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Existence of local solutions for differential equations with arbitrary fractional order

Abstract In this paper, we establish sufficient conditions for the existence of local solutions for a class of Cauchy type problems with arbitrary fractional order. The results are established by the application of the contraction mapping principle and Schaefer's fixed point theorem. An example is provided to illustrate the applicability of the results.

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الملخص

نقدم في هذه الورقة شروطاً لوجود حلول محلية لصف من المسائل من نوع كوشي ذات درجة كسرية اختيارية. يتم إثبات النتائج بتطبيق مبدأ راسم الانكماش ومبرهنة شرودر للقيمة الثابتة. تم تقديم مثال لتوضيح قابلية النتائج التي حصلنا عليها للتطبيق.

1 Introduction

In the recent years, there has been a great development in the study of fractional differential equations. This advancement is ranging from the theoretical analysis of the subject to analytical and numerical techniques. In fact, the extensive application of fractional differential equations appeared in many engineering and scientific disciplines, such as physics and engineering [15], diffusion processes ([14,24,29]), electrochemistry [17], electromagnetism [6], numerical analysis [7], optimal control ([12,16]), variational analysis [23], chaotic system [30], viscoelasticity [11], biology ([9,10]), biophysics [25], and economics [26]. The theoretical analysis of these kinds of differential equations is very important for the applicability on the reality. Therefore, as a part of theoretical analysis, the pre-knowledge of the existence of a solution to fractional differential equations is the first step for finding the analytic solution. Hence, an extensive research in existence of solution for different kinds of fractional differential equations is recently completed by many authors (see [2,4,5,8,18–21,27,31] and references therein). In [13] and [18], the authors obtained sufficient conditions for the existence of solutions of boundary value problem for differential equations of fractional order $\alpha \in (0, 1]$ and $\alpha \in (1, 2]$ involving the Caputo fractional derivative and nonlocal conditions. The researchers in the articles [4,21], and [27] considered the existence problem of solutions of boundary value problems for differential equations of fractional order $\alpha \in (2, 3]$. The existence and uniqueness of initial value problems of some differential equations of other

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fractional orders are investigated by many authors (see [2,22]). The general theory of Cauchy fractional differential equations is deeply introduced in the monograph [1] and in the survey [28]. In fact, the equivalent Volterra integral equation to Cauchy problem for nonlinear fractional differential equations introduced in the cited articles is essential to prove the existence of such systems. However, the generalization idea of existence problems to arbitrary fractional order with arbitrary inner point as initial condition has not been investigated by the researchers. Motivated by these ideas, we study in this paper the existence of a solution to the Cauchy problem

$$\begin{cases} {}^{C}D_{t_0}^{\alpha}x(t) = f(t, x(t)), & t \in [t_0, \theta) \cup (\theta, T] \\ x^{(k)}(\theta) = x_k, & k = 0, 1, 2, \ldots, n-1, \theta \in J, \end{cases} \tag{1.1}$$

at any inner point θ of a finite interval $J = [t_0, T]$ involving the Caputo fractional derivative ${}^{C}D_{t_0}^{\alpha}$, where $\alpha \in (n-1, n]$, $n \in \mathbb{N}$, and f is a given continuous function. The inner points of the interval involved in the problem can be used as impulses in a physical approach or sometimes nonlocal boundary condition, hence the problem may be considered as a case of nonlocal fractional differential model. However, the used technique of obtaining the solution of the problem is new compared with any previous works ([1]: Section 3.4.2) and the results on arbitrary fractional ordered differential equations generalize the existing problems.

2 Preliminaries

We introduce in this section some basic definitions and properties of fractional calculus (see [1]) which will be used in this paper.

Definition 2.1 A function f is said to be fractional integrable of order $\alpha > 0$ if

$$I_{t_0}^{\alpha} f(t) = \int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) \mathrm{d}s < \infty,$$

and if $\alpha = 0$, then $I^0 f(t) = f(t)$.

Next, we introduce the Caputo fractional derivative.

Definition 2.2 The Caputo fractional derivative of x is defined as:

$$ {}^{C}D_{t_0}^{\alpha}x(t) = I_{t_0}^{n-\alpha}\left(\frac{\mathrm{d}^n x}{\mathrm{d}t^n}\right)(t) = \int_{t_0}^{t} \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} x^{(n)}(s) \mathrm{d}s $$

for $t > t_0$.

In what follows, we assume that f and x are continuous functions, such that f and ${}^{C}D_{t_0}^{\alpha}x$ are fractional integrable of any order less than or equal α.

The compositions between the Caputo fractional derivative and fractional integrals are given by the following lemma.

Lemma 2.3 Let $t \in J$ and $c_k \in \mathbb{R}$. Then

$$\begin{cases} {}^{C}D_{t_0}^{\alpha} I_{t_0}^{\alpha}x(t) = x(t), \\ I_{t_0}^{\alpha} {}^{C}D_{t_0}^{\alpha}x(t) = x(t) + c_0 + c_1(t-t_0) + c_2(t-t_0)^2 + \cdots + c_{n-1}(t-t_0)^{n-1}, \\ {}^{C}D_{t_0}^{\alpha}x(t) = 0, \text{for } x(t) = c_0 + c_1(t-t_0) + c_2(t-t_0)^2 + \cdots + c_{n-1}(t-t_0)^{n-1}. \end{cases}$$

The following result will be used in the proof of the main theorem in the next section.

Lemma 2.4 Let (u_n) be a sequence of real numbers and $n, k \in \mathbb{N}$, such that $0 \leq k \leq n-1$. If v is a positive real number, then

$$\sum_{m=0}^{n-k-1} \sum_{r=0}^{n-k-m-1} (-1)^r \frac{v^{r+m}}{r!m!} u_{m+k+r} = u_k. \tag{2.1}$$

Proof The left-hand side of Eq. (2.1) can be rearranged as:

$$\sum_{m=0}^{n-k-1} \left(\sum_{r=0}^{m} \frac{(-1)^{m-r}}{r!(m-r)!} \right) v^m u_{k+m}.$$

Hence, by Binomial expansion, the inner sum can be reduced to

$$\frac{1}{m!} \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} = 0$$

for all $m \geq 1$, by which the result is obtained. $\square$

3 Existence problems for linear case

Consider the linear fractional differential equation

$$\begin{cases} {}^{C}D_{t_0}^{\alpha} x(t) = f(t), & t \in J - \{\theta\} \\ x^{(k)}(\theta) = x_k \in \mathbb{R}, & k = 0, 1, 2, \ldots, n-1, \theta \in J. \end{cases} \tag{3.1}$$

We introduce next the basic idea in this article, namely, the solution of (3.1) as an integral form.

Theorem 3.1 *Let f be a continuous real valued function. The fractional differential equation (3.1) is equivalent to the integral equation*

$$x(t) = \int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds + \sum_{k=0}^{n-1} \frac{(t-\theta)^k}{k!} \left(x_k - \int_{t_0}^{\theta} \frac{(\theta-s)^{\alpha-k-1}}{\Gamma(\alpha-k)} f(s) ds \right), \quad t \in J. \tag{3.2}$$

Proof Let $\alpha = n$. Then Eq. (3.1) is equivalent to nth order classical differential equation

$$\begin{cases} \frac{d^n}{dt^n} x(t) = f(t), & t \in J - \{\theta\} \\ x^{(k)}(\theta) = x_k, & k = 0, 1, 2, \ldots, n-1, \theta \in J \end{cases}$$

which can be integrated n times to have

$$x(t) = \sum_{k=0}^{n-1} \frac{(t-\theta)^k}{k!} x_k + \int_{\theta}^{t} \cdots \int_{\theta}^{s_1} f(s_0) ds_0 \cdots ds_{n-1}$$

that can be reduced to

$$x(t) = \sum_{k=0}^{n-1} \frac{(t-\theta)^k}{k!} x_k + \int_{\theta}^{t} \frac{(t-s)^{n-1}}{(n-1)!} f(s) ds. \tag{3.3}$$

Using binomial expansion, we have

$$\int_{\theta}^{t} \frac{(t-s)^{n-1}}{(n-1)!} f(s) ds$$

$$= \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!} f(s) ds - \int_{t_0}^{\theta} \frac{(t-s)^{n-1}}{(n-1)!} f(s) ds$$

$$= \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!} f(s) ds - \int_{t_0}^{\theta} \frac{(t-\theta+\theta-s)^{n-1}}{(n-1)!} f(s) ds$$

$$= \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!} f(s) ds - \sum_{k=0}^{n-1} \frac{(t-\theta)^k}{k!} \left(\int_{t_0}^{\theta} \frac{(\theta-s)^{n-k-1}}{(n-k-1)!} f(s) ds \right).$$

In accordance with (3.3), Eq. (3.2) follows. Now, let $n - 1 < \alpha < n$, Lemma 2.3, implies that

$$I_{t_0}^\alpha f(t) = I_{t_0}^\alpha \left({}^C D_{t_0}^\alpha \right) x(t) = x(t) + \sum_{k=0}^{n-1} c_k (t - t_0)^k. \tag{3.4}$$

Differentiating Eq. (3.4) k times, we have

$$\sum_{m=0}^{n-k-1} \frac{(k+m)!}{m!} c_{m+k} (t - t_0)^m = I_{t_0}^{\alpha-k} f(t) - x^{(k)}(t) \tag{3.5}$$

for $0 \le k \le n - 1$. In accordance with the given conditions in (3.1), Eq. (3.5) can be rewritten in the following array form (assuming $0! = 1$)

$$
\begin{bmatrix}
1 & \frac{1!}{1!}(\theta - t_0)^1 & \frac{2!}{2!}(\theta - t_0)^2 & \cdots & \frac{(n-2)!}{(n-2)!}(\theta - t_0)^{n-2} & \frac{(n-1)!}{(n-1)!}(\theta - t_0)^{n-1} \\
0 & \frac{1!}{0!}(\theta - t_0)^0 & \frac{2!}{1!}(\theta - t_0)^1 & \cdots & \frac{(n-2)!}{(n-3)!}(\theta - t_0)^{n-3} & \frac{(n-1)!}{(n-2)!}(\theta - t_0)^{n-2} \\
0 & 0 & \frac{2!}{0!}(\theta - t_0)^0 & \cdots & \frac{(n-2)!}{(n-4)!}(\theta - t_0)^{n-4} & \frac{(n-1)!}{(n-3)!}(\theta - t_0)^{n-3} \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \frac{(n-2)!}{0!}(\theta - t_0)^0 & \frac{(n-1)!}{1!}(\theta - t_0)^1 \\
0 & 0 & 0 & 0 & 0 & \frac{(n-1)!}{0!}(\theta - t_0)^0
\end{bmatrix}
\times
\begin{bmatrix}
c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-2} \\ c_{n-1}
\end{bmatrix}
$$

$$
=
\begin{bmatrix}
I_{t_0}^\alpha f(\theta) - x_0 \\
I_{t_0}^{\alpha-1} f(\theta) - x_1 \\
I_{t_0}^{\alpha-2} f(\theta) - x_2 \\
\vdots \\
I_{t_0}^{\alpha-n+2} f(\theta) - x_{n-2} \\
I_{t_0}^{\alpha-n+1} f(\theta) - x_{n-1}
\end{bmatrix}
$$

which can be algebraically solved to obtain, for $r = 1, 2, \ldots, n$,

$$c_{n-r} = \frac{1}{(n-r)!} \sum_{k=0}^{r-1} \frac{(-1)^k (\theta - t_0)^k}{k!} \left(I_{t_0}^{\alpha-n+r-k} f(\theta) - x_{n-r+k} \right).$$

Alternatively, for $m = 0, 1, 2 \cdots, n - 1$, it can be rewritten as:

$$c_m = \frac{1}{m!} \sum_{k=0}^{n-m-1} \frac{(-1)^k (\theta - t_0)^k}{k!} \left(I_{t_0}^{\alpha-m-k} f(\theta) - x_{m+k} \right). \tag{3.6}$$

Indeed, by Lemma 2.4, with $v = \theta - t_0$ and $u_k = I^{\alpha-k} f(\theta) - x_k$, Eq. (3.6) is a solution of (3.5) with $t = \theta$. In accordance with (3.6) and (3.4), we deduce that

$$x(t) = I_{t_0}^\alpha f(t) + \sum_{k=0}^{n-1} \sum_{m=0}^{n-k-1} \frac{(-1)^m (\theta - t_0)^m (t - t_0)^k}{m! k!} \left(x_{m+k} - I_{t_0}^{\alpha-m-k} f(\theta) \right).$$

The terms of this double summation can be rearranged to have

$$x(t) = \int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) \, ds + \sum_{k=0}^{n-1} \psi_k(t) \left(x_k - \int_{t_0}^{\theta} \frac{(\theta-s)^{\alpha-k-1}}{(\alpha-k)!} f(s) \, ds \right) \tag{3.7}$$

where

$$\psi_k(t) = \sum_{m=0}^{k} (-1)^{k-m} \frac{(\theta - t_0)^{k-m}}{(k-m)!} \frac{(t - t_0)^m}{m!}, \quad k = 0, 1, \ldots n - 1.$$

If $\theta = t_0$, then $\psi_k(t) = \frac{(t-t_0)^k}{k!}$, by which the solution of (3.1) is

$$x(t) = \int_{t_0}^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} f(s) \mathrm{d}s + \sum_{k=0}^{n-1} \frac{(t - t_0)^k}{k!} x_k. \tag{3.8}$$

Next, let $t_0 < \theta \le T$. Then binomial expansion can be applied to obtain

$$\psi_k(t) = \frac{(\theta - t_0)^k}{k!} \sum_{m=0}^{k} \binom{k}{m} \left(\frac{t - t_0}{\theta - t_0}\right)^m (-1)^{k-m}$$

$$= \frac{(\theta - t_0)^k}{k!} \left(\frac{t - \theta}{\theta - t_0}\right)^k = \frac{(t - \theta)^k}{k!}$$

by which, Eq. (3.7) leads to Eq. (3.2).

On the other hand, applying the operator $^C D_{t_0}^\alpha$, $n - 1 < \alpha \le n$ to Eq. (3.2), and using Lemma 2.3, we get Eq. (3.1) which completes the proof.   □

The following is a direct result of Theorem 3.1.

Corollary 3.2 *Let c be any real number. Then the fractional differential system*

$$\begin{cases} ^C D_{t_0}^\alpha x(t) = c, & n - 1 < \alpha \le n, \quad n \in \mathbb{N}, \\ x^{(k)}(\theta) = x_k \in \mathbb{R}, & k = 0, 1, 2, \ldots, n - 1, \quad \theta \in J \end{cases} \tag{3.9}$$

is equivalent to

$$x(t) = \frac{c(t - t_0)^\alpha}{\Gamma(\alpha)} + \sum_{k=0}^{n-1} \frac{(t - \theta)^k}{k!} \left(x_k - \frac{c(\theta - t_0)^{\alpha-k}}{k!\Gamma(\alpha - k)}\right).$$

In particular, if $c = 0$, then (3.9) is equivalent to

$$x(t) = \sum_{k=0}^{n-1} \frac{x_k}{k!} (t - \theta)^k.$$

4 Existence problems for nonlinear cases

We investigate in this section the existence of a local solution for the fractional systems (1.1) by applying Banach's and Schaefer's fixed point theorems.

Let $J_h = [\theta - h, \theta + h] \subset (t_0, T)$, where $0 < h < \min\{\theta - t_0, T - \theta\}$, and $Y_h = C(J_h, \mathbb{R})$ be the Banach space of all continuous functions y defined on J_h with values in $\mathbb{R}$, such that $^C D_{\theta-h}^\alpha y$ exists. Let $f \in C(J_h \times Y_h, Y_h)$ be a fractional integrable function of order $\alpha > 0$ that satisfies the following hypothesis:

(H1) There exists a positive constant A such that

$$\|f(t, x) - f(t, y)\| \le A\|x - y\|,$$

for any $t \in J_h$ and $x, y \in Y_h$. Moreover, let $B = \sup_{t \in J} \|f(t, 0)\|$ and $C = \max\{A, B\}$.

In accordance with Theorem 3.1, the fractional nonlinear system

$$
\begin{cases}
{}^{C}D_{\theta-h}^{\alpha}x(t) = f(t, x(t)), & t \in J_h - \{\theta\} \\
x^{(k)}(\theta) = x_k \in \mathbb{R}, & k = 0, 1, 2, \ldots, n - 1,
\end{cases}
\tag{4.1}
$$

is equivalent to the integral equation

$$
x(t) = \int_{\theta-h}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s)) \mathrm{d}s
$$

$$
+ \sum_{k=0}^{n-1} \frac{(t-\theta)^k}{k!} \left(x_k - \int_{\theta-h}^{\theta} \frac{(\theta-s)^{\alpha-k-1}}{\Gamma(\alpha-k)} f(s, x(s)) \mathrm{d}s \right).
\tag{4.2}
$$

Accordingly, we define the operator Ψ on Y_h as follows:

$$
\Psi x(t) = \int_{\theta-h}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s)) \mathrm{d}s
$$

$$
+ \sum_{k=0}^{n-1} \frac{(t-\theta)^k}{k!} \left(x_k - \int_{\theta-h}^{\theta} \frac{(\theta-s)^{\alpha-k-1}}{\Gamma(\alpha-k)} f(s, x(s)) \mathrm{d}s \right).
\tag{4.3}
$$

The next hypothesis is essential to state and prove the first main result in this section.

(H2) Let θ and r be positive real numbers such that

$$
\gamma = Ch^{\alpha} \left(\frac{2^{\alpha}}{\Gamma(\alpha+1)} + \sum_{k=0}^{n-1} \frac{1}{k!\Gamma(\alpha-k+1)} \right) < 1, \text{ and}
$$

$$
r \geq \frac{\gamma + \sum_{k=0}^{n-1} \frac{h^k}{k!} \|x_k\|}{1-\gamma}.
$$

Moreover, let $\Omega = \{x \in Y_h : \|x\| \leq r\}$.

Theorem 4.1 *Assume that (H1) and (H2) are satisfied. Then, there exists a unique solution for the fractional system* (4.1) *in* Y_h.

Proof We use the Banach fixed point theorem to show that Ψ defined by (4.3) has a fixed point on the closed subspace Ω of the Banach space Y_h. This fixed point satisfies the integral equation (4.2), hence is a solution of (4.1). Let $t \in J_h$. Then

$$
|\Psi x(t)| \leq \frac{(A\|x\|+B)}{\Gamma(\alpha+1)}(t-\theta+h)^{\alpha} + \sum_{k=0}^{n-1} \frac{|t-\theta|^k}{k!} \left(\|x_k\| + \frac{(A\|x\|+B)}{\Gamma(\alpha-k+1)} h^{\alpha-k} \right)
$$

$$
\leq \frac{(A\|x\|+B)}{\Gamma(\alpha+1)}(2h)^{\alpha} + \sum_{k=0}^{n-1} \frac{h^k}{k!}\|x_k\| + h^{\alpha} \sum_{k=0}^{n-1} \frac{(A\|x\|+B)}{k!\Gamma(\alpha-k+1)}
$$

$$
\leq \sum_{k=0}^{n-1} \frac{h^k}{k!}\|x_k\| + Bh^{\alpha} \left(\frac{2^{\alpha}}{\Gamma(\alpha+1)} + \sum_{k=0}^{n-1} \frac{1}{k!\Gamma(\alpha-k+1)} \right)
$$

$$
+ Ah^{\alpha} \left(\frac{2^{\alpha}}{\Gamma(\alpha+1)} + \sum_{k=0}^{n-1} \frac{1}{k!\Gamma(\alpha-k+1)} \right) \|x\|
$$

$$
\leq (1-\gamma)r + \gamma r = r.
$$

It is obviously by (H2) that Ψ maps Ω into itself. Next, let $x, y \in \Omega$. Then

$$|\Psi x(t) - \Psi y(t)| \leq \frac{A \|x - y\|}{\Gamma(\alpha + 1)} (t - \theta + h)^{\alpha} + \sum_{k=0}^{n-1} \frac{|t - \theta|^k}{k!} \frac{A \|x - y\|}{\Gamma(\alpha - k + 1)} h^{\alpha-k}$$

$$\leq Ah^{\alpha} \left(\frac{2^{\alpha}}{\Gamma(\alpha + 1)} + \sum_{k=0}^{n-1} \frac{1}{k!\Gamma(\alpha - k + 1)} \right) \|x - y\|$$

$$\leq \gamma \|x - y\|,$$

since $\gamma < 1$, then Ψ is a contraction mapping on Ω. Hence, Ψ has a fixed point which is the unique solution to (4.1). $\qquad \square$

Next, we show the existence of a local solution for the Cauchy problem

$$\begin{cases} {}^C D^{\alpha}_{T-h_1} x(t) = f(t, x(t)), & t \in [T - h_1, T), \\ x^{(k)}(T) = x_k \in \mathbb{R}, & k = 0, 1, 2, \ldots, n - 1. \end{cases} \tag{4.4}$$

Let $J_{h_1} = [T - h_1, T] \subset (t_0, T]$, where $0 < h_1 < T - t_0$, and $Y_{h_1} = C(J_{h_1}, \mathbb{R})$ be the Banach space of all continuous real valued functions defined on J_{h_1}, such that ${}^C D^{\alpha}_{T-h_1} y$ exists. Let $f \in C(J_{h_1} \times Y_{h_1}, Y_{h_1})$ be a fractional integrable function of order $\alpha > 0$. Hence, by Theorem 3.1, the system (**??**) is equivalent to the Fredholm–Volterra integral equation

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_{T-h_1}^{t} (t - s)^{\alpha-1} f(s, x(s)) ds$$

$$+ \sum_{k=0}^{n-1} (-1)^k \frac{(T - t)^k}{k!} \left(x_k - \int_{T-h_1}^{T} \frac{(T - s)^{\alpha-k-1}}{\Gamma(\alpha - k)} f(s, x(s)) ds \right),$$

for $x \in Y_{h_1}$ and $t \in J_{h_1}$.

We need to modify the hypothesis (H2) as the following:

(H3) Let β and r be positive real numbers such that

$$\beta = Ch_1^{\alpha} \left(\frac{1}{\Gamma(\alpha + 1)} + \sum_{k=0}^{n-1} \frac{1}{k!\Gamma(\alpha - k + 1)} \right) < 1,$$

$$r \geq \frac{\beta + \sum_{k=0}^{n-1} \frac{h_1^k}{k!} \|x_k\|}{1 - \beta}.$$

Moreover, let $\Omega = \left\{ x \in Y_{h_1} : \|x\| \leq r \right\}$.

The proof of the next result is similar to that one of Theorem 4.1, hence it is omitted.

Corollary 4.2 *Assume that (H1) and (H3) are satisfied. Then, there exists a unique solution for the fractional system (4.4) in Y_{h_1}.*

Let $f \in C(J_{h_2} \times Y_{h_2}, Y_{h_2})$ be a fractional integrable function of order $\alpha > 0$, where $J_{h_2} = [t_0, t_0 + h_2]$, $0 < h_2 < T - t_0$, and $Y_{h_2} = C(J_{h_2}, \mathbb{R})$ be the Banach space of all continuous real valued functions J_{h_2}, such that ${}^C D^{\alpha}_{t_0} y$ exists. Next result concerns with the existence of a local solution for the Cauchy problem

$$\begin{cases} {}^C D^{\alpha}_{t_0} x(t) = f(t, x(t)), t \in (t_0, \quad t_0 + h_2], \\ x^{(k)}(t_0) = x_k \in \mathbb{R}, \quad k = 0, 1, 2, \ldots, n - 1, \end{cases} \tag{4.5}$$

which is equivalent to Volterra integral equation (see Eq. (3.8))

$$x(t) = \int_{t_0}^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s)) ds + \sum_{k=0}^{n-1} \frac{(t - t_0)^k}{k!} x_k.$$

The hypothesis (H2) will be replaced by the following:

(H4) Let η and r be positive real numbers such that

$$\eta = \frac{Ch_2^{\alpha}}{\Gamma(\alpha+1)} < 1,$$

$$r \geq \frac{\eta + \sum_{k=0}^{n-1} \frac{h_2^k}{k!} \|x_k\|}{1 - \eta}.$$

Moreover, let $\Omega = \left\{ x \in Y_{h_2} : \|x\| \leq r \right\}.$

Corollary 4.3 *Assume that (H1) and (H4) are satisfied. Then, there exists a unique solution for the fractional system* (4.5) *in* Y_{h_2}.

The last result is devoted to solve the existence problem of the fractional system (4.1) which has equivalent integral Eq. (4.2). We define the operator $\Psi : C(J_h, \mathbb{R}) \to C(J_h, \mathbb{R})$, $J_h = [\theta - h, \theta + h]$, $0 < h < \min\{T - \theta, \theta - t_0\}$ as in Eq. (4.3), and using two versions of Schaefer's fixed point theorem to obtain the existence of the required local solution.

Theorem 4.4 [3] *If Ω is a closed bounded convex subset of a Banach space X and $\Psi : \Omega \to \Omega$ is completely continuous, then Ψ has a fixed point in Ω.*

Theorem 4.5 [3] *Let X be a Banach space. Assume that $\Psi : X \to X$ is completely continuous operator and the set $V = \{x \in X : x = \mu\Psi x, 0 < \mu < 1\}$ is bounded. Then, Ψ has a fixed point in X.*

The last result can be introduced now.

Theorem 4.6 *Let $f : J_h \times C(J_h, \mathbb{R}) \to C(J_h, \mathbb{R})$ be a continuous bounded function. Then, the fractional differential equation* (4.1) *has at least one solution.*

Proof The continuity of f on $J_h \times C(J_h, \mathbb{R})$ implies the continuity of Ψ on $C(J_h, \mathbb{R})$. Define the nonempty closed convex subset $\Omega = \{x \in C(J_h, \mathbb{R}) : \|x\| \leq r, r > 0\}$ of the Banach space $C(J_h, \mathbb{R})$. If $f_m = \max\{\|f(t, x)\| : (t, x) \in J_h \times \Omega\}$, then for any $x \in \Omega$, $t \in J_h$, we have

$$|\Psi x(t)| \leq \frac{f_m(t - \theta + h)^{\alpha}}{\Gamma(\alpha+1)} + \sum_{k=0}^{n-1} \frac{|t - \theta|^k}{k!} \left(\|x_k\| + \frac{f_m h^{\alpha-k}}{\Gamma(\alpha-k+1)} \right). \tag{4.6}$$

Hence $\|\Psi x\| \leq M$, where $M = \frac{f_m(2h)^{\alpha}}{\Gamma(\alpha+1)} + \sum_{k=0}^{n-1} \frac{|h|^k}{k!} \left(\|x_k\| + \frac{f_m h^{\alpha-k}}{\Gamma(\alpha-k+1)} \right)$. Further, we find that

$$\left| (\Psi x)'(t) \right| \leq \frac{f_m(t - \theta + h)^{\alpha-1}}{\Gamma(\alpha)} + \sum_{k=1}^{n-1} \frac{|t - \theta|^{k-1}}{(k-1)!} \left(\|x_k\| + \frac{f_m h^{\alpha-k}}{\Gamma(\alpha-k+1)} \right)$$

$$\leq \frac{f_m(2h)^{\alpha-1}}{\Gamma(\alpha)} + \sum_{k=1}^{n-1} \frac{h^{k-1}}{(k-1)!} \left(\|x_k\| + \frac{f_m h^{\alpha-k}}{\Gamma(\alpha-k+1)} \right)$$

$$= M'.$$

Hence for $t_1, t_2 \in J$, $t_1 < t_2$, we have

$$|\Psi x(t_2) - \Psi x(t_1)| \leq \int_{t_1}^{t_2} \left| (\Psi x)'(t) \right| \mathrm{d}t \leq M'(t_2 - t_1).$$

This implies that Ψ is equicontinuous on J_h. Thus, by the Arzela–Ascoli theorem, the operator Ψ is completely continuous. Next, let $x \in V = \{y \in \Omega : y = \mu\Psi y, 0 < \lambda < 1\}$. Then $x = \mu\Psi x$, for some $\mu \in (0, 1)$. Using (4.6), we have $|x(t)| = \mu|\Psi x(t)| \leq M$, for any $t \in J_h$. Hence $\|x\| \leq M$, which implies the boundedness of V. As a consequence of Theorems 4.4, or 4.5, the operator Ψ has at least one fixed point $x \in \Omega$, which is the solution of (4.1). This finishes the proof.   □

We give an example to explain the applicability of the above results.

Example 4.7 Consider the following nonlinear fractional differential equation

$$\begin{cases} {}^C D_0^{\sqrt{10}} x(t) = \frac{tx(t)+1}{4(1+x(t))}, & t \in [0, 0.5) \cup (0.5, 1] \\ x^{(k)}(0.5) = (0.5)^k, & k = 0, 1, 2, 3. \end{cases} \tag{4.7}$$

The function $f(t, x) = \frac{tx+1}{4(1+x)}$ is continuous and uniformly bounded on $[0, 1] \times [0, \infty)$. Hence, in view of Theorem 4.6, there exists a solution of (4.7). Moreover, f is globally Lipstchitz on $[0, 1] \times [0, \infty)$. By simple calculations, we have max $\{\eta, \beta, \gamma\} \leq 0.61$. Hence, for large values of r, all hypotheses (H1)–(H4) are satisfied. Therefore, using any above results, we can assert that (4.7) has a local solution for any $\theta \in J$.

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