proof It is well-known that (1), (2) and (3) are equivalent. It is clear that (3) implies (4). We see that (4) implies (3) by tensoring L^{-1} with $F_*^e \mathcal{O}_X \simeq L \oplus E$. □

We are interested in varieties such that $F_*^e \mathcal{O}_X$ is decomposed into line bundles. By the following lemma, such varieties are F-split.

Lemma 3.2 *Let X be a proper normal variety. Assume that $F_*^e\mathcal{O}_X \simeq \bigoplus_j M_j$ for some $e \in \mathbb{Z}_{>0}$ and divisorial sheaves M_j. Then, X is F-split.*

Proof We obtain the following

$$0 \neq H^0(X, F_*^e\mathcal{O}_X) \simeq \bigoplus_j H^0(X, M_j).$$

Therefore, we see $H^0(X, M_{j_0}) \neq 0$ for some j_0.

We have $M_{j_0} \simeq \mathcal{O}_X(E)$ for some effective divisor E on X. By Lemma 3.1, it is enough to show $E = 0$. Tensor $\mathcal{O}_X(-E)$ with

$$F_*^e\mathcal{O}_X \simeq \bigoplus_j M_j \simeq \mathcal{O}_X(E) \oplus \left(\bigoplus_{j\neq j_0} M_j\right)$$

and take the double dual. We obtain the following decomposition:

$$F_*^e(\mathcal{O}_X(-p^eE)) \simeq \mathcal{O}_X \oplus \left(\bigoplus_{j\neq j_0} \left(M_j \otimes_{\mathcal{O}_X} (-E)\right)^{**}\right).$$

Thus, $H^0(X, \mathcal{O}_X(-p^eE)) \neq 0$. This implies $E = 0$. □

The following result gives an upper bound of the number of p^e-torsion line bundles for F-split varieties.

Lemma 3.3 *Let X be a proper variety. Assume that X is F-split. Fix $e \in \mathbb{Z}_{>0}$. Let $F_*^e\mathcal{O}_X \simeq \bigoplus_{j\in J} M_j$ be a decomposition into indecomposable coherent sheaves M_j. Then, the following assertions hold.*

(1) *Let L be a line bundle with $L^{p^e} \simeq \mathcal{O}_X$. Then, $L \simeq M_{j_1}$ for some $j_1 \in J$.*
(2) *Let $j_1, j_2 \in J$ with $j_1 \neq j_2$. If M_{j_1} and M_{j_2} are line bundles and satisfy $M_{j_1}^{p^e} \simeq \mathcal{O}_X$ and $M_{j_2}^{p^e} \simeq \mathcal{O}_X$, then $M_{j_1} \not\simeq M_{j_2}$.*
(3) *The number of p^e-torsion line bundles on X is at most $p^{e\cdot\dim X}$.*

Proof (1) Tensor L^{-1} with $F_*^e\mathcal{O}_X \simeq \bigoplus_j M_j$ and we obtain

$$F_*^e\mathcal{O}_X \simeq F_*^e(L^{-p^e}) \simeq F_*^e\mathcal{O}_X \otimes_{\mathcal{O}_X} L^{-1} \simeq \bigoplus_j \left(M_j \otimes_{\mathcal{O}_X} L^{-1}\right).$$

Since X is F-split, we have

$$F_*^e\mathcal{O}_X \simeq \mathcal{O}_X \oplus \left(\bigoplus_i N_i\right)$$

where each N_i is an indecomposable sheaf. Then, the Krull–Schmidt theorem ([2, Theorem 2]) implies $M_{j_1} \otimes_{\mathcal{O}_X} L^{-1} \simeq \mathcal{O}_X$ for some j_1.

(2) Assume that, for some $j_1 \neq j_2$, M_{j_1} and M_{j_2} are line bundles such that $M_{j_1}^{p^e} \simeq \mathcal{O}_X$, $M_{j_2}^{p^e} \simeq \mathcal{O}_X$ and $M_{j_1} \simeq M_{j_2}$. Let us derive a contradiction. Tensor $M_{j_1}^{-1}$ and we obtain

$$F_*^e \mathcal{O}_X \simeq F_*^e(M_{j_1}^{-p^e}) \simeq \mathcal{O}_X \oplus \mathcal{O}_X \oplus \left(\bigoplus_{j \neq j_1, j_2} (M_j \otimes M_{j_1}^{-1}) \right).$$

Taking H^0, we obtain a contradiction.

(3) The assertion immediately follows from (1) and (2).

□

The following lemma is used in the next section and well-known for experts on F-singularities (cf. the proof of [16, Theorem 4.3]).

Lemma 3.4 *Let X be a smooth proper variety. Assume that X is F-split. Then, for every $e \in \mathbb{Z}_{>0}$,*

$$H^0(X, -(p^e - 1)K_X) \neq 0.$$

In particular, $\kappa(X) \leq 0$.

Proof By the Grothendieck duality, we can check

$$\mathcal{H}om_{\mathcal{O}_X}(F_*^e \mathcal{O}_X, \omega_X) \simeq F_*^e \omega_X.$$

This implies that ω_X is a direct summand of $F_*^e \omega_X$, which is equivalent to the assertion that $\mathcal{O}_X$ is a direct summand of $F_*^e(\omega_X^{1-p^e})$. □

4 A characterization of ordinary abelian varieties

In this section, we show the main theorem of this paper: Theorem 4.7. In the proof, we use [6, Theorem 1.1.1]. For this, it is necessary to show $\kappa_S(X) = 0$. We check this in Lemma 4.3. First, we recall the definition of $\kappa_S(X)$.

Definition 4.1 Let X be a smooth proper variety.

(1) Fix $m \in \mathbb{Z}_{>0}$. We define

$$S^0(X, mK_X) := \bigcap_{e \geq 0} \mathrm{Image} \left(\mathrm{Tr} : H^0(X, K_X + (m-1)p^e K_X) \to H^0(X, mK_X) \right).$$

where Tr is defined by the trace map $F_*^e \omega_X \to \omega_X$. For more details, see Remark 4.2 and [6, Lemma 2.2.3].

(2) We define

$$\kappa_S(X) := \max\{r \mid \dim S^0(X, mK_X) = O(m^r) \text{ for sufficiently divisible } m\}.$$

This definition is the same as the one of [6, Subsection 4.1].

Remark 4.2 The trace map $F^e_*\omega_X \to \omega_X$ in Definition 4.1 is obtained by applying the functor $\mathcal{H}om_{\mathcal{O}_X}(-, \omega_X)$ to the Frobenius $\mathcal{O}_X \to F^e_*\mathcal{O}_X$. Indeed, the Grothendieck duality implies $\mathcal{H}om_{\mathcal{O}_X}(F^e_*\mathcal{O}_X, \omega_X) \simeq F^e_*\omega_X$. Thus, we obtain the trace map $F^e_*\omega_X \to \omega_X$.

By the construction, if X is F-split, then the trace map $F^e_*\omega_X \to \omega_X$ is a split surjection. Therefore, in this case, $H^0(X, mK_X) \neq 0$ (resp. $\kappa(X) \geq 0$) implies $S^0(X, mK_X) \neq 0$ (resp. $\kappa_S(X) \geq 0$).

We check $\kappa_S(X) = 0$ to apply [6, Theorem 1.1.1] in the proof of Theorem 4.7.

Lemma 4.3 *Let X be a smooth projective variety. If X is F-split and K_X is pseudo-effective, then the following assertions hold.*

(1) $(p^e - 1)K_X \sim 0$ *for every* $e \in \mathbb{Z}_{>0}$.
(2) $\kappa_S(X) = 0$.

Proof (1) By Lemma 3.4, we obtain $-(p^e - 1)K_X \sim E$ where E is an effective divisor. Then, the pseudo-effectiveness of K_X implies that $E = 0$ (cf. [5, Lemma 5.4]).
(2) By (1), we obtain $\kappa(X) = 0$. By [6, Lemma 4.1.3], it suffices to show $\kappa_S(X) \geq 0$. By Remark 4.2, $\kappa(X) \geq 0$ implies $\kappa_S(X) \geq 0$.

□

The following lemma is a key to show Theorem 4.7.

Lemma 4.4 *Let X be a smooth projective variety. Fix $e \in \mathbb{Z}_{>0}$. Assume the following conditions.*

- $F^e_*\mathcal{O}_X \simeq \bigoplus_j M_j$ *where each M_j is a line bundle.*
- K_X *is pseudo-effective.*

Then, the following assertions hold.

(1) $M_j^{p^e} \simeq \mathcal{O}_X$ *for every* j.
(2) *The number of p^e-torsion line bundles on X is equal to* $p^{e\cdot\dim X}$.

Proof (1) By Lemma 3.2, X is F-split. Thus, Lemma 4.3 implies $(p^e - 1)K_X \sim 0$. Fix an index j_0 and we show $M_{j_0}^{p^e} \simeq \mathcal{O}_X$. We can write

$$F^e_*\mathcal{O}_X = M_{j_0} \oplus \left(\bigoplus_{j \neq j_0} M_j\right).$$

Tensor $M_{j_0}^{-1}$ and we obtain

$$H^0(X, M_{j_0}^{-p^e}) \simeq H^0(X, \mathcal{O}_X) \oplus \cdots .$$

In particular, we obtain $H^0(X, M_{j_0}^{-p^e}) \neq 0$. On the other hand, by applying $\mathcal{H}om_{\mathcal{O}_X}(-, \omega_X)$, we have

$$\begin{aligned} F_*^e \omega_X &\simeq \mathcal{H}om_{\mathcal{O}_X}(F_*^e \mathcal{O}_X, \omega_X) \\ &\simeq \mathcal{H}om_{\mathcal{O}_X}\left(\bigoplus_j M_j, \omega_X\right) \\ &\simeq \bigoplus_j (M_j^{-1} \otimes \omega_X) \end{aligned}$$

where the first isomorphism follows from the Grothendieck duality theorem for finite morphisms. Tensor ω_X^{-1} and we obtain

$$F_*^e \mathcal{O}_X \simeq F_*^e(\omega_X^{1-p^e}) \simeq (F_*^e \omega_X) \otimes_{\mathcal{O}_X} \omega_X^{-1} \simeq \bigoplus_j M_j^{-1}.$$

Then, tensor M_{j_0}, and we obtain $H^0(X, M_{j_0}^{p^e}) \neq 0$. Therefore, $M_{j_0}^{p^e} \simeq \mathcal{O}_X$.

(2) By Lemma 3.2, X is F-split. Then, the assertion follows from (1) and Lemma 3.3. □

Ordinary abelian varieties satisfy the condition that $F_*^e \mathcal{O}_X$ is decomposed into line bundles.

Lemma 4.5 *Let A be a d-dimensional ordinary abelian variety. Fix $e \in \mathbb{Z}_{>0}$. Let $\{M_j^{(e)}\}_{j \in J}$ be the set of the p^e-torsion line bundles on X. Then, the following assertions hold.*

(1) $F_*^e \mathcal{O}_A \simeq \bigoplus_{j \in J} M_j^{(e)}$.
(2) $M_j^{(e)} \in \mathrm{Pic}^0(A)$ *for every* $j \in J$.

Proof The number of p^e-torsion line bundles in $\mathrm{Pic}^0(X)$ is p^{ed}. Apply Lemma 3.3 and we obtain the assertion. □

We also need the following lemma.

Lemma 4.6 *Let X be a proper normal variety. Fix $e \in \mathbb{Z}_{>0}$. Assume that there are mutually distinct p^e-torsion line bundles $L_1, \ldots, L_{p^{e \dim X}}$ on X. Let $F_*^e \mathcal{O}_X \simeq E \oplus E'$ where $E \neq 0$ is an indecomposable coherent sheaf and E' is a coherent sheaf. Then, the following assertions hold.*

(1) *If* $\mathrm{rank}\, E < p$, *then* $F_*^e \mathcal{O}_X \simeq \bigoplus_{i=1}^{p^{e \dim X}} L_i$.

(2) *If* rank $E = p$*, then* $E \otimes_{\mathcal{O}_X} L_i \simeq E \otimes_{\mathcal{O}_X} L_j$ *for some* $1 \leq i < j \leq p^{e \dim X}$.

Proof Set $X_{\mathrm{reg}} \subset X$ to be the regular locus of X. Since $(F^e_* \mathcal{O}_X)|_{X_{\mathrm{reg}}}$ is locally free, $E|_{X_{\mathrm{reg}}}$ is also locally free.

We show that E is reflexive. Let

$$F^e_* \mathcal{O}_X \simeq E_1 \oplus \cdots \oplus E_s$$

be a decomposition into indecomposable sheaves with $E_1 \simeq E$. Take the double dual. Since $F^e_* \mathcal{O}_X$ is reflexive, each E_i is reflexive by the Krull–Schmidt theorem ([2, Theorem 2]).

(1) We show that

$$E \otimes_{\mathcal{O}_X} L_i \not\simeq E \otimes_{\mathcal{O}_X} L_j$$

for every $1 \leq i < j \leq p^{e \dim X}$. Assume $E \otimes_{\mathcal{O}_X} L_i \simeq E \otimes_{\mathcal{O}_X} L_j$ for some $1 \leq i < j \leq p^{e \dim X}$. Then, we obtain

$$\det(E|_{X_{\mathrm{reg}}}) \otimes_{\mathcal{O}_{X_{\mathrm{reg}}}} (L_i|_{X_{\mathrm{reg}}})^{\mathrm{rank}\, E} \simeq \det(E|_{X_{\mathrm{reg}}}) \otimes_{\mathcal{O}_{X_{\mathrm{reg}}}} (L_j|_{X_{\mathrm{reg}}})^{\mathrm{rank}\, E}.$$

By $1 \leq \mathrm{rank}\, E < p$, we obtain $L_i \simeq L_j$, which is a contradiction.

Thus $E \otimes_{\mathcal{O}_X} L_i$ is also an indecomposable direct summand of $F^e_* \mathcal{O}_X$. Therefore, we see rank $E = 1$ and

$$F^e_* \mathcal{O}_X \simeq \bigoplus_{i=1}^{p^{e \dim X}} E \otimes_{\mathcal{O}_X} L_i.$$

Since E is a divisorial sheaf, X is F-split by Lemma 3.2. Then, the assertion follows from Lemma 3.3.

(2) Assume that $E \otimes_{\mathcal{O}_X} L_i \not\simeq E \otimes_{\mathcal{O}_X} L_j$ for every $1 \leq i < j \leq p^{e \dim X}$. Let us derive a contradiction. Since E is indecomposable, so is $E \otimes_{\mathcal{O}_X} L_i$ for every i. Moreover, $E \otimes_{\mathcal{O}_X} L_i$ is also a direct summand of $F^e_* \mathcal{O}_X$. Thus, by the Krull–Schmidt theorem ([2, Theorem 2]), we obtain

$$F^e_* \mathcal{O}_X \simeq \bigoplus_{i=1}^{p^{e \dim X}} E \otimes_{\mathcal{O}_X} L_i \oplus \cdots .$$

Then, we obtain the following contradiction:

$$p^{e \dim X} = \mathrm{rank}(F^e_* \mathcal{O}_X) \geq p^{e \dim X} \times \mathrm{rank}\, E = p^{e \dim X} \times p.$$

□

We show the main theorem of this paper.

Theorem 4.7 *Let X be a smooth projective variety. Assume that the following conditions hold.*

- *For infinitely many $e \in \mathbb{Z}_{>0}$, $F^e_* \mathcal{O}_X \simeq \bigoplus_j M_j^{(e)}$ where each $M_j^{(e)}$ is a line bundle.*
- *K_X is pseudo-effective.*

Then, X is an ordinary abelian variety.

Proof Let

$$\alpha : X \to A := \mathrm{Alb}(X)$$

be the Albanese morphism.

Step 1. In this step, we show the following assertions.

(1) The Albanese morphism $\alpha : X \to A$ is surjective.
(2) The Albanese variety A is an ordinary abelian variety such that $\dim X = \dim A$.
(3) For every $e \in \mathbb{Z}_{>0}$, $F^e_* \mathcal{O}_X \simeq \bigoplus_j M_j^{(e)}$ where each $M_j^{(e)}$ is a p^e-torsion line bundle.

Proof of Step 1. (1) Lemma 3.2 implies that X is F-split. By Lemma 4.3, we see $\kappa_S(X) = 0$. Thus we can apply [6, Theorem 1.1.1(1)]. Then, the Albanese morphism $\alpha : X \to \mathrm{Alb}(X)$ is surjective.

(2) By (1), we obtain $\dim \mathrm{Pic}^0(X)_{\mathrm{red}} \le \dim X$. Set r_X to be the p-rank of $\mathrm{Pic}^0(X)_{\mathrm{red}}$. It suffices to show that $r_X = \dim X$. By Lemma 4.4 and an assumption, the number of p^e-torsion line bundles is equal to $p^{e \dim X}$ for infinitely many $e \in \mathbb{Z}_{>0}$. By Proposition 2.2(2), we can find an integer $\xi > 0$ such that

$$p^{er_X} \le p^{e \dim X} = |\mathrm{Pic}(X)[p^e]| \le p^{er_X} \times \xi,$$

for infinitely many $e > 0$. Taking the limit $e \to \infty$, we obtain $r_X = \dim X$.

(3) The assertion follows from (2) and Lemma 3.3. This completes the proof of Step 1.

□

By Step 1, the Albanese morphism $\alpha : X \to A$ is a generically finite surjective morphism and A is an ordinary abelian variety. We obtain the following decomposition

$$\alpha : X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} A$$

such that

- Y and Z are projective normal varieties.
- f is a birational morphism, and g and h are finite surjective morphisms.
- g is purely inseparable and h is separable.

Note that we can find such a decomposition as follows. First, we take the Stein factorization of α and we obtain Y. Then $f : X \to Y$ is birational and $Y \to A$ is finite. Second, take the separable closure L of $K(A)$ in $K(X) = K(Y)$ and consider the normalization Z of A in L.

Step 2. Y is smooth.

Proof of Step 2. Since $f_*\mathcal{O}_X = \mathcal{O}_Y$, Y is F-split. By Lemma 4.5, there are the mutually distinct p-torsion line bundles $M_1, \ldots, M_{p^{\dim X}}$ on A such that $M_i \in \mathrm{Pic}^0(A)$. By Sect. 2.2, $\alpha^* M_1, \ldots, \alpha^* M_{p^{\dim X}}$ are mutually distinct p-torsion line bundles on X. Thus, the number of p-torsion line bundles on Y is at least $p^{\dim X} = p^{\dim Y}$. Then, by Lemma 3.3, $F_*\mathcal{O}_Y \simeq \bigoplus_{j \in J} L_j$ for some p-torsion line bundles L_j on Y. Therefore Y is smooth by Kunz's criterion. □

Step 3. f is an isomorphism.

Proof of Step 3. We can write

$$K_X = f^* K_Y + E$$

where E is an f-exceptional divisor. Since Y is smooth and hence terminal (cf. [9, Section 2.3]), E is effective. Since $K_X \equiv 0$, we see that E is f-nef. By the negativity lemma (cf. [9, Lemma 3.39]), we see $E = 0$. Therefore, $K_X = f^* K_Y$. Thus, the codimension of $\mathrm{Ex}(f)$ in X is at least two. Since Y is smooth, f is an isomorphism. □

Now, we have

$$\alpha : X \xrightarrow{g} Z \xrightarrow{h} A$$

such that

- Z is projective normal variety.
- g is a finite surjective purely inseparable morphism.
- h is a finite surjective separable morphism.

Step 4. If g is an isomorphism, then α is also an isomorphism.

Proof of Step 4. We see that the albanese morphism

$$\alpha = h : X \to A$$

is a finite surjective separable morphism. Since K_X is numerically trivial and $K_A \sim 0$, $\alpha : X \to A$ is etale in codimension one. Then, by the Zariski–Nagata purity, α is etale. By [13, Section 18, Theorem], X is also an ordinary abelian variety. This completes the proof of Step 4. □

Step 5. g is an isomorphism.

Proof of Step 5. Assume that g is not an isomorphism. Then, we can find

$$\alpha : X \xrightarrow{\varphi} W \to Z \to A, \quad \beta : W \to A$$

which satisfies the following properties.

- W is a projective normal variety.
- $\varphi : X \to W$ and $W \to Z$ are finite surjective purely inseparable morphisms with $[K(X) : K(W)] = p$.

Since A is an ordinary abelian variety, there are mutually distinct p-torsion line bundles $M_1, \ldots, M_{p^{\dim X}}$ on A which form a subgroup of $\mathrm{Pic}^0 A$ (Lemma 4.5).

Claim *We prove the following assertions.*

(a) $F_*\mathcal{O}_W \simeq \varphi_*\mathcal{O}_X \oplus E$ *for some coherent sheaf* E.
(b) $F_*\mathcal{O}_W \simeq \beta^*M_1 \oplus \cdots \oplus \beta^*M_{p^{\dim X}}$.

Proof of Claim (a) Since $[K(X) : K(W)] = p$, the Frobenius map F_W factors through φ:

$$F_W : W \xrightarrow{\mu} X \xrightarrow{\varphi} W.$$

Since μ is a finite purely inseparable morphism, there is $e \in \mathbb{Z}_{>0}$ such that F_X^e factors through μ:

$$F_X^e : X \to W \xrightarrow{\mu} X.$$

Since X is F-split, the identity homomorphism $\mathrm{id}_{\mathcal{O}_X}$ factors through $\mu_*\mathcal{O}_W$:

$$\mathrm{id}_{\mathcal{O}_X} : \mathcal{O}_X \to \mu_*\mathcal{O}_W \to (F_X^e)_*\mathcal{O}_X \to \mathcal{O}_X.$$

Thus, we see

$$\mu_*\mathcal{O}_W \simeq \mathcal{O}_X \oplus E_1$$

for some coherent sheaf E_1 on X. Take the push-forward by φ and we obtain

$$(F_W)_*\mathcal{O}_W \simeq \varphi_*\mathcal{O}_X \oplus \varphi_*E_1.$$

(b) Set $L_i := \beta^*M_i$. By Sect. 2.2, $L_1, \ldots, L_{p^{\dim X}}$ are mutually distinct p-torsion line bundles on W such that $\{L_1, \ldots, L_{p^{\dim X}}\}$ forms a subgroup of $\mathrm{Pic}\, W$ and that

$$\varphi^*L_i \not\simeq \varphi^*L_j$$

for every $1 \le i < j \le p^{\dim X}$. There are the following two cases:

- $\varphi_*\mathcal{O}_X$ is not indecomposable.
- $\varphi_*\mathcal{O}_X$ is indecomposable.

Assume that $\varphi_*\mathcal{O}_X$ is not indecomposable. Then, $F_*\mathcal{O}_W$ has an indecomposable direct summand of rank $< p$. Therefore, by Lemma 4.6(1), we obtain

$$F_*\mathcal{O}_W \simeq \beta^*M_1 \oplus \cdots \oplus \beta^*M_{p^{\dim X}}.$$

This is what we want to show.

Assume that $\varphi_* \mathcal{O}_X$ is indecomposable. Since $\mathrm{rank}(\varphi_* \mathcal{O}_X) = p$, we can apply Lemma 4.6(2) and can find

$$\varphi_* \mathcal{O}_X \otimes L_i \simeq \varphi_* \mathcal{O}_X \otimes L_j$$

for some $1 \leq i < j \leq p^{\dim X}$. Since $\{L_1, \ldots, L_{p^{\dim X}}\}$ is a group, we obtain $L_i^{-1} \otimes_{\mathcal{O}_X} L_j \simeq L_r$ for some $1 \leq r \leq p^{\dim X}$ with $\varphi^* L_r \not\simeq \mathcal{O}_X$. Tensor L_i^{-1} and we see

$$\varphi_* \mathcal{O}_X \simeq \varphi_* \mathcal{O}_X \otimes L_r \simeq \varphi_*(\varphi^* L_r).$$

Then, taking H^0, we obtain the following contradiction

$$0 \neq H^0(X, \mathcal{O}_X) \simeq H^0(X, \varphi^* L_r) = 0,$$

where the last equality holds because $\varphi^* L_r$ is a non-trivial p-torsion line bundle. This completes the proof of Claim.

By the Krull–Schmidt theorem ([2, Theorem 2]), the assertions (a) and (b) in Claim imply

$$\varphi_* \mathcal{O}_X = \bigoplus_{j \in J} \beta^* M_j$$

for some $J \subset \{1, \ldots, p^{\dim X}\}$. Since $\# J = p$, we obtain $M_{j_0} \not\simeq \mathcal{O}_A$ for some $j_0 \in J$.

By Sect. 2.2, we see that $\alpha^* M_{j_0} \not\simeq \mathcal{O}_X$. Since $\alpha^* M_{j_0}$ is a non-trivial p-torsion line bundle, we obtain

$$H^0\big(X, \alpha^* M_{j_0}^{-1}\big) = 0.$$

On the other hand, we obtain

$$\varphi_* \alpha^* M_{j_0}^{-1} \simeq \varphi_* \varphi^* \beta^* M_{j_0}^{-1} \simeq \varphi_* \mathcal{O}_X \otimes \beta^* M_{j_0}^{-1}$$

$$\simeq \left(\bigoplus_{j \in J} \beta^* M_j \right) \otimes \beta^* M_{j_0}^{-1} \simeq \mathcal{O}_W \oplus \left(\bigoplus_{j \neq j_0} \beta^* M_j \otimes \beta^* M_{j_0}^{-1} \right),$$

which implies

$$H^0(X, \alpha^* M_{j_0}^{-1}) \neq 0.$$

This is a contradiction. Thus, $g : X \to Z$ is an isomorphism. This completes the proof of Step 5. □

Step 4 and Step 5 imply the assertion in the theorem. □

5 On the behavior of $F^e_*\mathcal{O}_X$ for some special varieties

In the former sections, we investigate varieties X such that $F_*\mathcal{O}_X$ is decomposed into line bundles. In this section, we study the behavior of $F_*\mathcal{O}_X$ for some special varieties.

5.1 Abelian varieties

In this subsection, we show Theorem 5.3. We recall some results essentially obtained by [15].

Theorem 5.1 (Oda) *Let $f : X \to Y$ be an isogeny of abelian varieties over k. Set $\hat{f} : \hat{Y} \to \hat{X}$ to be the dual of f. Let $L \in \mathrm{Pic}^0(X)$. Then,*

$$f_*L \simeq \mathrm{pr}_{1*}(\mathcal{P}_Y|_{Y\times \hat{f}^{-1}([L])})$$

where $\mathcal{P}_Y$ is the normalized Poincare line bundle of $(Y, 0)$ and pr_1 is the first projection.

Proof We can apply the same argument as [15, Corollary 1.7]. □

Theorem 5.2 (Oda) *Let X be an abelian variety. Let $S \subset \hat{X}$ be a closed subscheme of the dual abelian variety $\hat{X}$. If S is zero-dimensional and Gorenstein, then the following assertions hold.*

(1) *There exists an isomorphism between non-commutative k-algebras:*

$$\mathrm{End}_{\mathcal{O}_X}(\mathrm{pr}_{1*}(\mathcal{P}_X|_{X\times S})) \simeq \Gamma(S, \mathcal{O}_S).$$

In particular, $\mathrm{End}_{\mathcal{O}_X}(\mathrm{pr}_{1*}(\mathcal{P}_X|_{X\times S}))$ is a commutative ring.

(2) *If S is one point, that is, $\Gamma(S, \mathcal{O}_S)$ is a local ring, then $\mathrm{pr}_{1*}(\mathcal{P}_X|_{X\times S})$ is an indecomposable sheaf.*

Proof (1) holds from [15, Corollary 1.12]. We show (2). Assuming $\mathrm{pr}_{1*}(\mathcal{P}_X|_{X\times S}) \simeq E_1 \oplus E_2$ with $E_i \neq 0$, we derive a contradiction. By (1), the ring $\mathrm{End}_{\mathcal{O}_X}(\mathrm{pr}_{1*}(\mathcal{P}_X|_{X\times S}))$ is a commutative ring. We obtain idempotents $\mathrm{id}_{E_1} \times 0_{E_2}$ and $0_{E_1} \times \mathrm{id}_{E_2}$ such that $\mathrm{id}_{E_1} \times 0_{E_2} + 0_{E_1} \times \mathrm{id}_{E_2}$ is the unity of the ring $\mathrm{End}_{\mathcal{O}_X}(\mathrm{pr}_{1*}(\mathcal{P}_X|_{X\times S}))$. Therefore, we obtain

$$\Gamma(S, \mathcal{O}_S) \simeq \mathrm{End}_{\mathcal{O}_X}(\mathrm{pr}_{1*}(\mathcal{P}_X|_{X\times S})) \simeq A \times B$$

for some non-zero rings A and B. But, $\Gamma(S, \mathcal{O}_S)$ is a local ring. This is a contradiction. □

We show the main theorem of this subsection.

Theorem 5.3 *Let X be an abelian variety. Set r_X to be the p-rank of X. Let $L \in \mathrm{Pic}^0(X)$. Then, for every $e \in \mathbb{Z}_{>0}$, we obtain*

$$F^e_*L \simeq E_1 \oplus \cdots \oplus E_{p^{er_X}}$$

where each E_i is an indecomposable locally free sheaf of rank $p^{e(\dim X - r_X)}$.

Proof Fix $e \in \mathbb{Z}_{>0}$. Consider the absolute Frobenius morphism $F_X^e : X \to X$. Set $X^{(p^e)} := X \times_{k, F_k^e} k$ and we obtain

$$F_X^e : X \xrightarrow{F_X^{e,\mathrm{rel}}} X^{(p^e)} \xrightarrow{\beta} X.$$

where β is a non-k-linear isomorphism of schemes and

$$F_X^{e,\mathrm{rel}} : X \to X^{(p^e)}$$

is k-linear. Thus, it suffices to show that

$$(F_X^{e,\mathrm{rel}})_* L \simeq E'_1 \oplus \cdots \oplus E'_{p^{er_X}}$$

for some indecomposable locally free sheaves E'_i of rank $p^{e(\dim X - r_X)}$. Take the dual of $F_X^{e,\mathrm{rel}}$:

$$\widehat{(F_X^{e,\mathrm{rel}})} : \widehat{X^{(p^e)}} \to \hat{X}.$$

We show that the number of the fiber of every closed point of $\widehat{\left(F_X^{e,\mathrm{rel}}\right)}$ is p^{er_X}. Since $\widehat{\left(F_X^{e,\mathrm{rel}}\right)}(k)$ is a group homomorphism, the numbers of all the fibers are the same. Thus, it suffices to prove that the number of $\widehat{\left(F_X^{e,\mathrm{rel}}\right)}^{-1}(0_{\hat{X}}) = \mathrm{Ker}\left(\widehat{\left(F_X^{e,\mathrm{rel}}\right)}(k)\right)$ is p^{er_X}. This is equivalent to show that the number of line bundles $M \in \mathrm{Pic}^0(X^{(p^e)}) = \widehat{X^{(p^e)}}(k)$ such that $(F_X^{e,\mathrm{rel}})^* M \simeq \mathcal{O}_X$ is p^{er_X}. Since $\beta : X^{(p^e)} \to X$ is an isomorphism, we prove that the number of line bundles $N \in \mathrm{Pic}^0(X)$ such that $N^{p^e} = (F_X^e)^* N \simeq \mathcal{O}_X$ is p^{er_X}. This follows from the definition of the p-rank.

Taking the separable closure, we obtain

$$\widehat{\left(F_X^{e,\mathrm{rel}}\right)} : \widehat{X^{(p^e)}} \xrightarrow{g} Y \xrightarrow{h} \hat{X},$$

where Y is a normal projective variety, g is a finite surjective purely inseparable morphism and h is a finite surjective separable morphism. Since the numbers of every fiber of $\widehat{\left(F_X^{e,\mathrm{rel}}\right)}$ are the same, h is an etale morphism. In particular, Y is an abelian variety ([13, Section 18, Theorem]) and we may assume that g and h are isogenies. Take the duals again and we obtain

$$F_X^{e,\mathrm{rel}} : X \xrightarrow{\hat{h}} \hat{Y} \xrightarrow{\hat{g}} X^{(p^e)}.$$

Let

$$\mathrm{Ker}(\widehat{F_X^{e,\mathrm{rel}}}) = g^{-1}([M_1]) \amalg \cdots \amalg g^{-1}([M_{p^{er_X}}])$$

be the decomposition into one point schemes. By Theorem 5.1, we obtain

$$\begin{aligned}(F_X^{e,\mathrm{rel}})_*\mathcal{O}_X &\simeq \mathrm{pr}_{1*}(\mathcal{P}_{X^{(p^e)}}|_{X^{(p^e)}\times \widehat{\mathrm{Ker}(F_X^{e,\mathrm{rel}})}}) \\ &\simeq \mathrm{pr}_{1*}(\mathcal{P}_{X^{(p^e)}}|_{X^{(p^e)}\times g^{-1}([M_1])}) \oplus \cdots \oplus \mathrm{pr}_{1*}(\mathcal{P}_{X^{(p^e)}}|_{X^{(p^e)}\times g^{-1}([M_{p^{er_X}}])}) \\ &\simeq \hat{g}_* M_1 \oplus \cdots \oplus \hat{g}_* M_{p^{er_X}}.\end{aligned}$$

Thus, it suffices to show that each locally free sheaf

$$\mathrm{pr}_{1*}(\mathcal{P}_{X^{(p^e)}}|_{X^{(p^e)}\times g^{-1}([M_j])}) \simeq \hat{g}_* M_j$$

is indecomposable. We see that $g^{-1}([M_j])$ is one point. Thus, if $g^{-1}([M_j])$ is Gorenstein, then $\mathrm{pr}_{1*}(\mathcal{P}_{X^{(p^e)}}|_{X^{(p^e)}\times g^{-1}([M_j])})$ is indecomposable by Theorem 5.2(2). Since g is finite and Y is smooth, $g^{-1}([M_j])$ is a local complete intersection scheme. In particular, $g^{-1}([M_j])$ is Gorenstein. □

5.2 Curves

In this subsection, we show Theorem 5.5. We need the following result from the theory of stable vector bundles.

Theorem 5.4 *Let X be a smooth projective curve of genus $g \geq 2$. Let L be a line bundle on X. Then, $F_*^e L$ is indecomposable for every $e \in \mathbb{Z}_{>0}$.*

Proof Since L is a line bundle, L is a stable vector bundle. Then, by [17, Theorem 2.2], $F_*^e L$ is also a stable vector bundle. Since stable vector bundles are indecomposable, $F_*^e L$ is indecomposable. □

We show the main theorem of this subsection.

Theorem 5.5 *Let X be a smooth projective curve of genus g. Fix an arbitrary positive integer e. Then the following assertions hold.*

- (0) *If $g = 0$, then $F_*^e \mathcal{O}_X \simeq \bigoplus L_j$ where every L_j is a line bundle.*
- (1or) *If $g = 1$ and X is an ordinary elliptic curve, then $F_*^e \mathcal{O}_X \simeq \bigoplus L_j$ where every L_j is a line bundle.*
- (1ss) *If $g = 1$ and X is a supersingular elliptic curve, then $F_*^e \mathcal{O}_X$ is indecomposable.*
- (2) *If $g \geq 2$, then $F_*^e \mathcal{O}_X$ is indecomposable.*

Proof The assertion (0) immediately follows from the fact that every locally free sheaf of finite rank on $\mathbb{P}^1$ is decomposed into the direct sum of line bundles.

The assertions (1or) and (1ss) hold by Theorem 5.3. The assertion (2) follows from Theorem 5.4. □

By Theorem 5.5, it is natural to ask the following question.

Question 5.6 *If X is a smooth projective surface X of general type, then is $F_*\mathcal{O}_X$ indecomposable?*

As far as the authors know, this question is open. On the other hand, if we drop the assumption that X is smooth, then there exists a counter-example as follows. For a related result, see also [7, Example 3.5].

Theorem 5.7 *There exists a projective normal surface X which satisfies the following properties.*

(1) *The singularities of X are at worst canonical.*
(2) *K_X is ample.*
(3) *$F_*\mathcal{O}_X$ is not indecomposable.*

Proof Let S be an ordinary abelian surface. Fix a very ample line bundle H on S. Let $s \in H^0(X, H^p)$ be a general element and set

$$\pi : X := \mathrm{Spec}_S\big(\mathcal{O}_S \oplus H^{-1} \oplus \cdots \oplus H^{-(p-1)}\big) \to S$$

to be the finite purely inseparable morphism where the $\mathcal{O}_S$-algebra $\mathcal{O}_S \oplus H^{-1} \oplus \cdots \oplus H^{-(p-1)}$ is defined by $s \in H^0(X, H^p)$. By [10, Remark 3.5(1)], we can apply [10, Theorem 3.4] for $\mathcal{L} := H$. Since the scheme X constructed above is the same as the $\alpha_{\mathcal{L}}$-torsor $\delta(s)$ appearing in [10, Theorem 3.4]. Therefore, X is normal and has at worst A_{p-1}-singularities. Thus (1) holds. We see

$$K_X = \pi^* K_S + (p-1)\pi^* H \sim (p-1)\pi^* H,$$

which implies (2).

We show (3). Since $\pi : X \to S$ is a finite purely inseparable morphism of degree p, the absolute Frobenius morphisms of X and S factors through π:

$$F_S : S \to X \xrightarrow{\pi} S, \quad F_X : X \xrightarrow{\pi} S \xrightarrow{\varphi} X.$$

Since S is F-split, the identity homomorphism $\mathrm{id}_{\mathcal{O}_S}$ factors through $\pi_*\mathcal{O}_X$:

$$\mathrm{id}_{\mathcal{O}_S} : \mathcal{O}_S \to \pi_*\mathcal{O}_X \to (F_S)_*\mathcal{O}_S \to \mathcal{O}_S.$$

This implies

$$\pi_*\mathcal{O}_X \simeq \mathcal{O}_S \oplus E$$

for some coherent sheaf E. Taking the push-forward by φ, we see

$$(F_X)_*\mathcal{O}_X = \varphi_*\pi_*\mathcal{O}_X \simeq \varphi_*\mathcal{O}_S \oplus \varphi_* E.$$

This implies (3). □

Remark 5.8 If X is a smooth projective curve of general type, then $F_*\mathcal{O}_X$ is indecomposable by Theorem 5.4. Theorem 5.4 depends on the theory of the stable vector bundles. For the 2-dimensional case, a similar result is obtained by Kitadai–Sumihiro [8], Liu–Zhou [11], and Sun [18]. For example, [18, Theorem 4.9 and Remark 4.10] imply that $F_*\mathcal{O}_X$ is indecomposable under the assumptions that $\mu(\Omega^1_X) > 0$ and Ω^1_X is semi-stable.

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