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Additive Coloring of Planar Graphs

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Abstract   An *additive coloring* of a graph G is an assignment of positive integers $\{1, 2, \ldots, k\}$ to the vertices of G such that for every two adjacent vertices the sums of numbers assigned to their neighbors are different. The minimum number k for which

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there exists an additive coloring of G is denoted by $\eta(G)$. We prove that $\eta(G) \leqslant 468$ for every planar graph G. This improves a previous bound $\eta(G) \leqslant 5544$ due to Norin. The proof uses Combinatorial Nullstellensatz and the coloring number of planar hypergraphs. We also demonstrate that $\eta(G) \leqslant 36$ for 3-colorable planar graphs, and $\eta(G) \leqslant 4$ for every planar graph of girth at least 13. In a group theoretic version of the problem we show that for each $r \geqslant 2$ there is an r-chromatic graph G_r with no additive coloring by elements of any abelian group of order r.

Keywords   Additive coloring · Planar graphs

1 Introduction

Let G be a simple graph, and let k be a positive integer. By a *coloring* of G we mean any function f from the set of vertices $V(G)$ to the set $\{1, 2, \ldots, k\}$. Given a coloring f, consider the induced function $S = S(f)$ on the set $V(G)$ defined by the formula

$$S(v) = \sum_{x \in N(v)} f(x),$$

where $N(v)$ denotes the set of neighbors of the vertex v in G. The initial coloring f is called an *additive coloring* of G if $S(u) \neq S(v)$ for every pair of adjacent vertices u and v. The minimum number k for which there exists an additive coloring of G is denoted by $\eta(G)$.

The notion of additive coloring was introduced in [5] as a vertex version of the 1–2–3-conjecture of Karoński et al. [8]. In the original problem the numbers are assigned to the edges of a graph, and prospective color of a vertex v is derived as the sum of numbers assigned to the edges incident to v. It is conjectured that for every connected graph (except K_2) one can produce a proper vertex coloring in this way using only three numbers: 1, 2, and 3. Currently best bound is 5, as proved by Kalkowski et al. [7].

In the related additive coloring problem no finite bound is possible since for cliques we have $\eta(K_n) = n$. We conjecture however, that perhaps $\eta(G) \leqslant \chi(G)$ for every graph G, where $\chi(G)$ denotes the usual chromatic number. This conjecture is widely open as it is not known whether $\eta(G)$ is bounded for bipartite graphs. In [5] we proved that $\eta(G) \leqslant 3$ for planar bipartite graphs, and also that $\eta(G) \leqslant 100280245065$ for general planar graphs. The later bound was improved to 5544 by Norin (personal communication). We present his argument in Sect. 2 for completeness.

In this note we obtain a further improvement of this bound. Our main result asserts that $\eta(G) \leqslant 468$ for every planar graph G. The proof uses Combinatorial Nullstellensatz of Alon [1], and the coloring number of hyperhraphs represented by planar bipartite graphs. For planar graphs of girth at least 13 we get a much better bound by 4, using a decomposition theorem of Bu et al. [3].

Let us finally mention that additive colorings have been recently introduced independently by Chartrand et al. [4], under a different name of *sigma colorings* (but with the same inspiration coming from 1–2–3-conjecture). However, the main parameter

$\sigma(G)$ studied there is different than $\eta(G)$, as it is defined by the number of different colors used by the coloring f rather than the range of a color set.

2 Coloring Number of Graphs and Hypergraphs

We start with presenting an unpublished result of Norin. Recall that the coloring number col(G) of a graph G is the least integer k such that there exists a linear ordering of the vertices $v_1, \ldots, v_n$ such that the number of *backward* neighbors of v_i (those contained in the set $\{v_1, \ldots, v_{i-1}\}$) is at most $k-1$, for every $i = 1, 2, \ldots, n$. It is well known that col$(G) \leqslant 6$ for every planar graph G.

Theorem 1 *(S. Norin) Let G be a graph with chromatic number $\chi(G) = r$ and coloring number* col$(G) = k$. *Let $n_1, \ldots, n_r$ be r pairwise coprime integers, with $n_i \geqslant k$ for all $i = 1, 2, \ldots, k$. Then $\eta(G) \leqslant n_1 \times \ldots \times n_r$. In particular, $\eta(G) \leqslant 5544$ for every planar graph G (by taking $n_1 = 7$, $n_2 = 8$, $n_3 = 9$, and $n_4 = 11$).*

Proof Fix a proper coloring c of a graph G using colors $\{1, 2, \ldots, r\}$. Also, fix a linear ordering of the vertices realizing col$(G) = k$. Let $n_1, \ldots, n_r$ be any positive integers such that $\gcd(n_i, n_j) = 1$ for every pair $i \neq j$, with $n_i \geqslant k$ for all $i = 1, 2, \ldots, r$. Suppose now that each vertex v is assigned with a certain weight $n(v) \in \mathbb{Z}_{n_j}$, with $j = c(v)$. Denote by $S_i(v)$ the sum of weights of all the neighbors of v in color i. More formally,

$$S_i(v) = \sum_{x \in N(v) \cap c^{-1}(i)} n(x),$$

where the summation is in the group $\mathbb{Z}_{n_i}$. Finally, let $S(v) = (S_1(v), \ldots, S_r(v))$.

Since no neighbor of v is colored with $c(v)$, we have $S_j(v) = 0$ for $j = c(v)$. Our aim is to modify weights $n(v)$ greedily so that $S_{c(v)}(u) \neq 0$ for every backward neighbor u of v. This will imply that $S(u) \neq S(v)$ for every pair of adjacent vertices u and v.

Suppose we have achieved this property for all vertices up to v_{i-1} by choosing appropriate weights $n(v_1), \ldots, n(v_{i-1})$. Now we have to find a weight for the vertex v_i. Let $j = c(v_i)$. For every backward neighbor u of v_i there is only one value of $n(v_i)$ making $S_j(u) \equiv 0 \pmod{n_j}$. Since there are at most $k-1$ backward neighbors of v_i, there are only $k-1$ forbidden values for $n(v_i)$. Since $n_j > k-1$, there is a free element of $\mathbb{Z}_{n_j}$ for the weight $n(v_i)$.

To get an additive coloring of the graph G we assign to every vertex v, an element $f(v) = (f_1(v), \ldots, f_r(v))$ of the group $\mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_r}$, defined by $f_j(v) = n(v)$ if $j = c(v)$, and $f_j(v) = 0$, otherwise. This completes the proof, as the group $\mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_r}$ is isomorphic to $\mathbb{Z}_N$, where $N = n_1 \times \ldots \times n_r$. □

The notion of coloring number can be generalized in a natural way for hypergraphs. Given a hypergraph H and a linear ordering of its vertices $v_1, \ldots, v_n$, define the *backward degree* of a vertex v_i as the number of *different* hyperedges of the form $\{v_i\} \cup A$, with $A \subseteq \{v_1, \ldots, v_{i-1}\}$ (we allow A to be empty). The *coloring number*

col(H) of the hypergraph H is the minimum k such that in some linear ordering of the vertices all backward degrees are at most $k - 1$. This definition differs slightly from the one given in [9], but it is appropriate for our purposes.

Lemma 1 *Let H be a hypergraph with* col(H) $= k$. *Then there is a function f :* $V(H) \to \mathbb{Z}_k$ *such that every hyperedge B satisfies*

$$\sum_{v \in B} f(v) \not\equiv 0 \pmod{k}.$$

Proof Start with a linear ordering of the vertices realizing col(H) and proceed greedily with respect to it. At each step there are at most $k - 1$ partial sums we have to take into account, and each of them is reset by exactly one value. Hence, there is always a good choice for the next value of f. □

Now we give an upper bound for the coloring number of hypergraphs arising from bipartite planar graphs.

Lemma 2 *Let G be a bipartite planar graph with bipartition classes X and Y. Let H be a hypergraph on the set of vertices X whose incidence graph is G. Then* col(H) $\leqslant 12$. *In particular, there exists a coloring $f : X \to \mathbb{Z}_{12}$ satisfying condition:*

$$\sum_{x \in N(y)} f(x) \not\equiv 0 \pmod{12}$$

*for every non-isolated vertex $y \in Y$.*

Proof We may assume that no two vertices in Y are *twins* (have exactly the same nonempty neighborhood), as multiple hyperedges do not count in backward degree. We shall prove that the hypergraph H always contains a vertex of the usual degree at most 11. This is sufficient since a hypergraph $H - x$ still does not contain multiple hyperedges (therefore the incidence graph of $H - x$ does not contain twins), and we may order the vertices of H by a sequential deletion of such vertices.

Fix an embedding of G in the plane. Transform this embedding into a new plane graph P in the following way. For every vertex $y \in Y$, draw a simple closed curve $C(y)$ through the neighbors of y within ε-distance from the connecting edges (see Fig. 1), so that a simply connected region $F(y)$ arises with the following properties:

1. All neighbors of y belong to $C(y)$.
2. All other points of the edges connecting y to its neighbors (and y itself) are in the interior of $F(y)$.
3. No other points of the embedding of G are in $F(y)$.

Forget now about y's and their edges inside regions $F(y)$. In this way we get a plane (pseudo)graph P on the set of vertices X whose faces can be properly 2-colored: color the faces $F(y)$ by black and all other faces by white. Notice that hyperedges of H turned into black faces in P. Hence, $\deg_H(v)$ is just the number of black faces incident to v.

Fig. 1 *Filled closed curves*
$C(y)$ routed around every *white*
vertex $y \in Y$
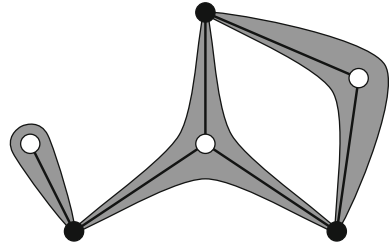
We claim that there is always a vertex in P incident to at most 11 black faces. First, shrink all loops and all 2-sided faces of P to get a new pseudograph Q whose faces have at least three vertices. Let v, e, and f denote the number of vertices, edges, and faces in Q, respectively. So, we have $3f \leqslant 2e$, and by Euler's formula we get $e \leqslant 3v - 6$. Hence, there must be a vertex x of degree at most 5 in Q. Now, by the lack of twins in G, each edge incident to x in Q has multiplicity at most 4 in P. Also, there can be at most one loop at each vertex in P, by the same reason. Therefore the degree of x in P is at most 22, and there are at most 11 black faces incident to x. The proof of the lemma is complete.   □

It is worth noticing that the above lemma is tight. To see this take the icosahedron on the vertex set X and modify it in the following way: (1) subdivide each edge and each face of the icosahedron with one new vertex, (2) append a hanging edge to each vertex from X. The resulting graph is a twin-free planar bipartite graph in which every vertex in X has degree 11.

3 Combinatorial Nullstellensatz

For the proof of our main result we will need a simple consequence of the celebrated Combinatorial Nullstellensatz of Alon.

Theorem 2 *(Combinatorial Nullstellensatz) Let $\mathbb{F}$ be an arbitrary field, and let $P(x_1, \ldots, x_n)$ be a polynomial in the ring of polynomials $\mathbb{F}[x_1, \ldots, x_n]$. Suppose that there is a nonvanishing monomial $x_1^{k_1} \ldots x_n^{k_n}$ in P such that $k_1 + \ldots + k_n = \deg(P)$. Then for every subsets $A_1, \ldots, A_n$ of the field $\mathbb{F}$, with $|A_i| \geqslant k_i + 1$, there are elements $a_i \in A_i$ such that $P(a_1, \ldots, a_n) \neq 0$.*

The above theorem has many surprising applications in geometry, combinatorics, and number theory [1]. An elegant and simple proof of this can be also found in [10]. We used it in [5] to prove that every planar bipartite graph has an additive coloring from arbitrary lists of size at least three. Below we give a slight extension of this result, which will be useful later.

Theorem 3 *Let G be a bipartite graph whose edges can be oriented so that each vertex has indegree at most k. Suppose that each vertex v is assigned with a list $L(v)$ of $k + 1$ real numbers. Then for every function $q : V(G) \to \mathbb{R}$ there is a coloring f of the vertices such that*

$$q(u) + \sum_{x \in N(u)} f(x) \neq q(v) + \sum_{x \in N(v)} f(x)$$

for every pair of adjacent vertices u and v.

Proof Let $U = \{u_1, \ldots, u_m\}$ and $V = \{v_1, \ldots, v_n\}$ be the bipartition classes of a graph G. Let $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ be the variables assigned to the vertices of these classes, respectively. Denote by $S(u)$ the sum of variables assigned to the neighbors of u. Consider a polynomial P over the field of reals defined by

$$P(x_1, \ldots, x_m, y_1, \ldots, y_n) = \prod_{u_i v_j \in E(G)} (q(u_i) + S(u_i) - q(v_j) - S(v_j)).$$

We claim that P contains a nonvanishing monomial with exponents bounded by k. Let $\vec{G}$ be an orientation of G with indegrees bounded by k. In every factor of P corresponding to an edge $u_i v_j$ choose one of the variables x_i or y_j—the one that corresponds to the vertex on which the arrow points. In this way we obtain the monomial $M = x_1^{k_1} \ldots x_m^{k_m} y_1^{l_1} \ldots y_n^{l_n}$ satisfying $0 \leqslant k_i, l_j \leqslant k$. Why is this monomial nonvanishing in P It is because each variable occurs in factors of P with a uniform sign (x_i with minus sign, y_j with plus sign). Hence, the sign of the monomial M in P is uniquely determined by the sequence of exponents, and therefore its copies cannot cancel. Finally, to apply Combinatorial Nullstellensatz, notice that $\deg(P)$ equals the number of edges in G, which is the same as $k_1 + \ldots + k_m + l_1 + \ldots + l_n$ since $q(u_i) - q(v_j)$ are constants. $\qquad\square$

Corollary 1 *Every tree has an additive coloring from arbitrary lists of size two. Every bipartite planar graph has an additive coloring from arbitrary lists of size three.*

Proof Every tree has an orientation with at most one incoming edge to every vertex. Every bipartite planar graph has an orientation with indegrees bounded by two. $\qquad\square$

4 Main Results

Let us start with a simpler case of planar 3-colorable graphs.

Theorem 4 *Every planar graph G with $\chi(G) \leqslant 3$ satisfies $\eta(G) \leqslant 36$.*

Proof Let $V(G) = A \cup B \cup C$ be a partition of the vertex set of G into three independent sets. Let H be a subgraph of G on the set of vertices $V(H) = A \cup B \cup C$ with the edge set

$$E(H) = \{uv \in E(G) : u \in A \cup B \text{ and } v \in C\}.$$

Clearly H is a bipartite graph. Hence, by Lemma 2, there is a function $h : C \to \{1, 2, \ldots, 12\}$ such that the sum

$$S_h(u) = \sum_{x \in N_H(u)} h(x)$$

satisfies $S_h(u) \not\equiv 0 \pmod{12}$ for every vertex $u \in A \cup B$ having at least one neighbor in C. For other vertices the above sum is empty and we adopt $S_h(u) = 0$ by convention.

Consider now a bipartite subgraph F of G induced by the vertices $A \cup B$. Assign to each vertex u in F the list $L(u) = \{12, 24, 36\}$, and apply Theorem 3 with function $q(u) = S_h(u)$. Let f be a coloring satisfying the assertion of Theorem 3. That is, f satisfies the condition $S_f(u) + S_h(u) \neq S_f(v) + S_h(v)$ for every edge $uv \in E(F)$, where

$$S_f(u) = \sum_{x \in N_F(u)} f(x).$$

Finally, let g be a function defined on the whole set of vertices $V(G)$ by joining f and h:

$$g(x) = \begin{cases} h(x) & \text{if } x \in C \\ f(x) & \text{if } x \in A \cup B \end{cases}.$$

We claim that g is an additive coloring of G over the set $\{1, 2, \ldots, 36\}$. Let $S(u)$ be the sum of g-labels over the whole neighborhood $N(u)$, that is, $S(u) = S_h(u) + S_f(u)$. Let uv be any edge in G. If $u \in A \cup B$ and $v \in C$, then $S_h(u) \not\equiv 0 \pmod{12}$ and $S_f(u) \equiv 0 \pmod{12}$, thus $S(u) \not\equiv 0 \pmod{12}$. On the other hand, $S_h(v) \equiv S_f(v) \equiv 0 \pmod{12}$, so $S(v) \equiv 0 \pmod{12}$. In the other case, if $u \in A$ and $v \in B$, the condition $S(u) \neq S(v)$ is guaranteed by the construction of f. This completes the proof. $\qquad\square$

The proof for 4-colorable planar graphs is similar in spirit, though a bit more technical.

Theorem 5 *Every planar graph satisfies $\eta(G) \leqslant 468$.*

Proof Let $V(G) = A \cup B \cup C \cup D$ be a partition of the vertex set of G into four independent sets. Let H_1 be a subgraph of G on the set of vertices $(A \cup B) \cup C$ with the edge set

$$E(H_1) = \{uv \in E(G) : u \in A \cup B \text{ and } v \in C\}.$$

Clearly H_1 is a bipartite graph. Hence, by Lemma 2, there is a function $h_1 : C \to \mathbb{Z}_{12}$ such that the sum

$$S_{h_1}(u) = \sum_{x \in N_{H_1}(u)} h_1(x)$$

satisfies $S_{h_1}(u) \not\equiv 0 \pmod{12}$ for every vertex $u \in A \cup B$ with at least one neighbor in C. Now, let H_2 be a subgraph of G on the set of vertices $(A \cup B \cup C) \cup D$ with the edge set

$$E(H_2) = \{uv \in E(G) : u \in A \cup B \cup C \text{ and } v \in D\}.$$

Clearly H_2 is a bipartite graph. Hence, by Lemma 2, there is a function $h_2 : D \to \mathbb{Z}_{13}$ such that the sum

$$S_{h_2}(u) = \sum_{x \in N_{H_2}(u)} h_2(x)$$

satisfies $S_{h_2}(u) \not\equiv 0 \pmod{13}$ for every vertex $u \in (A \cup B \cup C)$ having a neighbor in D.

Now, using functions h_1 and h_2, we define a new function $h : C \cup D \to \{1, 2, \ldots, 156\}$ as follows. First we extend h_1 and h_2 to the whole set $C \cup D$ by putting $h_1(x) = 0$ for $x \in D$ and $h_2(x) = 0$ for $x \in C$. Let σ be a group isomorphism from $\mathbb{Z}_{12} \times \mathbb{Z}_{13}$ to $\mathbb{Z}_{156}$. Notice that $\sigma(a, b) \equiv a \pmod{12}$ and $\sigma(a, b) \equiv b \pmod{13}$ ($\sigma(a, b) \equiv 13a - 12b \pmod{156}$ is such an isomorphism). For each $x \in C \cup D$ define $h(x)$ as the unique number in the range $\{1, 2, \ldots, 156\}$ satisfying the congruence

$$h(x) \equiv \sigma(h_1(x), h_2(x)) \pmod{156}.$$

In particular, $h(u) \equiv h_1(u) \pmod{12}$ and $h(u) \equiv h_2(u) \pmod{13}$ for every $u \in C \cup D$. Let

$$S_h(u) = \sum_{x \in N(u) \cap (C \cup D)} h(x)$$

for every $u \in A \cup B \cup C$, where, as before, $S_h(u) = 0$ if $N(u) \cap (C \cup D) = \emptyset$. Consequently, $S_h(u)$ satisfies

$$S_h(u) \equiv \begin{cases} S_{h_1}(u) \pmod{12}, & \text{for } u \in (A \cup B) \cap N(C), \\ S_{h_2}(u) \pmod{13}, & \text{for } u \in (A \cup B \cup C) \cap N(D), \\ 0 \pmod{12}, & \text{for } u \in C, \\ 0 \pmod{13}, & \text{for } u \in D. \end{cases}$$

Consider now a bipartite subgraph F of G induced by the vertices $A \cup B$. Assign to each vertex u in F the list $L(u) = \{156, 312, 468\}$, and apply Theorem 3 with function $q(u) = S_h(u)$. Let f be a coloring satisfying the assertion of Theorem 3. That is, f satisfies the condition $S_f(u) + S_h(u) \neq S_f(v) + S_h(v)$ for every edge $uv \in E(F)$, where

$$S_f(u) = \sum_{x \in N_F(u)} f(x).$$

Putting things together we define a function g on the whole set of vertices $V(G)$ by joining f and h:

$$g(x) = \begin{cases} h(x) & \text{if } x \in C \cup D \\ f(x) & \text{if } x \in A \cup B \end{cases}.$$

We claim that g is an additive coloring of G over the set $\{1, 2, \ldots, 468\}$. Let $S(u)$ be the sum of g-labels over the whole neighborhood $N(u)$, that is, $S(u) = S_h(u) + S_f(u)$. Let uv be any edge in G. If $u \in A \cup B$ and $v \in C$, then $S_h(u) \equiv S_{h_1}(u) \not\equiv 0 \pmod{12}$ while $S_f(u) \equiv 0 \pmod{12}$. Thus, $S(u) \not\equiv 0 \pmod{12}$; the other end of the edge satisfies $S(v) = S_h(v) + S_g(v) \equiv S_f(v) \equiv 0 \pmod{12}$. Likewise, if $u \in A \cup B \cup C$ and $v \in D$, then $S(u) \equiv S_{h_2}(u) \not\equiv 0 \pmod{13}$ while $S(v) \equiv 0 \pmod{13}$. If $u \in A$ and $v \in B$, condition $S(u) \neq S(v)$ is guaranteed by construction of f. This completes the proof.   □

A set of vertices I in a graph G is called *two-independent* if the distance between any two vertices of I is at least three. In [3] it was proved that every planar graph of girth at least 13 has a vertex decomposition into two sets I and F such that I is two-independent and F induces a forest. Our last theorem follows easily from this result.

Theorem 6 *Every planar graph of girth at least 13 satisfies $\eta(G) \leqslant 4$.*

Proof Let $V(G) = I \cup F$, where I is 2-independent and F induces a forest. By Corollary 1 there is an additive coloring f of the forest F using labels $\{2, 4\}$. Extend this coloring to the whole graph G by putting $f(i) = 1$ for each vertex $i \in I$. It is easy to see that f is an additive coloring of G.   □

5 Finite Abelian Groups

The problem of additive coloring can be considered in a more general setting of abelian (additive) groups. We may use elements of any such group Γ as the labels of vertices and define the additive coloring the same way as before. Accordingly to our main conjecture, as well as to the methods we have develop so far, one could expect that perhaps every graph has an additive coloring over some group whose order is equal to the chromatic number of the graph. We prove below that this is not true.

Theorem 7 *For every $r \geqslant 2$ there is a graph G_r such that $\chi(G_r) = r$, and there is no additive coloring of G_r over any finite abelian group of order r, but there is an additive coloring of G_r in $\mathbb{Z}_{r+1}$.*

Proof Let P denote a path on five vertices a, x, b, y, c (in that order). Consider a graph $H = H(r)$ obtained by blowing up each of the two vertices x and y to the clique K_{r-1}. Now, take r copies of H, chose one vertex v_i in any of the two cliques K_{r-1} in each copy of H, and join all these vertices mutually to form a new clique K_r. We claim that in this way we constructed a graph G_r satisfying the assertion of the theorem. It is not hard to see that $\chi(G_r) = r$. To prove the first part of the theorem, suppose that Γ is any abelian group of order r, and there is a coloring $f : V(G_r) \to \Gamma$ such that the sums $S(v)$ form a proper coloring of G_r. Notice that in any proper coloring of H with r colors, the vertices a, b, and c must have the same color. Thus $S(a) = S(b) = S(c)$. Notice also that, by the definition of additive coloring we have $S(b) = S(a) + S(c)$, which implies that $S(a) = S(b) = S(c) = 0$ in every copy of H in G_r. This implies in

turn that $S(v) \neq 0$ for all other vertices of G_r. In particular, we get a proper coloring of the clique K_r by non-zero elements of $\mathbb{Z}_r$, which is not possible.

For the second assertion we define explicitly an additive coloring function $f : V(G_r) \to \mathbb{Z}_{r+1}$ as follows. Denote by H_i the ith copy of the graph H in G_r. Let X_i and Y_i denote the two cliques K_{r-1} in H_i obtained by blowing up the vertices x and y, respectively. Also, let a_i, b_i, and c_i be the respective copies of the end vertices and the middle vertex of the path P in H_i. Finally, let v_i denote the unique vertex of H_i belonging to the clique K_r. We may assume that $v_i \in V(X_i)$. We have to distinguish two cases.

1. *(The number $r + 1$ is odd.)* Put $f(v_i) = f(b_i) = 0$ and $f(a_i) = f(c_i) = i$ for all $i = 1, 2, \ldots, r$. Then label any vertex of Y_i with 0 and extend injectively the coloring using all labels from the set $\{1, 2, \ldots, r\} \setminus \{i, -i\}$ on each of the two cliques X_i and Y_i. So, the total sum of labels in each of the cliques X_i and Y_i is zero. Hence, we get $S(v_i) = i$ and $S(a_i) = S(b_i) = S(c_i) = 0$. For any other vertex u we get $S(u) \neq 0$. Also, we cannot have conflicts inside cliques X_i and Y_i by injectivity.

2. *(The number $r + 1$ is even.)* Let $r + 1 = 2k$. First we construct our coloring on all copies H_i for $i \neq k$. Put $f(v_i) = f(b_i) = f(c_i) = 0$ and $f(a_i) = i$. Extend injectively the coloring on the clique X_i using all labels from the set $\{1, 2, \ldots, r\} \setminus \{i, -i\}$. So, the total sum of labels on X_i is equal to k. Next, extend the coloring injectively to cliques Y_i using all labels from the set $\{1, 2, \ldots, r\} \setminus \{k\}$. Hence, the total sum of labels over Y_i is zero. Thus we get $S(v_i) = k+i$, $S(a_i) = S(b_i) = k$, and $S(c_i) = 0$ for all $i \neq k$. For $u \in X_i$ we have $S(u) = k+i - f(u) \neq k$, since $f(u) \neq i$. For $u \in Y_i$ we have $S(u) = -f(u) \neq 0, k$. Also there are no conflicts inside cliques X_i and Y_i by injectivity. It remains to extend the coloring to the copy H_k. Put $f(v_k) = 0$, $f(a_k) = 1$, $f(b_k) = k$, and $f(c_k) = k - 1$. Next put injectively all labels from the set $\{1, 2, \ldots, r\} \setminus \{k, k+1\}$ to the vertices of X_k, and similarly for Y_k using the set $\{0, 1, \ldots, r\} \setminus \{k, r\}$. So, the total sum over X_k is $k-1$ and the total sum over Y_k is 1. Hence, we get $S(a_k) = k-1$, $S(c_k) = 1$, $S(b_k) = k$, and $S(v_k) = 0$. Since each vertex $u \in X_k \cup Y_k$ satisfies $S(u) = -f(u)$, no other conflicts could appear.

The proof is complete. □

Notice that the graph G_4 from the proof above is planar, so the correspondent of our main conjecture is false even for planar graphs in the finite groups environment. Notice also, that G_2 is a tree, and G_3 is an outer planar graph, so the same difficulty arises for planar graphs with smaller chromatic numbers. Perhaps every r-colorable graph has an additive coloring modulo $r + 1$.

We conclude this section with the following simple result.

Theorem 8 *Let A be a fixed abelian group. The problem of deciding whether a given graph G has an additive coloring over A is NP-complete if $|A| \geqslant 3$, and polynomial for $A = \mathbb{Z}_2$.*

Proof Let $|A| = k \geqslant 3$. For a given graph G, whose vertex set is $V(G) = \{v_1, \ldots, v_n\}$, consider a new graph G' obtained by adding n new vertices $\{v'_1, \ldots, v'_n\}$ and n new

edges $v_i v_i'$ for $i = 1, \ldots, n$. We prove that G is k-colorable (in the usual sense) if and only if G' is additively colorable over A. This will prove the first assertion of the theorem.

Obviously, if G' has an additive coloring over A, then G is k-colorable in the usual sense. For the other implication, assume that G is k-colorable, and fix a proper coloring c of G using A as the set of colors. Now fix a nonzero element $a \in A$ and define a new coloring f of G' in the following way:

1. If $c(v_i) = 0$, then $f(v_i) = a$.
2. If $c(v_i) \neq 0$, then $f(v_i) = 0$.
3. $f(v_i') = c(v_i) - \displaystyle\sum_{x \in N_G(v_i)} f(v_i)$.

We claim that f is a desired additive coloring of G' over A. Indeed, the sum of colors around each vertex v_i satisfies

$$S(v_i) = \sum_{x \in N_G(v_i)} f(v_i) + f(v_i') = c(v_i),$$

so there are no conflicts in G. Also by definition of f we have

$$S(v_i') = f(v_i) \neq c(v_i) = S(v_i)$$

for each vertex v_i'. This proves the claim.

For the second assertion just notice that the problem reduces to recognizing if a given graph G is bipartite, and then checking solvability of a system of linear equations of the form $Mx = y$ over $\mathbb{Z}_2$, where M is the adjacency matrix of G, and y is a binary vector encoding a proper coloring of G. There are actually two possible such vectors for a connected bipartite graph G. This completes the proof. □

6 Open Problems

We conclude the paper with a short list of open questions concerning additive coloring of graphs.

conjecture 1 Every graph G satisfies $\eta(G) \leqslant \chi(G)$.

It is not known whether this is true for bipartite graphs. It is not even known if $\eta(G)$ is bounded for bipartite graphs. A heuristic argument is that the statement of the conjecture holds trivially if we extend the set of labels to real numbers. Indeed, any proper coloring of a k-colorable graph G with a set of k real numbers which is independent over rationals, gives an additive coloring of G. Another direction is to consider additive colorings in finite abelian groups.

conjecture 2 Every graph G has an additive coloring modulo $\chi(G) + 1$.

If true this is best possible, as we proved in section 5.

Our last problem arose as a vertex analog of the famous *antimagic labeling conjecture* of Ringel [6].

conjecture 3 Let G be a simple graph on n vertices in which no two vertices have the same neighborhood. Then there is a bijection $f : V(G) \rightarrow \{1, 2, \ldots, n\}$ such that

$$\sum_{x \in N(u)} f(x) \neq \sum_{x \in N(v)} f(x)$$

for any two distinct vertices u and v.

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