

# Analysis of the Elements of Drag in Three-Dimensional Viscous and Inviscid Flows

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## Abstract

This paper examines the analytical, experimental, and computational aspects of the determination of the drag acting on an aircraft in flight, with or without powered engines, for subsonic/transonic flow. Using a momentum balance approach, the drag is represented by an integral over a cross-flow plane at an arbitrary distance behind the aircraft. Asymptotic evaluation of the integral shows the drag can be decomposed into three components corresponding to streamwise vorticity and variations in entropy and stagnation enthalpy. These are shown to be related to the established engineering concepts of induced drag, wave drag, profile drag and engine power and efficiency. This decomposition of the components of drag is useful in formulating techniques for accurately evaluating drag using computational fluid dynamics calculations or experimental data.

## 1 Introduction

The two most important aerodynamic quantities affecting an aircraft in flight are lift and drag. Nearly all aerodynamic analysis is an attempt to maximize the lift for a given amount of drag, or conversely to minimize the drag for a given amount of lift. The analysis of these quantities for various aircraft configurations forms the basis of most aerodynamic research. Because of this, reliable methods to compute these forces from available experimental or computational data are essential.

Traditionally, aerodynamic forces have been measured in wind tunnels using strain-gauge balances. This approach is very good for measuring the lift, but the drag of a typical aircraft at reasonable angles of incidence is often an order of magnitude less than the lift, and therefore more difficult to measure. In particular, the presence of the model sting or support rails accurate drag measurement very difficult using this approach. This led to attempts to measure drag using techniques based on a control volume approach. The simplest application of this is to measure the momentum deficit parallel to the freestream within the wake of a model. The main drawback to this approach, however, was the need to perform the wake survey throughout the downstream flowfield, as well as various difficulties associated with the presence of the wind tunnel walls. An approach developed by Betz [1] modified the integral formulation to take into account the presence of the wind tunnel walls, and reduced the area of integration to the region directly behind the model. Unfortunately, Betz did not include terms which would account for the drag due to vortices, an important aspect of measuring the drag of a finite span wing. His approach was also found to have certain measurement difficulties as shown by Maull and Bearman [10]. In an attempt to correct some of the problems in Betz's approach, Maslcell [8] showed that an integral formulation could be obtained which would allow the measurement of both profile and vortex drag, both of which could be obtained from measurements in a reduced region behind the aircraft. Since that time various improvements to the model have been made for experimental measurements of drag; these include formulations due to Wu *et al* [14] and Brune and Bogataj [2].

As computational fluid dynamics (CFD) has matured over the years, it has become a goal of CFD researchers to be able to predict aerodynamic drag from numerical

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simulations. Early attempts at  $\tau$  were usually met with frustration, as most approaches involving the pressure  $p$  friction over the surface of the body in order to calculate forces (the computational equivalent of force measurement in  $\tau$ ). Surface  $\tau$  met clue to the need to approximate the curved surface with flat facets, and the difficulty in accurately predicting  $\tau$  friction. This has  $\tau$  to attempt to  $\tau$  experimental wake integral methods to CFD computations. Methods involving wake integration have been shown to  $\tau$  accurate at predicting profile and vortex drag, as shown by van Dam and Nilcfetrat [12] and Janus [3]. An equivalent lifting-line approach by Mathias *et al* [9] has also been shown to be able to accurately compute

The problem with the current approaches used to compute aerodynamic forces from CFD solutions is that various terms are usually neglected. These terms are known to be small far downstream of the aircraft, but in CFD calculations the wake becomes increasingly diffuse downstream because of numerical smoothing, and so the integral methods need to be applied much closer to the aircraft. This  $\tau$  carefully at the drag wake-survey methods and comes to an improved understanding of the importance of the various integrals and the terms that are often neglected. The first approach is to take the cross-flow plane to be far downstream of the aircraft so that all flow components can be assumed to be approximately invariant in the freestream direction. This leads very simply to an integral form of the drag showing the different contributions due to streamwise vorticity and variations in entropy and stagnation enthalpy. Next, an analysis is performed for a plane which is much closer to the aircraft, and at which there is still significant flow variation in the freestream direction. The same drag result is eventually obtained after careful analysis and appropriate asymptotic approximations. The purpose of this section is to relate the current analysis to the work of Betz [1], Maskell [8], Wu *et al* [14], Lock [7], van der Vooren and Slooff [13], van Dam [12], Mathias *et al* [9] and others. In practice, experimental measurement planes are always in this near-field region and there has been considerable discussion in the literature regarding the terms which should be included in the drag computation. It is shown in the analysis presented here that the terms due to the potential flow component of the velocity field cancel. A connection is also shown between the control volume formulation and the classical lifting-line theory of induced drag, showing that the current analysis reduces to the classical analysis under certain limiting conditions. The

discuss the application of the theory to the calculation of drag from experimental measurements or computational results.

## 2 Control volume formulation

The combined aerodynamic force can be written as an integral over the surface of an aircraft as,

$$\mathbf{F} = \int_S -p \mathbf{n} + \boldsymbol{\tau} \cdot \mathbf{n} \, dA, \quad (1)$$

where  $p$  is the pressure,  $\mathbf{n}$  is the surface normal,  $\boldsymbol{\tau}$  is the stress tensor. Using the integral form of the momentum equations the force can also be expressed as an integral over the surface of any control volume enclosing the aircraft,

$$\mathbf{F} = \int_S -p \mathbf{n} - \rho(\mathbf{u} \cdot \mathbf{n}) \mathbf{u} + \boldsymbol{\tau} \cdot \mathbf{n} \, dA. \quad (2)$$

Conservation of mass for the same control volume requires that

$$\int_S \rho(\mathbf{u} \cdot \mathbf{n}) \, dA = 0, \quad (3)$$

and for any closed surface

$$\int_S \mathbf{n} \, dA = 0. \quad (4)$$

Therefore, if the far-field velocity relative to the aircraft is  $\mathbf{U}_\infty$  aligned with the x-coordinate direction, an equivalent form of the force integral is

$$\mathbf{F} = \int_S -(p - p_\infty) \mathbf{n} - \rho(\mathbf{u} \cdot \mathbf{n})(\mathbf{u} - \mathbf{U}_\infty) + \boldsymbol{\tau} \cdot \mathbf{n} \, dA. \quad (5)$$

If the control volume surface is sufficiently far from the aircraft, the viscous stress terms may be neglected and so the integral becomes

$$\mathbf{F} = - \int_S (p - p_\infty) \mathbf{n} + \rho(\mathbf{u} \cdot \mathbf{n})(\mathbf{u} - \mathbf{U}_\infty) \, dA. \quad (6)$$

The control volume is  $\tau$  to be a cube aligned with the  $(x, y, z)$  coordinate axes and with the downstream face a fixed distance downstream of the aircraft. As the size of the cube increases, the contribution to the drag component of the integral from the other five faces tends to zero. Therefore the final expression for the drag is

$$D = - \int \int \int (p - p_\infty + \rho u(u - U_\infty)) \, dy \, dz. \quad (7)$$

is the common starting point for the development of methods of experimental data. [6]. The lift is directly lly mounting the aircraft model on a sting and force using a drag is substantially smaller than the lift, its direct measurement is much so methods based on this control volume approach are often more accurate.

When using CFD methods, the aerodynamic forces on the aircraft can be evaluated by direct numerical approximation of the integral in Equation (1), but even here there are benefits in using the drag integrals that result from the cross-flow plane analysis. These include elimination of spurious drag due to numerical smoothing; potentially faster steady-state convergence of the drag estimate in time-marching computations; avoidance of possible errors due to far-field boundary conditions; improved physical insight into the sources of drag for a particular aircraft configuration. These aspects are all discussed later in the relevant sections.

An additional integral which will be important for powered engines comes from the principle of energy conservation. If thermal diffusion and work due to viscous stresses are both negligible in the far-field, then energy conservation over the control volume surface  $S$  gives

$$E = \int_S \rho(\mathbf{u} \cdot \mathbf{n}) H \, dA, \quad (8)$$

where  $H$  is the stagnation enthalpy and  $E$  is the rate of energy input due to fuel combustion. Because of mass conservation, an equivalent form is

$$E = \int_S \rho(\mathbf{u} \cdot \mathbf{n}) \Delta H \, dA, \quad (9)$$

where  $\Delta H \equiv H - H_\infty$ . Taking the control volume to be cube as before, this leads to the integral

$$E = \iiint \rho u \Delta H \, dy \, dz, \quad (10)$$

evaluated on the downstream cross-flow plane.

### 3 Far-field analysis

downstream of the aircraft, the flow is approximately invariant in the  $x$ -direction. First, we consider a flow in is no streamwise vorticity. In this case, the flow velocity is purely in the  $x$ -direction

$$\mathbf{u} = (u(y, z), 0, 0), \quad (11)$$

and so  $p(y, z) = p_\infty$  to  $y$  and  $z$  components of the momentum equations. Using the definitions of the stagnation enthalpy a copy,

$$\frac{1}{(\gamma-1)\rho} + \frac{1}{2}(u^2 + v^2 + w^2) \quad (12)$$

$$\log\left(\frac{p}{p_\infty}\right) - \gamma \log\left(\frac{\rho}{\rho_\infty}\right) \quad (13)$$

with the freestream entropy defined to be zero, it follows that

$$\rho = \rho_\infty \exp(-s/c_p) \quad (14)$$

$$n = (U_\infty^2 \exp(s/c_p) + 2(H - H_\infty \exp(s/c_p)))^{1/2}. \quad (15)$$

These values can then be used to obtain the drag,

$$D = - \iint \rho u (u - U_\infty) \, dy \, dz. \quad (16)$$

If the entropy,  $s$ , and the perturbation in stagnation enthalpy,  $\Delta H \equiv H - H_\infty$ , are both small then

$$u \approx \left( U_\infty^2 + U_\infty^2 \frac{s}{R} + 2\Delta H - 2H_\infty \frac{s}{R} + \frac{\Delta H}{U_\infty^2} - \frac{p_\infty}{\rho_\infty U_\infty^2} \frac{s}{R} \right), \quad (17)$$

neglecting terms which are  $O(s^2, s\Delta H, (\Delta H)')$  and hence the drag is

$$D \approx \iint p_\infty \frac{s}{R} - \rho_\infty \Delta H \, dy \, dz. \quad (18)$$

In inviscid flow without powered engines  $\Delta H$  is zero and this reduces to the standard integral for transonic wave drag, first derived by Oswatitsch [11]. In viscous flow without powered engines  $\Delta H$  is usually still negligible. The increased entropy associated with the drag now comes from both the shocks and dissipation in the boundary layer and wake, and so the drag integral is the combination of what is usually referred to as wave drag and profile drag. In the outflow from powered engines,  $\Delta H$  is positive corresponding to the by the engine. The entropy will also be positive due to the inevitable thermodynamic cycle inefficiency and aerodynamic losses in the engine.

We now consider a flow with uniform entropy and stagnation enthalpy and streamwise vorticity  $\zeta(y, z)$ . The velocity field now has

$$\mathbf{u} = (U_\infty, v(y, z), w(y, z)). \quad (19)$$

To leading order, the density is uniform and so the mass equation requires that

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (20)$$

It is therefore possible to define  $v$  and  $w$  in terms of a cross-flow streamfunction  $\psi$ ,

$$v = \frac{\partial \psi}{\partial z}, \quad w = -\frac{\partial \psi}{\partial y} \quad (21)$$

which must satisfy the streamfunction/vorticity equation

$$\nabla^2 \psi = -\zeta. \quad (22)$$

When the entropy and stagnation enthalpy are both uniform, the pressure is related to the flow speed,  $q \equiv \sqrt{u^2 + v^2 + w^2}$ , by

$$\frac{p}{p_\infty} = \left( \frac{H_\infty - \frac{1}{2}q^2}{H_\infty - \frac{1}{2}U_\infty^2} \right)^{\frac{\gamma-1}{\gamma}} \quad (23)$$

and hence

$$\frac{dp}{d(q^2)} = -\frac{\gamma-1}{2\gamma} \frac{p}{H_\infty - \frac{1}{2}q^2} = -\frac{1}{2}\rho. \quad (24)$$

Since  $q^2 = U_\infty^2 + v^2 + w^2$ , this gives

$$\Delta p \approx -\frac{1}{2}\rho_\infty(v^2 + w^2) \quad (25)$$

and so the drag is

$$D = \frac{1}{2}\rho_\infty \iint v^2 + w^2 \, dy \, dz \quad (26)$$

The simple physical interpretation of this equation is that the moving aircraft is doing work on the surrounding air at rate  $DU$ , which must equal the rate at which it is leaving, in its wake, kinetic energy associated with the cross-flow.

It is possible to leave the integral in this form, but it is more convenient to express the velocity components in terms of the streamfunction and integrate by parts to obtain the following result, first obtained by Maskell [8],

$$\begin{aligned} D &= \frac{1}{2}\rho_\infty \iint \left( \frac{\partial \psi}{\partial y} \right)^2 + \left( \frac{\partial \psi}{\partial z} \right)^2 \, dy \, dz \\ &= -\frac{1}{2}\rho_\infty \iint \psi \frac{\partial^2 \psi}{\partial y^2} + \psi \frac{\partial^2 \psi}{\partial z^2} \, dy \, dz \\ &= \frac{1}{2}\rho_\infty \iint \psi \zeta \, dy \, dz. \end{aligned} \quad (27)$$

There are three attractive features to the first, is 11011-zero in only a limited area of the cross-flow plane so the be performed over a This is particularly important for experimental purposes, allowing a reduction in the area survey required to determine the vorticity, from which the corresponding streamfunction is computed and then the integral is approximated. The second attraction is that the value of this integral is fairly insensitive to the streamwise location of the plane on which it is evaluated. Therefore, although it has been derived based on the assumption that the plane is in the far-field of the aircraft flow field, it can in fact be evaluated on a plane which is well within the near-field. The third feature is that it shows clearly the relationship between this component of drag and the shed vorticity associated with the lift on a finite-span aircraft. This corresponds to the 'induced drag' of classical lifting-line theory; this relationship is further developed in a later section.

For a flow field variations in and stagnation enthalpy in addition to streamwise vorticity, the two analyses can be approximately combined by adding the respective drag components, neglecting higher order terms, to obtain

$$D \approx D_1 + D_2 + D_3, \quad (28)$$

where

$$\begin{aligned} D_1 &= p_\infty \iint \frac{s}{R} \, dy \, dz \\ D_2 &= -\rho_\infty \iint \Delta H \, dy \, dz \\ D_3 &= \frac{1}{2}\rho_\infty \iint \psi \zeta \, dy \, dz. \end{aligned} \quad (29)$$

This equation corresponds to Equation (10.28) in reference [13], if  $D_3$  is kept in its cross-flow kinetic energy form, as in Equation (26) above.

In an experiment or a computation, each of the three integrals will be a weak function of the streamwise position of the plane on which they are evaluated. As explained in Section 2, while moving downstream  $D_2$  will approach a constant value  $-E/U_\infty$ , where  $E$  is the rate of energy addition in the engines.  $D_3$  will decay very slowly to zero as the streamwise vorticity diffuses until the vorticity shed by one wing cancels the vorticity of the opposite sign shed by the other wing. In a CFD computation, because of numerical smoothing and coarse grids in the far-field this will take place within the first, 100 aircraft, lengths; in reality it would take very much longer'. As  $D_3$  decreases there is a corresponding increase in  $D_1$  since the total drag remains a constant.

In fact, the sum of the three components will be approximately well into the near-field of the aircraft. This is fortunate because experimental measurements will usually have to be taken in the near-field. Also, if a detailed breakdown of the sources of drag in a CFD calculation is required it is best to evaluate the three integrals in the near-field before numerical smoothing causes a shift from  $D_3$  to  $D_1$ .

The lift can also be related, approximately, to the streamwise vorticity in the far-field. This result was first obtained for flow by Mascell [8]. The starting point is the of the total force vector as an integral over the surface of a cubic control volume enclosing the aircraft, as in Equation (6),

$$\mathbf{F} = - \int_S (p - p_\infty) \mathbf{n} + \rho (\mathbf{u} \cdot \mathbf{n}) (\mathbf{u} - \mathbf{U}_\infty) dA. \quad (30)$$

Neglecting are quadratic in the perturbation velocities, the pressure perturbation on the side planes is related to the flow velocity perturbation by

$$\Delta p \approx -\rho_\infty U_\infty \Delta u \quad (31)$$

and hence the lift is

$$L \approx \rho_\infty U_\infty \int_S (\Delta u n_z - w n_x) dA. \quad (32)$$

Using the following identity, in which  $\mathbf{j}$  is the unit vector in and  $\mathbf{v}$  is an arbitrary vector field.

$$\begin{aligned} \nabla \times (\mathbf{y} \mathbf{v}) &= (\nabla \mathbf{y}) \times \mathbf{v} + \mathbf{y} (\nabla \times \mathbf{v}) \\ &= \mathbf{j} \times \mathbf{v} + \mathbf{y} (\nabla \times \mathbf{v}), \end{aligned} \quad (33)$$

it follows that

$$\begin{aligned} L &\approx -\rho_\infty U_\infty \int_S \mathbf{n} \cdot (\mathbf{j} \times \Delta \mathbf{u}) dA \\ &= \rho_\infty U_\infty \int_S (\mathbf{y} \mathbf{n} \cdot (\nabla \times \mathbf{u}) - \mathbf{n} \cdot (\nabla \times (\mathbf{y} \Delta \mathbf{u}))) dA. \end{aligned} \quad (34)$$

For any vector field  $\mathbf{v}$  and any closed surface  $S$ ,

$$\int_S \mathbf{n} \cdot (\nabla \times \mathbf{v}) dA = 0 \quad (35)$$

Also, the flow vorticity is non-zero only on stream face of the cube. Therefore, result is

$$L \approx \rho_\infty U_\infty \iint y \zeta dy dz, \quad (36)$$

with the integration being over just the downstream cross-

## 4 Near-field analysis

In the near field in are significant variations in the x-direction the velocity field can be expressed using a Clebsch decomposition as

$$\mathbf{u} = \nabla \phi + \nabla \times \boldsymbol{\psi}, \quad (37)$$

where  $\boldsymbol{\psi}$  is now a vector function which satisfies the equation

$$\nabla^2 \boldsymbol{\psi} = -\boldsymbol{\zeta}, \quad (38)$$

with  $\boldsymbol{\zeta}$  being the vorticity vector. It is convenient to split  $\boldsymbol{\psi}$  into the streamwise part  $\boldsymbol{\psi}i$  and the remainder, so that

$$\mathbf{u} = \nabla \phi + \nabla \times (\boldsymbol{\psi}i) + \mathbf{u}_w. \quad (39)$$

The term  $\mathbf{u}_w$  associated with the transverse vorticity is non-zero only in Its dominant component is in the streamwise direction and so it corresponds to the velocity defect related to the variations in entropy and stagnation enthalpy, as discussed in the previous section. The link between transverse vorticity, entropy and stagnation enthalpy is also explicit in Crocco's theorem for steady flow,

$$\mathbf{u} \times \boldsymbol{\zeta} = UH - T\nabla s. \quad (40)$$

The drag due to this written as a function of stagnation enthalpy and entropy variations as before. Removing this term we now concentrate on the drag associated with the velocity field

$$\mathbf{u} = \nabla \phi + \nabla \times (\boldsymbol{\psi}i), \quad (41)$$

entropy and stagnation enthalpy.

Considering the pressure as a function of the flow speed, it was shown in the previous section that

$$\frac{dp}{d(q^2)} = -\frac{1}{2}\rho \quad (42)$$

Differentiating this again, gives

$$\frac{d^2 p}{d(q^2)^2} = -\frac{1}{2} \frac{d\rho}{d(q^2)} = -\frac{1}{2c^2} \frac{dp}{d(q^2)} = \frac{\rho}{4c^2}. \quad (43)$$

The change in freestream speed is

$$\Delta(q^2) = (U_\infty + \Delta u)^2 + v^2 + w^2 - U_\infty^2 \quad (44)$$

so performing a second-order Taylor series expansion about freestream conditions gives

$$\begin{aligned} \Delta p &\approx -\frac{1}{2}\rho_\infty \Delta(q^2) + \frac{\rho_\infty}{8c_\infty^2} (\Delta(q^2))^2 \\ &\approx -\frac{1}{2}\rho_\infty (v^2 + w^2 + 2U_\infty \Delta u + (1 - M_\infty^2)(\Delta u)^2) \end{aligned} \quad (45)$$

To first order, the corresponding change in density is

$$\Delta\rho \approx \frac{\Delta p}{c_\infty^2} \approx -\frac{\rho_\infty U_\infty}{c_\infty^2} \Delta u \quad (46)$$

and so

$$\begin{aligned} \rho u \Delta u &\approx \rho_\infty U_\infty \Delta u + \rho_\infty (\Delta u)^2 + \Delta\rho U_\infty \Delta u \\ &\approx \rho_\infty U_\infty \Delta u + \rho_\infty (1-M_\infty^2) (\Delta u)^2 \end{aligned} \quad (47)$$

Putting these into the drag integral gives

$$D = \frac{1}{2} \iint (v^2 + w^2 - (1-M_\infty^2)(\Delta u)^2) dy dz. \quad (48)$$

This equation corresponds to Equation (10.20) of reference [13] when there is no variation in entropy or stagnation enthalpy. Following the approach of Maskell [8],

$$v = -\frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial z}, \quad (49)$$

$$w = \frac{\partial\phi}{\partial z} + \frac{\partial\psi}{\partial y}, \quad (50)$$

so integrating by parts gives

$$\iint (v^2 + w^2) dy dz = \iint (\psi\zeta - \phi\sigma) dy dz, \quad (51)$$

where

$$\sigma \equiv \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}. \quad (52)$$

Combining these gives

$$\begin{aligned} D &= \frac{1}{2}\rho_\infty \iint \psi\zeta dy dz \\ &\quad - \frac{1}{2}\rho_\infty \iint \phi\sigma dy dz \\ &\quad - \frac{1}{2}\rho_\infty \iint (1-M_\infty^2)(\Delta u)^2 dy dz. \end{aligned} \quad (53)$$

The first integral is exactly the same as appeared in the far-field analysis. The second integral appears in the analyses of Masllet [8] and Wu *et al* [14] but is usually ignored in practice on the grounds that  $\sigma$  is small; this is essentially just the far-field argument used in the previous section. The third integral has been derived previously by Betz [1] for incompressible flow, and by Lock [7] and van der Vooren and Slooff [13] for compressible flow. Again it is usually argued that it is negligible.

In fact, to leading order the second and third integrals cancel. To prove this requires use of the mass equation which, to leading order, can be written as

$$U_\infty \frac{\partial\rho}{\partial x} + \rho_\infty \nabla \cdot \mathbf{u} = 0. \quad (54)$$

From Equation (46),

$$\frac{\partial\rho}{\partial x} \approx -\frac{\rho_\infty U_\infty}{c_\infty^2} \frac{\partial u}{\partial x}, \quad (55)$$

and so it follows that

$$(1-M_\infty^2) \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \approx 0 \quad (56)$$

Hence, using integration by parts in both the  $y$  and  $z$ -directions,

$$\begin{aligned} \frac{d}{dx} \iint \phi\sigma dy dz &= \frac{d}{dx} \iint \phi \left( \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \right) dy dz \\ &= -\frac{d}{dx} \iint \left( \frac{\partial\phi}{\partial y} \right)^2 + \left( \frac{\partial\phi}{\partial z} \right)^2 dy dz \\ &= -2 \iint \frac{\partial^2\phi}{\partial y\partial x} \frac{\partial\phi}{\partial y} + \frac{\partial^2\phi}{\partial z\partial x} \frac{\partial\phi}{\partial z} dy dz \\ &= 2 \iint \frac{\partial\phi}{\partial x} \frac{\partial^2\phi}{\partial y^2} + \frac{\partial\phi}{\partial x} \frac{\partial^2\phi}{\partial z^2} dy dz \\ &= -2 \iint 2(1-M_\infty^2) \frac{\partial^2\phi}{\partial x^2} \frac{\partial\phi}{\partial x} dy dz \\ &= -\frac{d}{dx} \iint (1-M_\infty^2) \left( \frac{\partial\phi}{\partial x} \right)^2 dy dz. \end{aligned} \quad (57)$$

Integrating this o.d.e. in the  $x$ -direction with the boundary condition that both integrals tend to zero as  $x \rightarrow \infty$  gives the final result that

$$\iint \phi\sigma dy dz + \iint (1-M_\infty^2)(\Delta u)^2 dy dz = 0. \quad (58)$$

Thus, this analysis shows that it is correct to drop the potential flow term in Maskell's analysis, and keep only the terms due to the trailing axial vorticity and the entropy and stagnation enthalpy variations, as derived in the previous section. This result should not be surprising. In the absence of any shed vorticity or variation in entropy or stagnation enthalpy, all flow approach freestream conditions in the far-field and so there must be zero drag. As a consequence, the drag integral at any axial location in the near-field or the far-field must be identically zero.

## 5 Connection to lifting-line theory

In classical lifting line theory, the wing is assumed to have an extremely high aspect ratio and sheds a flat sheet of streamwise vorticity behind it. To satisfy Kelvin's circulation theorem, the strength of the

shed vortex sheet  $\gamma(y)$  is related to the circulation  $\Gamma(y)$  around the airfoil section,

$$\gamma(y) = -\frac{d\Gamma}{dy}. \quad (59)$$

Using standard lifting-line theory, the drag is given by

$$D = \rho_\infty U_\infty \int_{-b/2}^{b/2} \Gamma(y_o) \alpha_i(y_o) dy_o, \quad (60)$$

where  $\alpha_i(y_o)$  is the induced angle of attack at  $y_o$  due to the shed vorticity along the span. Using the Biot-Savart Law for the semi-infinite vortex lines trailing behind the wing, the induced downwash is given by

$$\alpha_i(y_o) = -\frac{1}{4\pi U_\infty} \int_{-b/2}^{b/2} \frac{\gamma(y)}{y_o - y} dy. \quad (61)$$

The integrand is singular at  $y = y_o$ ; only the principal part is retained. Combining these two equations gives

$$\begin{aligned} D &= -\frac{\rho_\infty}{4\pi} \int_{-b/2}^{b/2} \int_{-b/2}^{b/2} \frac{\Gamma(y_o) \gamma(y)}{y_o - y} dy dy_o \\ &= \int_{-b/2}^{b/2} \gamma(y) f(y) dy \end{aligned} \quad (62)$$

where

$$f(y) = -\frac{\rho_\infty}{4\pi} \int_{-b/2}^{b/2} \frac{\Gamma(y_o)}{y_o - y} dy_o \quad (63)$$

We now consider the drag given by the formula

$$D = \frac{1}{2} \rho_\infty \iint \psi \zeta dy dz \quad (64)$$

where the streamfunction  $\psi$  satisfies Poisson's equation

$$\nabla^2 \psi = -\zeta \quad (65)$$

The general solution to this equation, subject to the boundary condition that  $\nabla \psi \rightarrow 0$  as  $y^2 + z^2 \rightarrow \infty$ , is

$$\psi(y, z) = -\frac{1}{4\pi} \iint \zeta(y_o, z_o) \log((y-y_o)^2 + (z-z_o)^2) dy_o dz_o \quad (66)$$

When  $\zeta$  is a distributed line source of strength  $\gamma(y)$  along  $z_o = 0$ , the integral becomes

$$\psi(y, z) = -\frac{1}{4\pi} \int_{-b/2}^{b/2} \gamma(y_o) \log((y-y_o)^2 + z^2) dy_o. \quad (67)$$

Integrating by parts, and using the fact that the circulation goes to zero at each wingtip, gives

$$\psi(y, z) = -\frac{1}{2\pi} \int_{-b/2}^{b/2} \frac{\Gamma(y_o) (y_o - y)}{(y - y_o)^2 + z^2} dy_o. \quad (68)$$

Taking the limit  $z \rightarrow 0$ , this gives the induced drag as

$$D = \int_{-b/2}^{b/2} \gamma(y) f(y) dy, \quad (69)$$

where, as before,

$$f(y) = -\frac{\rho_\infty}{4\pi} \int_{-b/2}^{b/2} \frac{\Gamma(y_o)}{y_o - y} dy_o. \quad (70)$$

per unit span is  $\rho_\infty U_\infty \Gamma(y)$ , and so

$$L = \rho_\infty U_\infty \int_{-b/2}^{b/2} \Gamma(y) dy. \quad (71)$$

Integrating by parts once gives

$$L = \rho_\infty U_\infty \int_{-b/2}^{b/2} y \gamma(y) dy. \quad (72)$$

This corresponds precisely to the lift integral derived earlier for a general distribution of streamwise vorticity at the cross-flow plane, Equation (36), in the limit in which the vorticity is concentrated into a vortex sheet.

Thus, in the case of a planar vortex sheet the streamfunction-vorticity lift and drag integrals give the same result as classical lifting-line theory. The advantage of the streamfunction-vorticity approach over the lifting-line theory is that it is much more general in its ability to handle non-planar trailing edge geometry, due to winglets, pylons, complex boundary layer separations, etc. The advantage of the lifting-line theory is its extreme simplicity for simple applications and its ability to directly prove that an elliptic lift distribution gives the induced drag of a wing of fixed span [9].

## 6 Experimental Measurements

Experimental wake surveys have traditionally employed four-hole or five-hole probes from which one obtains the static and stagnation pressures, as well as all three velocity components. They are also being considered as a way to obtain these properties [2]. The generally accepted method for computed induced drag based on such wake surveys is to compute the streamwise vorticity by differentiating the velocity field using

$$\zeta = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}. \quad (73)$$

This method of computing the streamwise vorticity can lead to errors in the prediction of the induced drag

due to the differentiation of the discrete velocity measurements. One solution to this problem involves curve-fitting the measurements in order to obtain more accurate derivatives [2]. An alternative approach is to relate the crossflow vorticity to the circulation, and thus re-integration.

For a region  $C$  in a cross-flow area, the integral of the streamwise vorticity is equal to the circulation around the boundary of  $C$ ,

$$\int_C \zeta \, dy \, dz = \oint_{\partial C} v \, dy + w \, dz. \quad (74)$$

Therefore, given the values of the cross-flow velocity components  $v_{j,k}$  and  $w_{j,k}$  at a uniform grid of measurement points in a cross-plane (as shown in Figure I) the streamwise component of vorticity  $\zeta$  in each measurement 'cell' can be approximated by

$$\zeta_{j+\frac{1}{2},k+\frac{1}{2}} = \frac{\Gamma_{j+\frac{1}{2},k+\frac{1}{2}}}{\Delta y \Delta z} \quad (75)$$

with the circulation  $\Gamma_{j+\frac{1}{2},k+\frac{1}{2}}$  defined by

$$\begin{aligned} \Gamma_{j+\frac{1}{2},k+\frac{1}{2}} = & \frac{1}{2}(v_{j,k} + v_{j+1,k}) \Delta y \\ & + \frac{1}{2}(w_{j+1,k} + w_{j+1,k+1}) \Delta z \\ & - \frac{1}{2}(v_{j+1,k+1} + v_{j,k+1}) \Delta y \\ & - \frac{1}{2}(w_{j,k+1} + w_{j,k}) \Delta z. \end{aligned} \quad (76)$$

For an arbitrary distribution of vorticity, the streamfunction  $\psi(y, z)$  is

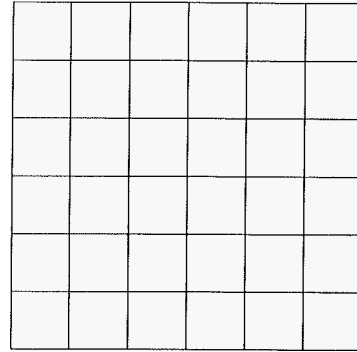
$$\psi(y, z) = -\frac{1}{4\pi} \iint \zeta(y_o, z_o) \log((y-y_o)^2 + (z-z_o)^2) \, dy_o \, dz_o \quad (77)$$

Approximating the integral using the formulation first described by Lamb [5] gives

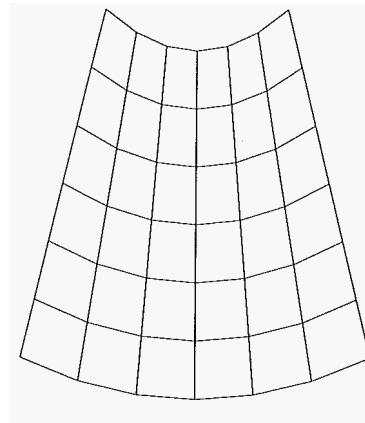
$$\begin{aligned} \psi_{j,k} = & -\frac{1}{4\pi} \sum_{j_o, k_o} \Delta y \Delta z \zeta_{j_o+\frac{1}{2}, k_o+\frac{1}{2}} \log \left( (j_o + \frac{1}{2} - j)^2 \Delta y^2 \right. \\ & \left. + (k_o + \frac{1}{2} - k)^2 \Delta z^2 \right) \\ = & -\frac{1}{4\pi} \sum_{j_o, k_o} \Gamma_{j_o+\frac{1}{2}, k_o+\frac{1}{2}} \log \left( (j_o + \frac{1}{2} - j)^2 \Delta y^2 \right. \\ & \left. + (k_o + \frac{1}{2} - k)^2 \Delta z^2 \right). \end{aligned} \quad (78)$$

Finally, the drag integral can be approximated by summing over each cell to give the induced drag as

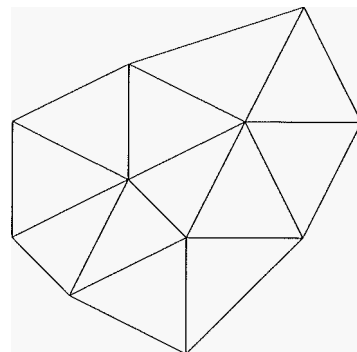
$$D = \frac{1}{2} \rho_\infty \sum_{j,k} \frac{1}{4} (\psi_{j,k} + \psi_{j+1,k} + \psi_{j,k+1} + \psi_{j+1,k+1}) \Gamma_{j+\frac{1}{2}, k+\frac{1}{2}}. \quad (79)$$



a) Cartesian



b) structured



c) unstructured

Figure 1: Cross-flow grids for the evaluation of drag integrals



of powered any significant level of surface heat transfer, there is negligible variation in stagnation enthalpy. Therefore, the be deduced directly from the stagnation pressure and the corresponding drag integral is easily approximated.

If there are powered engines, then stagnation enthalpy (or stagnation temperature) measurements may need to be made. Alternatively, if  $\Delta H$  can be assumed to be small, the integral corresponding to the stagnation enthalpy related to the engines' fuel consumption; by energy conservation the net outflow of energy from volume surrounding the aircraft must match the of energy in the form of heat release due to fuel combustion.

## 7 CFD computations

For CFD calculations using unstructured grids, there is no crossflow plane in the computational grid, and so the approach for the evaluation of the crossflow drag integrals is to adopt techniques from flow visualisation. A cross-flow 'cutting' be defined to be orthogonal to the freestream flow and at a fixed of the aircraft. The grid nodes on this cutting are defined by the intersection of the the edges of the 3D grid and all flow also be defined at the new grid nodes by along the cut edges. The nodes of the cutting planes are triangles, based on of the cutting plane to cut cells. The full details for unstructured grids composed of tetrahedra, prisms, hexahedra are a paper by [4].

ample of the grid is Figure 1.

Once the triangular cutting-plane grid has been constructed, the evaluation of the drag integral is quite straightforward. The circulation around a triangular cell is

$$\Gamma_\alpha = \sum_{\text{edges}} (\bar{v}\Delta y + \bar{w}\Delta z), \quad (80)$$

where  $\bar{v}$  and  $\bar{w}$  are the average velocity along an edge, and  $\Delta y$  and  $\Delta z$  are the changes in  $y$  and  $z$  along the edge (going around the cell in the direction as viewed from  $\mathbf{x} = \mathbf{m}$ ).

$\psi_j$  is given by

$$\psi_j = \sum \Gamma_\beta \log((y_j - y_\beta)^2 + (z_j - z_\beta)^2), \quad (81)$$

where  $y_\beta, z_\beta$  are the coordinates of the cell. The induced drag integral is from a

summation over all of the triangular cells,

$$D = \frac{1}{2} \rho_\infty \sum_\alpha \bar{\psi}_\alpha \Gamma_\alpha, \quad (82)$$

where  $\bar{\psi}_\alpha$  is the average of  $\psi$  values at the three corner nodes.

Two refinements to the above formulation reduce the computational cost of evaluating the drag. The first addresses the problem that each streamfunction value requires a loop over all of the cells in the cross-flow plane. Therefore the total computational cost is proportional to the square of the number of cells, which can be large for very fine grids. However, in a few cells have significant levels of circulation, and it is only these cells which are needed for an accurate drag evaluation. Substituting Equation (81) into Equation (82) gives

$$D = \frac{1}{2} \rho_\infty \sum_{\alpha, \beta} \Gamma_\alpha \Gamma_\beta D_{\alpha\beta} \quad (83)$$

where

$$D_{\alpha\beta} = \frac{1}{2\pi} \sum_j \log((y_j - y_\beta)^2 + (z_j - z_\beta)^2) \quad (84)$$

with the  $j$  summation being over the 3 nodes at the corners of cell  $\alpha$ . The drag summation, Equation (83), can be restricted to those values of  $\alpha$  and  $\beta$  for which the magnitudes of  $\Gamma_\alpha$  and  $\Gamma_\beta$  exceed a threshold. Setting the threshold to be 0.1% of the maximum circulation in the cell leads to a negligible error in the drag summation but can give a huge reduction in the computational cost.

The above formulation is for the common case in which the CFD computation is performed for one half of a flow which is symmetric about  $y=0$ . Rather than constructing the other half of the flow field and then applying the above procedure, it is simpler to account for the image vorticity by doubling the streamfunction as

$$\psi_j = -\frac{1}{4\pi} \sum_\beta \Gamma_\beta \left\{ \log((y_j - y_\beta)^2 + (z_j - z_\beta)^2) - \log((y_j + y_\beta)^2 + (z_j - z_\beta)^2) \right\}. \quad (85)$$

For calculations on single-block and multi-block structured grids, it is unlikely that there exists a suitable grid coordinate plane which is at a uniform streamwise location relative to the aircraft. One option is to use the same 'cutting plane' approach just presented, creating an unstructured triangular grid on the cross-flow plane. The streamfunction values are data interpolated along the cut edges of the structured grid.

is possible when there are  $i$  and  $i+1$  which are approximately cross-flow planes. For 3D volume  $D$ , the of vorticity is related to a surface integral of the velocity,

$$\int_D \zeta dV = \int_{\partial D} \mathbf{u} \times d\mathbf{S}. \quad (86)$$

Therefore, using the 3D cell lying between coordinate planes  $i$  and  $i+1$ ,  $j$  and  $j+1$ , and  $k$  and  $k+1$ , a value for the streamwise vorticity is given by

$$\zeta_{j+\frac{1}{2},k+\frac{1}{2}} = \frac{1}{V} \sum_{\text{faces}} \mathbf{i} \cdot (\bar{\mathbf{u}} \times \mathbf{S}), \quad (87)$$

where  $V$  is the volume of the cell,  $\bar{\mathbf{u}}$  is the average velocity on the face and  $\mathbf{S}$  is the face area vector. This vorticity is a nominal cross-flow plane on which the grid coordinates are

$$y_{j,k} = \frac{1}{2}(y_{i,j,k} + y_{i+1,j,k}), \quad z_{j,k} = \frac{1}{2}(z_{i,j,k} + z_{i+1,j,k}). \quad (88)$$

This non-orthogonal structured grid is also illustrated in Figure 1. The streamfunction  $\psi_{j,k}$  is obtained from

$$\psi_{j,k} = -\frac{1}{4\pi} \sum_{j_o, k_o} \Gamma_{j_o+\frac{1}{2}, k_o+\frac{1}{2}} \log \left( (y_{j,k} - y_{j_o+\frac{1}{2}, k_o+\frac{1}{2}})^2 + (z_{j,k} - z_{j_o+\frac{1}{2}, k_o+\frac{1}{2}})^2 \right) \quad (89)$$

where the circulation of the cell is defined as

$$\Gamma_{j_o+\frac{1}{2}, k_o+\frac{1}{2}} = A_{j_o+\frac{1}{2}, k_o+\frac{1}{2}} \zeta_{j_o+\frac{1}{2}, k_o+\frac{1}{2}}, \quad (90)$$

the coordinates at the centre of the cell are

$$\begin{aligned} y_{j+\frac{1}{2}, k+\frac{1}{2}} &= \frac{1}{4}(y_{j,k} + y_{j+1,k} + y_{j,k+1} + y_{j+1,k+1}), \\ z_{j+\frac{1}{2}, k+\frac{1}{2}} &= \frac{1}{4}(z_{j,k} + z_{j+1,k} + z_{j,k+1} + z_{j+1,k+1}), \end{aligned} \quad (91)$$

and  $A$  is the cell area.

Finally, the induced drag integral is approximated by

$$D = \frac{1}{2} \rho_\infty \sum_{j,k} \frac{1}{4} (\psi_{j,k} + \psi_{j+1,k} + \psi_{j,k+1} + \psi_{j+1,k+1}) \Gamma_{j+\frac{1}{2}, k+\frac{1}{2}}. \quad (92)$$

The same two refinements which were described for the unstructured grid also be used for this structured grid analysis.

The next issue is the interpretation of the values obtained from the drag integrals. Using CFD methods, it is possible to directly evaluate the aerodynamic force on the aircraft using a numerical approximation of the surface integral of Equation (1).

CFD methods are conservative, so if the surface force integration is performed in a manner consistent with

the CFD discretisation of the cells with surface faces, then it is possible to sum over a very large number of computational cells surrounding the aircraft and deduce numerical surface force integral is exactly equal to that which would be obtained by a numerical force/momentum to Equation (5) applied on the enclosing In the far-field, numerical smoothing the real viscous effects, are very small. Therefore, the far-field asymptotic showing that the numerical force integral on the be equated to the drag integrals on the cross-flow plane.

This raises the question of what is to be gained from evaluating the drag using the cross-flow plane integrals rather than the direct surface integration. There are in fact four benefits in using the cross-flow integrals:

1. In subsonic Euler far-field drag analysis shows one due to the streamwise vorticity arising as a consequence of the spanwise lift distribution is physically meaningful and should have very nearly the correct physical Euler calculations give relatively accurate lift predictions. The second contribution due to entropy variations is almost entirely spurious. Physically there should be a slight level of entropy rise due to some early diffusion of the shed vorticity, but in the CFD computation almost all of the entropy will be due to numerical smoothing in regions with high flow gradients and inadequate grid resolution, especially near the leading edge of the wing. As a consequence, a more accurate prediction of the real aircraft drag is obtained by entirely neglecting the entropy drag integral, keeping only the induced drag streamwise vorticity integral.
2. For transonic Euler calculations with shocks, and for Navier-Stokes calculations with entropy generation in the boundary layer, it is much harder to distinguish between physically correct entropy generation and spurious numerical generation, so it may not be possible to apply such a correction.
3. If the boundaries are not sufficiently far from the aircraft, or if the are not sufficiently accurate (e.g. do not incorporate the far-field correction due to the lift on the aircraft) then there may be a very small error in the effective freestream flow angle. This will produce only a small error in lift but can produce a more significant error in drag since the effective rotation of the lift vector means that the lift will contribute an apparent drag component of magnitude  $L\Delta\alpha$ . This problem is totally avoided by use of

of the drag. The streamwise component of vorticity is only very slightly altered by a slight error in the freestream flow angle, so the relative drag error will be extremely small.

When there are no powered engines, or when the stagnation enthalpy variation is sufficiently mixed out that it can be equated to the energy input to the engines, the drag depends solely on the entropy variations and the streamwise vorticity. These quantities change very little during the final stages of time-marching convergence to the steady-state solution. Therefore, the drag integral based on the downstream cross-flow plane will converge to the final steady-state value quicker than the force integral over the surface of the aircraft. In practical CFD computations, this should allow fewer computational iterations to be required to obtain a given level of convergence of both the lift and drag.

4. Even if there were no quantitative advantages in expressing the drag in terms of the cross-flow integrals, there is still a major qualitative benefit. Engineering analysis is simply one step in the process of engineering design, creating a better product. From this design viewpoint, it is important to not only know the value of overall drag but to also understand the causes of that drag so that design decisions can be made to hopefully reduce it. For example, a high level of induced drag for a given span and overall lift would suggest a poor spanwise lift distribution which might be improved by changing the spanwise variation in the angle of attack or re-cambering certain parts of the wing. Alternatively, a large entropy drag might be clue to either poor wave drag due to shocks or poor profile drag due to a boundary layer separation. This would therefore suggest areas of further study of the detailed CFD computation.

## 8 Evaluation of drag computations

Two test cases are used to validate the numerical discretisation and programming implementation of the induced drag integral. The first is the wake behind an elliptically loaded planar wing. Using a unit semi-span, the spanwise lift distribution is taken to be  $\Gamma(\theta) = \sin \theta$ , where the spanwise coordinate is  $y = \cos \theta$ . The flow velocity field in the wake is then given by

$$v(y, z) = -\frac{1}{2\pi} \int_0^\pi \frac{\sin \theta}{(y - \cos \theta)^2 + z^2} \cos \theta \, d\theta$$

$$w(y, z) = \frac{1}{2\pi} \int_0^\pi \frac{y - \cos \theta}{(y - \cos \theta)^2 + z^2} \cos \theta \, d\theta \quad (93)$$

and the exact value for the drag is  $\pi/8$ , assuming unit freestream density.

Using a Cartesian grid of size  $20 \times 40$  for the region  $0 \leq y \leq 2, -1 \leq z \leq 1$ , with clustering to accurately capture the vortex sheet and the large velocity gradients around the wing tip (as shown in Figure 2), the error from the numerical induced drag integral is only 1.1%. With a uniform Cartesian grid of the same size over the same region, the error increases to 15% showing the effect of the decreased wake resolution.

The second case is the wake behind an engine whose exhaust is not aligned with the freestream. Using polar coordinates,  $y = r \cos \theta$ ,  $z = r \sin \theta$  and assuming a unit radius for the engine, the cross-flow velocity field is

$$v = \begin{cases} 0, & r < 1 \\ -\frac{\sin 2\theta}{r^2}, & r > 1 \end{cases}$$

$$w = \begin{cases} -1, & r < 1 \\ \frac{\cos 2\theta}{r^2}, & r > 1 \end{cases} \quad (94)$$

By integrating the cross-flow kinetic energy, the exact value for the drag is found to be  $\pi$ , again assuming a unit freestream density.

Using a polar grid of size  $20 \times 40$  for  $0 \leq r \leq 2$ , with clustering to accurately capture the vortex sheet at  $r = 1$ , (as shown in Figure 3), the error from the numerical induced drag integral is only 1.4%. With a uniform Cartesian grid of the same size over the region  $0 \leq y \leq 1.5, -1.5 \leq z \leq 1.5$ , the error increases to 4.4%, again because of the effective smoothing of the velocity discontinuity across the vortex sheet.

The drag calculation methods were next tested on a case which had been both experimentally and computationally predicted. A simple rectangular wing model which used an untwisted NACA 0016 airfoil section was tested by Brune and Bogataj [2]. The wing had an aspect ratio of six with rounded wing tip fairings, and was tested at  $M = 0.18$ ,  $\alpha = 8.22^\circ$  and  $Re = 1.27 \times 10^6$ . The model had boundary layer trip strips to ensure that the flow was turbulent over the majority of the wing surface. Table 1 presents averaged lift and drag coefficients obtained from both wake surveys and balance measurements.  $C_{di}$  is the induced drag computed from the streamwise vorticity in the wake survey, and  $C_{dp}$  is the profile drag computed from the total pressure measured in the wake survey. Table 1 also shows lift and drag estimates calculated by Mathias *et al* using integration and an equivalent lifting line

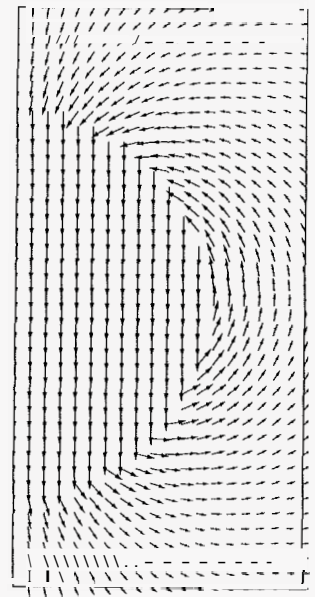
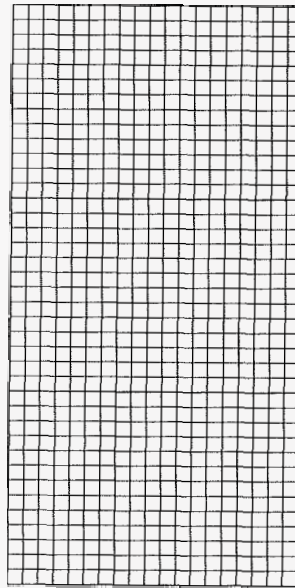
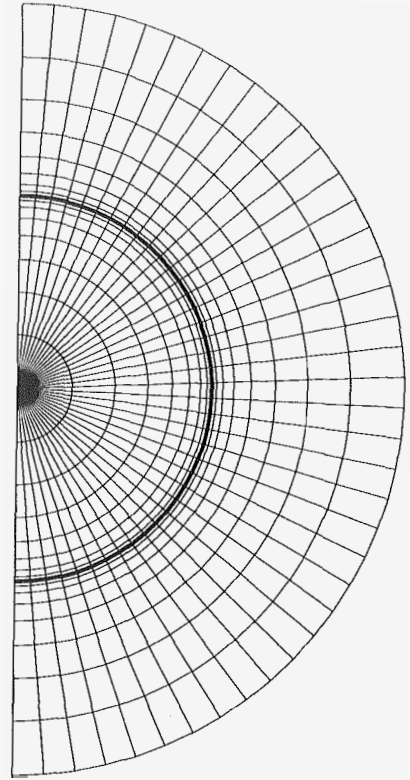
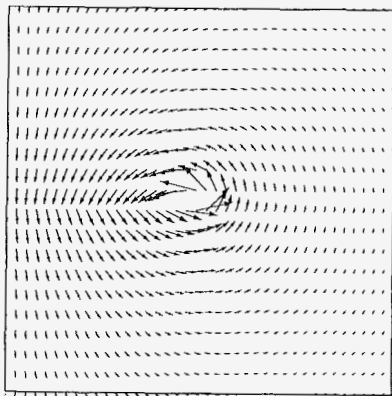
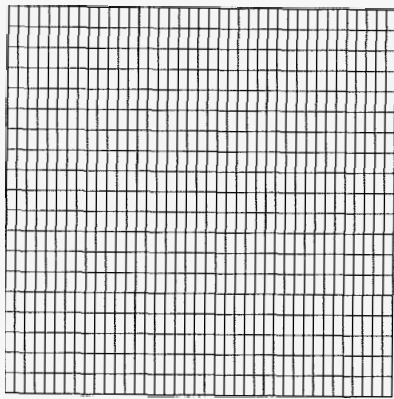
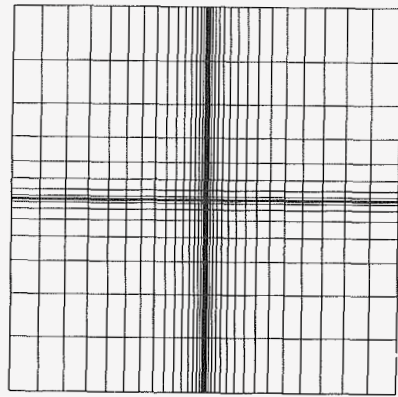


Figure 2: First test case: a) stretched grid, b) uniform grid, c) velocity vectors

Figure 3: Second test case: a) stretched grid, b) uniform grid, c) velocity vectors

model, based on CFD results obtained using an incompressible Navier-Stokes method [9]. As can be seen, the equivalent lifting line model predicts lift and induced drag coefficients that are within 5% of the wake survey results. The surface integration, however, badly overpredicts the total drag coefficient, probably due to a combination of turbulence modeling effects and numerical truncation errors.

This same geometry and CFD flow solution was used to evaluate the methods developed in this paper. The grid and wing geometry for the CFD calculation are shown in Fig. 4; the mesh is a single zone  $81 \times 81 \times 81$  structured C-H grid. The integrals in Equation 29 were evaluated using the CFD data from a crossflow grid plane. Figure 5 shows the resulting predictions of lift, induced drag and profile drag coefficients at various distances downstream from the wing trailing edge. The predictions are compared with the data from the wind tunnel test. In general, the results show that reasonable predictions of the lift and induced drag are made within three chords of the trailing edge. The profile drag coefficient is computed to be 0.022 at three chords behind the trailing edge; this is higher than the experimental value of 0.015, but is in line with the results from the surface integration giving a total drag coefficient of 0.0413.

Another item of interest was to verify that wake surveys predicting lift and induced drag do not need to take place across the full span and height of the wind tunnel or CFD solution. Various experimentalists have verified that one of the advantages of integrating the vorticity is that the size of the wake survey can be greatly reduced. Figure 6 shows the vorticity from the CFD solution in a crossflow plane which is 0.2 chords behind the trailing edge. The results clearly show the vast majority of the vorticity is confined to a very small area behind the wing tip and trailing edge. To further verify this, the wake survey calculations were performed again at 0.2 chords behind the trailing edge, (corresponding to Figure 6) with the vertical height of the integration area being restricted to values ranging from 6 chords above and below the wing surface, to within 0.1 chords of the wing. The results for lift and induced drag coefficient for these restricted integration areas are shown in Figure 7 and are compared with the available experimental data. Both coefficients are seen to be well predicted in regions as small as one chord height above and below the wing.

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	$C_l$	$C_d$	$C_{dp}$	$C_{di}$
experiment: wake survey	0.5657	0.0321	0.0151	0.0170
experiment: balance data	0.5723	0.0319		
CFD: lifting line model	0.551			0.016
CFD: surface integration	0.560	0.0413		

Table 1: Experimental and CFD lift and drag coefficients for Brune and Bogataj test case

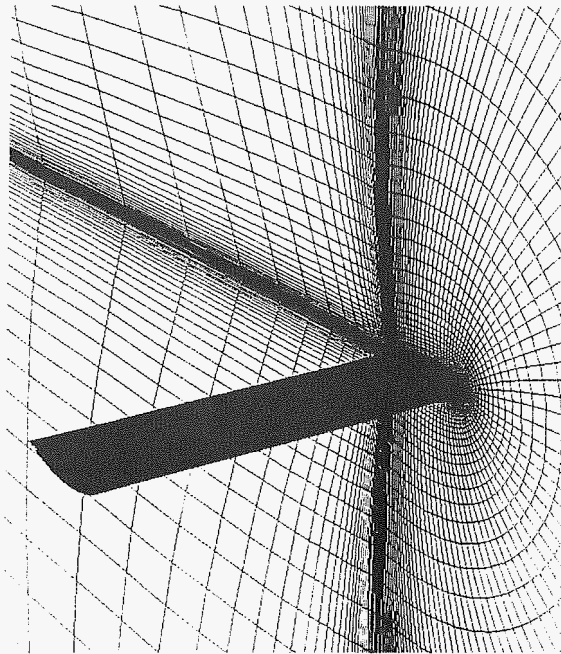


Figure 4: NACA 0016 wing surface and C-H grid (81 x 81 x 81)

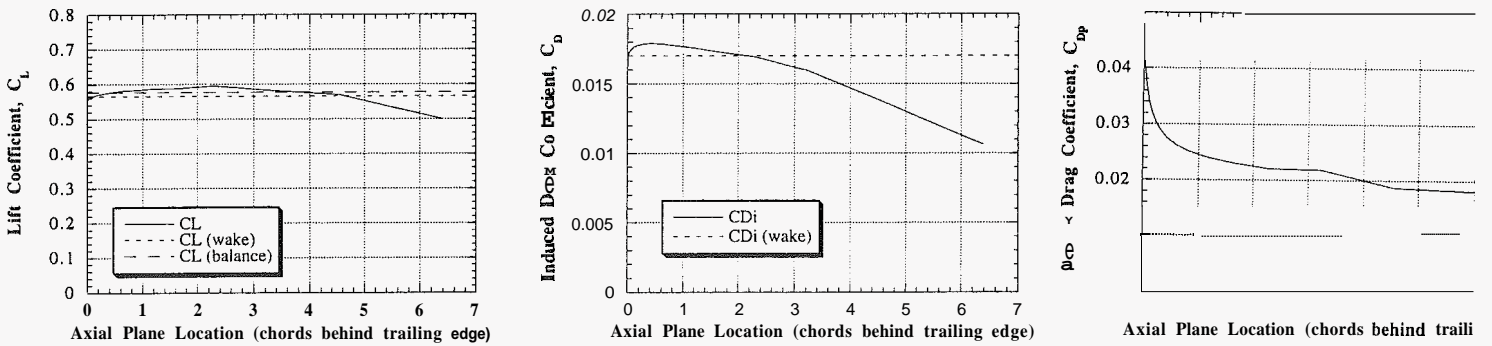


Figure 5: Lift, induced drag, and profile drag coefficients at various axial plane locations

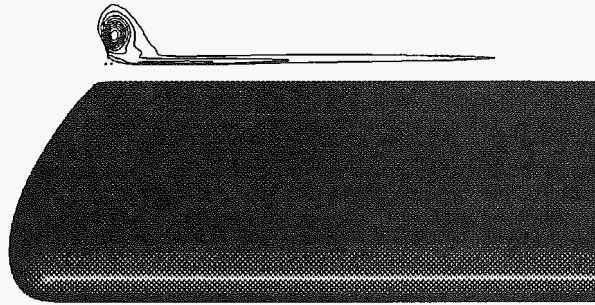


Figure 6: Vorticity in cross-flow plane at  $x/c = 0.2$  chords behind trailing edge

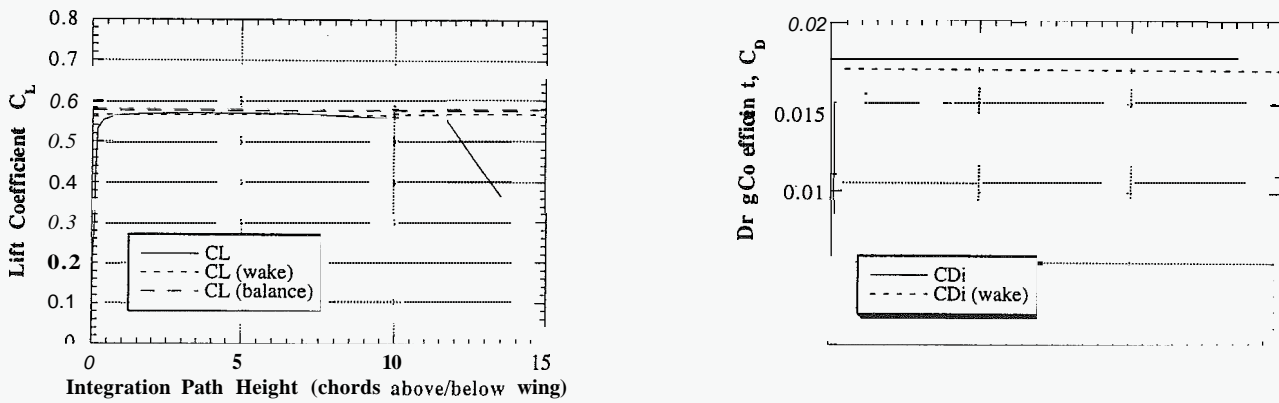


Figure 7: Lift and induced drag coefficients for varying integration heights