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Frustration index and Cheeger inequalities for discrete and continuous magnetic Laplacians

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Abstract We discuss a Cheeger constant as a mixture of the frustration index and the expansion rate, and prove the related Cheeger inequalities and higher order Cheeger inequalities for graph Laplacians with cyclic signatures, discrete magnetic Laplacians on finite graphs and magnetic Laplacians on closed Riemannian manifolds. In this process, we develop spectral clustering algorithms for partially oriented graphs and multi-way spectral clustering algorithms via metrics in lens spaces and complex projective spaces. As a byproduct, we give a unified viewpoint of Harary's structural balance theory of signed graphs and the gauge invariance of magnetic potentials.

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1 Introduction

Cheeger's inequality is one of the most fundamental and important estimates in spectral geometry. It was first proved by Cheeger for the Laplace-Beltrami operator on a Riemannian manifold [7] and later extended to the setting of discrete graphs, see e.g., [1,2,6,11], demonstrating the close relationship between the spectrum and the geometry of the underlying space. This inequality has a tremendous impact in discrete and continuous theories and is an important intersection point for interactions between both communities. For example, it stimulated research in discrete mathematics such as spectral clustering algorithms for data mining [36], or the construction of expander graphs [25]. Cheeger inequalities have also been considered on metric graphs, see, e.g., [42] and, using a coarea formula in the proof [44]. We recently witness several fruitful interactions in the other direction: Lee, Oveis Gharan and Trevisan's higher order Cheeger inequalities [28,29] on finite graphs were used by Miclo [40] to prove that hyperbounded, ergodic, and self-adjoint Markov operators admit a spectral gap, solving a 40-year-old conjecture of Simon and Høegh–Krohn [50]. For further developments, see [32,57]. Another example is an improved Cheeger's inequality for finite graphs by Kwok et al. [27], which was subsequently used to establish an optimal dimension-free upper bound of eigenvalue ratios for weighted closed Riemannian manifolds with nonnegative Ricci curvature [33] (see also [34]). This answers open questions of Funano and Shioya [15,16].

Spectral theory of discrete and continuous magnetic Laplacians attracted a lot of attention and literature on this subject developed rapidly, see, e.g., [9,13,14,17,24,31,41,43,47–49, 53]. Shigekawa proved the following comparison result in [47]: the least eigenvalue of the magnetic Laplacian on a closed Riemannian manifold is bounded from above by the least eigenvalue of a related Schrödinger operator. He also proved Weyl's asymptotic formula for magnetic Laplacians. Paternain [43] obtained an upper bound of the least eigenvalue in terms of the so-called harmonic value and Mañé's critical value of the corresponding Lagrangian. On finite planar graphs, Lieb and Loss [31] solved physically motivated extremality problems for eigenvalue expressions of the discrete magnetic Laplacian.

In this paper, we discuss a definition of Cheeger constants (Definitions 3.5, 3.6 and 7.3) reflecting the nontriviality of the magnetic potentials in terms of the frustration index (see Definitions 3.4 and 7.2) and the global connectivity of the underlying space. This definition works for both discrete and continuous magnetic Laplacians, and graph Laplacians with k-cyclic signatures ($k \in \mathbb{N}$). Recall that discrete magnetic Laplacians can be considered as graph Laplacians with a $U(1)$-signature. We would like to point out that our definition of Cheeger constants provides invariances under switching operations (Definition 2.3) or gauge transformations (Eq. 7.8). Furthermore, we prove the corresponding Cheeger inequalities and higher order Cheeger inequalities (Theorems 4.1, 4.6, 5.1, 7.4, and 7.7). We notice that our Theorem 4.6, the Cheeger inequality for discrete magnetic Laplacian, overlaps with a Cheeger inequality of Bandeira, Singer and Spielman [4, Theorem 4.1] in the framework of graph connection Laplacians [51]. See Remark 4.9 for a more detailed explanation. It is known in physics that "a magnetic field raises the energy" [31]. Roughly speaking, our estimates tell us that a magnetic field raises the energy via raising the frustration index. We focus on finite graphs and compact Riemannian manifolds in this paper.

Cheeger inequalities are essentially coarea inequalities. In the proof, we obtain in particular coarea inequalities related to the frustration index on graphs as well as on manifolds (Lemmas 4.3 and 7.5).

In fact, we were led to our Cheeger constant definition by an investigation of graph Laplacians with k-cyclic signatures, aiming at extending a previous spectral interpretation [3] of Harary's structural balance theory [21,22] for graphs with (± 1)-signatures. It turns out that the Cheeger inequalities for graph Laplacians with k-cyclic signatures and their proofs provide spectral clustering algorithms for partially oriented graphs (alternatively called mixed graphs without loops and multiple edges [23,45,46,60]), aiming at detecting interesting substructures. A partially oriented graph may contain both oriented and unoriented edges. In the proof of such inequalities, we develop a random k-partition argument, which is algorithmic (see Lemma 4.2 and Proposition 6.6). Recall that, in the setting of (± 1)-signed graphs (i.e., $k = 2$), the eigenfunctions are real valued and a bipartition of the underlying graph can be given naturally according to the sign of the eigenfunction. But here we have complex valued eigenfunctions. Hence we do not have any natural k-partitions. That is why new ideas are needed. The generally non-symmetric graph Laplacians of partially oriented graphs are hardly useful for the purpose of spectral clustering. Our idea is to associate to a partially oriented graph and a natural number $k \in \mathbb{N}$ an unoriented graph with a special k-cyclic signature. We then perform spectral clustering algorithms employing eigenfunctions of the graph Laplacian with the associated signature. According to our Cheeger constant definition, we can obtain interesting k-cyclic substructures. See Sect. 6 for details.

To prove higher order Cheeger inequalities, we develop new multi-way spectral clustering algorithms using metrics on *lens spaces* and *complex projective spaces*. This provides a deeper understanding of earlier spectral clustering algorithms via metrics on real projective spaces presented in [32] and [3]. These clustering algorithms were initially designed to find almost bipartite subgraphs of a given graph [32], and then extended to find almost balanced subgraphs of a signed graph [3]. While all operators studied in [3,32] are bounded, we show that finding proper metrics for clustering is also useful for unbounded operators: the spectral clustering algorithms via metrics on complex projective spaces are crucial to prove the higher order Cheeger inequalities of the magnetic Laplacian on a closed Riemannian manifold (Lemma 7.8).

The paper is organized as follows. In Sect. 2, we set up notation for the discrete setting and recall basic spectral theory of related graph operators. In Sect. 3, we define the frustration index and the (multi-way) Cheeger constants. We prove the corresponding Cheeger's inequality in Sect. 4 and higher order Cheeger inequalities in Sect. 5. In Sect. 6, we discuss applications of Cheeger inequalities for spectral clustering on partially oriented graphs. In Sect. 7, we extend the results developed on discrete graphs to magnetic Laplacians on closed Riemannian manifolds.

2 Notations and basic spectral theory

Throughout the paper, $G = (V, E)$ denotes an undirected simple finite graph on N vertices with vertex set V and edge set E. We denote edges of G by $\{u, v\}$, and $u \sim v$ means that $u \in V$ and $v \in V$ are connected by an edge. For any subset $\widetilde{V} \subseteq V$, let $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ be the subgraph of G induced by $\widetilde{V}$, that is, an edge $\{u, v\}$ of $\widetilde{G}$ is an edge of G with $u, v \in \widetilde{V}$. We tacitly associate to every edge $e = \{u, v\} \in E$ a positive symmetric weight $w_{uv} = w_{vu} = w_e$ and define the weighted degree d_u of a vertex $u \in V$ by $d_u := \sum_{v, v \sim u} w_{uv}$. For a positive measure $\mu : V \to \mathbb{R}^+$ on V, we define the *maximal μ-degree* of the graph G as

$$d_\mu := \max_{u \in V} \left\{ \frac{\sum_{v, v \sim u} w_{uv}}{\mu(u)} \right\} = \max_{u \in V} \left\{ \frac{d_u}{\mu(u)} \right\}. \tag{2.1}$$

Henceforth we always consider weighted graphs, unless stated otherwise, but refer to them simply as graphs. We denote by $e = (u, v)$ the oriented edge starting at u and terminating at v, and by $\bar{e} = (v, u)$ the oriented edge with the reversed orientation. Let $E^{or} := \{(u, v), (v, u) \mid \{u, v\} \in E\}$ be the set of all oriented edges.

Definition 2.1 Let G be a graph and Γ be a group. A *signature* of G is a map $s : E^{or} \to \Gamma$ such that

$$s(\bar{e}) = s(e)^{-1}, \tag{2.2}$$

where $s(e)^{-1}$ is the inverse of $s(e)$ in Γ. The *trivial signature* $s \equiv 1$, where 1 stands for the identity element of Γ, is denoted by s_1. For an oriented edge $e = (u, v) \in E^{or}$, we will also write $s_{uv} := s(e)$ for convenience.

For $k \in \mathbb{N}$, we use the standard combinatorial notation $[k] = \{1, 2, \ldots, k\}$. In this paper, we will restrict ourselves to the case that the signature group Γ is the cyclic group $S_k^1 := \{\xi^j \mid j \in [k]\}$ of order k, generated by the primitive k-th root of unity $\xi := e^{2\pi i/k} \in \mathbb{C}$, and the case that Γ is the unitary group $U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$. The notation S_k^1 emphasizes the fact that the elements in S_k^1 lie on the unit circle.

We consider the following Laplacian Δ_μ^s associated to the weighted graph (G, w) with signature $s : E^{or} \to \Gamma$ and vertex measure $\mu : V \to \mathbb{R}^+$. For any function $f : V \to \mathbb{C}$, and any vertex $u \in V$, we have

$$\Delta_\mu^s f(u) := \frac{1}{\mu(u)} \sum_{v, v \sim u} w_{uv}(f(u) - s_{uv} f(v)). \tag{2.3}$$

Note that the summation in (2.3) over the vertices v adjacent to u can also be understood as a summation over the oriented edges $e = (u, v) \in E^{or}$, and the signature is evaluated at (u, v).

The Laplacian Δ_μ^s has the following decomposition

$$\Delta_\mu^s = (D_\mu)^{-1}(D - A^s)$$

where D and D_μ are the diagonal matrices with $D_{uu} = d_u$ and $(D_\mu)_{uu} = \mu(u)$ for all $u \in V$ while A^s is the (weighted) signed adjacency matrix with

$$A_{uv}^s := \begin{cases} 0, & u = v \text{ or } \{u, v\} \notin E, \\ w_{uv} s_{uv}, & \{u, v\} \in E. \end{cases}$$

When $\Gamma = S_k^1$, we call this operator the graph Laplacian with the *k-cyclic signature*. When $\Gamma = U(1)$, this is the discrete magnetic Laplacian studied in Sunada [53] (see also Shubin [48]). By (2.2), the matrix Δ_μ^s is Hermitian, and hence all its eigenvalues are real which can be listed with multiplicity as follows:

$$0 \leq \lambda_1(\Delta_\mu^s) \leq \lambda_2(\Delta_\mu^s) \leq \cdots \leq \lambda_N(\Delta_\mu^s) \leq 2d_\mu. \tag{2.4}$$

For any two functions $f, g : V \to \mathbb{C}$, we define their inner product as

$$\langle f, g \rangle_\mu := \sum_{u \in V} f(u)\overline{g(u)}\mu(u). \tag{2.5}$$

It is easy to check that

$$\langle \Delta_\mu^s f, g \rangle_\mu = \sum_{\{u,v\} \in E} w_{uv}(f(u) - s_{uv} f(v))(\overline{g(u) - s_{uv} g(v)}). \tag{2.6}$$

Note that the right hand side of the above equality is well-defined since $\Gamma \subseteq U(1)$. The corresponding *Rayleigh quotient* $\mathcal{R}_\mu^s(f)$ of a function $f : V \to \mathbb{C}$ is

$$\mathcal{R}_\mu^s(f) := \frac{\sum_{\{u,v\} \in E} w_{uv} |f(u) - s_{uv} f(v)|^2}{\sum_{u \in V} |f(u)|^2 \mu(u)}. \tag{2.7}$$

The Courant–Fisher–Weyl min–max principle tells that, for any $n \in [N]$,

$$\lambda_n(\Delta_\mu^s) = \min_{\substack{\langle f_p, f_q \rangle_\mu = 0, \\ p,q \in [n], p \neq q}} \max_{f \in \mathrm{span}\{f_1, \ldots, f_n\}} \mathcal{R}_\mu^s(f), \tag{2.8}$$

where $f_1, \ldots, f_n$, $f \not\equiv 0$.

Remark 2.2 In the case of a graph G with measure $\mu_d(u) := d_u$ for all $u \in V$ and signature group $\Gamma = U(1)$ or $\Gamma = S_k^1$, k even, Eq. (2.8) implies the following relations between eigenvalues

$$2 - \lambda_{N-k+1}(\Delta_{\mu_d}^{-s}) = \lambda_k(\Delta_{\mu_d}^s). \tag{2.9}$$

Here $-s$ is the signature obtained by taking the negative values of s (as complex numbers). This generalizes [3, Lemma 1] where $\Gamma = S_2^1 = \{\pm 1\}$.

There is a natural operation, called switching, acting on the signatures [58,59].

Definition 2.3 Let G be a graph with signature s. For any function $\tau : V \to \Gamma$ we can define a new signature $s^\tau : E^{or} \to \Gamma$ as follows:

$$s^\tau(e) = \tau(u)s(e)\tau(v)^{-1} \quad \forall e = (u, v) \in E^{or}. \tag{2.10}$$

We call the function τ a *switching function*. The signature s and s' are said to be *switching equivalent* if there exists a switching function τ such that $s' = s^\tau$.

One can check that switching is indeed an equivalence relation on the set of signatures. An important invariant of the switching operation is the spectrum of Δ_μ^s. In fact, it holds that (see e.g. [59])

$$\Delta_\mu^{s^\tau} = D(\tau)\Delta_\mu^s D(\tau)^{-1}, \tag{2.11}$$

where $D(\tau)$ is the diagonal matrix with entries $D(\tau)_{uu} = \tau(u)$. This means that $\Delta_\mu^{s^\tau}$ and Δ_μ^s are unitarily equivalent and have the same spectrum. If the signature $s : E^{or} \to \Gamma$ is switching equivalent to the trivial signature s_1, the operator Δ_μ^s is unitarily equivalent to the classical graph Laplacian. In this case we have $\lambda_1(\Delta_\mu^s) = 0$. We will show in Sect. 4 that this is the only case where the first eigenvalue vanishes. Observe that on a tree, any signature is switching equivalent to the trivial signature.

Remark 2.4 The concept of switching is developed in the study of Harary's balance theory for signed graphs [21], i.e. graphs with signatures $s : E^{or} \to S_2^1 = \{+1, -1\}$, which we briefly review in the next section. The corresponding terminology in the magnetic theory is the gauge transformation, see, e.g., [9,47]. Note that switching is an operation acting on the signatures $s_{uv} := e^{i\alpha_{uv}}$, while the gauge transformation is acting on the magnetic potentials α_{uv}, where $(u, v) \in E^{or}$. We will only use the terminology of the magnetic theory in the manifold case, see Sect. 7. Switching equivalent signatures are called cohomologous weight functions in [53].

3 Frustration index and Cheeger constants

One of our motivations for introducing the Cheeger constants is Harary's structural balance theory [21]. Let G be a finite graph with (possibly non-abelian) signature group Γ and signature $s : E^{or} \rightarrow \Gamma$, and $\mathcal{C}$ be a cycle, which is a graph of the sequence $(u_1, u_2), (u_2, u_3), \ldots, (u_{l-1}, u_l), (u_l, u_1)$ of distinct edges. Then the signature of $\mathcal{C}$ is the conjugacy class of the element

$$s_{u_1 u_2} s_{u_2, u_3} \cdots s_{u_{l-1} u_l} s_{u_l u_1} \in \Gamma.$$

Note that the signature of a cycle is switching invariant.

Definition 3.1 A signature $s : E^{or} \rightarrow \Gamma$ is said to be *balanced* if the signature of every cycle of G is (the conjugacy class of the) identity element $1 \in \Gamma$.

For convenience, we will also say that the graph G or a subgraph of G is balanced if the signature restricted on it is balanced. Since the signature of a cycle is switching invariant, the property of being balanced is also switching invariant. We have the following characterization of being balanced using switching operations.

Proposition 3.2 ([[58], Corollary 3.3]) *A signature $s : E^{or} \rightarrow \Gamma$ is balanced if and only if it is switching equivalent to the trivial signature s_1.*

Remark 3.3 The concept of balance has been studied in the literature under various terminologies. For example, a balanced cycle is said to be satisfying Kirchhoff's Voltage Law in [19]. In [9], the related concept to the signature of a cycle is the holonomy map. In magnetic theory, it is related to the magnetic flux [31].

We define the following frustration index to quantify how far a signature on a subset is from being balanced.

Definition 3.4 Let G be a finite graph with signature s and $V_1 \subseteq V$ nonempty with induced subgraph (V_1, E_1). The *frustration index* $\iota^s(V_1)$ of V_1 is defined as

$$\iota^s(V_1) := \min_{\tau : V_1 \to \Gamma} \sum_{\{u,v\} \in E_1} w_{uv} |\tau(u) - s_{uv} \tau(v)| \tag{3.1}$$

$$= \min_{\tau : V_1 \to \Gamma} \sum_{\{u,v\} \in E_1} w_{uv} |1 - \tau(u)^{-1} s_{uv} \tau(v)|. \tag{3.2}$$

A direct computation shows that the frustration index of a set is switching invariant and, according to Proposition 3.2, we have

$$\iota^s(V_1) = 0 \Leftrightarrow \text{ the subgraph induced by } V_1 \text{ is balanced.} \tag{3.3}$$

If G is unweighted and $\Gamma = \{+1, -1\}$, then

$$\iota^s(V) = 2e_{min}^s(V), \tag{3.4}$$

where $e_{min}^s(V)$ is the minimal number of edges that need to be removed from E in order to make $G = (V, E)$ balanced. The quantity $e_{min}^s(V)$ is exactly the *line index of balance* of Harary [22]. Having the work of Vannimenus and Toulouse [56] in mind, Zaslavsky suggested later the term "frustration index" to Harary (T. Zaslavsky 2014, private communication).

We denote the *boundary measure* of V_1 by

$$|E(V_1, V_1^c)| := \sum_{u \in V_1} \sum_{v \in V_1^c} w_{uv}, \tag{3.5}$$

where V_1^c is the complement of V_1 in V. The *μ-volume* of V_1 is given by

$$\mathrm{vol}_\mu(V_1) := \sum_{u \in V_1} \mu(u). \tag{3.6}$$

Definition 3.5 Let G be a finite graph with a signature s. The *Cheeger constant $h_1^s(\mu)$* is defined as

$$h_1^s(\mu) := \min_{\emptyset \neq V_1 \subseteq V} \phi_\mu^s(V_1), \tag{3.7}$$

where

$$\phi_\mu^s(V_1) := \frac{\iota^s(V_1) + |E(V_1, V_1^c)|}{\mathrm{vol}_\mu(V_1)}. \tag{3.8}$$

The choice of V_1 achieving the minimum in (3.7) can be viewed as a subset of vertices which balances the two complementary goals of minimizing its frustration index and its expansion, measured by the edges $E(V_1, V_1^c)$ connecting V_1 with its complement.

A *nontrivial n-subpartition* of V is given by n pairwise disjoint nonempty subsets $V_1, \ldots, V_n \subset V$ and a *nontrivial n-partition* additionally satisfies $\bigcup_{p \in [n]} V_p = V$. We abbreviate a nontrivial n-(sub)partition $\{V_1, \ldots, V_n\}$ by $\{V_p\}_{[n]}$. In the spirit of Miclo [39], we define the multi-way Cheeger constants as follows.

Definition 3.6 Let G be a finite graph with a signature s. The *n-way Cheeger constant $h_n^s(\mu)$* of G is defined as

$$h_n^s(\mu) := \min_{\{V_p\}_{[n]}} \max_{p \in [n]} \phi_\mu^s(V_p), \tag{3.9}$$

where the minimum is taken over all nontrivial n-subpartitions $\{V_p\}_{[n]}$ of V.

Observe that the n-way Cheeger constant of a graph G is monotone with respect to n, that is, $h_n^s(\mu) \leq h_{n+1}^s(\mu)$.

Using (3.3) and the fact that the frustration index is switching invariant, we obtain the following properties of the Cheeger constants.

Proposition 3.7 *The n-way Cheeger constants $h_n^s(\mu)$ of a graph G are switching invariant. Moreover, $h_n^s(\mu) = 0$ if and only if G consists of at least n connected components and at least n of them are balanced.*

If $s_b : E^{or} \to \Gamma$ denotes a balanced signature, then $h_1^{s_b}(\mu) = 0$ becomes trivial and

$$h_2^{s_b}(\mu) = \min_{\{V_1, V_2\}} \max_{p \in [2]} \phi_\mu^{s_b}(V_p) = \min_{\substack{\emptyset \neq V_1 \subseteq V \\ \mathrm{vol}_\mu(V_1) \leq \frac{1}{2} \mathrm{vol}_\mu(V)}} \frac{|E(V_1, V_1^c)|}{\mathrm{vol}_\mu(V_1)}, \tag{3.10}$$

that is, $h_2^{s_b}(\mu)$ reduces to the classical Cheeger constant.

Remark 3.8 Due to equation (3.4), the n-way Cheeger constant in (3.9) reduces to the signed Cheeger constant introduced on signed graphs [3] with signature group $\Gamma = \{+1, -1\}$. We mention that the signed Cheeger constant in [3] is a unification of the classical Cheeger constant, the non-bipartiteness parameter in [10], the bipartiteness ratio in [54,55], and the dual Cheeger constant in [5].

For $n \in [N]$ and any signature $s : E^{or} \to \Gamma$, we observe

$$h_n^{s_b}(\mu) \leq h_n^s(\mu). \tag{3.11}$$

In fact, if $h_n^s(\mu) = \max_{p \in [n]} \phi_\mu^s(\widetilde{V}_p)$, i.e. $\{\widetilde{V}_p\}_{[n]}$ is the nontrivial n subpartition of V that achieves $h_n^s(\mu)$, we have $\phi_\mu^{s_b}(\widetilde{V}_p) \leq \phi_\mu^s(\widetilde{V}_p)$ since $\iota^{s_b}(\widetilde{V}_p) = 0 \leq \iota^{s_b}(\widetilde{V}_p)$. Hence, (3.11) follows by Definition 3.6. The inequality (3.11) is similar, in spirit, with Kato's inequality for noncompact spaces [12, Lemma 1.2, Corollary 1.3] (alternatively, also called the diamagnetic inequality for both compact and noncompact spaces in [31]) where the bottom of the spectrum increases when a balanced signature is replaced by an unbalanced signature.

For $n = 1$ we have the following result. Recalling $h_1^{s_b}(\mu) = 0$, Proposition 3.9 tells us that this change of the first Cheeger constant (by choosing an unbalanced signature) can be quite large.

Proposition 3.9 *Let G be an unweighted connected finite d-regular graph and $M = \max_{v \in V} \mu(v)$. Then, for every $k \geq 2$, there exists a k-cyclic signature $s_0 : E^{or} \to S_k^1$ such that*

$$h_1^{s_0}(\mu) \geq \frac{d - 2\sqrt{d-1}}{2M}. \tag{3.12}$$

Proof Extending a result of [37,38], it is shown in [35, Theorem 2] that there exists a k-cyclic signature s_0 such that the maximal eigenvalue of the matrix A^{s_0} is no greater than $2\sqrt{d-1}$. The estimate (3.12) is then an immediate consequence of this result, combined with Cheeger's inequality (4.1), given at the beginning of the next section. □

4 Cheeger's inequality

In this section, we prove Cheeger's inequality relating $\lambda_1(\Delta_\mu^s)$ to the first Cheeger constant $h_1^s(\mu)$ for graph Laplacians with cyclic signatures (Theorem 4.1) and for discrete magnetic Laplacians (Theorem 4.6).

Theorem 4.1 *Let G be a finite graph with signature $s : E^{or} \to S_k^1$. Then we have*

$$\frac{1}{2}\lambda_1(\Delta_\mu^s) \leq h_1^s(\mu) \leq 2\sqrt{2d_\mu \lambda_1(\Delta_\mu^s)}. \tag{4.1}$$

We start with preparations for the proof of Theorem 4.1. Let $B_r(0) := \{z \in \mathbb{C} \mid |z| < r\}$ be the open disk in $\mathbb{C}$ with center 0 and radius r. For $\theta \in [0, 2\pi)$ and $k \in \mathbb{N}$, we define the following k disjoint sectorial regions

$$Q_j^\theta := \left\{ re^{i\alpha} \in \overline{B_1(0)} \;\middle|\; r \in (0, 1], \alpha \in \left[\theta + \frac{2\pi j}{k}, \theta + \frac{2\pi(j+1)}{k}\right) \right\}, \tag{4.2}$$

where $j = 0, 1, \ldots, k-1$. Then for any $t \in (0, 1]$, we define the function $Y_{t,\theta} : \overline{B_1(0)} \to \mathbb{C}$ as

$$Y_{t,\theta}(z) := \begin{cases} \xi^j, & \text{if } z \in Q_j^\theta \backslash B_t(0), \\ 0, & \text{if } z \in B_t(0), \end{cases} \tag{4.3}$$

where ξ denotes the k-th primitve root of unity.

The following lemma plays a key role.

Lemma 4.2 *For any two points $z_1, z_2 \in \overline{B_1(0)}$, we have*

$$\frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \left| Y_{\sqrt{t},\theta}(z_1) - Y_{\sqrt{t},\theta}(z_2) \right| dt \, d\theta \leq 2 \left| z_1 - z_2 \right| (\left| z_1 \right| + \left| z_2 \right|). \qquad (4.4)$$

Proof W.l.o.g., we can assume that $|z_1| \geq |z_2|$ with $z_1 \in Q_{j_1}^\theta$ and $z_2 \in Q_{j_2}^\theta$. Then we have

$$|Y_{\sqrt{t},\theta}(z_1) - Y_{\sqrt{t},\theta}(z_2)| = \begin{cases} |\xi^{j_1} - \xi^{j_2}|, & \text{if } \sqrt{t} \leq |z_2|, \\ 1, & \text{if } |z_2| < \sqrt{t} \leq |z_1|, \\ 0, & \text{if } |z_1| < \sqrt{t}. \end{cases} \qquad (4.5)$$

Hence,

$$\int_0^1 \left| Y_{\sqrt{t},\theta}(z_1) - Y_{\sqrt{t},\theta}(z_2) \right| dt = |\xi^{j_1} - \xi^{j_2}| \cdot |z_2|^2 + (|z_1|^2 - |z_2|^2). \qquad (4.6)$$

Let $\alpha_{z_1 z_2} \in [0, \pi]$ be the angle between the two rays joining z_1, z_2 to the origin. If $2\pi l/k \leq \alpha_{z_1 z_2} < 2\pi(l+1)/k$ for some integer $0 \leq l < k/2$, the term $|\xi^{j_1} - \xi^{j_2}|$ is equal to either $|1 - \xi^l|$ or $|1 - \xi^{l+1}|$, hence we calculate

$$\frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \left| Y_{\sqrt{t},\theta}(z_1) - Y_{\sqrt{t},\theta}(z_2) \right| dt \, d\theta$$

$$= \left(\frac{k\alpha_{z_1 z_2}}{2\pi} - l \right) \left(|1 - \xi^{l+1}| \cdot |z_2|^2 + |z_1|^2 - |z_2|^2 \right)$$

$$+ \left(l + 1 - \frac{k\alpha_{z_1 z_2}}{2\pi} \right) \left(|1 - \xi^l| \cdot |z_2|^2 + |z_1|^2 - |z_2|^2 \right)$$

$$\leq 2|1 - \xi^l| \cdot |z_2|^2 + \left(|z_1|^2 - |z_2|^2 \right),$$

where we used $|1 - \xi^{l+1}| \leq |1 - \xi| + |1 - \xi^l| \leq 2|1 - \xi^l|$. Observe that we have

$$|z_1 - z_2| \geq \left| \frac{z_1}{|z_1|} |z_2| - z_2 \right| \geq |z_2| \cdot |1 - \xi^l| \qquad (4.7)$$

and

$$|z_1|^2 - |z_2|^2 = (|z_1| - |z_2|) \cdot (|z_1| + |z_2|) \leq |z_1 - z_2| \cdot (|z_1| + |z_2|). \qquad (4.8)$$

Therefore, we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \left| Y_{\sqrt{t},\theta}(z_1) - Y_{\sqrt{t},\theta}(z_2) \right| dt \, d\theta \leq 2|z_1 - z_2| \cdot |z_2| + |z_1 - z_2| \cdot (|z_1| + |z_2|), \qquad (4.9)$$

which implies (4.4). $\qquad \square$

Lemma 4.2 can be considered as an extension of [3, Lemma 5] and [54,55, Section 3.2]. The novel point here is that we introduce an extra degree of randomness in the argument of z in order to handle the difficulty caused by cyclic signatures. Actually, this provides a random k-partition parametrized by an angle θ, which will be discussed further in Sect. 6. This lemma is a version of a coarea inequality, which becomes transparent from the following direct consequence.

For any non-zero function $f : V \to \mathbb{C}$ defined on the vertices of a graph G and any $t \in [0, \max_{u \in V} |f(u)|]$, we define the following non-empty subset of V:

$$V^f(t) := \{u \in V \mid t \leq |f(u)|\}. \qquad (4.10)$$

Lemma 4.3 (Coarea inequality) *Let* $s : E^{or} \to S_k^1$ *be a signature of G. For any function* $f : V \to \mathbb{C}$ *with* $\max_{u \in V} |f(u)| = 1$, *we have*

$$\int_0^1 \iota^s \left(V^f(\sqrt{t}) \right) + \left| E \left(V^f(\sqrt{t}), (V^f(\sqrt{t}))^c \right) \right| dt$$

$$\leq 2 \sum_{\{u,v\} \in E} w_{uv} |f(u) - s_{uv} f(v)| \cdot (|f(u)| + |f(v)|). \tag{4.11}$$

Proof First observe that

$$\frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \sum_{\{u,v\} \in E} w_{uv} \left| Y_{\sqrt{t},\theta}(f(u)) - s_{uv} Y_{\sqrt{t},\theta}(f(v)) \right| dt \, d\theta$$

$$\geq \int_0^1 \iota^s \left(V^f(\sqrt{t}) \right) + \left| E(V^f(\sqrt{t}), (V^f(\sqrt{t}))^c) \right| dt. \tag{4.12}$$

In fact, the summation in the integrand of the LHS of the above inequality can be split into two parts: (i) The summation over edges connecting two vertices from $V^f(\sqrt{t})$ and $V^f(\sqrt{t})^c$, respectively. This part equals to $\left| E(V^f(\sqrt{t}), (V^f(\sqrt{t}))^c) \right|$; (ii) The summation over edges connecting two vertices from $V^f(\sqrt{t})$. This part is bounded from below by $\iota^s \left(V^f(\sqrt{t}) \right)$ by Definition 3.4.

Notice further that

$$s_{uv} Y_{\sqrt{t},\theta}(f(v)) = Y_{\sqrt{t},\theta}(s_{uv} f(v)), \tag{4.13}$$

the inequality (4.11) follows directly from Lemma 4.2. □

The coarea inequality is particularly useful to prove Lemma 4.4.

Lemma 4.4 *Let* $s : E^{or} \to S_k^1$ *be a signature of G and* $f : V \to \mathbb{C}$ *be a nonzero function. Then there exists* $t' \in [0, \max_{u \in V} |f(u)|^2]$ *such that*

$$\phi_\mu^s(V^f(\sqrt{t'})) \leq 2\sqrt{2d_\mu \mathcal{R}_\mu^s(f)}, \tag{4.14}$$

where $\mathcal{R}_\mu^s(f)$ *was defined in* (2.7).

Proof Since f is non-zero, we may assume (after rescaling) that $\max_{u \in V} |f(u)| = 1$. Moreover,

$$|Y_{\sqrt{t},\theta}(f(u))| = \begin{cases} 1, & \text{if } |f(u)| \geq \sqrt{t}, \\ 0, & \text{otherwise}, \end{cases} \tag{4.15}$$

implies

$$\int_0^1 \text{vol}_\mu(V^f(\sqrt{t})) \, dt = \int_0^1 \sum_{u \in V} \left| Y_{\sqrt{t},\theta}(f(u)) \right| \mu(u) \, dt = \sum_{u \in V} |f(u)|^2 \mu(u). \tag{4.16}$$

Now we consider the quotient

$$I := \frac{\int_0^1 \iota^s(V^f(\sqrt{t})) + \left| E(V^f(\sqrt{t}), (V^f(\sqrt{t}))^c) \right| dt}{\int_0^1 \text{vol}_\mu(V^f(\sqrt{t})) dt}. \tag{4.17}$$

Therefore, there exists $t' \in [0, 1]$ such that

$$I \geq \phi_\mu^s(V^f(\sqrt{t'})). \tag{4.18}$$

On the other hand, Lemma 4.3, (4.16), and the Cauchy–Schwarz inequality imply

$$I \leq \frac{2\sum_{\{u,v\}\in E} w_{uv}|f(u) - s_{uv}f(v)| \cdot (|f(u)| + |f(v)|)}{\sum_{u\in V}|f(u)|^2\mu(u)}$$

$$\leq \frac{2\sqrt{\sum_{\{u,v\}\in E} w_{uv}|f(u) - s_{uv}f(v)|^2}\sqrt{\sum_{\{u,v\}\in E} w_{uv}(|f(u)| + |f(v)|)^2}}{\sum_{u\in V}|f(u)|^2\mu(u)}.$$

Since

$$\sum_{\{u,v\}\in E} w_{uv}(|f(u)| + |f(v)|)^2 \leq 2\sum_{\{u,v\}\in E} w_{uv}(|f(u)|^2 + |f(v)|^2)$$

$$= 2\sum_{u\in V}\sum_{v,v\sim u} w_{uv}|f(u)|^2,$$

we conclude that

$$I \leq 2\sqrt{2d_\mu \mathcal{R}_\mu^s(f)}. \tag{4.19}$$

Combining the estimates (4.18) and (4.19) proves the lemma. □

Proof of Theorem 4.1 The upper estimate in (4.1) follows from Lemma 4.4 by setting f to be the eigenfunction corresponding to the eigenvalue $\lambda_1(\Delta_\mu^s)$.

It remains to prove the lower estimate of $h_1^s(\mu)$ in (4.1). Let $\widetilde{V}$ be the subset of V that achieves the Cheeger constant $h_1^s(\mu)$ in (3.7) with induced subgraph $(\widetilde{V}, \widetilde{E})$ and $\widetilde{\tau}: \widetilde{V} \to S_k^1$ be the switching function that achieves the frustration index $\iota^s(\widetilde{V})$ in (3.1). Define the function $\widetilde{f}: V \to \mathbb{C}$ via:

$$\widetilde{f}(u) := \begin{cases} \widetilde{\tau}(u), & \text{if } u \in \widetilde{V}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.20}$$

Using (2.8) and the estimate $|\widetilde{\tau}(u) - s_{uv}\widetilde{\tau}(v)| \leq 2$, we obtain

$$\lambda_1(\Delta_\mu^s) \leq \mathcal{R}_\mu^s(\widetilde{f})$$

$$= \frac{\sum_{\{u,v\}\in\widetilde{E}} w_{uv}|\widetilde{\tau}(u) - s_{uv}\widetilde{\tau}(v)|^2 + |E(\widetilde{V}, \widetilde{V}^c)|}{\text{vol}_\mu(\widetilde{V})}$$

$$\leq \frac{2\iota^s(\widetilde{V}) + |E(\widetilde{V}, \widetilde{V}^c)|}{\text{vol}_\mu(\widetilde{V})}$$

$$\leq 2h_1^s(\mu). \tag{4.21}$$

□

Remark 4.5 Since the signature is S_k^1-valued, the constant 2 in (4.21) can be slightly improved to be $|1 - \xi^{(k-1)/2}|$ when k is odd.

For $\Gamma = U(1)$ we have the following Cheeger's inequality.

Theorem 4.6 *Let G be a finite graph with signature $s: E^{or} \to U(1)$. Then*

$$\frac{1}{2}\lambda_1(\Delta_\mu^s) \leq h_1^s(\mu) \leq \frac{3}{2}\sqrt{2d_\mu\lambda_1(\Delta_\mu^s)}. \tag{4.22}$$

The constant in the upper bound of (4.22) is slightly better than the constant in (4.1). This is due to Lemma 4.7 below.

For any $t \in (0, 1]$, we define $X_t : \overline{B_1(0)} \to \mathbb{C}$ as

$$X_t(z) := \begin{cases} z/|z|, & \text{if } z \in \overline{B_1(0)} \setminus B_t(0), \\ 0, & \text{if } z \in B_t(0). \end{cases} \tag{4.23}$$

Lemma 4.7 *For any two points $z_1, z_2 \in \overline{B_1(0)}$, we have*

$$\int_0^1 \left| X_{\sqrt{t}}(z_1) - X_{\sqrt{t}}(z_2) \right| dt \leq \frac{3}{2}|z_1 - z_2|(|z_1| + |z_2|). \tag{4.24}$$

Proof W.l.o.g., we assume that $|z_1| \geq |z_2| > 0$. Observe that

$$\int_0^1 \left| X_{\sqrt{t}}(z_1) - X_{\sqrt{t}}(z_2) \right| dt \leq \left| \frac{z_1}{|z_1|} - \frac{z_2}{|z_2|} \right| |z_2|^2 + (|z_1|^2 - |z_2|^2). \tag{4.25}$$

Recalling (4.7), we have

$$\left| \frac{z_1}{|z_1|} - \frac{z_2}{|z_2|} \right| |z_2|^2 \leq |z_1 - z_2||z_2| \leq \frac{1}{2}|z_1 - z_2|(|z_1| + |z_2|). \tag{4.26}$$

Combining this with (4.8) proves the lemma. $\square$

With this lemma at hand, the proofs of Theorems 4.1 and 4.6 are very similar. We omit the details but mention the following analogue of Lemma 4.4.

Lemma 4.8 *Let $s : E^{or} \to U(1)$ be a signature of G and $f : V \to \mathbb{C}$ be a nonzero function. Then there exists $t' \in [0, \max_{u \in V} |f(u)|^2]$ such that*

$$\phi_\mu^s(V^f(\sqrt{t'})) \leq \frac{3}{2}\sqrt{2d_\mu \mathcal{R}_\mu^s(f)}. \tag{4.27}$$

Remark 4.9 We notice that the inequality (4.22) for $\Gamma = U(1)$ overlaps with a Cheeger inequality for a connection Laplacian of G discussed by Bandeira, Singer and Spielman [4] to solve a partial synchronization problem. The connection Laplacian $\mathcal{L}$ is defined for a simple graph G where a matrix $O_{uv} \in O(l)$ is assigned to each $(u, v) \in E^{or}$ such that $O_{vu} = (O_{uv})^{-1}$. For any vector-valued function $f : V \to \mathbb{R}^l$ and any vertex $u \in V$, we then have

$$\mathcal{L}f(u) := \frac{1}{d_u} \sum_{v, v \sim u} w_{uv}(f(u) - O_{uv}f(v)) \in \mathbb{R}^l. \tag{4.28}$$

For a graph G with signature $s : E^{or} \to U(1)$ we consider the particular positive measure μ on V defined as $\mu(u) := d_u$ and rewrite the value $s_{uv} := a_{uv} + ib_{uv} \in U(1)$ for each $(u, v) \in E^{or}$ as

$$\begin{pmatrix} a_{uv} & -b_{uv} \\ b_{uv} & a_{uv} \end{pmatrix} \in SO(2). \tag{4.29}$$

If we also rewrite a complex valued function $f := f_1 + if_2$ as an $\mathbb{R}^2$-valued function $f := (f_1, f_2)^T$, the discrete magnetic Laplacian Δ_μ^s translates into a connection Laplacian $\mathcal{L}^s$ with eigenvalues

$$0 \leq \lambda_1(\Delta_\mu^s) = \lambda_1(\Delta_\mu^s) \leq \cdots \leq \lambda_N(\Delta_\mu^s) = \lambda_N(\Delta_\mu^s). \tag{4.30}$$

Thus, each eigenvalue $\lambda_i(\Delta_\mu^s)$ of Δ_μ^s is an eigenvalue of $\mathcal{L}^s$ with doubled multiplicity. If we denote the Euclidean norm in $\mathbb{R}^l$ by $\|\cdot\|$, Bandeira, Singer and Spielman define a *(partial)* ℓ_1-*frustration constant* as

$$\eta_{G,1}^* := \min_{\tau:V\to\mathbb{S}^{l-1}\cup\{0\}} \frac{\sum_{u,v\in V} w_{uv}\|\tau(u) - O_{uv}\tau(v)\|}{\sum_{u\in V} d_u\|\tau(u)\|}, \qquad (4.31)$$

and prove that

$$\lambda_1(\mathcal{L}) \le \eta_{G,1}^* \le \sqrt{10\lambda_1(\mathcal{L})}. \qquad (4.32)$$

If we assign elements of *SO(2)* to edges of G (instead of $O(2)$), we observe that

$$\eta_{G,1}^* = 2h_1^s(\mu), \quad \text{and} \quad \lambda_1(\mathcal{L}^s) = \lambda_1(\Delta_\mu^s). \qquad (4.33)$$

Hence, inequality (4.32) leads to inequality (4.22). Finally, Bandeira, Singer and Spielman have a refined analysis for (4.24) that improves the constant $3/2$ in (4.24) and (4.22) to $\sqrt{5}/2$, [4, Appendix A].

A direct corollary of Theorems 4.1 and 4.6 as well as Proposition 3.7 is the following characterization of the case that the first eigenvalue vanishes.

Corollary 4.10 $\lambda_1(\Delta_\mu^s) = 0$ *if and only if the underlying graph has a balanced connected component.*

We remark that Corollary 4.10 can also be easily derived by the min–max principle (2.8).

5 Spectral clustering via lens spaces and complex projective spaces

In this section, we prove the following higher order Cheeger inequalities.

Theorem 5.1 *There exists an absolute constant $C > 0$ such that for any finite graph G with signature s and all $n \in [N]$, we have*

$$\frac{1}{2}\lambda_n(\Delta_\mu^s) \le h_n^s(\mu) \le Cn^3\sqrt{d_\mu\lambda_n(\Delta_\mu^s)}. \qquad (5.1)$$

Note that in Theorem 5.1 the signature group Γ can be either S_k^1 or $U(1)$.

The upper bound of $h_n^s(\mu)$ in (5.1) is the essential part of Theorem 5.1 and its proof relies on the development of a proper spectral clustering algorithm for the operator Δ_μ^s. In other words, we aim to find an n-subpartition $\{V_p\}_{[n]}$ with small constants $\phi_\mu^s(V_p)$, based on the information contained in the eigenfunctions of the operator Δ_μ^s.

Let f_i be an orthonormal family of eigenfunctions corresponding to $\lambda_i(\Delta_\mu^s)$ for $i \in [n]$. We consider the following map:

$$F : V \to \mathbb{C}^n, \quad F(u) = (f_1(u), f_2(u), \dots, f_n(u)). \qquad (5.2)$$

Since $\lambda_n(\Delta_\mu^s) = \mathcal{R}_\mu^s(f_n)$, the Rayleigh quotient of F is also bounded by $\lambda_n(\Delta_\mu^s)$:

$$\begin{aligned}
\mathcal{R}_\mu^s(F) &:= \frac{\sum_{\{u,v\}\in E} w_{uv}\|F(u) - s_{uv}F(v)\|^2}{\sum_{u\in V} \mu(u)\|F(u)\|^2} \\
&= \frac{\sum_{p\in[n]}\sum_{\{u,v\}\in E} w_{uv}|f_p(u) - s_{uv}f_p(v)|^2}{\sum_{p\in[n]}\sum_{u\in V} \mu(u)|f_p(u)|^2} \\
&\le \lambda_n(\Delta_\mu^s), \qquad (5.3)
\end{aligned}$$

where $\| \cdot \|$ stands for the standard Hermitian norm in $\mathbb{C}^n$. Our goal is to construct n maps $\Psi_p : V \to \mathbb{C}^n$, $p \in [n]$, with pairwise disjoint supports such that

(1) each Ψ_p can be viewed as a localization of F, i.e., Ψ_p is the product of F and a cut-off function $\eta : V \to \mathbb{R}$ (see 5.13 below),
(2) each Rayleigh quotient satisfies $\mathcal{R}_\mu^s(\Psi_p) \leq C(n) \mathcal{R}_\mu^s(F)$, where $C(n)$ is a constant only depending on n.

Then, applying Lemmas 4.4 and 4.8 will finish the proof.

This strategy is adapted from the proof of the higher order Cheeger inequalities for unsigned graphs due to Lee et al. [28,29]. A critical new point here is to find a proper metric on the space of points $\{F(u) | u \in V\} \subset \mathbb{C}^n$ for the spectral clustering algorithm. In other words, we need a proper metric to localize the map F. The original algorithm in [28,29] used a spherical metric. The second author [32] studied a spectral clustering via metrics on real projective spaces to prove higher order dual Cheeger inequalities for unsigned graphs. Later in [3], the above two algorithms and, hence, the corresponding two kinds of inequalities, were unified in the framework of Harary's signed graphs, i.e., graphs with signatures $s : E^{or} \to \{+1, -1\}$. In particular, the metrics on real projective spaces were shown to be the proper metrics for clustering in the framework of signed graphs. In our current more general setting of graphs with signatures $s : E^{or} \to \Gamma$, where $\Gamma = S_k^1$ or $\Gamma = U(1)$, the new metrics will be defined on lens spaces and complex projective spaces.

5.1 Lens spaces and complex projective spaces

In this subsection, we provide metrics of lens spaces and complex projective spaces for the spectral clustering algorithms in the case of $\Gamma = S_k^1$ and $\Gamma = U(1)$, respectively. Both lens spaces and complex projective spaces are important objects in geometry and topology. See, e.g., [26, Chapter 5] for details about these spaces.

Let $\mathbb{S}^{2n-1} := \{\mathbf{z} \in \mathbb{C}^n \mid \|\mathbf{z}\| = 1\}$ be the unit sphere in the space $\mathbb{C}^n$. Then $\Gamma \subset \mathbb{C}$ acts on $\mathbb{S}^{2n-1}$ by scalar multiplication. For any two points $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{S}^{2n-1} \subset \mathbb{C}^n$, we define the following equivalence relation:

$$\mathbf{z}_1 \sim \mathbf{z}_2 \Leftrightarrow \exists \gamma \in \Gamma \quad \text{such that } \mathbf{z}_1 = \gamma \mathbf{z}_2. \tag{5.4}$$

For $\Gamma = S_k^1$, the corresponding quotient space $\mathbb{S}^{2n-1}/\Gamma$ is the lens space $L(k; 1, \ldots, 1)$, while for $\Gamma = U(1)$, the quotient space $\mathbb{S}^{2n-1}/\Gamma$ is the complex projective space $\mathbb{C}P^{n-1}$. Let $[\mathbf{z}]$ denote the equivalence class of $\mathbf{z} \in \mathbb{S}^{2n-1}$. We consider the following metric on $\mathbb{S}^{2n-1}/\Gamma$:

$$d([\mathbf{z}_1], [\mathbf{z}_2]) := \min_{\gamma \in \Gamma} \|\mathbf{z}_1 - \gamma \mathbf{z}_2\|. \tag{5.5}$$

The space $\mathbb{S}^{2n-1}/\Gamma$ can also be endowed with a distance d_{quot} which is induced from the standard Riemannian metric on $\mathbb{S}^{2n-1} \subset \mathbb{R}^{2n}$. This induced metric has positive Ricci curvature. If $\Gamma = S_k^1$, the sectional curvature of this metric is constant equal to 1, and if $\Gamma = U(1)$, this metric is the well-known Fubini-Study metric. The two metrics d and d_{quot} on $\mathbb{S}^{2n-1}/\Gamma$ are equivalent, i.e., there exist two constants $c_1, c_2 > 0$ such that for all $[z_1], [z_2] \in S^{2n-1}/\Gamma$,

$$c_1 d_{quot}([\mathbf{z}_1], [\mathbf{z}_2]) \leq d([\mathbf{z}_1], [\mathbf{z}_2]) \leq c_2 d_{quot}([\mathbf{z}_1], [\mathbf{z}_2]). \tag{5.6}$$

Recall the concept of the metric doubling constant $\rho_{\mathbb{X}}$ of a metric space $(\mathbb{X}, d_{\mathbb{X}})$. This constant is the infimum of all numbers ρ such that every ball B in $\mathbb{X}$ can be covered by ρ balls of half the radius of B.

Proposition 5.2 *The metric doubling constant ρ_Γ of $(\mathbb{S}^{2n-1}/\Gamma, d)$ satisfies*

$$\log_2 \rho_\Gamma \leq Cn, \tag{5.7}$$

where C is an absolute constant.

Proof By equivalence (5.6), we only need to consider the metric space $(\mathbb{S}^{2n-1}/\Gamma, d_{quot})$. Since $\mathbb{S}^{2n-1}/\Gamma$ with its standard metric has nonnegative Ricci curvature, the Bishop–Gromov comparison theorem guarantees

$$\frac{\text{vol}(B_r([\mathbf{z_1}]))}{\text{vol}(B_{r/2}([\mathbf{z_1}]))} \leq \bar{C}^n, \tag{5.8}$$

for some absolute constant $\bar{C}$. (Note that the real dimension of the lens space is $2n - 1$ and of the complex projective space is $2n - 2$.) A standard argument implies now the claim of the proposition. For details see, e.g., [8, p. 67] or [32, Section 2.2]. □

The metric d on $\mathbb{S}^{2n-1}/\Gamma$ induces a pseudo metric on the space $\mathbb{C}^n \setminus \{0\}$, which—by abuse of notation—will again be denoted by d:

$$d(\mathbf{z}_1, \mathbf{z}_2) := d\left(\left[\frac{\mathbf{z}_1}{\|\mathbf{z}_1\|}\right], \left[\frac{\mathbf{z}_2}{\|\mathbf{z}_2\|}\right]\right). \tag{5.9}$$

The following obvious property is the reason why we use the metric d on S^{2n-1}/Γ from (5.5). This reason will become clear in the next Sect. 5.2.

Proposition 5.3 *For every pair $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{C}^n \setminus \{0\}$ and every $\gamma \in \Gamma$, we have*

$$d(\mathbf{z}_1, \mathbf{z}_2) = d(\mathbf{z}_1, \gamma \mathbf{z}_2). \tag{5.10}$$

The considerations of the next two subsections prepare the ground for the study of the Rayleigh quotient $\mathcal{R}^s_\mu(F)$ of the map $F : V \to \mathbb{C}^n$ defined in (5.2).

5.2 Localization of the map F

We endow the support $V_F := \{u \in V | F(u) \neq 0\}$ with the pseudo metric d_F induced by d via

$$d_F(u, v) := d(F(u), F(v)). \tag{5.11}$$

Given a subset $S \subseteq V$ and $\epsilon > 0$, we first define a cut-off function $\eta : V \to \mathbb{R}$ by

$$\eta(u) := \begin{cases} 0, & \text{if } F(u) = 0, \\ \max\{0, 1 - \frac{1}{\epsilon}d_F(u, S \cap V_F)\}, & \text{otherwise} \end{cases} \tag{5.12}$$

and then localize F via η as

$$\Psi := \eta F : V \to \mathbb{C}^n. \tag{5.13}$$

Note that the ϵ-neighborhood $N_\epsilon(S \cap V_F, d_F) := \{u \in V | d_F(u, S \cap V_F) < \epsilon\}$ of $S \cap V_F$ contains the support of the map Ψ.

In the next lemma, $G_F = (V_F, E_F)$ denotes the induced subgraph on V_F of G.

Lemma 5.4 *If $\{u, v\} \in E_F$ and $\|F(v)\| \leq \|F(u)\|$ then*

$$d(F(u), F(v))\|F(v)\| \leq \|F(u) - s_{uv}F(v)\|. \tag{5.14}$$

Proof Observe that we only need to prove

$$d(F(u), F(v))\|F(v)\| \leq \|F(u) - F(v)\| \tag{5.15}$$

for any pair of points $F(u), F(v) \in \mathbb{C}^n \setminus \{0\}$ with $\|F(v)\| \leq \|F(u)\|$: we can replace $F(v)$ in (5.15) by $s_{uv}F(v)$ and use Proposition 5.3 to obtain (5.14). By the definition of the metric d, we obtain (5.15) as follows:

$$d(F(u), F(v))\|F(v)\| \leq \left\| \frac{F(u)}{\|F(u)\|} - \frac{F(v)}{\|F(v)\|} \right\| \|F(v)\| \leq \|F(u) - F(v)\|,$$

where we used the estimate (4.7) for the latter inequality. □

Lemma 5.4 enables us to prove the following result.

Lemma 5.5 *For any $\{u, v\} \in E$, we have*

$$\|\Psi(u) - s_{uv}\Psi(v)\| \leq \left(1 + \frac{1}{\epsilon}\right) \|F(u) - s_{uv}F(v)\|. \tag{5.16}$$

Proof If at least one of $F(u)$ and $F(v)$ is equal to zero, then the estimate (5.16) holds trivially. Hence, we suppose that $u, v \in V_F$. W.l.o.g., we can assume that $\|F(u)\| \leq \|F(v)\|$ and calculate

$$\begin{aligned}
\|\Psi(u) - s_{uv}\Psi(v)\| &= \|\eta(u)F(u) - s_{uv}\eta(v)F(v)\| \\
&\leq |\eta(u)| \cdot \|F(u) - s_{uv}F(v)\| + |\eta(u) - \eta(v)| \cdot \|F(v)\| \\
&\leq \|F(u) - s_{uv}F(v)\| + \frac{d_F(u, v)\|F(v)\|}{\epsilon}.
\end{aligned}$$

Applying Lemma 5.4 completes the proof. □

Note that the inequality (5.16) is useful for the estimate of the numerator of the Rayleigh quotient of Ψ.

5.3 Decomposition of the underlying space via orthonormal functions

For later purposes, we work on a general measure space $(\mathcal{V}, \mu)$ in this subsection, where $\mathcal{V}$ is a topological space and μ is a Borel measure. Two particular cases we have in mind are a vertex set V of a finite graph with a measure $\mu : V \to \mathbb{R}^+$, and a closed Riemannian manifold with its Riemannian volume measure. We will apply the results in this subsection to the latter case in Sect. 7.

On $(\mathcal{V}, \mu)$, we further assume that there exist n measurable functions

$$f_1, f_2, \ldots, f_n : \mathcal{V} \to \mathbb{C},$$

which are orthonormal, i.e., for any $i, j \in [n]$,

$$\langle f_i, f_j \rangle := \int_{\mathcal{V}} f_i \overline{f_j} d\mu = \delta_{ij}.$$

Then the map $F : \mathcal{V} \to \mathbb{C}^n$ is given accordingly as in (5.2).

We consider the measure μ_F on $\mathcal{V}$ given by

$$d\mu_F = \|F\|^2 d\mu.$$

For any two points x, y in $\mathcal{V}_F := \{x \in \mathcal{V} : F(x) \neq 0\}$, we have the distance between them

$$d_F(x, y) := \min_{\gamma \in \Gamma} \left\| \frac{F(x)}{\|F(x)\|} - \gamma \frac{F(y)}{\|F(y)\|} \right\|. \tag{5.17}$$

The main result of this subsection is the following theorem.

Theorem 5.6 *Let $(\mathcal{V}_F, d_F, \mu_F)$ be as above. There exist an absolute constant C_0 and a nontrivial n-subpartition $\{T_i\}_{[n]}$ of $\mathcal{V}_F$ such that*

(i) $d_F(T_p, T_q) \geq \frac{2}{C_0 n^{5/2}}$, *for all $p, q \in [n]$, $p \neq q$,*

(ii) $\mu_F(T_p) \geq \frac{1}{2n} \mu_F(\mathcal{V}_F)$, *for all $p \in [n]$.*

The difficulty for the construction of the above n-subpartition is to achieve the property (ii). That is, we have to find a subpartition which possesses large enough measure. When $d_F(x, y)$ is given by the spherical distance $\left\| \frac{F(x)}{\|F(x)\|} - \frac{F(y)}{\|F(y)\|} \right\|$, Theorem 5.6 was proved in [28,29, Lemma 3.5]. In our situation, we have to deal with the metrics, given in (5.17), of lens spaces or complex projective spaces. We refer the reader to [18] for another interesting decomposition result.

An important ingredient of the proof is the following lemma derived from the random partition theory [20,30]. Note that a partition of a set A can also be considered as a map $P : A \to 2^A$, where $x \in A$ is mapped to the unique set $P(x)$ of the partition that contains x. A random partition $\mathcal{P}$ of A is a probability measure ν on a set of partitions of A. Then $\mathcal{P}(x)$ is understood as a random variable from the probability space to subsets of A containing x.

Lemma 5.7 *Let A be a subset of the metric space $(\mathbb{S}^{2n-1}/\Gamma, d)$ (for d recall 5.5). Then for every $r > 0$ and $\delta \in (0, 1)$, there exists a random partition $\mathcal{P}$ of A, i.e., a distribution ν over partitions of A such that*

(i) $\mathrm{diam}(S) \leq r$ *for any S in every partition P in the support of ν,*

(ii) $\mathbb{P}_\nu \left[B_{r/\alpha}(x) \subseteq \mathcal{P}(x) \right] \geq 1 - \delta$ *for all $x \in A$, where $\alpha = 32 \log_2(\rho_\Gamma)/\delta$.*

We refer to [20, Theorem 3.2] and [30, Lemma 3.11] for the proof, see also [32, Theorem 2.4]. For convenience, we describe briefly the construction of the random partition claimed in Lemma 5.7. Let $\{x_i\}_{[m]}$ be a $r/4$-net of $\mathbb{S}^{2n-1}/\Gamma$, that is, $d(x_i, x_j) \geq r/4$, for any $i \neq j$, and $\mathbb{S}^{2n-1}/\Gamma = \bigcup_{i \in [m]} B_{r/4}(x_i)$. Since $(\mathbb{S}^{2n-1}/\Gamma, d)$ is compact, m is a finite number. For $R \in [r/4, r/2]$, we construct a partition of $(\mathbb{S}^{2n-1}/\Gamma, d)$ as follows. A permutation σ of the set $[m]$ provides an order for all points in the net which is used to define, for every $i \in [m]$,

$$S_i^{R,\sigma} := \left\{ x \in \mathbb{S}^{2n-1}/\Gamma \mid x \in B_R(x_i) \text{ and } \sigma(i) < \sigma(j) \text{ for all } j \in [m] \text{ with } x \in B_R(x_j) \right\}.$$

That is, we have $x \in S_i^{R,\sigma}$ if $\sigma(i)$ is the smallest number for which x is contained in $B_R(x_i)$. Then $P^{R,\sigma} = \{S_i^{R,\sigma}\}_{[m]}$ constitutes a partition of $\mathbb{S}^{2n-1}/\Gamma$. Now let σ be a uniformly random permutation of $[m]$, and R be chosen uniformly random from the interval $[r/4, r/2]$. These choices define a random partition $\mathcal{P}$. If we choose R uniformly from a fine enough discretization of the interval $[r/4, r/2]$, we can make $\mathcal{P}$ to be finitely supported. In fact, this random partition fulfills the two properties in Lemma 5.7.

Remark 5.8 Lemma 5.7 holds true for any metric space. In particular, the finiteness of the $r/4$-net is not necessary. This is shown in [30, Lemma 3.11].

Lemma 5.7 leads to the following result. Note that, the property (ii) in Lemma 5.7 ensures the existence of at least one subpartition which captures a large fraction of the whole measure.

Lemma 5.9 *On* $(\mathcal{V}_F, d_F, \mu_F)$, *for any* $r > 0$ *and* $\delta \in (0, 1)$, *there exists a nontrivial subpartition* $\{\widehat{S}_i\}_{[m]}$ *such that*

(i) $\operatorname{diam}(\widehat{S}_i, d_F) \leq r$ *for any* $i \in [m]$,
(ii) $d_F(\widehat{S}_i, \widehat{S}_j) \geq 2r/\alpha$, *where* $\alpha = 32 \log_2(\rho_\Gamma)/\delta$,
(iii) $\sum_{i \in [m]} \mu_F(\widehat{S}_i) \geq (1 - \delta)\mu_F(\mathcal{V}_F)$.

Proof Let $\mathcal{P}$ be the random partition on $\mathcal{V}_F$ induced from the one constructed in Lemma 5.7 via the map F. Let $I_{B_{r/\alpha}(x) \subseteq \mathcal{P}(x)}$ be the indicator function for the event that $B_{r/\alpha}(x) \subseteq \mathcal{P}(x)$ happens. Then we obtain from Lemma 5.7 (ii)

$$\mathbb{E}_\mathcal{P} \left(\int_\mathcal{V} I_{B_{r/\alpha}(x) \subseteq \mathcal{P}(x)} d\mu_F(x) \right) \geq (1 - \delta)\mu_F(\mathcal{V}) \tag{5.18}$$

by interchanging the expectation and the integral. On the other hand, we have

$$\mathbb{E}_\mathcal{P} \left(\int_\mathcal{V} I_{B_{r/\alpha}(x) \subseteq \mathcal{P}(x)} d\mu_F(x) \right)$$

$$= \sum_{P \in \mathcal{P}} \sum_{S \in P} \int_S I_{B_{r/\alpha}(x) \subseteq \mathcal{P}(x)} d\mu_F(x) \mathbb{P}_\nu(P)$$

$$= \sum_{P \in \mathcal{P}} \sum_{S \in P} \int_{\widehat{S}} d\mu_F(x) \mathbb{P}_\nu(P), \tag{5.19}$$

where $\widehat{S} := \{x \in S : B_{r/\alpha}(x) \subseteq S\}$. Hence, there exists a partition $P = \{S_i\}_{[m]}$ of $\mathcal{V}_F$ for some natural number m such that

$$\sum_{i \in [m]} \mu_F(\widehat{S}_i) \geq (1 - \delta)\mu_F(\mathcal{V}). \tag{5.20}$$

This completes the proof. □

In order to prove Theorem 5.6, we also need the following result.

Lemma 5.10 *If a subset* $S \subseteq \mathcal{V}$ *satisfies* $\operatorname{diam}(S \cap \mathcal{V}_F, d_F) \leq r$ *for some* $r \in (0, 1)$, *then*

$$\mu_F(S) \leq \frac{1}{n(1 - r^2)} \mu_F(\mathcal{V}). \tag{5.21}$$

Proof W.l.o.g., we can assume that $S \subseteq \mathcal{V}_F$. Using the fact that $f_1, \ldots, f_n$ are orthonormal, we obtain the following two properties. First, we have

$$\mu_F(\mathcal{V}) = \int_\mathcal{V} \sum_{p \in [n]} |f_p|^2 d\mu = n. \tag{5.22}$$

Second, we have for any $\mathbf{z} := (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ with $\|\mathbf{z}\| = 1$,

$$\int_\mathcal{V} |\langle \mathbf{z}, F(x) \rangle|^2 d\mu(x) = \int_\mathcal{V} \sum_{p, q \in [n]} z_p \overline{z_q} \, \overline{f_p(x)} f_q(x) d\mu(x) = 1. \tag{5.23}$$

Combining (5.22) and (5.23), we conclude for any $y \in S$,

$$\frac{\mu_F(\mathcal{V})}{n} = \int_\mathcal{V} \left| \left\langle \frac{F(y)}{\|F(y)\|}, F(x) \right\rangle \right|^2 d\mu(x)$$

$$= \int_\mathcal{V} \left| \left\langle \frac{F(y)}{\|F(y)\|}, \frac{F(x)}{\|F(x)\|} \right\rangle \right|^2 d\mu_F(x). \tag{5.24}$$

Since $|z|^2 \geq (z + \overline{z})^2 / 4$ for each $z \in \mathbb{C}$, we obtain that for any $\gamma \in \Gamma$:

$$\left| \left\langle \frac{F(y)}{\|F(y)\|}, \frac{F(x)}{\|F(x)\|} \right\rangle \right|^2 = \left| \left\langle \frac{F(y)}{\|F(y)\|}, \gamma \frac{F(x)}{\|F(x)\|} \right\rangle \right|^2$$

$$\geq \frac{1}{4} \left(2 - \left\| \frac{F(y)}{\|F(y)\|} - \gamma \frac{F(x)}{\|F(x)\|} \right\|^2 \right)^2. \tag{5.25}$$

Recalling (5.17), the definition of d_F, we arrive at

$$\frac{\mu_F(\mathcal{V})}{n} \geq \int_S \left(1 - \frac{1}{2} d_F(y, x)^2 \right)^2 d\mu_F(x) \geq (1 - r^2)\mu_F(S). \tag{5.26}$$

$\square$

Proof of Theorem 5.6 With Lemma 5.9 and 5.10 at hand, Theorem 5.6 can be proved similarly as [28,29, Lemma 3.5], see also [32, Lemma 6.2]. For convenience, we recall it here. Let $\{\widehat{S}_i\}_{[m]}$ be the subpartition constructed in Lemma 5.9. Then by Lemma 5.10, we have for each $i \in [m]$,

$$\mu_F(\widehat{S}_i) \leq \frac{1}{n(1 - r^2)} \mu_F(\mathcal{V}). \tag{5.27}$$

We apply the following procedure to $\{\widehat{S}_i\}_{[m]}$. If we can find two sets of the subpartition, say $\widehat{S}_i$ and $\widehat{S}_j$, such that

$$\mu_F(\widehat{S}_i) \leq \frac{1}{2n} \mu_F(\mathcal{V}), \quad \mu_F(\widehat{S}_j) \leq \frac{1}{2n} \mu_F(\mathcal{V}),$$

then replace them by $\widehat{S}_i \cup \widehat{S}_j$. Thus, when we stop, we obtain the sets $T_1, T_2, \ldots, T_l$ for some number l, such that

$$\mu_F(T_i) \leq \frac{1}{n(1 - r^2)} \mu_F(\mathcal{V}), \quad \forall i \in [l],$$

and

$$\mu_F(T_i) \geq \frac{1}{2n} \mu_F(\mathcal{V}), \quad \forall i \in [l - 1].$$

Setting $r = \frac{1}{3\sqrt{n}}$ and $\delta = \frac{1}{4n}$, we check that

$$(n - 1) \cdot \frac{1}{n(1 - r^2)} < 1 - \delta - \frac{1}{2n}. \tag{5.28}$$

This implies that $l \geq n$. Moreover, if we redefine $T_n := \bigcup_{j=n}^{l} T_j$, we have

$$\mu_F(T_n) \geq \frac{1}{2n} \mu_F(\mathcal{V}). \tag{5.29}$$

Thus the subpartition $\{T_i\}_{[n]}$ satisfies the property (ii). One can then verify the property (i) by Proposition 5.2 and Lemma 5.9. $\square$

5.4 Proof of Theorem 5.1

We first prove the upper bound of (5.1). Let $\{T_i\}_{[n]}$ be the subpartition of V_F obtained from Theorem 5.6. Choosing $\epsilon = \frac{1}{C_0 n^{5/2}}$, we define the cut-off functions η_p as in (5.12) (replacing the set S there by T_p). Then the maps $\Psi_p := \eta_p F$, $p \in [n]$, have pairwise disjoint support.

Recalling that $\Psi_p|_{T_p} = F|_{T_p}$, and applying Lemma 5.5 as well as fact (ii) of Theorem 5.6, we obtain that for any $p \in [n]$,

$$\mathcal{R}^s_\mu(\Psi_p) \leq \left(1 + \frac{1}{\epsilon}\right)^2 \frac{\sum_{\{u,v\} \in E} w_{uv} \|F(u) - s_{uv} F(v)\|^2}{\sum_{u \in T_p} \mu(u) \|F(u)\|^2}$$

$$\leq 2n(1 + C_0 n^{5/2})^2 \mathcal{R}^s_\mu(F) \leq C n^6 \mathcal{R}^s_\mu(F), \tag{5.30}$$

where C is an absolute constant. For every $p \in [n]$, the map Ψ_p has at least one coordinate function ψ_p that satisfies $\mathcal{R}^s_\mu(\psi_p) \leq \mathcal{R}^s_\mu(\Psi_p)$. In particular, we find functions ψ_p, $p \in [n]$, with pairwise disjoint support and an absolute constant C such that

$$\mathcal{R}^s_\mu(\psi_p) \leq C n^6 \mathcal{R}^s_\mu(F). \tag{5.31}$$

Now inequality (5.3) and Lemma 4.4 for $\Gamma = S^1_k$ or Lemma 4.8 for $\Gamma = U(1)$ yield the desired upper bound of (5.1).

Now we prove the lower bound of (5.1). Suppose that the n-way Cheeger constant $h^s_n(\mu)$ is achieved by the nontrivial n-subpartition $\{\widetilde{V}_p\}_{[n]}$ and that the function $\widetilde{\tau}_p : \widetilde{V}_p \to \Gamma$ achieves the frustration index $\iota^s(\widetilde{V}_p)$ for each $p \in [n]$. Moreover, consider functions $\widetilde{f}_p : V \to \mathbb{C}$ with pairwise disjoint support given for $p \in [n]$ by:

$$\widetilde{f}_p(u) := \begin{cases} \widetilde{\tau}_p(u), & \text{if } u \in \widetilde{V}_p; \\ 0, & \text{otherwise.} \end{cases} \tag{5.32}$$

By the min–max principle (2.8), we know

$$\lambda_n(\Delta^s_\mu) \leq \max_{a_1,\ldots,a_n} \mathcal{R}^s_\mu(\widetilde{f}_a), \tag{5.33}$$

where the maximum is taken over all complex numbers $a_1, \ldots, a_n \in \mathbb{C}$ such that the linear combination $\widetilde{f}_a := \sum_{p \in [n]} a_p \widetilde{f}_p$ of $\widetilde{f}_1, \ldots, \widetilde{f}_n$ is nontrivial. This implies

$$\sum_{u \in V} \mu(u) |\widetilde{f}_a(u)|^2 = \sum_{p \in [n]} |a_p|^2 \mathrm{vol}_\mu(\widetilde{V}_p). \tag{5.34}$$

We now want to relate (5.33) and (5.34) to the frustration index and the boundary measure. To that direction, we set $B_{uv} := w_{uv} |\widetilde{f}_a(u) - s_{uv} \widetilde{f}_a(v)|^2$ and obtain

$$\sum_{\{u,v\} \in E} B_{uv} = \frac{1}{2} \sum_{p,q \in [n]} \sum_{\substack{u \in \widetilde{V}_p \\ v \in \widetilde{V}_q}} B_{uv} + \sum_{p \in [n]} \sum_{\substack{u \in \widetilde{V}_p \\ v \in V^*}} B_{uv} + \frac{1}{2} \sum_{u,v \in V^*} B_{uv},$$

where $V^* = \left(\bigcup_{p \in [n]} \widetilde{V}_p\right)^c$. For $u, v \in \widetilde{V}_p$, $p \in [n]$, we have

$$|\widetilde{f}_a(u) - s_{uv} \widetilde{f}_a(v)|^2 = |a_p|^2 \cdot |\widetilde{\tau}_p(u) - s_{uv} \widetilde{\tau}_p(v)|^2, \tag{5.35}$$

while for $u \in \widetilde{V}_p$ and $v \in \widetilde{V}_q$ with $p, q \in [n]$ and $p \neq q$ we have

$$|\widetilde{f}_a(u) - s_{uv} \widetilde{f}_a(v)|^2 = |a_p \widetilde{\tau}_p(u) - s_{uv} a_q \widetilde{\tau}_q(v)|^2 \leq 2(|a_p|^2 + |a_q|^2). \tag{5.36}$$

Now the definitions of the frustration index and of the boundary measure yield

$$
\sum_{\{u,v\}\in E} B_{uv} \leq \sum_{p\in[n]} |a_p|^2 \left(2\iota^s(\widetilde{V}_p) + 2\Big|E(\widetilde{V}_p, \bigcup_{q\neq p} \widetilde{V}_q)\Big| + \Big|E(\widetilde{V}_p, V^*)\Big| \right)
$$

$$
\leq 2 \sum_{p\in[n]} |a_p|^2 \left(\iota^s(\widetilde{V}_p) + \Big|E(\widetilde{V}_p, \widetilde{V}_p^c)\Big| \right). \tag{5.37}
$$

If we now combine the estimates (5.33), (5.34), and (5.37), we arrive at

$$
\lambda_n(\Delta_\mu^s) \leq 2 \max_{p\in[n]} \phi_\mu^s(\widetilde{V}_p) = 2h_n^s(\mu). \tag{5.38}
$$

6 Application: spectral clustering on oriented graphs and mixed graphs

In this section, we discuss an application of the Cheeger inequalities (and their proofs) in the case $\Gamma = S_k^1$. These results indicate algorithms to find interesting substructures in an oriented graph or a mixed graph.

6.1 Generalization of Harary's balance theorem

Let us first discuss an equivalent definition of the Cheeger constant $h_1^s(\mu)$ if $\Gamma = S_k^1$. For a nonempty subset $\widetilde{V}$ of V, let $\widetilde{V}_0, \ldots, \widetilde{V}_{k-1}$ be an *ordered k-partition* of $\widetilde{V}$, that is, $\widetilde{V}_i$ are pairwise disjoint sets and their union is $\widetilde{V}$. In contrast to a nontrivial k-partition, all but one $\widetilde{V}_i$ may be empty. We write $\mathcal{V}_k(\widetilde{V})$ for an ordered k-partition $\widetilde{V}_0, \ldots, \widetilde{V}_{k-1}$ of $\widetilde{V}$.

Given an ordered k-partition $\mathcal{V}_k(\widetilde{V})$ of $\widetilde{V} \subseteq V$, we define, for $0 \leq i, j \leq k-1$ and $l \in \mathbb{Z}$,

$$
|E^l(\widetilde{V}_i, \widetilde{V}_j)| := \sum_{u\in\widetilde{V}_i} \sum_{\substack{v\in\widetilde{V}_j \text{ s.t.} \\ s_{uv}=\xi^l}} w_{uv} \tag{6.1}
$$

as the (weighted) cardinality of oriented edges with signature ξ^l that begin in $\widetilde{V}_i$ and terminate in $\widetilde{V}_j$.

Definition 6.1 Let G be a finite graph with signature $s : E^{or} \to S_k^1$. For any nonempty subset $\widetilde{V}$ of V, the *k-partiteness ratio* of an ordered k-partition $\mathcal{V}_k(\widetilde{V})$ of $\widetilde{V}$ is defined as

$$
\beta_\mu^s\left(\mathcal{V}_k(\widetilde{V})\right) = \frac{\frac{1}{2}\sum_{i,j=0}^{k-1}\sum_{l=1}^{k-1} |1-\xi^l| \cdot |E^{i-j+l}(\widetilde{V}_i, \widetilde{V}_j)| + |E(\widetilde{V}, \widetilde{V}^c)|}{\mathrm{vol}_\mu(\widetilde{V})}. \tag{6.2}
$$

The *minimal k-partiteness ratio* $\beta_\mu^s(\widetilde{V}, k)$ of $\widetilde{V}$ is defined as

$$
\beta_\mu^s(\widetilde{V}, k) := \min_{\mathcal{V}_k(\widetilde{V})} \beta_\mu^s\left(\mathcal{V}_k(\widetilde{V})\right), \tag{6.3}
$$

where the minimum is taken over all ordered k-partitions $\mathcal{V}_k(\widetilde{V})$ of $\widetilde{V}$.

The next goal is to prove that the Cheeger constant for $\Gamma = S_k^1$ can also be expressed in terms of the k-partiteness ratio, see Corollary 6.3 below.

Lemma 6.2 *Let G be a finite graph with signature $s : E^{or} \to S_k^1$. For any nonempty $\widetilde{V} \subseteq V$, we have*

$$
\phi_\mu^s(\widetilde{V}) = \beta_\mu^s(\widetilde{V}, k). \tag{6.4}
$$

Proof For any function $\tau : V_1 \to S_k^1$, we have a natural k-partition $\mathcal{V}_k(\widetilde{V})$ of $\widetilde{V}$ given by

$$\widetilde{V}_i := \{u \in \widetilde{V} \mid \tau(u) = \xi^i\} \tag{6.5}$$

for $i = 0, 1, \ldots, k - 1$. We can check that

$$\sum_{\{u,v\}\in\widetilde{E}} w_{uv}|\tau(u) - s_{uv}\tau(v)| = \frac{1}{2} \sum_{i,j=0}^{k-1} \sum_{l=1}^{k-1} |1 - \xi^l| \cdot |E^{i-j+l}(\widetilde{V}_i, \widetilde{V}_j)|. \tag{6.6}$$

Observe that the correspondence between the set of S_k^1-valued functions on $\widetilde{V}$ and the set of ordered k-partitions of $\widetilde{V}$ given by (6.5) is one-to-one. Hence, we obtain by definition of the frustration index

$$\iota^s(\widetilde{V}) = \min_{\mathcal{V}_k(\widetilde{V})} \frac{1}{2} \sum_{i,j=0}^{k-1} \sum_{l=1}^{k-1} |1 - \xi^l| \cdot |E^{i-j+l}(\widetilde{V}_i, \widetilde{V}_j)|. \tag{6.7}$$

This proves the lemma. $\qquad\square$

Corollary 6.3 *Let G be a finite graph with signature $s : E^{or} \to S_k^1$. Then*

$$h_1^s(\mu) = \min_{\emptyset \neq \widetilde{V} \subseteq V} \beta_\mu^s(\widetilde{V}, k). \tag{6.8}$$

This enables us to prove the following structural balance theorem.

Theorem 6.4 *Let G be a finite connected graph with a signature $s : E^{or} \to S_k^1$. Then the following statements are equivalent:*

(i) *The signature s is balanced.*
(ii) *There exists an ordered k-partition $V_0, \ldots, V_{k-1}$ of V such that all edges that begin in V_i and terminate in V_j have signature ξ^{i-j} for all $0 \leq i, j \leq k - 1$.*

Proof Recall that $h_1^s(\mu) = 0$ if and only if the signature is balanced. The theorem is then a direct consequence of (6.8). $\qquad\square$

Remark 6.5 Harary's balance theorem [21] states that a signature $s : E^{or} \to \{\pm 1\}$ is balanced if and only if there exists a bipartition V_0, V_1 of V such that an edge has signature -1 if and only if it has one end point in V_0 and one in V_1. Theorem 6.4 is a natural generalization of Harary's theorem.

In Figure 1, we schematically illustrate the situation of Theorem 6.4 if $k \in \{3, 4\}$. The class of edges that begin and terminate in V_i are represented by one unoriented edge labeled by $\xi^0 = 1$. For distinct i, j, the class of edges with endpoints in V_i and V_j are represented by an oriented edge that begins in V_i and terminates in V_j with $i < j$. These oriented edges are labeled by ξ^{i-j}.

6.2 Finding a good substructure

The proof of Cheeger's inequality in Sect. 4, especially Lemma 4.4, actually indicates an algorithm to find a subset $\widetilde{V} \subseteq V$ with a constant $\phi_\mu^s(\widetilde{V})$ close to the Cheeger constant $h_1^s(\mu)$ of G. In other words, $\phi_\mu^s(\widetilde{V})$ is not larger than the upper bound for $h_1^s(\mu)$ given in Cheeger's inequality (Theorem 4.1): for every nonzero function $f : V \to \mathbb{C}$, Lemma 4.4 provides a nonempty subset $\widetilde{V} := V^f(\sqrt{t'}) \subseteq V$ satisfying (4.14). If we choose f to be the
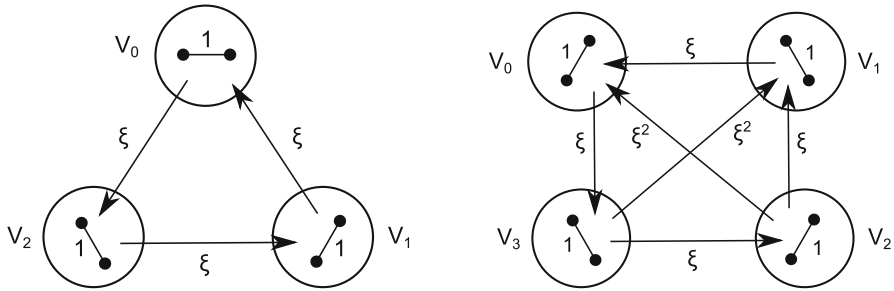
Fig. 1 Schematic illustration of Theorem 6.4 for $k = 3$ (*left*) and $k = 4$ (*right*)

eigenfunction corresponding to $\lambda_1(\Delta_\mu^s)$, we see that $\widetilde{V}$ is a nonempty subset of V with the required property.

Now consider a finite graph G with a k-cyclic signature s. From Lemma 6.2, we know that $\phi_\mu^s(\widetilde{V})$ agrees with the minimum of the k-partiteness ratios of all ordered k-partitions $\mathscr{V}_k(\widetilde{V})$. Having found a nonempty subset $\widetilde{V} := V^f(\sqrt{t'}) \subseteq V$ satisfying (4.14), we explain in this subsection, how to find a finer substructure of $\widetilde{V}$, namely an ordered k-partition $\mathscr{V}_k(\widetilde{V})$ with a k-partiteness ratio that is at most the upper bound given in (4.14). The precise statement is given in Proposition 6.6 below.

Recall the notation Q_j^θ and $V^f(t)$ of (4.2) and (4.10), respectively. Given $t \in [0, 1]$ and $\theta \in [0, 2\pi)$, we define an ordered k-partition $\mathscr{V}_k(V^f(\sqrt{t}, \theta))$ of $V^f(\sqrt{t}, \theta) \subseteq V$ by

$$V_j^f(\sqrt{t}, \theta) := \{u \in V \mid \sqrt{t} \leq |f(u)| \text{ and } f(u) \in Q_j^\theta\} \tag{6.9}$$

for $0 \leq j \leq k - 1$ and modify Lemma 4.4 into the following result.

Proposition 6.6 *Let* $s : E^{or} \to S_k^1$ *be a signature of* G. *For any nonzero function* $f : V \to \mathbb{C}$ *with* $\max_{u \in V} |f(u)| = 1$, *there exist* $t' \in [0, 1]$ *and* $\theta' \in [0, 2\pi)$ *such that*

$$\beta_\mu^s\left(\mathscr{V}_k(V^f(\sqrt{t'}, \theta'))\right) \leq 2\sqrt{2d_\mu \mathcal{R}_\mu^s(f)}. \tag{6.10}$$

Proof Instead of inequality (4.12), we consider the equality

$$\frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \sum_{\{u,v\} \in E} w_{uv} \left| Y_{\sqrt{t},\theta}(f(u)) - s_{uv} Y_{\sqrt{t},\theta}(f(v)) \right| dt \, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \left(\frac{1}{2} \sum_{i,j=0}^{k-1} \sum_{l=1}^{k-1} \left| 1 - \xi^l \right| \cdot \left| E^{i-j+l}(W_i, W_j) \right| + \left| E(\widetilde{V}, \widetilde{V}^c) \right| \right) dt \, d\theta.$$

where $W_j := V_j^f(\sqrt{t}, \theta)$ and $\widetilde{V} := V^f(\sqrt{t})$. The remaining proof follows along similar arguments as the ones given in the proof of Lemma 4.4. □

This Proposition provides the following spectral clustering algorithm to find an ordered k-subpartition of V with a k-partiteness ratio bounded above by the upper bound in Cheeger's inequality. Firstly, find the eigenfunction $f_1 : V \to \mathbb{C}$ corresponding to $\lambda_1(\Delta_\mu^s)$. For convenience, we can normalize f_1 such that $\max_{u \in V} |f(u)| = 1$. Secondly, find the required ordered k-subpartition from the sets (6.9) by running over fine enough discretizations of the parameters t and θ.
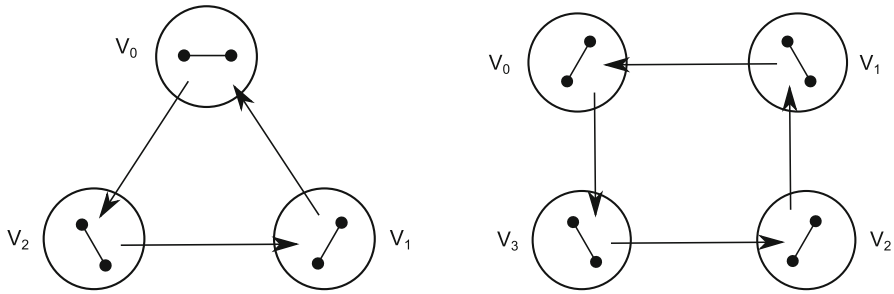
Fig. 2 Ideal substructures of partially oriented graphs in case of $k = 3$ (*left*) $k = 4$ (*right*) that are approximated in Sect. 6.3

6.3 Applications to partially oriented graphs

In this subsection, we consider *mixed graphs* instead of undirected graphs which are studied in scheduling problems, for example [45,52]. Recall that a *mixed graph* is a graph $G = (V, E_U \cup E_O)$ that consists of unoriented edges (the set E_U) as well as oriented edges (the set E_O) such that no two vertices $u, v \in V$ form more than one edge of $E_U \cup E_O$. As mentioned in the introduction, we call such a graph also *partially oriented*. Clearly, a partially oriented graph is an *oriented graph* if and only if $E_U = \emptyset$. The algorithm discussed in the previous subsection has interesting applications for partially oriented graphs.

Given a partially oriented graph $G = (V, E_U \cup E_O)$ and a natural number k, we now want to find a nonempty subset $\widetilde{V} \subseteq V$ and an ordered k-subpartition $\mathscr{V}_k(\widetilde{V}) = \{V_0, V_1, \ldots, V_{k-1}\}$ of $\widetilde{V}$ which approximates the following ideal substructure:

(i) The subset $\widetilde{V}$ has empty boundary.
(ii) An edge $e \in E_U \cup E_O$ with endpoints $u, v \in V_i$ for some $0 \leq i \leq k - 1$ is unoriented, that is, $e \in E_U$.
(iii) The partially oriented subgraph $G_{\widetilde{V}}$ induced by $\widetilde{V}$ has the following *cyclic property*: the only oriented edges of $G_{\widetilde{V}}$ begin in V_i and end in V_{i-1} for some $0 \leq i \leq k - 1$ where we identify V_{-1} and V_{k-1}.

Such ideal substructures are schematically illustrated in Figure 2 for $k = 3$ and $k = 4$.

Our approach to this problem is to construct an unoriented graph $G = (V, E)$ with a k-cyclic signature s from a given partially oriented graph $G = (V, E_U \cup E_O)$. More precisely, we consider the new edge set $E := E_U \cup E_O$ where the orientations in E_O are dropped and define a signature $s : E^{or} \rightarrow S_k^1$ by assigning to every edge $\{u, v\} \in E$ the value

$$s_{uv} := \begin{cases} 1, & \text{if } \{u, v\} \in E_U; \\ \xi, & \text{if } (u, v) \in E_O; \\ \xi^{-1} & \text{if } (v, u) \in E_O. \end{cases} \tag{6.11}$$

This construction to transform a connected partially oriented graph G is set up in such a way that the signature is balanced if and only if G has the above ideal structure. Using the eigenfunction of the eigenvalue $\lambda_1(\Delta_\mu^s)$, we apply the spectral clustering algorithm discussed in the Sect. 6.2 to find a k-subpartition $\mathscr{V}_k(\widetilde{V})$ of some $\widetilde{V} \subseteq V$ with k-partiteness ratio $\beta_\mu^s(\mathscr{V}_k(\widetilde{V}))$ at most the upper bound given in Cheeger's inequality. Note that the k-partiteness ratio can be viewed as a measure to quantify the quality of an approximation to the ideal case

which is achieved if and only if $\beta_\mu^s(\mathscr{V}_k(\widetilde{V})) = 0$. By Corollary 6.3, the k-partiteness ratio $\beta_\mu^s(\mathscr{V}_k(\widetilde{V}))$ is bounded from below by the Cheeger constant $h_1^s(\mu)$.

We remark that in the special situation were we start with an oriented graph, the ordered k-subpartition $V_0, V_1, \ldots, V_{k-1}$ of V approximates an ideal substructure with no edges having both endpoints in V_i for some $0 \leq i \leq k-1$.

These considerations can clearly be extended to obtain multi-way spectral clustering algorithms. Combining the method here with the spectral clustering via metrics on lens spaces in Sect. 5, we can find n subgraphs where each subgraph defines a sparse cut and approximates an ideal substructure as described above.

7 Magnetic Laplacians on Riemannian manifolds

In this section, we transfer the ideas related to Cheeger constants and Cheeger inequalities from discrete magnetic Laplacians to the Riemannian setting.

Let M be a closed connected Riemannian manifold. We consider a real smooth 1-form $\boldsymbol{\alpha}$ and the corresponding *magnetic Laplacian* $\Delta^{\boldsymbol{\alpha}}$ on M, defined as

$$\Delta^{\boldsymbol{\alpha}} = D^*D, \tag{7.1}$$

where the operator $D := d + i\boldsymbol{\alpha}$, d is the exterior differential, maps smooth complex valued functions to smooth complex valued 1-forms and D^* is the formal adjoint of D w.r.t. the L^2-inner product of functions and 1-forms:

$$\int_M \langle Df, \boldsymbol{\eta} \rangle dx = \int_M f \overline{D^*\boldsymbol{\eta}} dx. \tag{7.2}$$

The 1-form $\boldsymbol{\alpha}$ is called the *magnetic potential*. One can check that for any smooth function $f : M \to \mathbb{C}$,

$$\Delta^{\boldsymbol{\alpha}} f := \Delta f - 2i\langle df, \boldsymbol{\alpha} \rangle + \left(id^*\boldsymbol{\alpha} + |\boldsymbol{\alpha}|^2\right) f, \tag{7.3}$$

where d^*, $\Delta := d^*d$ is the Laplace–Beltrami operator, $\langle \cdot, \cdot \rangle$ the Hermitian inner product in the cotangent bundle T^*M induced by the Riemannian metric, and $|\boldsymbol{\alpha}|^2 := \langle \boldsymbol{\alpha}, \boldsymbol{\alpha} \rangle$.

We recall some basic spectral properties of the magnetic Laplacian from [47] (see also [43, Section 4]). The operator $\Delta^{\boldsymbol{\alpha}}$ is essentially self-adjoint as an operator defined on smooth complex valued functions (with compact support). Its self-adjoint extension is defined on a dense subset of the Hilbert space $L^2(M, \mathbb{C})$ of complex valued square integrable functions w.r.t the Riemannian measure. In the sequel, we will use the same notation for both, the essentially self-adjoint operator and its closed self-adjoint extension. Since M is compact, $\Delta^{\boldsymbol{\alpha}}$ has only discrete spectrum, and the eigenvalues can be listed with multiplicity as follows (see [47, Theorem 2.1])

$$0 \leq \lambda_1(\Delta^{\boldsymbol{\alpha}}) \leq \lambda_2(\Delta^{\boldsymbol{\alpha}}) \leq \cdots \nearrow \infty. \tag{7.4}$$

Due to (7.1), the corresponding Rayleigh quotient of a smooth function $f : M \to \mathbb{C}$ is given by

$$\mathcal{R}^{\boldsymbol{\alpha}}(f) := \frac{\int_M |(d + i\boldsymbol{\alpha})f|^2 dx}{\int_M |f|^2 dx}. \tag{7.5}$$

The min–max principle (2.8) still holds in this setting. In particular, we have

$$\lambda_1(\Delta^{\boldsymbol{\alpha}}) = \inf_{\substack{f \in C^\infty(M, \mathbb{C}) \\ \text{s.t. } f \not\equiv 0}} \mathcal{R}^{\boldsymbol{\alpha}}(f), \tag{7.6}$$

where $C^\infty(M, \mathbb{C})$ is the set of smooth complex valued functions.

Consider $U(1)$ as a subset $\{z \in \mathbb{C} \mid |z| = 1\}$ of $\mathbb{C}$ and denote the set of smooth maps from M to $U(1)$ by $C^\infty(M, U(1))$. For $\tau \in C^\infty(M, U(1))$, we then define by

$$\boldsymbol{\alpha}_\tau := \frac{d\tau}{i\tau} \tag{7.7}$$

a smooth 1-form. The set $\mathfrak{B} := \{\boldsymbol{\alpha}_\tau \mid \tau \in C^\infty(M, U(1))\}$ has the following characterization due to Shigekawa, [47, Proposition 3.1 and Theorem 4.2]. Since $a\boldsymbol{\alpha}_\tau = \boldsymbol{\alpha}_{\tau^a}$ for $a \in \mathbb{R}$ and $\boldsymbol{\alpha}_\tau + \boldsymbol{\alpha}_{\tau'} = \boldsymbol{\alpha}_{\tau\tau'}$, $\mathfrak{B}$ is in fact a real vector space.

Theorem 7.1 *(Shigekawa) The following statements are equivalent:*

  (i) $\lambda_1(\Delta^{\boldsymbol{\alpha}}) = 0$;
  (ii) $\boldsymbol{\alpha} \in \mathfrak{B}$;
 (iii) $d\boldsymbol{\alpha} = 0$ *and* $\int_C \boldsymbol{\alpha} = 0 \mod 2\pi$, *for any closed curve C in M.*

This result can be compared with Corollary 4.10: the set $\mathfrak{B}$ is comparable to the set of balanced signatures in the discrete setting. Locally, we can find a smooth real-valued function θ such that $\tau = e^{i\theta}$ and $\boldsymbol{\alpha}_\tau = d\theta$.

In the discrete setting, Laplacians Δ^s_μ with switching equivalent signatures are unitarily equivalent by (2.11) while magnetic Laplacians $\Delta^{\boldsymbol{\alpha}}$ are unitarily equivalent under *gauge transformations* in the smooth setting. Recall that a gauge transformation

$$\boldsymbol{\alpha} \mapsto \boldsymbol{\alpha} + \boldsymbol{\alpha}_\tau \tag{7.8}$$

is associated to any $\tau \in C^\infty(M, U(1))$. We have ([47, Proposition 3.2])

$$\overline{\tau}\Delta^{\boldsymbol{\alpha}}\tau = \Delta^{\boldsymbol{\alpha}+\boldsymbol{\alpha}_\tau}. \tag{7.9}$$

In particular, if $\boldsymbol{\alpha} \in \mathfrak{B}$, then $\Delta^{\boldsymbol{\alpha}}$ is unitarily equivalent to Δ. In other words, $\mathfrak{B}$ is the set of magnetic potentials which "can be gauged away".

Definition 7.2 Let $\boldsymbol{\alpha}$ be a magnetic potential on M. For any nonempty Borel subset $\Omega \subseteq M$, the *frustration index* $\iota^{\boldsymbol{\alpha}}(\Omega)$ of Ω is defined as

$$\iota^{\boldsymbol{\alpha}}(\Omega) = \inf_{\tau \in C^\infty(\Omega, U(1))} \int_\Omega |(d + i\boldsymbol{\alpha})\tau| dx = \inf_{\eta \in \mathfrak{B}_\Omega} \int_\Omega |\eta + \boldsymbol{\alpha}| dx, \tag{7.10}$$

where $\mathfrak{B}_\Omega := \{\boldsymbol{\alpha}_\tau \mid \tau \in C^\infty(\Omega, U(1))\}$.

Clearly, the frustration index $\iota^{\boldsymbol{\alpha}}(\Omega)$ is invariant under gauge transformations of the potential $\boldsymbol{\alpha}$. Roughly speaking, the frustration index measures how far the potential $\boldsymbol{\alpha}$ is from the set $\mathfrak{B}_\Omega$.

For any Borel subset $\Omega \subseteq M$, we denote by $\mathrm{vol}(\Omega)$ its Riemannian volume. Its boundary measure $\mathrm{area}(\partial\Omega)$ is defined as

$$\mathrm{area}(\partial\Omega) := \liminf_{r \to 0} \frac{\mathrm{vol}(\Omega_r) - \mathrm{vol}(\Omega)}{r}, \tag{7.11}$$

where Ω_r is the open r-neighborhood of Ω. Let us denote

$$\phi^{\boldsymbol{\alpha}}(\Omega) := \frac{\iota^{\boldsymbol{\alpha}}(\Omega) + \mathrm{area}(\partial\Omega)}{\mathrm{vol}(\Omega)}. \tag{7.12}$$

Definition 7.3 Let M be a closed Riemannian manifold with a magnetic potential $\boldsymbol{\alpha}$. The *n-way Cheeger constant* h_n^{α} is defined as

$$h_n^{\alpha} := \inf_{\{\Omega_p\}_{[n]}} \max_{p \in [n]} \phi^{\alpha}(\Omega_p), \tag{7.13}$$

where the infimum is taken over all n-subpartitions $\{\Omega_p\}_{[n]}$ of M with $\mathrm{vol}(\Omega_p) > 0$ for every $p \in [n]$.

In particular, the Cheeger constant h_1^{α} vanishes if and only if $\boldsymbol{\alpha} \in \mathfrak{B}$. We prove the following lower bound for the first eigenvalue $\lambda_1(\Delta^{\alpha})$.

Theorem 7.4 *Let $\boldsymbol{\alpha}$ be a magnetic potential on a closed connected Riemannian manifold M. Then we have*

$$h_1^{\alpha} \leq 2\sqrt{2\lambda_1(\Delta^{\alpha})}. \tag{7.14}$$

We first prove the following Lemma which is an analogue of Lemma 4.3.

Lemma 7.5 *(Coarea inequality) Let $\boldsymbol{\alpha}$ be a magnetic potential on M. For any nonzero smooth function $f : M \to \mathbb{C}$, we have*

$$\int_0^{\infty} \left(\iota^{\alpha}(\Omega^f(\sqrt{t})) + \mathrm{area}(\partial\Omega^f(\sqrt{t})) \right) dt \leq 2\sqrt{2} \int_M |f| \cdot |(d + i\boldsymbol{\alpha})f| dx, \tag{7.15}$$

where we use the notation $\Omega^f(\sqrt{t}) := \{x \in M \mid \sqrt{t} \leq |f(x)|\}$.

Proof For convenience, we denote $f_0 := |f|$. W.l.o.g., we assume that $f_0(x) > 0$, for any $x \in M$. Otherwise, we first consider integration over $\Omega^f(\varepsilon)$ in the right hand side of (7.15), $\varepsilon > 0$, and then let $\varepsilon \to 0$.

For the function f, we have the following associated 1-form in $\mathfrak{B}$:

$$\boldsymbol{\eta}_f := \boldsymbol{\alpha}_{\frac{f}{f_0}}. \tag{7.16}$$

Locally, there is a smooth real-valued function θ such that $f/f_0 = e^{i\theta}$ and $\boldsymbol{\eta}_f = d\theta$. Therefore, we have locally

$$|(d + i\boldsymbol{\alpha})f| = |(d + i\boldsymbol{\alpha})(f_0 e^{i\theta})| = |df_0 + if_0(d\theta + \boldsymbol{\alpha})|. \tag{7.17}$$

This implies that

$$|(d + i\boldsymbol{\alpha})f| = |df_0 + if_0(\boldsymbol{\eta}_f + \boldsymbol{\alpha})|. \tag{7.18}$$

Note that both df_0 and $f_0(\boldsymbol{\eta}_f + \boldsymbol{\alpha})$ are real-valued 1-forms. We estimate

$$|(d + i\boldsymbol{\alpha})f| = \sqrt{|df_0|^2 + |f_0(\boldsymbol{\eta}_f + \boldsymbol{\alpha})|^2} \geq \frac{1}{\sqrt{2}} \left(|df_0| + |f_0(\boldsymbol{\eta}_f + \boldsymbol{\alpha})| \right). \tag{7.19}$$

By the coarea formula, we have

$$\int_M f_0 |df_0| dx = \int_0^{\infty} t \cdot \mathrm{area}(\partial\Omega^{f_0}(t)) dt. \tag{7.20}$$

We also have

$$\int_M f_0^2 |\boldsymbol{\eta}_f + \boldsymbol{\alpha}| dx = 2 \int_0^{\infty} t \int_{\Omega^{f_0}(t)} |\boldsymbol{\eta}_f + \boldsymbol{\alpha}| dx \, dt$$

$$\geq \int_0^{\infty} t \int_{\Omega^{f_0}(t)} |\boldsymbol{\eta}_f + \boldsymbol{\alpha}| dx \, dt. \tag{7.21}$$

Combining (7.19), (7.20), and (7.21), we obtain

$$\int_M |f| \cdot |(d + i\boldsymbol{\alpha})f| dx \geq \frac{1}{2\sqrt{2}} \int_0^\infty 2t \left(\text{area}(\partial\Omega^f(t)) + \int_{\Omega^f(t)} |\boldsymbol{\eta}_f + \boldsymbol{\alpha}| dx \right) dt$$

$$= \frac{1}{2\sqrt{2}} \int_0^\infty \left(\text{area}(\partial\Omega^f(\sqrt{t})) + \int_{\Omega^f(\sqrt{t})} |\boldsymbol{\eta}_f + \boldsymbol{\alpha}| dx \right) dt$$

Recalling the definition of the frustration index (7.10), this proves the lemma. $\qquad\square$

Similarly as in Sect. 4 for the discrete setting, we derive the following lemma from the coarea inequality, which is the continuous analogue of Lemma 4.8.

Lemma 7.6 *Let $\boldsymbol{\alpha}$ be a magnetic potential on M. For any nonzero smooth function $f : M \to \mathbb{C}$, there exists $t' \in [0, \max_{x \in M} |f(x)|^2]$ such that*

$$\phi^{\boldsymbol{\alpha}}(\Omega^f(\sqrt{t'})) \leq 2\sqrt{2\mathcal{R}^{\boldsymbol{\alpha}}(f)}. \tag{7.22}$$

Proof First observe that there exists t' such that

$$\phi^{\boldsymbol{\alpha}}(\Omega^f(\sqrt{t'})) \leq \frac{\int_0^\infty \left(\iota^{\boldsymbol{\alpha}}(\Omega^f(\sqrt{t})) + \text{area}(\partial\Omega^f(\sqrt{t})) \right) dt}{\int_0^\infty \text{vol}(\Omega^f(\sqrt{t})) dt}. \tag{7.23}$$

Note that $\int_M |f(x)|^2 dx = \int_0^\infty \text{vol}(\Omega^f(\sqrt{t})) dt$. Then the lemma follows from applying the coarea inequality and Cauchy–Schwarz inequality. $\qquad\square$

Theorem 7.4 is proved by applying Lemma 7.6 to the corresponding eigenfunction of $\lambda_1(\Delta^{\boldsymbol{\alpha}})$. We also have the following higher order Cheeger inequalities for the magnetic Laplacian $\Delta^{\boldsymbol{\alpha}}$.

Theorem 7.7 *There exists an absolute constant $C > 0$ such that for any closed connected Riemannian manifold M with a magnetic potential $\boldsymbol{\alpha}$ and $n \in \mathbb{N}$, we have*

$$h_n^{\boldsymbol{\alpha}} \leq Cn^3 \sqrt{\lambda_n(\Delta^{\boldsymbol{\alpha}})}. \tag{7.24}$$

For the proof, first consider Lemma 7.8 below which is an analogue of Lemma 5.5. Let $F : M \to \mathbb{C}$ be the map given by

$$F(x) = (f_1(x), f_2(x), \ldots, f_n(x)) \in \mathbb{C}^n, \tag{7.25}$$

where f_i are orthonormal eigenfunctions that correspond to the eigenvalues $\lambda_i(\Delta^{\boldsymbol{\alpha}})$ for $i \in [n]$. The pseudometric d_F on $M_F := \{x \in M \mid F(x) \neq 0\}$ is defined by (5.11) via

$$d_F(x, y) := \inf_{\gamma \in U(1)} \left\| \frac{F(x)}{\|F(x)\|} - \gamma \frac{F(y)}{\|F(y)\|} \right\|. \tag{7.26}$$

For $\epsilon > 0$, the cut-off function η from (5.12) is directly transferred to the manifold setting and yields a localized function ηF.

Lemma 7.8 *For almost every $x \in M$, we have*

$$\|(d + i\boldsymbol{\alpha})(\eta F)(x)\|^2 \leq 2 \left(1 + \frac{4}{\epsilon^2} \right) \|(d + i\boldsymbol{\alpha})F(x)\|^2. \tag{7.27}$$

Proof If $F(x) = 0$, the estimate (7.27) follows directly from $|\eta| \leq 1$. We therefore assume $F(x) \neq 0$ in the following and set $f_{p,0} := |f_p|$ for every $p \in [n]$. Then there is a real-valued function θ_p that is defined in a small neighborhood of $x \in M$ such that $f_p = f_{p,0}e^{i\theta_p}$. We now obtain at x

$$
\begin{aligned}
\|(d + i\boldsymbol{\alpha})(\eta F)\|^2 &= \sum_{p \in [n]} |(d + i\boldsymbol{\alpha})(\eta f_{p,0}e^{i\theta_p})|^2 \\
&= \sum_{p \in [n]} |f_{p,0}\, d\eta + \eta df_{p,0} + i(\eta f_{p,0})(\boldsymbol{\alpha} + d\theta_p)|^2 \\
&\leq \sum_{p \in [n]} \left(2f_{p,0}^2 |d\eta|^2 + 2|\eta|^2 |df_{p,0}|^2 + |f_{p,0}(\boldsymbol{\alpha} + d\theta_p)|^2 \right) \\
&\leq 2|d\eta|^2 \sum_{p \in [n]} f_{p,0}^2 + 2 \sum_{p \in [n]} |df_{p,0} + if_{p,0}(\boldsymbol{\alpha} + d\theta_p)|^2 \\
&= 2|d\eta|^2 \|F\|^2 + 2\|(d + i\boldsymbol{\alpha})F\|^2.
\end{aligned}
\tag{7.28}
$$

There exist a unit tangent vector $\sigma'(0) \in T_x M$ such that

$$
|d\eta(x)| = \lim_{t \to 0} \frac{|\eta(\sigma(t)) - \eta(\sigma(0))|}{t},
\tag{7.29}
$$

where $\sigma(t) := \exp_x(t\sigma'(0))$ is the geodesic with $\sigma(0) = x$. Since we have

$$
|\eta(\sigma(t)) - \eta(\sigma(0))| \leq \frac{1}{\epsilon} \cdot d_F(\sigma(t), \sigma(0)),
\tag{7.30}
$$

we conclude

$$
|d\eta(x)| \cdot \|F(x)\| \leq \frac{1}{\epsilon} \cdot \lim_{t \to 0} \frac{d_F(\sigma(t), \sigma(0)) \cdot \|F(x)\|}{t}.
\tag{7.31}
$$

Using (7.26) and setting

$$
\gamma(t) := e^{i \int_0^t \langle \alpha_{|\sigma(t)}, \sigma'(t)\rangle dt},
\tag{7.32}
$$

we obtain

$$
\begin{aligned}
d_F(\sigma(t), \sigma(0))\|F(x)\| &\leq \left\| \gamma(t)\frac{F(\sigma(t))}{\|F(\sigma(t))\|} - \frac{F(\sigma(0))}{\|F(\sigma(0))\|} \right\| \cdot \|F(x)\| \\
&= \left\| \frac{G(t)}{\|G(t)\|} - \frac{G(0)}{\|G(0)\|} \right\| \cdot \|G(0)\|,
\end{aligned}
\tag{7.33}
$$

where $G(t) := \gamma(t)F(\sigma(t))$. Now we can carry out similar estimates as in Lemma 5.4. Although we do not know whether $\|G(0)\|$ is smaller than $\|G(t)\|$, we still obtain

$$
\begin{aligned}
\left\| \frac{G(t)}{\|G(t)\|} - \frac{G(0)}{\|G(0)\|} \right\| \|G(0)\| &\leq \left\| \frac{\|G(0)\|}{\|G(t)\|} \cdot G(t) - G(t) \right\| + \|G(t) - G(0)\| \\
&\leq 2 \cdot \|G(t) - G(0)\|.
\end{aligned}
\tag{7.34}
$$

Inserting (7.33) and (7.34) into (7.31), we obtain

$$
|d\eta(x)| \cdot \|F(x)\| \leq \frac{2}{\epsilon} \cdot \lim_{t \to 0} \frac{\|G(t) - G(0)\|}{t}
$$

$$
= \frac{2}{\epsilon} \cdot \lim_{t \to 0} \frac{\sqrt{\sum_{p \in [n]} |\gamma(t) f_p(\sigma(t)) - \gamma(0) f_p(\sigma(0))|^2}}{t}
$$

$$
= \frac{2}{\epsilon} \cdot \sqrt{\sum_{p \in [n]} \left| \lim_{t \to 0} \frac{\gamma(t) f_p(\sigma(t)) - \gamma(0) f_p(\sigma(0))}{t} \right|^2}
$$

$$
= \frac{2}{\epsilon} \cdot \sqrt{\sum_{p \in [n]} \left| \langle (d + i\boldsymbol{\alpha}) f_p(x), \sigma'(0) \rangle \right|^2}. \tag{7.35}
$$

In the last equality above, we used the fact that $\frac{d\gamma(t)}{dt}\big|_{t=0} = i \langle \boldsymbol{\alpha}(x), \sigma'(0) \rangle$. Since $|\sigma'(0)| = 1$, we conclude

$$
|d\eta(x)| \cdot \|F(x)\| \leq \frac{2}{\epsilon} \|(d + i\boldsymbol{\alpha}) F(x)\|. \tag{7.36}
$$

Combining (7.36) and (7.28), we finally obtain (7.27). $\qquad\square$

Note that the pseudometric (7.26) induced from the metric on a complex projective space played an important role in the proof.

Proof of Theorem 7.7 Applying Theorem 5.6 to $(M_F, d_F, \|F(x)\|^2 dx)$, we obtain a subpartition $\{T_i\}_{[n]}$ of M_F, such that

(i) $d_F(T_p, T_q) \geq \frac{2}{C_0 n^{5/2}}$, for all $p, q \in [n]$, $p \neq q$,
(ii) $\int_{T_p} \|F(x)\|^2 dx \geq \frac{1}{2n} \int_M \|F(x)\|^2 dx$, for all $p \in [n]$,

where C_0 is an absolute constant. Employing further Lemmas 7.6 and 7.8, the proof of the theorem can be done via the same arguments as in Sect. 5.4. $\qquad\square$

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