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On Coefficients Problems for Typically Real Functions Related to Gegenbauer Polynomials

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Abstract. We solve problems concerning the coefficients of functions in the class $\mathcal{T}(\lambda)$ of typically real functions associated with Gegenbauer polynomials. The main aim is to determine the estimates of two expressions: $|a_4 - a_2a_3|$ and $|a_2a_4 - {a_3}^2|$. The second one is known as the second Hankel determinant. In order to obtain these bounds, we consider the regions of variability of selected pairs of coefficients for functions in $\mathcal{T}(\lambda)$. Furthermore, we find the upper and the lower bounds of functionals of Fekete–Szegö type. Finally, we present some conclusions for the classes $\mathcal{T}$ and $\mathcal{T}(1/2)$.

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1. Introduction

Let Δ denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ be the class of all functions f analytic in Δ, normalized by the condition $f(0) = f'(0) - 1 = 0$. This means that $f \in \mathcal{A}$ has the expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \,. \tag{1}$$

In 1994 Szynal [15] introduced the class $\mathcal{T}(\lambda)$, $\lambda \geq 0$ as the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = \int_{-1}^{1} k(z,t)\mathrm{d}\mu(t), \tag{2}$$

where

$$k(z,t) = \frac{z}{(1-2tz+z^2)^{\lambda}}, \quad z \in \Delta, \quad t \in [-1,1] \tag{3}$$

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and μ is a probability measure on the interval $[-1,1]$. The collection of such measures on $[a,b]$ is denoted by $P_{[a,b]}$.

The function $k(z,t)$ has the Taylor series expansion

$$k(z,t) = z + C_1^{(\lambda)}(t)z^2 + C_2^{(\lambda)}(t)z^3 + \cdots \ , \tag{4}$$

where $C_n^{(\lambda)}(t)$ denotes the Gegenbauer polynomial of degree n (for details about the Gegenbauer polynomials, see [1,14]).

First polynomials of this type are following:

$$\begin{aligned}
C_0^{(\lambda)}(t) &= 1,\\
C_1^{(\lambda)}(t) &= 2\lambda t,\\
C_2^{(\lambda)}(t) &= 2\lambda(\lambda+1)t^2 - \lambda,\\
C_3^{(\lambda)}(t) &= \frac{4}{3}\lambda(\lambda+1)(\lambda+2)t^3 - 2\lambda(\lambda+1)t.
\end{aligned} \tag{5}$$

If $f \in \mathcal{T}(\lambda)$ is given by (2), then the coefficients of this function can be written as follows:

$$a_n = \int_{-1}^{1} C_{n-1}^{(\lambda)}(t)\mathrm{d}\mu(t) \ . \tag{6}$$

Note that $\mathcal{T}(1) = \mathcal{T}$ is the well-known class of typically real functions (for details, see e.g., [2,8,9]). For $\lambda = 1/2$ we obtain the class of typically real functions related to Legendre polynomials $P_n(t) = C_n^{(1/2)}(t)$. As it was shown in [15] (see also, [12]), this class is a proper superclass for $\mathcal{S}^*_{\mathbb{R}}(1/2)$, the class of starlike functions of order $1/2$ and for $\mathcal{K}_{\mathbb{R}}(i)$, the class of convex functions in the direction of the imaginary axis; both of them consist of functions with real coefficients. More precisely, we have two chains of inclusions:

$$\mathcal{K}_{\mathbb{R}} \subset \mathcal{K}_{\mathbb{R}}(i) \subset \mathcal{T}(1/2)$$

and

$$\mathcal{K}_{\mathbb{R}} \subset \mathcal{S}^*_{\mathbb{R}}(1/2) \subset \mathcal{T}(1/2) \ .$$

This observation is significant because it happens that the results in $\mathcal{T}(1/2)$ can be transferred to its subclasses (see [12,16]).

In this paper we consider various problems concerning the coefficients of functions in $\mathcal{T}(\lambda)$. We discuss the Fekete–Szegö functional $a_3 - \mu {a_2}^2$ and its modification, i.e., $a_5 - \mu {a_3}^2$. Moreover, we find the sharp bounds of two expressions: $|a_4 - a_2a_3|$ and $|a_2a_4 - {a_3}^2|$. The second is called the second Hankel determinant. It was Pommerenke the first who studied this determinant in the geometric theory of analytic functions ([6,7]).

Recently, the Hankel determinant has been studied by many mathematicians. They discussed the second Hankel determinants for various classes of univalent functions. Some results in this direction can be found in [3–5,10,11].

2. Regions of Variability of a Pair of Coefficients for $\mathcal{T}(\lambda)$

Let $A_{n,m}$ denote the set of variability of the point (a_n, a_m), where a_n and a_m are the coefficients of a given function $f \in \mathcal{T}(\lambda)$ with the series expansion (1). The set $A_{n,m}$ coincides with the closed convex hull of the curve

$$\gamma_{n,m} : [-1,1] \ni t \mapsto \left(C_{n-1}^{(\lambda)}(t), C_{m-1}^{(\lambda)}(t)\right) .$$

By the Caratheodory theorem we conclude that it is sufficient to discuss only the functions

$$f(z) = \alpha k(z,t_1) + (1-\alpha)k(z,t_2) ,$$

where $-1 \leq t_1 \leq t_2 \leq 1$, $\alpha \in [0,1]$.

Now, we shall establish a few simple lemmas concerning the sets $A_{n,m}$ for initial integers n and m.

Lemma 2.1. *For the class $\mathcal{T}(\lambda)$, $\lambda > 0$, we have*

$$A_{2,3} = \left\{(x,y) \in \mathbb{R}^2 : \ -2\lambda \leq x \leq 2\lambda, \frac{\lambda+1}{2\lambda}x^2 - \lambda \leq y \leq \lambda(2\lambda+1)\right\} .$$

Proof. Since $A_{2,3} = \operatorname{conv}\{\gamma_{2,3}(t) : t \in [-1,1]\}$, taking into account (5) we conclude that $\gamma_{2,3}([-1,1])$ is a part of parabola $y = \frac{\lambda+1}{2\lambda}x^2 - \lambda$ for $x \in [-2\lambda, 2\lambda]$. Hence, the boundary of the convex hull of $\gamma_{2,3}([-1,1])$ consists of this curve and the line segment that connects two endpoints of this arc. $\square$

The similar result can be obtained for $A_{3,5}$. It is enough to determine the convex hull of the arc of the parabola $\gamma_{3,5}([-1,1])$. In this way we get the following lemma.

Lemma 2.2. *For the class $\mathcal{T}(\lambda)$, $\lambda > 0$, we have*

$$\begin{aligned} A_{3,5} = \Big\{&(x,y) \in \mathbb{R}^2 : \ -\lambda \leq x \leq \lambda(2\lambda+1), \\ &\frac{(\lambda+2)(\lambda+3)}{6\lambda(\lambda+1)}x^2 - \frac{2\lambda(\lambda+2)}{3(\lambda+1)}x - \frac{\lambda(\lambda+3)(2\lambda+1)}{6(\lambda+1)} \leq y \\ &\leq \frac{1}{3}\lambda(\lambda+2)x + \frac{1}{6}\lambda(\lambda+3)(2\lambda+1)\Big\}. \end{aligned}$$

Remark 1. In both lemmas the boundaries of the discussed sets consist of an arc of a parabola and a line segment. The points lying on the parabolas correspond to the coefficients of $k(z,t)$, $t \in [-1,1]$. On the other hand, the points of the line segments are the coefficients of a function $\alpha k(z,-1) + (1-\alpha)k(z,1)$ with suitably taken $\alpha \in [0,1]$.

Lemma 2.3. *For the class $\mathcal{T}(\lambda)$, $\lambda > 0$, we have*

$$A_{2,4} = \left\{(x,y) \in \mathbb{R}^2 : \ -2\lambda \leq x \leq 2\lambda, -g(-x) \leq y \leq g(x)\right\},$$

where

$$g(x) = \begin{cases} \frac{(\lambda+1)(\lambda+2)}{6\lambda^2}x^3 - (\lambda+1)x, & x \in [-2\lambda, -\lambda] \\ \frac{1}{2}\lambda(\lambda+1)x + \frac{1}{3}\lambda(\lambda+1)(\lambda+2), & x \in [-\lambda, 2\lambda] \ . \end{cases}$$

Proof. Since $A_{2,4} = \mathrm{conv}\{\gamma_{2,4}(t) : t \in [-1,1]\}$, we write $x = 2\lambda t$ and $y = \frac{4}{3}\lambda(\lambda+1)(\lambda+2)t^3 - 2\lambda(\lambda+1)t$. The curve $\gamma_{2,4}([-1,1])$ is given by the explicit formula $y = \frac{(\lambda+1)(\lambda+2)}{6\lambda^2}x^3 - (\lambda+1)x$. It can be verified that its convex hull is bounded by parts of this curve for $x \in [-2\lambda, -\lambda]$ and $x \in [\lambda, 2\lambda]$ and two parallel line segments: l_1 connecting points $\gamma_{2,4}(-1/2)$ and $\gamma_{2,4}(1)$ and l_2 with endpoints in $\gamma_{2,4}(-1)$ and $\gamma_{2,4}(1/2)$. □

Applying this lemma we can derive the bounds of the product of the second and the fourth coefficients of $f \in \mathcal{T}(\lambda)$.

Theorem 2.1. *For $f \in \mathcal{T}(\lambda)$, $\lambda > 0$, the following sharp bounds hold:*

$$a_2a_4 \le \tfrac{4}{3}\lambda^2(\lambda+1)(2\lambda+1) \tag{7}$$

and

$$a_2a_4 \ge \begin{cases} -\frac{3\lambda^2(\lambda+1)}{2(\lambda+2)}, & \lambda \in (0,1] \\ -\frac{1}{18}\lambda(\lambda+1)(\lambda+2)^2, & \lambda \ge 1\ . \end{cases} \tag{8}$$

Proof. The upper bound is obvious. In order to calculate the lower bound of a_2a_4, we consider only these points (a_2, a_4) which belong to the boundary of $A_{2,4}$. Since $A_{2,4}$ is symmetric with respect to the origin, it is enough to discuss the points lying on the curve $y = g(x)$, where g is the function described in Lemma 2.3.

Let us denote by $h_1(x)$ and $h_2(x)$ two functions corresponding to the values of a_2a_4 on each part of the curve $y = g(x)$. Namely,

$$h_1(x) \equiv \frac{(\lambda+1)(\lambda+2)}{6\lambda^2}x^4 - (\lambda+1)x^2, \quad x \in [-2\lambda, -\lambda]$$

and

$$h_2(x) \equiv \tfrac{1}{2}\lambda(\lambda+1)x^2 + \tfrac{1}{3}\lambda(\lambda+1)(\lambda+2)x \quad x \in [-\lambda, 2\lambda]\ .$$

Only one critical point $x_1 = -\lambda\sqrt{\frac{3}{\lambda+2}}$ of h_1 is in $[-2\lambda, -\lambda]$, but only if $\lambda \le 1$. Hence,

$$\begin{aligned} &\min\{h_1(x) : x \in [-2\lambda, -\lambda]\} \\ &\quad = \begin{cases} h_1(x_1) = -\frac{3\lambda^2(\lambda+1)}{2(\lambda+2)}, & 0 < \lambda \le 1 \\ h_1(-\lambda) = \frac{1}{6}\lambda^2(\lambda+1)(\lambda-4), & \lambda \ge 1\ . \end{cases} \end{aligned} \tag{9}$$

It is easy to check that

$$\begin{aligned} &\min\{h_2(x) : x \in [-\lambda, 2\lambda]\} \\ &\quad = \begin{cases} h_2(-\lambda) = \frac{1}{6}\lambda^2(\lambda+1)(\lambda-4), & 0 < \lambda \le 1 \\ h_2(x_2) = -\frac{1}{18}\lambda(\lambda+1)(\lambda+2)^2, & \lambda \ge 1\ , \end{cases} \end{aligned} \tag{10}$$

where $x_2 = -(\lambda+2)/3$.

Combining (9) and (10) we obtain the desired result. □

For $\lambda = 1$ this result reduces to the one proved in [17] in Theorem 13.

3. Functionals of the Fekete–Szegö Type

Now, we are ready to find the sharp bounds of two functionals defined for $f \in \mathcal{T}(\lambda)$ of the form (1): $a_3 - \mu {a_2}^2$ and $a_5 - \mu {a_3}^2$. The first one is very well known and often discussed the so-called Fekete–Szegö functional.

Theorem 3.1. *If $f \in \mathcal{T}(\lambda)$, $\lambda > 0$, then the following sharp bounds hold:*

$$a_3 - \mu {a_2}^2 \le \begin{cases} \lambda(2\lambda + 1 - 4\lambda\mu), & \mu \le 0 \\ \lambda(2\lambda + 1), & \mu \ge 0 \end{cases} \tag{11}$$

and

$$a_3 - \mu {a_2}^2 \ge \begin{cases} -\lambda, & \mu \le \frac{\lambda+1}{2\lambda} \\ \lambda(2\lambda + 1 - 4\lambda\mu), & \mu \ge \frac{\lambda+1}{2\lambda} \end{cases} . \tag{12}$$

Proof. Let $f \in \mathcal{T}(\lambda)$. From Lemma 2.1,

$$a_3 - \mu {a_2}^2 \le a_3 \le \lambda(2\lambda + 1) \quad \text{for} \quad \mu \ge 0$$

and

$$a_3 - \mu {a_2}^2 \le \lambda(2\lambda + 1) - \mu\,(2\lambda)^2 \quad \text{for} \quad \mu \le 0 \,.$$

The equality holds if $f(z) = \frac{1}{2}\,[k(z,-1) + k(z,1)] = z + C_2^{(\lambda)}(t) z^3 + \cdots$ for $\mu \ge 0$ and $k(z, \pm 1)$ for $\mu \le 0$.

On the other hand, by Lemma 2.1,

$$a_3 - \mu {a_2}^2 \ge \left(\frac{\lambda + 1}{2\lambda} - \mu \right) {a_2}^2 - \lambda \,.$$

Hence, for $h(x) = \left(\frac{\lambda+1}{2\lambda} - \mu\right) x^2 - \lambda$, we get

$$\min\{h(x) : x \in [-2\lambda, 2\lambda]\} = \begin{cases} h(0), & \mu \le \frac{\lambda+1}{2\lambda} \\ h(\pm 2\lambda), & \mu \ge \frac{\lambda+1}{2\lambda} \end{cases}$$

with equality for $k(z, 0) = z - \lambda z^3 + \cdots$ in first case and for $k(z, \pm 1)$ in the second one. □

Theorem 3.2. *If $f \in \mathcal{T}(\lambda)$, $\lambda > 0$, then the following sharp bounds hold:*

$$a_5 - \mu {a_3}^2 \le \begin{cases} \frac{1}{6}\lambda(2\lambda + 1)\,[(\lambda + 1)(2\lambda + 3) - 6\lambda(2\lambda + 1)\mu]\,, & \mu \le \frac{\lambda+2}{6(2\lambda+1)} \\ \frac{1}{36}\lambda \left[\frac{\lambda(\lambda+2)^2}{\mu} + 6(\lambda + 3)(2\lambda + 1) \right], & \mu \ge \frac{\lambda+2}{6(2\lambda+1)} \end{cases} \tag{13}$$

and

$$a_5 - \mu {a_3}^2 \ge \begin{cases} -\frac{\lambda}{6(\lambda+1)} \left[\frac{4\lambda^2(\lambda+2)^2}{(\lambda+2)(\lambda+3) - 6\lambda(\lambda+1)\mu} + (\lambda + 3)(2\lambda + 1) \right], & \mu \le \frac{(\lambda+2)(2\lambda+3)}{6\lambda(2\lambda+1)} \\ \frac{1}{6}\lambda(2\lambda + 1)\,[(\lambda + 1)(2\lambda + 3) - 6\lambda(2\lambda + 1)\mu]\,, & \mu \ge \frac{(\lambda+2)(2\lambda+3)}{6\lambda(2\lambda+1)}. \end{cases} \tag{14}$$

Proof. Let $f \in \mathcal{T}(\lambda)$. Taking into account the bounds of a_5 proved in Lemma 2.2, we have $a_5 - \mu {a_3}^2 \le h_1(a_3)$, where

$$h_1(x) = -\mu x^2 + \tfrac{1}{3}\lambda(\lambda+2)x + \tfrac{1}{6}\lambda(\lambda+3)(2\lambda+1), \quad x \in [-\lambda, \lambda(2\lambda+1)] .$$

If $\mu \le 0$, then $h_1(x) \le \max\{h_1(-\lambda), h_1(\lambda(2\lambda+1))\}$. But for $\mu \le 0$ and $\lambda > 0$ we know that $-\lambda^2\mu \le -\lambda^2(2\lambda+1)^2\mu$ and $-\frac{1}{3}\lambda^2(\lambda+2) < \frac{1}{3}\lambda^2(\lambda+2)(2\lambda+1)$, so

$$\begin{aligned} h_1(-\lambda) &= -\lambda^2\mu - \tfrac{1}{3}\lambda^2(\lambda+2) + \tfrac{1}{6}\lambda(\lambda+3)(2\lambda+1) \\ &< -\lambda^2(2\lambda+1)^2\mu + \tfrac{1}{3}\lambda^2(\lambda+2)(2\lambda+1) + \tfrac{1}{6}\lambda(\lambda+3)(2\lambda+1) \\ &= h_1(\lambda(2\lambda+1)) . \end{aligned}$$

For $\mu > 0$ we obtain

$$h_1(x) \le \begin{cases} h_1(x_1), & x_1 < \lambda(2\lambda+1) \\ h_1(\lambda(2\lambda+1)), & x_1 \ge \lambda(2\lambda+1) , \end{cases}$$

where $x_1 = \frac{\lambda(\lambda+2)}{6\mu}$, $x_1 > 0$. Combining these two cases we obtain (13).

By Lemma 2.2, $a_5 - \mu {a_3}^2 \ge h_2(a_3)$, where

$$h_2(x) = qx^2 - \frac{2\lambda(\lambda+2)}{3(\lambda+1)}x - \frac{\lambda(\lambda+3)(2\lambda+1)}{6(\lambda+1)}, \; x \in [-\lambda, \lambda(2\lambda+1)]$$

and $q = \frac{(\lambda+2)(\lambda+3)}{6\lambda(\lambda+1)} - \mu$.

If $q \le 0$, then $q\lambda^2 \ge q\lambda^2(2\lambda+1)^2$ and $\frac{2\lambda^2(\lambda+2)}{3(\lambda+1)} > -\frac{2\lambda^2(\lambda+2)(2\lambda+1)}{3(\lambda+1)}$. Hence,

$$\begin{aligned} h_2(-\lambda) &= q\lambda^2 + \frac{2\lambda^2(\lambda+2)}{3(\lambda+1)} - \frac{\lambda(\lambda+3)(2\lambda+1)}{6(\lambda+1)} \\ &> q\lambda^2(2\lambda+1)^2 - \frac{2\lambda^2(\lambda+2)(2\lambda+1)}{3(\lambda+1)} - \frac{\lambda(\lambda+3)(2\lambda+1)}{6(\lambda+1)} \\ &= h_2(\lambda(2\lambda+1)) . \end{aligned}$$

For $q > 0$,

$$h_2(x) \ge \begin{cases} h_2(x_2), & x_2 < \lambda(2\lambda+1) \\ h_2(\lambda(2\lambda+1)), & x_2 \ge \lambda(2\lambda+1) , \end{cases}$$

where $x_2 = \frac{\lambda(\lambda+2)}{3(\lambda+1)q}$, $x_2 > 0$.

Once again, combining both cases for q, the inequality (14) follows.

By Remark 1, the extremal functions in (13) are: $k(z, 1)$ and a function $\alpha_0 k(z, -1) + (1 - \alpha_0)k(z, 1)$ with suitably taken $\alpha_0 \in [0, 1]$. The equality in (14) holds for $k(z, t_0)$ for some $t_0 \in [-1, 1)$ if $\mu < \frac{(\lambda+2)(2\lambda+3)}{6\lambda(2\lambda+1)}$ and for $k(z, 1)$ if $\mu \ge \frac{(\lambda+2)(2\lambda+3)}{6\lambda(2\lambda+1)}$. □

From Theorems 3.1 and 3.2, taking $\lambda = 1$ and $\lambda = 1/2$, we conclude the results for the class $\mathcal{T}$ and for $\mathcal{T}(1/2)$.

Corollary 1. *For any $f \in \mathcal{T}$ we have*

$$\left.\begin{array}{ll} -1, & \mu \le 1 \\ 3-4\mu, & \mu \ge 1 \end{array}\right\} \le a_3 - \mu {a_2}^2 \le \left\{\begin{array}{ll} 3-4\mu, & \mu \le 0 \\ 3, & \mu \ge 0 \end{array}\right.$$

and

$$\left.\begin{array}{ll} -\frac{1}{4(1-\mu)} - 1, & \mu \le 5/6 \\ 5-9\mu, & \mu \ge 5/6 \end{array}\right\} \le a_5 - \mu {a_3}^2 \le \left\{\begin{array}{ll} 5-9\mu, & \mu \le 1/6 \\ 2+\frac{1}{4\mu}, & \mu \ge 1/6 \ . \end{array}\right.$$

Corollary 2. *For any $f \in \mathcal{T}(1/2)$ we have*

$$\left.\begin{array}{ll} -1/2, & \mu \le 3/2 \\ 1-\mu, & \mu \ge 3/2 \end{array}\right\} \le a_3 - \mu {a_2}^2 \le \left\{\begin{array}{ll} 1-\mu, & \mu \le 0 \\ 1, & \mu \ge 0 \end{array}\right.$$

and

$$\left.\begin{array}{ll} -\frac{1}{18}\left(\frac{25}{35-18\mu} + 7\right), & \mu \le 5/3 \\ 1-\mu, & \mu \ge 5/3 \end{array}\right\} \le a_5 - \mu {a_3}^2 \le \left\{\begin{array}{ll} 1-\mu, & \mu \le 5/24 \\ \frac{7}{12}+\frac{25}{576\mu}, & \mu \ge 5/24 \ . \end{array}\right.$$

4. Bounds of $|a_4 - a_2a_3|$

Let us denote by $\Omega_n(\mathcal{T}(\lambda))$, $n \ge 1$ the region of variability of three succeeding coefficients of functions in $\mathcal{T}(\lambda)$, i.e., the set $\{(a_n(f), a_{n+1}(f), a_{n+2}(f)) : f \in \mathcal{T}(\lambda)\}$. Therefore, $\Omega_n(\mathcal{T}(\lambda))$ is the closed convex hull of the curve

$$\gamma_n : [-1, 1] \ni t \mapsto \left(C^{(\lambda)}_{n-1}(t), C^{(\lambda)}_{n}(t), C^{(\lambda)}_{n+1}(t)\right) \ .$$

Let X be a compact Hausdorff space and $J_\mu = \int_X J(t)d\mu(t)$. Szapiel in [13] (Thm.1.40) proved the following theorem.

Theorem 4.1 ([13]). *Let $J : [\alpha, \beta] \to \mathbb{R}^n$ be continuous. Suppose that there exists a positive integer k, such that for each non-zero p in $\mathbb{R}^n$ the number of solutions of any equation $\langle \vec{J(t)}, \vec{p}\ \rangle = const$, $\alpha \le t \le \beta$ is not greater than k. Then, for every $\mu \in P_{[\alpha,\beta]}$ such that J_μ belongs to the boundary of the convex hull of $J([\alpha, \beta])$ the following statements are true:*

1. *if $k = 2m$, then*
   (a) $|\mathrm{supp}(\mu)| \le m$ *or*
   (b) $|\mathrm{supp}(\mu)| = m + 1$ *and* $\{\alpha, \beta\} \subset \mathrm{supp}(\mu)$,

2. *if $k = 2m + 1$, then*
   (a) $|\mathrm{supp}(\mu)| \le m$ *or*
   (b) $|\mathrm{supp}(\mu)| = m + 1$ *and one of the points α, β belongs to* $\mathrm{supp}(\mu)$.

In the above, the symbol $\langle \vec{u}, \vec{v}\ \rangle$ means the scalar product of vectors $\vec{u}$ and $\vec{v}$, whereas the symbols P_X and $|\mathrm{supp}(\mu)|$ describe the set of probability measures on X and the cardinality of the support of μ, respectively.

According to Theorem 4.1, the boundary of the convex hull of $\gamma_2([-1, 1])$ is determined by atomic measures μ for which support consists of at most 2 points, where one of them is -1 or 1. In this way we have the following lemma.

Lemma 4.1. *The boundary of $\Omega_2(\mathcal{T}(\lambda))$ consists of points (a_2, a_3, a_4) that correspond to the following functions:*

$$f(z) = \alpha\frac{z}{(1-2tz+z^2)^\lambda} + (1-\alpha)\frac{z}{(1-z)^{2\lambda}}\ , \quad \alpha \in [0,1], \quad t \in [-1,1] \tag{15}$$

or

$$f(z) = \alpha\frac{z}{(1-2tz+z^2)^\lambda} + (1-\alpha)\frac{z}{(1+z)^{2\lambda}}\ , \quad \alpha \in [0,1], \quad t \in [-1,1]\ . \tag{16}$$

Now, we can establish the main theorem of this section.

Theorem 4.2. *If $f \in \mathcal{T}(\lambda)$ and $\lambda \geq 1$, then $|a_4 - a_2a_3| \leq \frac{2}{3}\lambda(4\lambda^2 - 1)$. The result is sharp.*

Proof. The function (15) has the Taylor series expansion

$$f(z) = z + \sum_{k=2}^{\infty}\left[\alpha C_{k-1}^{(\lambda)}(t) + (1-\alpha)C_{k-1}^{(\lambda)}(1)\right] z^k\ . \tag{17}$$

Hence, $a_4 - a_2a_3 = h_1(\alpha, t)$, where

$$\begin{aligned} h_1(\alpha,t) = {} & [\alpha C_3(t) + (1-\alpha)C_3(1)] - [\alpha C_1(t) \\ & +(1-\alpha)C_1(1)] \cdot [\alpha C_2(t) + (1-\alpha)C_2(1)]\ . \end{aligned} \tag{18}$$

From (5) we obtain

$$\begin{aligned} h_1(\alpha,t) = {} & \tfrac{2}{3}\lambda(\lambda+1)[\alpha(2\lambda(t^3-1) + (t-1)(2t+1)^2) + 2\lambda + 1] \\ & -2\lambda^2(\alpha(t-1)+1)(2\alpha(\lambda+1)(t^2-1) + 2\lambda + 1)\ . \end{aligned} \tag{19}$$

Similarly, for the function given by (16), $a_4 - a_2a_3 = h_2(\alpha, t)$. Moreover, it is easy to conclude from (18) that $h_2(\alpha, t) = -h_1(\alpha, -t)$. Taking into account the symmetry of the range of variability of t, we obtain the same estimates of $|a_4 - a_2a_3|$ for the functions given by (15) and (16).

For the function (15), we can write

$$\begin{aligned} h_1(\alpha,t) = {} & 4\lambda^2(\lambda+1)\alpha(1-\alpha)(1-t)(1-t^2) + \frac{8}{3}\lambda(\lambda^2-1)(1-t^3)\alpha \\ & + 2\lambda(1-t)\alpha + \frac{2}{3}\lambda(1-4\lambda^2)\ . \end{aligned}$$

First three expressions are positive, so $h_1(\alpha, t) \geq \frac{2}{3}\lambda(1 - 4\lambda^2)$. In this case the equality holds for $t = 1$ or $\alpha = 0$; consequently, the extremal function is $f(z) = \frac{z}{(1-z)^{2\lambda}}$. Therefore, the estimate is sharp.

Now, we will prove that

$$h_1(\alpha,t) \leq \frac{2}{3}\lambda(4\lambda^2 - 1)\ . \tag{20}$$

Inequality (20) is equivalent to $w(\lambda) \leq 0$, where

$$w(\lambda) = B_3\lambda^3 + B_2\lambda^2 + B_1\lambda$$

and

$$\begin{aligned} B_1 &= -2\alpha t(1-t^2) + \frac{2}{3}[2-\alpha(1-t^3)] \ , \\ B_2 &= 4\alpha(1-\alpha)(1-t)(1-t^2) \ , \\ B_3 &= -4\alpha(1-\alpha)(1+t)t + \frac{4}{3}\alpha(1-3\alpha)t^3 - 4(1-\alpha)^2 - \frac{4}{3}(1+\alpha) \ . \end{aligned}$$

Obviously, B_2 is non-negative. Since B_1 can be rewritten as $B_1 = \frac{4}{3} - \frac{2}{3}\alpha(1-t)(1+2t)^2$, the minimum of B_1 is attained for $\alpha = 1$, so $B_1 \geq \frac{2}{3}(1+t)(1-2t)^2 \geq 0$.

We shall prove that $B_3 \leq 0$. Indeed,

$$B_3 = -\frac{4}{3}\alpha(1-\alpha)(1+t)^3 + \frac{8}{3}b(t) \ ,$$

where

$$b(t) = \alpha(1-2\alpha)t^3 - (2-3\alpha+2\alpha^2) \ .$$

But, if $\alpha \in \left[0, \frac{1}{2}\right]$, then $b(t) \leq b(1) = -2 + 4\alpha(1-\alpha) \leq -1$ and if $b \in \left[\frac{1}{2}, 1\right]$, then $b(t) \leq b(-1) = -2(1-\alpha) \leq 0$. Moreover, $-\frac{4}{3}\alpha(1-\alpha)(1+t)^3 \leq 0$ for all $t \in [-1,1], \ \alpha \in [0,1]$. It means that $B_3 \leq 0$. The inequality $B_3 \leq 0$ implies that in order to prove that $w(\lambda) \leq 0$, it is enough to show $w(1) \leq 0$.

Let us denote by $P(\alpha, t)$, where $\alpha \in [0,1]$, $t \in [-1,1]$, the value $w(1) = -8(1-t)(1-t^2)\alpha^2 + 2(1-t)(5-4t^2)\alpha - 4$. We shall derive the greatest value of $P(\alpha,t)$ for $\alpha \in [0,1]$, $t \in [-1,1]$.

Since $\frac{\partial P}{\partial \alpha} = -16(1-t)(1-t^2)\alpha + 2(1-t)(5-4t^2)$, as a function of a variable α, vanishes only for $\alpha_0 = \frac{5-4t^2}{8(1-t^2)}$, we conclude that $P(\alpha,t) \leq \frac{(5-4t^2)^2}{8(1+t)} - 4$, but only if $\alpha_0 \in [0,1]$. It occurs for $t^2 \leq \frac{3}{4}$.

If we consider the function $h(t) = \frac{(5-4t^2)^2}{8(1+t)}$, we get $\max\{h(t) : t^2 \leq \frac{3}{4}\} = h(-\frac{1}{2}) = 4$. Hence, we obtain $P(\alpha,t) \leq 0$ for $t^2 \leq \frac{3}{4}$, $\alpha \in [0,1]$.

Now, we need to verify this inequality on the boundary of the set $[0,1] \times [-1,1]$. We have the following:

- $P(0,t) = -4$,
- $P(1,t) = -2 - 2t \in [-4,0]$,
- $P(\alpha,-1) = 4(\alpha-1) \in [-4,0]$,
- $P(\alpha,1) = -4$.

Considering both parts of this discussion we have $P(\alpha,t) \leq 0$ in the whole set $[0,1] \times [-1,1]$, so $w(1) \leq 0$ as we required. Therefore, the inequality (20) is also satisfied. Finally, we obtain $|a_4 - a_2 a_3| \leq \frac{2}{3}\lambda(4\lambda^2 - 1)$. □

From Theorem 4.2, taking $\lambda = 1$, we conclude the result for the class $\mathcal{T}$.

Corollary 3. *For any $f \in \mathcal{T}$, we have*

$$|a_4 - a_2 a_3| \leq 2 \ .$$

5. Bounds of $|a_2a_4 - {a_3}^2|$

Theorem 5.1. *If $f \in \mathcal{T}(\lambda)$ and $\lambda \geq 1$, then $|a_2a_4 - {a_3}^2| \leq \lambda^2(1+2\lambda)^2$. The result is sharp.*

Proof. At the beginning, observe that for $k(z,t)$ given by (3) we have

$$a_2a_4 - {a_3}^2 = C_1^{(\lambda)}(t)C_3^{(\lambda)}(t) - \left[C_2^{(\lambda)}(t)\right]^2 = -\frac{4}{3}\lambda^2(\lambda^2-1)t^4 - \lambda^2\ . \qquad (21)$$

The same argument as in the proof of Theorem 4.2 yields that only the functions given by (15) and (16) should be discussed in order to obtain the desired estimates.

Denote by $g_1(\alpha,t)$ and $g_2(\alpha,t)$ the second Hankel determinant for the functions (15) and (16), respectively. It is easy to see that $g_1(\alpha,t) = g_2(\alpha,-t)$. For this reason we discuss only the first function.

From (15) it follows that

$$\begin{aligned} g_1(\alpha,t) = \alpha^2\left[C_1(t)C_3(t) - [C_2(t)]^2\right] + (1-\alpha)^2\left[C_1(1)C_3(1) - [C_2(1)]^2\right] \\ + \alpha(1-\alpha)\left[C_1(t)C_3(1) + C_1(1)C_3(t) - 2C_2(t)C_2(1)\right]\ . \end{aligned}$$

Hence, (5) and (21) result in

$$\begin{aligned} g_1(\alpha,t) = -\alpha^2\lambda^2\left[\frac{4}{3}(\lambda^2-1)t^4+1\right] - (1-\alpha)^2\lambda^2\left[\frac{4}{3}(\lambda^2-1)+1\right] \\ +\alpha(1-\alpha)w(t)\ , \end{aligned} \qquad (22)$$

where

$$\begin{aligned} w(t) = \frac{8}{3}\lambda^2(\lambda+1)(\lambda+2)t^3 - 4\lambda^2(\lambda+1)(2\lambda+1)t^2 + \frac{8}{3}\lambda^2(\lambda^2-1)t \\ +2\lambda^2(2\lambda+1). \end{aligned}$$

If $\lambda \geq 1$, then $w(t) \geq w(-1)$ and $-\left[\frac{4}{3}(\lambda^2-1)t^4+1\right] \geq -\left[\frac{4}{3}(\lambda^2-1)+1\right]$ with equality for $t=-1$ or $t=1$. Thus, for $\lambda \geq 1$, we have

$$\begin{aligned} g_1(\alpha,t) &\geq -\left[\frac{4}{3}(\lambda^2-1)+1\right]\lambda^2(\alpha^2+(1-\alpha)^2) + \alpha(1-\alpha)w(-1) \\ &= \frac{1}{3}\lambda^2(2\lambda+1)\left[1-2\lambda-16\alpha(1-\alpha)(1+\lambda)\right] \\ &\geq \frac{1}{3}\lambda^2(2\lambda+1)\left[1-2\lambda-4(1+\lambda)\right] = -\lambda^2(1+2\lambda)^2\ . \end{aligned}$$

Now, we shall show that $g_1(\alpha,t) < \lambda^2(1+2\lambda)^2$. From (22) we obtain the rough inequality

$$g_1(\alpha,t) < \alpha(1-\alpha)\left[\frac{8}{3}\lambda^2(\lambda+1)(\lambda+2) + \frac{8}{3}\lambda^2(\lambda^2-1) + 2\lambda^2(2\lambda+1)\right]\ .$$

Hence

$$\begin{aligned} g_1(\alpha,t) &< \frac{2}{3}\lambda^2(\lambda+1)(\lambda+2) + \frac{2}{3}\lambda^2(\lambda^2-1) + \frac{1}{2}\lambda^2(2\lambda+1) \\ &= \lambda^2(2\lambda+1)\left(\frac{2}{3}\lambda+\frac{7}{6}\right)\ , \end{aligned}$$

which is clearly less than $\lambda^2(2\lambda+1)^2$ for $\lambda \geq 1$.

Therefore, for $\lambda \geq 1$, we have $|a_2a_4 - {a_3}^2| \leq \lambda^2(1+2\lambda)^2$. The equality holds if $t = 1$ and $\alpha = \frac{1}{2}$, so we obtain the extremal function $f(z) = \frac{1}{2}\left[\frac{z}{(1+z)^{2\lambda}} + \frac{z}{(1-z)^{2\lambda}}\right]$. Thus, the estimate is sharp. □

From Theorem 5.1, taking $\lambda = 1$, we obtain the result for the class $\mathcal{T}$ presented in [17].

Corollary 4. *For any* $f \in \mathcal{T}$ *we have*

$$|a_2a_4 - {a_3}^2| \leq 9 \; .$$

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