



Symmetry, quantitative Liouville theorems and analysis of large solutions of conformally invariant fully nonlinear elliptic equations

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Abstract We establish blow-up profiles for any blowing-up sequence of solutions of general conformally invariant fully nonlinear elliptic equations on Euclidean domains. We prove that (i) the distance between blow-up points is bounded from below by a universal positive number, (ii) the solutions are very close to a single standard bubble in a universal positive distance around each blow-up point, and (iii) the heights of these bubbles are comparable by a universal factor. As an application of this result, we establish a quantitative Liouville theorem.

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1 Introduction

The main goal of this paper is to give a fine analysis of blow-up solutions of conformally invariant fully nonlinear second order elliptic equations.

Let $n \geq 3$ be an integer and

$$\Gamma \subset \mathbb{R}^n \text{ be an open convex symmetric cone with vertex at the origin} \tag{1}$$

satisfying

$$\Gamma_n \subset \Gamma \subset \Gamma_1, \tag{2}$$

where

$$\Gamma_n := \{\lambda \in \mathbb{R}^n \mid \lambda_i > 0 \forall i\}, \quad \Gamma_1 := \left\{ \lambda \in \mathbb{R}^n \mid \sum_{i=1}^n \lambda_i > 0 \right\}.$$

We assume that

$$f \in C^1(\Gamma) \cap C^0(\overline{\Gamma}) \text{ is symmetric in } \lambda_i, \tag{3}$$

$$f > 0, \quad \frac{\partial f}{\partial \lambda_i} > 0 \text{ in } \Gamma \forall i, \quad f = 0 \text{ on } \partial\Gamma, \tag{4}$$

$$f(\lambda) < 1 \quad \forall \lambda \in \Gamma \text{ satisfying } \sum_{i=1}^n \lambda_i < \delta. \tag{5}$$

In (1) and (3), the symmetric property of Γ and f is understood in the sense that if $\lambda \in \Gamma$ and $\tilde{\lambda}$ is a permutation of λ , then $\tilde{\lambda} \in \Gamma$ and $f(\tilde{\lambda}) = f(\lambda)$. Also, throughout the paper,

whenever we write $f(\lambda)$, we implicitly assume that $\lambda \in \overline{\Gamma}$.

When $\Gamma \neq \Gamma_1$, (5) is a consequence of (3) and (4) (cf. [13, Proposition B.1]). However, this does not have to be the case when $\Gamma = \Gamma_1$, for example when

$$f(\lambda) = \left(\sum_{i=1}^n \lambda_i \right)^{\frac{n-1}{n+3}} \left(\sum_{i=1}^n \lambda_i^2 \right)^{\frac{2}{n+3}}.$$

Illuminating examples of (f, Γ) are $(f, \Gamma) = (\sigma_k^{\frac{1}{k}}, \Gamma_k)$ where $\sigma_k(\lambda) = \sum \lambda_{i_1} \dots \lambda_{i_k}$ is the k -th elementary symmetric function and

$$\begin{aligned} \Gamma_k &= \text{the connected component of } \{\lambda \in \mathbb{R}^n : \sigma_k(\lambda) > 0\} \text{ containing } \Gamma_n \\ &= \{\lambda \in \mathbb{R}^n : \sigma_l(\lambda) > 0 \text{ for all } 1 \leq l \leq k\}. \end{aligned}$$

Besides (1)–(5), $(\sigma_k^{\frac{1}{k}}, \Gamma_k)$ enjoys other nice and helpful properties, such as concavity and homogeneity properties of $\sigma_k^{1/k}$, Newton’s inequalities, divergence and variational structures, etc., which we do not assume in this paper. In particular, we would like to note that no concavity or homogeneity assumption on f is being made in the present paper.

For a positive C^2 function u , let A^u be the $n \times n$ matrix with entries

$$(A^u)_{ij} = -\frac{2}{n-2}u^{-\frac{n+2}{n-2}}\nabla_i\nabla_ju + \frac{2n}{(n-2)^2}u^{-\frac{2n}{n-2}}\nabla_iu\nabla_ju - \frac{2}{(n-2)^2}u^{-\frac{2n}{n-2}}|\nabla u|^2\delta_{ij}.$$

This is sometimes referred to as the conformal Hessian of u .

The conformal Hessian A^u arises naturally in conformal geometry as follows. Recall that the Riemann curvature $Riem_g$ of a Riemannian metric g can be decomposed into traced and traceless parts as

$$Riem_g = A_g \circledast g + W_g,$$

where $A_g = \frac{1}{n-2}(\text{Ric}_g - \frac{1}{2(n-1)}R_g g)$, Ric_g , R_g and W_g are the Schouten curvature, the Ricci curvature, the scalar curvature and the Weyl curvature of g and \circledast denotes the Kulkarni–Nomizu product. While the $(1, 3)$ -valent Weyl curvature remains unchanged under a conformal change of the metric, the Schouten curvature is adjusted by a second order operator of the conformal factor. In particular, if we consider the metric $g_u := u^{\frac{4}{n-2}}g_{flat}$ conformal to the flat metric g_{flat} on \mathbb{R}^n , then the Schouten curvature A_{g_u} of g_u is given by the conformal Hessian in the form

$$A_{g_u} = u^{\frac{4}{n-2}}(A^u)_{ij}dx^i dx^j.$$

Consequently, we have

$$\lambda(A_{g_u}) = \lambda(A^u)$$

where $\lambda(A_{g_u})$ denotes the eigenvalues of A_{g_u} with respect to the metric g_u and $\lambda(A^u)$ denotes those of the matrix A^u .

A^u enjoys a conformal invariance property, inherited from the conformal structure of \mathbb{R}^n , which will be of special importance in our treatment. Recall that a map $\varphi : \mathbb{R}^n \cup \{\infty\} \rightarrow \mathbb{R}^n \cup \{\infty\}$ is called a Möbius transformation if it is the composition of finitely many of the following types of transformations:

- a translation: $x \mapsto x + \bar{x}$ where \bar{x} is a given vector in \mathbb{R}^n ,
- a dilation: $x \mapsto ax$ where a is a given positive scalar,
- a Kelvin transformation: $x \mapsto \frac{x}{|x|^2}$.

For a function u and a Möbius transformation φ , let

$$u_\varphi = |J_\varphi|^{\frac{n-2}{2n}}u \circ \varphi, \tag{6}$$

where J_φ is the Jacobian of φ . A calculation gives

$$A^{u_\varphi}(x) = O_\varphi(x)^t A^u(\varphi(x)) O_\varphi(x)$$

for some orthogonal $n \times n$ matrix $O_\varphi(x)$. In particular,

$$\lambda(A^{u_\varphi}(x)) = \lambda(A^u(\varphi(x))). \tag{7}$$

The main result of this paper concerns an analysis on the behavior of a sequence $\{u_k\} \in C^2(B_3(0))$ satisfying

$$f(\lambda(A^{u_k})) = 1, \quad u_k > 0, \quad \text{in } B_3(0), \tag{8}$$

and

$$\sup_{B_1(0)} u_k \rightarrow \infty, \tag{9}$$

where (f, Γ) satisfies (1)–(5). Note that no other assumptions on u_k is made.

As is known, Eq. (8) is a fully nonlinear elliptic equation. Fully nonlinear elliptic equations involving $f(\lambda(\nabla^2 u))$ were investigated in the classic paper of Caffarelli et al. [2].

Our paper appears to be the first fine blow-up analysis in this fully nonlinear context. We expect this to serve as a crucial step in the study of the problem on Riemannian manifolds.

To obtain our result on fine analysis of blow-up solutions, we make use of the following Liouville theorems.

Theorem A ([8]) *Let (f, Γ) satisfy (1)–(4) and let $0 < v \in C^2(\mathbb{R}^n)$ satisfy*

$$f(\lambda(A^v)) = 1 \quad \text{in } \mathbb{R}^n. \tag{10}$$

Then

$$v(x) \equiv \left(\frac{a}{1 + b^2|x - \bar{x}|^2} \right)^{\frac{n-2}{2}}, \quad x \in \mathbb{R}^n \tag{11}$$

for some $\bar{x} \in \mathbb{R}^n$ and some positive constants a and b satisfying

$$f(2b^2a^{-2}, \dots, 2b^2a^{-2}) = 1.$$

Theorem B ([12]) *Let Γ satisfy (1) and (2), and let $0 < v \in C_{loc}^{0,1}(\mathbb{R}^n \setminus \{0\})$ satisfy*

$$\lambda(A^v) \in \partial\Gamma \text{ on } \mathbb{R}^n \setminus \{0\} \tag{12}$$

in the viscosity sense (see Definition 1.2 below). Then v is radially symmetric about the origin and $v(r)$ is non-increasing in r .

For $(f, \Gamma) = (\frac{1}{2n}\sigma_1, \Gamma_1)$, Eq. (8) is the critical exponent equation $-\Delta u = n(n - 2)u^{(n+2)/(n-2)}$ and Theorem A was proved by Caffarelli et al. [1]. See also Gidas et al. [5] under some decay assumption of u at infinity. For $(f, \Gamma) = (\sigma_2^{1/2}, \Gamma_2)$ in \mathbb{R}^4 and $v \in C_{loc}^{1,1}(\mathbb{R}^4)$, the result was proved by Chang et al. [3].

In fact we need a stronger version of Theorem A (see Theorem 1.1) and a variant of Theorem B (see Theorem 1.2). For simplicity, readers are advised that in the main body of the paper

all theorems, propositions and lemmas hold under (1)–(5),
instead of the stated weaker hypotheses on (f, Γ) .

Theorem 1.1 *Let (f, Γ) satisfy*

$$\Gamma \subset \mathbb{R}^n \text{ be an open symmetric set,} \tag{13}$$

$$\Gamma \subset \Gamma_1 \text{ and } \Gamma \cap \{\lambda + t\mu : t > 0\} \text{ is convex for all } \lambda \in \Gamma, \mu \in \Gamma_n, \tag{14}$$

$$f \in C^1(\Gamma) \text{ is symmetric in } \lambda_i \text{ and } \frac{\partial f}{\partial \lambda_i} > 0 \text{ in } \Gamma \forall i. \tag{15}$$

Assume that $0 < v \in C^0(\mathbb{R}^n)$, $0 < v_k \in C^2(B_{R_k}(0))$, $R_k \rightarrow \infty$,

$$f(\lambda(A^{v_k})) = 1 \quad \text{in } B_{R_k}(0), \tag{16}$$

and

$$v_k \rightarrow v \text{ in } C_{loc}^0(\mathbb{R}^n). \tag{17}$$

Then either v is constant or v is of the form (11) for some $\bar{x} \in \mathbb{R}^n$ and some positive constants a and b .

If it holds in addition that

$$\text{there exists } t_0 > 0 \text{ such that } f(t_0, \dots, t_0) < 1, \tag{18}$$

then v cannot be constant. If it holds further that

$$\Gamma + \Gamma_n = \{\lambda + \mu : \lambda \in \Gamma, \mu \in \Gamma_n\} \subset \Gamma, \tag{19}$$

then the constants a and b in (11) satisfy $(2b^2a^{-2}, \dots, 2b^2a^{-2}) \in \Gamma$ and

$$f(2b^2a^{-2}, \dots, 2b^2a^{-2}) = 1.$$

Remark 1.1 In Theorem 1.1, if condition (18) is dropped, the case that v is constant can occur. See the counterexample in Remark 2.1.

Theorem 1.2 Let (f, Γ) satisfy (13)–(15). Assume that $v_* \in C^0(\mathbb{R}^n \setminus \{0\})$, $0 < v_k \in C^2(B_{R_k}(0) \setminus \{0\})$, $R_k \rightarrow \infty$,

$$f(\lambda(A^{v_k})) = 1 \quad \text{in } B_{R_k}(0) \setminus \{0\},$$

and, for some $M_k \rightarrow \infty$,

$$M_k v_k \rightarrow v_* \text{ in } C^0_{loc}(\mathbb{R}^n \setminus \{0\}). \tag{20}$$

Then v_* is radially symmetric about the origin, i.e. $v_*(x) = v_*(|x|)$. In particular, if $v_k \in C^2(B_{R_k}(0))$, $f(\lambda(A^{v_k})) = 1$ in $B_{R_k}(0)$ and $M_k v_k$ converges to v_* in $C^0_{loc}(\mathbb{R}^n)$, then v_* is constant.

Remark 1.2 When (f, Γ) satisfies (1)–(4) and an additional hypothesis that f is homogeneous of positive degree, the function v_* in Theorem 1.2 is a viscosity solution of (12) and the conclusion follows from Theorem B. However, when f is not homogeneous, v_* is not necessarily a viscosity solution of (12).

It is not difficult to see that, under (1)–(4), the function v in Theorem 1.1 is a viscosity solution of (10) (see Remark B.2). We have the following conjecture.

Conjecture Let (f, Γ) satisfy (1)–(4), and let $0 < v \in C^0_{loc}(\mathbb{R}^n)$ be a viscosity solution of (10). Then v is of the form (11) for some $\bar{x} \in \mathbb{R}^n$ and some positive constants a and b .

The notion of viscosity solutions given below is consistent with that in [12].

Definition 1.1 A positive continuous function v in an open set $\Omega \subset \mathbb{R}^n$ is a viscosity supersolution (respectively, subsolution) of

$$f(\lambda(A^v)) = 1, \quad \text{in } \Omega,$$

when the following holds: if $x_0 \in \Omega$, $\varphi \in C^2(\Omega)$, $(v - \varphi)(x_0) = 0$, and $v - \varphi \geq 0$ near x_0 , then

$$f(\lambda(A^\varphi(x_0))) \geq 1$$

(respectively, if $(v - \varphi)(x_0) = 0$, and $v - \varphi \leq 0$ near x_0 , then either $\lambda(A^\varphi(x_0)) \in \mathbb{R}^n \setminus \bar{\Gamma}$ or $f(\lambda(A^\varphi(x_0))) \leq 1$). We say that v is a viscosity solution if it is both a viscosity supersolution and a viscosity subsolution.

Definition 1.2 A positive continuous function v in an open set $\Omega \subset \mathbb{R}^n$ is a viscosity supersolution (respectively, subsolution) of

$$\lambda(A^v) \in \partial\Gamma, \text{ in } \Omega,$$

when the following holds: if $x_0 \in \Omega$, $\varphi \in C^2(\Omega)$, $(v - \varphi)(x_0) = 0$, and $v - \varphi \geq 0$ near x_0 , then

$$\lambda(A^\varphi(x_0)) \in \bar{\Gamma}$$

(respectively, if $(v - \varphi)(x_0) = 0$, and $v - \varphi \leq 0$ near x_0 , then either $\lambda(A^\varphi(x_0)) \in \mathbb{R}^n \setminus \bar{\Gamma}$). We say that v is a viscosity solution if it is both a viscosity supersolution and a viscosity subsolution.

It is clear that for C^2 functions the notions of viscosity solutions and classical solutions coincide. Also, viscosity super- and sub-solutions are stable under uniform convergence, see ‘‘Appendix B’’.

Note that for any $\lambda = (\lambda_1, \dots, \lambda_n) \in \Gamma$, $t_1 = \max \lambda_i + 1 > 0$ and $(t_1, \dots, t_1) \in \lambda + \Gamma_n$. In other words, the sets $\lambda + \Gamma_n$ have non-empty intersection with the ray $\{(t, \dots, t) : t > 0\}$. Thus, if $f^{-1}(1) \neq \emptyset$, then, in view of (4), there exists some $c > 0$ such that $f(c, \dots, c) = 1$. In such situation, working with $\tilde{f}(\lambda) := f(\frac{c}{2}\lambda)$ instead of f , we may assume without loss of generality the following normalization condition

$$f(2, \dots, 2) = 1. \tag{21}$$

Let

$$U(x) := \left(\frac{1}{1 + |x|^2} \right)^{\frac{n-2}{2}}, \quad x \in \mathbb{R}^n.$$

A calculation gives

$$A^U \equiv 2I.$$

With the normalization (21), U satisfies

$$f(\lambda(A^U)) = 1 \text{ on } \mathbb{R}^n.$$

For $\bar{x} \in \mathbb{R}^n$ and $\mu > 0$, let

$$U^{\bar{x}, \mu}(x) = \mu U(\mu^{\frac{2}{n-2}}(x - \bar{x})) = \left(\frac{\mu^{\frac{2}{n-2}}}{1 + \mu^{\frac{4}{n-2}}|x - \bar{x}|^2} \right)^{\frac{n-2}{2}}.$$

Note that, in the sense of (6), $U^{\bar{x}, \mu} = U_\varphi$ with $\varphi(x) = \mu^{\frac{2}{n-2}}(x - \bar{x})$. Hence, by the conformal invariance (7), for any $\bar{x} \in \mathbb{R}^n$ and $\mu > 0$,

$$f(\lambda(A^{U^{\bar{x}, \mu}})) = 1 \text{ on } \mathbb{R}^n.$$

Theorem 1.3 *Let (f, Γ) satisfy (13)–(15), (18)–(19), (5) and the normalization condition (21). Let $\epsilon \in (0, 1/2]$. There exist constants $\bar{m} = \bar{m}(f, \Gamma) \geq 1$, $K = K(f, \Gamma) > 1$, $\delta_* = \delta_*(\epsilon, f, \Gamma) > 0$, $C_* = C_*(\epsilon, f, \Gamma) > 1$ such that for any positive $u \in C^2(B_3(0))$ satisfying¹*

$$f(\lambda(A^u)) = 1 \text{ in } B_3(0) \text{ and } \sup_{B_1(0)} u \geq C_*,$$

¹ Note that, \bar{m} and K are independent of ϵ .

there exists $\{x^1, \dots, x^m\} \subset B_2(0)$ with $1 \leq m \leq \bar{m}$ satisfying

- (i) $u(x^1) \geq \sup_{B_1(0)} u$,
- (ii) $|x^i - x^j| \geq \frac{1}{K}$ for all $1 \leq i \neq j \leq m$,
- (iii) $\frac{1}{K} \leq \frac{u(x^i)}{u(x^j)} \leq K$ for all $1 \leq i, j \leq m$,
- (iv) $|u(x) - U^{x^i, u(x^i)}(x)| \leq \epsilon U^{x^i, u(x^i)}(x)$ for all $1 \leq i \leq m, x \in B_{\delta_*}(x^i)$,
- (v) $\frac{1}{K \delta_*^{n-2}} \leq u(x) \leq \frac{K}{\delta_*^{n-2}} u(x^1)$ for all $x \in B_{\frac{3}{2}}(0) \setminus \cup_{i=1}^m B_{\delta_*}(x^i)$,
- (vi) $u(x^i) = \sup_{B_{\delta_*}(x^i)} u$.

Remark 1.3 If it holds further that $\sup_{B_1(0)} u > \frac{K^{1/2}}{\delta_*^{\frac{n-2}{2}}}$, then

$$\min_{1 \leq i \leq m} \text{dist}(x^i, B_1(0)) \leq \left[\sup_{B_1(0)} u \right]^{-\frac{2}{n-2}}.$$

To see this, let x_* be a point in $\bar{B}_1(0)$ such that $u(x_*) = \sup_{B_1(0)} u$. In view of (v) and the stated condition on $\sup_{B_1(0)} u$, x_* belongs to some ball $B_{\delta_*}(x^{i_0})$. By (iv), we then have

$$u(x_*) \leq 2U^{x^{i_0}, u(x^{i_0})}(x_*) \leq \frac{1}{|x_* - x^{i_0}|^{\frac{n-2}{2}}},$$

which implies the assertion.

Theorem 1.3 can be stated equivalently as follows.

Theorem 1.4 *Let (f, Γ) satisfy (13)–(15), (18)–(19), (5) and the normalization condition (21). Assume that $0 < u_k \in C^2(B_3(0))$ satisfy (8) and (9). Let $\epsilon \in (0, 1/2]$. Then there exist $\bar{m} = \bar{m}(f, \Gamma) \geq 1, K(f, \Gamma) > 1$ and $\delta_* = \delta_*(\epsilon, f, \Gamma) > 0$ such that, after passing to a subsequence, still denoted by u_k , there exists $\{x_k^1, \dots, x_k^m\} \subset B_2(0)$ ($1 \leq m \leq \bar{m}$) satisfying²*

- (i) $u_k(x_k^1) \geq \sup_{B_1(0)} u_k$,
- (ii) $|x_k^i - x_k^j| \geq \frac{1}{K}$ for all $k \geq 1, 1 \leq i \neq j \leq m$,
- (iii) $\frac{1}{K} \leq \frac{u_k(x_k^i)}{u_k(x_k^j)} \leq K$ for all $k \geq 1, 1 \leq i, j \leq m$,
- (iv) $|u_k(x) - U^{x_k^i, u_k(x_k^i)}(x)| \leq \epsilon U^{x_k^i, u_k(x_k^i)}(x)$ for all $k \geq 1, 1 \leq i \leq m, x \in B_{\delta_*}(x_k^i)$,
- (v) $\frac{1}{K \delta_*^{n-2}} \leq u_k(x) \leq \frac{K}{\delta_*^{n-2}} u_k(x_k^1)$ for all $k \geq 1, x \in B_{\frac{3}{2}}(0) \setminus \cup_{i=1}^m B_{\delta_*}(x_k^i)$,
- (vi) $u_k(x_k^i) = \sup_{B_{\delta_*}(x_k^i)} u_k$.

Remark 1.4 By Remark 1.3, we have

$$\text{dist}(\{x_k^1, \dots, x_k^m\}, B_1(0)) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

When $(f, \Gamma) = (\frac{1}{2^n} \sigma_1, \Gamma_1)$, Eq. (8) is $-\Delta u_k = n(n-2)u_k^{\frac{n+2}{n-2}}$ and Theorem 1.4 in this case was proved by Schoen [16].

See Li [9] and Chen and Lin [4] for analogous results for the equation $-\Delta u_k = K(x)u_k^{\frac{n+2}{n-2}}$.

In Theorems 1.3 and 1.4, $B_1(0), B_2(0)$ and $B_3(0)$ can be replaced respectively by $B_{r_1}(0), B_{r_2}(0)$ and $B_{r_3}(0), 0 < r_1 < r_2 < r_3$, and in this case the constants \bar{m}, K, δ_* and C_* depend also on r_1, r_2 and r_3 .

The following is a quantitative version of Theorem A, and is related to Theorems 1.1 and 1.3.

² The constant m is independent of k .

Theorem 1.5 (Quantitative Liouville Theorem) *Let (f, Γ) satisfy (13)–(15), (18)–(19), (5) and the normalization condition (21), and let $\gamma, r_1 > 0$ be constants. Then, for every $\epsilon \in (0, 1/2]$, there exist some constants $\delta_* > 0, R^* > 0$, depending only on $(f, \Gamma), \gamma, r_1$ and ϵ , such that if $0 < v \in C^2(B_R(0))$ for some $R \geq R^*$,*

$$f(\lambda(A^v)) = 1 \quad \text{in } B_R(0), \tag{22}$$

and

$$v \geq \gamma \quad \text{in } B_{r_1}(0), \tag{23}$$

then, for some $\bar{x} \in \mathbb{R}^n$ satisfying

$$v(\bar{x}) = \max_{B_{\delta_* R}(0)} v \leq \frac{2^{n-1}}{\gamma r_1^{n-2}}, \quad |\bar{x}| \leq 2^{\frac{1}{n-2}} \gamma^{-\frac{2}{n-2}}, \tag{24}$$

there holds

$$|v(y) - U^{\bar{x}, v(\bar{x})}(y)| \leq \epsilon U^{\bar{x}, v(\bar{x})}(y), \quad \forall |y - \bar{x}| \leq \delta_* R. \tag{25}$$

Remark 1.5 The constant δ_* in Theorems 1.3 and 1.5 can be chosen the same.

Remark 1.6 An analogous result for the degenerate elliptic equation $\lambda(A^v) \in \partial\Gamma$ is a consequence of the local gradient estimate [13, Theorem 1.5].

An ingredient in our proof of Theorems 1.3 and 1.4 is the following local gradient estimate, which follows from Theorem 1.2 and the proof of [12, Theorem 1.10].

Theorem 1.6 *Let (f, Γ) satisfy (13)–(15) and let $v \in C^2(B_2(0))$ satisfy, for some constant $b > 0$,*

$$0 < v \leq b \quad \text{in } B_2(0), \tag{26}$$

and

$$f(\lambda(A^v)) = 1 \quad \text{in } B_2(0). \tag{27}$$

Then, for some constant C depending only on (f, Γ) and b ,

$$|\nabla \ln v| \leq C \quad \text{in } B_1(0). \tag{28}$$

For $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$, the result was proved by Guan and Wang [6].

When (f, Γ) satisfies (1)–(4) and is homogeneous of positive degree, Theorem 1.6 was proved in [12].

The rest of the paper is organized as follows. We start in Sect. 2 with the proof of Theorems 1.1 and 1.2. We then prove Theorem 1.6 in Sect. 3. In Sect. 4, we first establish an intermediate quantitative Liouville result and then use it to prove Theorem 1.3. In Sect. 5, we prove Theorem 1.5 as an application of Theorem 1.3. In ‘‘Appendix A’’, we present a lemma about super-harmonic functions which is used in the body of the paper. In ‘‘Appendix B’’, we include a relevant remark on the limit of viscosity solutions of elliptic PDE. Finally we collect in ‘‘Appendix C’’ some relevant calculus lemmas.

2 Non-quantitative Liouville theorems

In this section, we prove Theorems 1.1 and 1.2. We use the method of moving spheres and establish along the way, as a tool, a gradient estimate which is in a sense weaker than that in Theorem 1.6 but suffices for the moment. (Note that the proof of Theorem 1.6 relies on Theorem 1.2.)

2.1 A gradient estimate

Theorem 2.1 *Let (f, Γ) satisfy (13), (15) and*

$$\Gamma \cap \{\lambda + t\mu : t > 0\} \text{ is convex for all } \lambda \in \Gamma, \mu \in \Gamma_n.$$

Let $0 < v \in C^2(B_2(0))$ satisfy, for some constant $\theta > 1$,

$$\sup_{B_2(0)} v \leq \theta \inf_{B_2(0)} v, \quad (29)$$

and

$$f(\lambda(A^v)) = 1 \quad \text{in } B_2(0).$$

Then, for some constant C depending only on n and θ ,

$$|\nabla \ln v| \leq C \quad \text{in } B_1(0).$$

This type of gradient estimate was established and used in various work of the first named author and his collaborators under less general hypothesis on (f, Γ) . It turns out that the same proof works in the current situation. We give a detailed sketch here for completeness.

We use the method of moving spheres as in [7, 8, 14, 15]. For a function w defined on a subset of \mathbb{R}^n , we define

$$w_{x,\lambda}(y) = \frac{\lambda^{n-2}}{|y-x|^{n-2}} w\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right)$$

wherever the expression makes sense. We will use w_λ to denote $w_{0,\lambda}$. We start with a simple result.

Lemma 2.1 *Let $R > 0$ and w be a positive Lipschitz function in $\bar{B}_R(0)$ such that, for some $L > 0$,*

$$|\ln w(y) - \ln w(z)| \leq L|y - z| \text{ for all } y, z \in \bar{B}_R(0).$$

Then for $\underline{\lambda} = \min(\frac{n-2}{2L}, \frac{R}{2})$ we have

$$w_\lambda \leq w \text{ in } B_{\underline{\lambda}}(0) \setminus B_\lambda, \forall 0 < \lambda < \underline{\lambda}. \quad (30)$$

Proof Write w in polar coordinates $w(r, \theta)$. It is easy to see that (30) is equivalent to

$$r^{\frac{n-2}{2}} w(r, \theta) \leq s^{\frac{n-2}{2}} w(s, \theta), \quad \forall 0 < r < s < \underline{\lambda}, \forall \theta. \quad (31)$$

Estimate (31) is readily seen from the estimates

$$\begin{aligned} \ln w(s, \theta) - \ln w(r, \theta) &\geq -L|s - r| \geq -\frac{n-2}{2\underline{\lambda}}(s - r) \\ &\geq -\frac{n-2}{2}[\ln s - \ln r]. \end{aligned}$$

Lemma 2.1 is established. \square

Proof of Theorem 2.1 By Lemma 2.1, there exists some $r_0 \in (0, 1/3)$ such that

$$v_{x,\lambda} \leq v \text{ in } B_{r_0}(x) \setminus B_\lambda(x) \text{ for all } \lambda \in (0, r_0) \text{ and } x \in B_{4/3}(0).$$

It is easy to see that, for some $r_1 \in (0, r_0)$,

$$v_{x,\lambda} \leq v \text{ in } B_{5/3}(0) \setminus B_{r_0}(x) \text{ for all } \lambda \in (0, r_1) \text{ and } x \in B_{4/3}(0).$$

We then define, for $x \in B_{4/3}(0)$,

$$\bar{\lambda}(x) = \sup \left\{ \lambda \in (0, 5/3 - |x|) : v_{x,\lambda} \leq v \text{ in } B_{5/3}(0) \setminus B_\lambda(x) \right\}.$$

We have

$$v_{x,\bar{\lambda}(x)} \leq v \text{ in } B_{5/3}(0) \setminus B_{\bar{\lambda}(x)}(x),$$

By the conformal invariance (7), $v_{x,\bar{\lambda}(x)}$ satisfies

$$f(\lambda(A^{v_{x,\bar{\lambda}(x)}})) = 1, \quad \text{in } B_{5/3}(0) \setminus \overline{B_{\bar{\lambda}(x)}(x)}.$$

Using the above two displayed equations, the definition of $\bar{\lambda}(x)$, and using the ellipticity of the equation satisfied by v and $v_{x,\bar{\lambda}(x)}$, we can apply the strong maximum principle and Hopf Lemma to infer that either $\bar{\lambda}(x) = 5/3 - |x|$ or there exists some $y \in \partial B_{5/3}(0)$ such that

$$v_{x,\bar{\lambda}(x)}(y) = v(y)$$

—see the proof of [8, Lemma 4.5].

In the latter case, (29) implies that

$$\theta \geq \frac{v\left(x + \frac{\bar{\lambda}(x)^2(y-x)}{|y-x|^2}\right)}{v(y)} = \frac{|y-x|^{n-2}}{\bar{\lambda}(x)^{n-2}} \geq \frac{(5/3 - |x|)^{n-2}}{\bar{\lambda}(x)^{n-2}}.$$

In either case, we obtain that

$$\bar{\lambda}(x) \geq c(n, \theta) > 0 \text{ for all } x \in B_{4/3}(0).$$

The conclusion then follows from [8, Lemma A.2]. □

2.2 Proof of Theorem 1.1

Remark 2.1 If we drop condition (18), the case that v is constant in Theorem 1.1 can occur. For example, consider $n \geq 3$ and

$$f(\lambda) = \sigma_2(\lambda) + 1, \quad \Gamma = \left\{ \lambda \in \mathbb{R}^n \mid \sum_{j=1}^n \lambda_j - \lambda_i > 0 \text{ for all } i = 1, \dots, n \right\},$$

$$v_k(x) = \left(\frac{1}{R_k^{n+4}} |x - x_k|^{-\frac{n-4}{2}} + 1 \right)^{\frac{2(n-2)}{n-4}} \text{ for } |x| < \frac{|x_k|}{2} = R_k \rightarrow \infty.$$

It is readily seen that $f(t, \dots, t) > 1$ for all $(t, \dots, t) \in \Gamma$, v_k satisfies (16) (cf. [13, Theorem 1.6]) and $v_k \rightarrow 1$ in $C_{loc}^0(\mathbb{R}^n)$.

Proof of Theorem 1.1 We may assume that $R_k \geq 5$ for all k .

Clearly, for every $\beta > 1$, there exists some positive constant $C(\beta)$, independent of k , such that $1/C(\beta) \leq v_k \leq C(\beta)$ in $B_\beta(0)$. It follows from Theorem 2.1 that $|\nabla \ln v_k| \leq C'(\beta)$ in

$B_{\beta/2}(0)$. It follows, after passing to a subsequence, that for every $0 < \alpha < 1$, $v_k \rightarrow v$ in $C_{loc}^\alpha(\mathbb{R}^n)$, $v \in C_{loc}^{0,1}(\mathbb{R}^n)$ and v is super-harmonic on \mathbb{R}^n .

Using the positivity, the superharmonic of v , and the maximum principle, we can find $c_0 > 0$ such that

$$v(y) \geq 2c_0|y|^{2-n}, \quad \forall |y| \geq 1. \tag{32}$$

Passing to a subsequence and shrinking R_k and $c_0 > 0$, if necessary, we may assume that

$$|v_k(y) - v(y)| \leq R_k^{-n}, \quad \forall |y| \leq R_k, \tag{33}$$

and

$$v_k(y) \geq c_0(1 + |y|)^{2-n}, \quad \forall |y| < R_k. \tag{34}$$

Lemma 2.2 *Under the hypotheses of Theorem 1.1, there exists a function $\lambda^{(0)} : \mathbb{R}^n \rightarrow (0, \infty)$ such that, for all k ,*

$$(v_k)_{x,\lambda} \leq v_k \text{ in } B_{R_k}(0) \setminus B_\lambda(x), \quad \forall 0 < \lambda < \lambda^{(0)}(x) \text{ and } |x| \leq \frac{R_k}{5}.$$

Proof For $|x| \leq \frac{R_k}{5}$, we have, by (33) and (34), for all k that

$$\frac{1}{c_1(x)} \leq v_k \leq c_1(x) \text{ in } B_{4r_1(x)}(x) \subset B_{R_k}(0), \tag{35}$$

where

$$r_1(x) = \frac{1}{4} + |x|, \text{ and } c_1(x) = \max \left\{ 1 + \sup_{B_{4r_1(x)}(x)} v, \frac{1}{c_0}(1 + |x| + 4r_1(x))^{n-2} \right\}.$$

By (35) and Theorem 2.1, there exists $c_2(x) > 0$, independent of k , such that

$$|\nabla \ln v_k| \leq c_2(x) \text{ in } B_{2r_1(x)}(x).$$

Thus, by Lemma 2.1, we can find $0 < \lambda_1(x) < r_1(x)$ independent of k such that

$$(v_k)_{x,\lambda} \leq v_k \text{ in } B_{\lambda_1(x)}(x) \setminus B_\lambda(x) \quad \forall 0 < \lambda < \lambda_1(x). \tag{36}$$

For $0 < \lambda < \lambda_1(x)$, we have, using (35), that

$$(v_k)_{x,\lambda}(y) \leq \frac{\lambda^{n-2}c_1(x)}{|y-x|^{n-2}}, \quad \forall y \in B_{R_k}(0) \setminus B_\lambda(x). \tag{37}$$

When $y \in B_{R_k}(0) \setminus B_{4r_1(x)}(x)$, $\frac{1}{2}(1 + |y|) < |y-x|$ and we obtain, using (37) and (34), that

$$(v_k)_{x,\lambda}(y) \leq \frac{(2\lambda)^{n-2}c_1(x)}{(1 + |y|)^{n-2}} \leq \frac{(2\lambda)^{n-2}c_1(x)}{c_0} v_k(y). \tag{38}$$

When $y \in B_{4r_1(x)}(x) \setminus B_{\lambda_1(x)}(x)$, $1 + |y| \leq 2(1 + 3|x|)$, $|y-x| \geq \lambda_1(x)$ and we obtain, using (37) and (34), that

$$(v_k)_{x,\lambda}(y) \leq \frac{\lambda^{n-2}c_1(x)}{\lambda_1(x)^{n-2}} \leq \frac{(2\lambda)^{n-2}c_1(x)(1 + 3|x|)^{n-2}}{\lambda_1(x)^{n-2}c_0} v_k(y). \tag{39}$$

Letting

$$\lambda^{(0)}(x) = \min \left\{ \lambda_1(x), \frac{\lambda_1(x)}{2(1 + 3|x|)} \left[\frac{c_1(x)}{c_0} \right]^{\frac{1}{n-2}} \right\} \leq \lambda_1(x),$$

we derive from (38) and (39) that

$$(v_k)_{x,\lambda} \leq v_k \text{ in } B_{R_k}(0) \setminus B_{\lambda_1(x)}(x) \text{ and } 0 < \lambda < \lambda^{(0)}(x). \tag{40}$$

Lemma 2.2 follows from (36) and (40). □

Define, for $x \in \mathbb{R}^n$ and $|x| \leq R_k/5$, that

$$\bar{\lambda}_k(x) = \sup \left\{ 0 < \mu \leq \frac{R_k}{5} \mid (v_k)_{x,\lambda} \leq v_k \text{ in } B_{R_k}(0) \setminus B_\lambda(x), \forall 0 < \lambda < \mu \right\}.$$

By Lemma 2.2,

$$\bar{\lambda}(x) := \liminf_{k \rightarrow \infty} \bar{\lambda}_k(x) \in [\lambda^{(0)}(x), \infty], \quad x \in \mathbb{R}^n.$$

By (32),

$$\alpha := \liminf_{|y| \rightarrow \infty} |y|^{n-2} v(y) \in (0, \infty].$$

Lemma 2.3 *Assume (13)–(15). Then either v is constant or*

$$\alpha = \lim_{|y| \rightarrow \infty} |y|^{n-2} v_{x,\bar{\lambda}(x)}(y) = \bar{\lambda}(x)^{n-2} v(x) < \infty, \quad \forall x \in \mathbb{R}^n.$$

Proof Step 1. If $\bar{\lambda}(x) < \infty$ for some $x \in \mathbb{R}^n$, then

$$\alpha = \lim_{|y| \rightarrow \infty} |y|^{n-2} v_{x,\bar{\lambda}(x)}(y) = \bar{\lambda}(x)^{n-2} v(x) < \infty.$$

Since $\bar{\lambda}(x) < \infty$, we have, along a subsequence, $\bar{\lambda}_k(x) \rightarrow \bar{\lambda}(x)$ —but for simplicity, we still use $\{\bar{\lambda}_k(x)\}$, $\{v_k\}$, etc to denote the subsequence. By the definition of $\bar{\lambda}_k(x)$, we have

$$(v_k)_{x,\bar{\lambda}_k(x)} \leq v_k \text{ in } B_{R_k}(0) \setminus B_{\bar{\lambda}_k(x)}(x), \tag{41}$$

By the conformal invariance (7), $(v_k)_{x,\bar{\lambda}_k(x)}$ satisfies

$$f(\lambda(A^{(v_k)_{x,\bar{\lambda}_k(x)}})) = 1, \quad \text{in } B_{R_k}(0) \setminus \overline{B_{\bar{\lambda}_k(x)}(x)}. \tag{42}$$

Using (16), (41), (42), the definition of $\bar{\lambda}_k(x)$, and using the ellipticity of the equation satisfied by v_k and $(v_k)_{x,\bar{\lambda}_k(x)}$, we can apply the strong maximum principle and Hopf Lemma to infer the existence of some $y_k \in \partial B_{R_k}(0)$ such that

$$(v_k)_{x,\bar{\lambda}_k(x)}(y_k) = v_k(y_k)$$

—see the proof of [8, Lemma 4.5].

It follows that

$$\lim_{k \rightarrow \infty} |y_k|^{n-2} v_k(y_k) = \lim_{k \rightarrow \infty} |y_k|^{n-2} (v_k)_{x,\bar{\lambda}_k(x)}(y_k) = (\bar{\lambda}(x))^{n-2} v(x).$$

This implies, in view of (33), that

$$\alpha \leq \lim_{k \rightarrow \infty} |y_k|^{n-2} v(y_k) = \bar{\lambda}(x)^{n-2} v(x) = \lim_{|y| \rightarrow \infty} |y|^{n-2} v_{x,\bar{\lambda}(x)}(y) < \infty.$$

On the other hand, if \hat{y}_i is such that $|\hat{y}_i| \rightarrow \infty$ and

$$\alpha = \lim_{i \rightarrow \infty} |y_i|^{n-2} v(y_i),$$

then, since $v_{x, \bar{\lambda}(x)} \leq v$ in $\mathbb{R}^n \setminus B_{\bar{\lambda}(x)}(x)$, we have

$$v(y_i) \geq \frac{\bar{\lambda}(x)^{n-2}}{|y_i - x|^{n-2}} v\left(x + \frac{\bar{\lambda}(x)^2(y_i - x)}{|y_i - x|^2}\right).$$

This gives

$$\alpha = \lim_{i \rightarrow \infty} |y_i|^{n-2} v(y_i) \geq \bar{\lambda}(x)^{n-2} v(x).$$

Step 1 is established.

Step 2. It remains to show that either v is constant or, for every $x \in \mathbb{R}^n$, $\bar{\lambda}(x) < \infty$.

To this end, we show that if $\bar{\lambda}(x) = \infty$ for some $x \in \mathbb{R}^n$, then v is constant. Indeed, assume that $\bar{\lambda}_k(x) \rightarrow \infty$ as $k \rightarrow \infty$. We easily derive from this and the convergence of v_k to v that

$$v_{x, \lambda} \leq v \quad \text{in } \mathbb{R}^n \setminus B_\lambda(x) \quad \forall \lambda > 0. \tag{43}$$

The above is equivalent to the property that for every fixed unit vector e , $r^{\frac{n-2}{2}} v(x + re)$ is non-decreasing in r . Thus

$$r^{n-2} \min_{\partial B_r(x)} v \geq r^{\frac{n-2}{2}} \min_{\partial B_1(x)} v \quad \forall r \geq 1.$$

In particular, $\alpha = \liminf_{|y| \rightarrow \infty} |y|^{n-2} v(y) = \infty$. This implies, by Step 1, that $\bar{\lambda}(x) = \infty$ for every $x \in \mathbb{R}^n$, and therefore (43) holds for every $x \in \mathbb{R}^n$. This implies that v is a constant, see Corollary C.1. □

Lemma 2.4 *Assume (13)–(15) and (18). Then the function v in Theorem 1.1 cannot be constant.*

Proof Fix some $t > 0$ for the moment. Set $\varphi(x) = v(0) - t|x|^2$ and fix some $r > 0$ such that $\varphi > 0$ in $B_r(0)$ and $\varphi < v_k$ on $\partial B_r(0)$ for all sufficiently large k . Let

$$\gamma_k = \sup_{B_r(0)} (\varphi - v_k) \text{ and } \varphi_k = \varphi - \gamma_k.$$

Then $\varphi_k \leq v_k$ in $B_r(0)$ and $\varphi_k(x_k) = v_k(x_k)$ for some $x_k \in \bar{B}_r(0)$. Noting that

$$\gamma_k = (\varphi - v)(x_k) - (v_k - v)(x_k) = -t|x_k|^2 - (v_k - v)(x_k)$$

and

$$\lim_{k \rightarrow \infty} \gamma_k = \sup_{B_r(0)} (\varphi - v) = 0$$

we deduce that $x_k \rightarrow 0$. This leads to

$$\varphi_k(x_k) = v_k(x_k), \nabla \varphi_k(x_k) = \nabla v_k(x_k), \nabla^2 \varphi_k(x_k) \leq \nabla^2 v_k(x_k)$$

and

$$A^{\varphi_k}(x_k) \geq A^{v_k}(x_k).$$

Noting that there is some $C > 0$ independent of δ and k such that, for large k ,

$$A^{\varphi_k}(x_k) \leq \left(\frac{4}{n-2} v(0)^{-\frac{n+2}{n-2}} t + C\delta\right) I.$$

Thus, we can select t and δ such that

$$A^{vk}(x_k) \leq A^{\varphi k}(x_k) \leq t_0 I.$$

where t_0 is the constant in (18). Since $f(\lambda(A^{vk}(x_k))) = 1$, this contradicts (14), (15) and (18). \square

Recall that $0 < v \in C_{loc}^{0,1}(\mathbb{R}^n)$, $\Delta v \leq 0$ in \mathbb{R}^n , and it remains to consider the case that, for every $x \in \mathbb{R}^n$, there exists $0 < \bar{\lambda}(x) < \infty$ such that

$$v_{x, \bar{\lambda}(x)}(y) \leq v(y), \quad \forall |y - x| \geq \bar{\lambda}(x),$$

and

$$\lim_{|y| \rightarrow \infty} |y|^{n-2} v_{x, \bar{\lambda}(x)}(y) = \alpha := \liminf_{|y| \rightarrow \infty} |y|^{n-2} v(y) < \infty.$$

If v is in $C^2(\mathbb{R}^n)$, the conclusion of Theorem 1.1 follows from the proof of Theorem 1.3 in [8]. An observation made in [11] easily allows the proof to hold for $v \in C_{loc}^{0,1}(\mathbb{R}^n)$. For readers' convenience, we outline the proof below.

Let $\psi(y) = \frac{y}{|y|^2}$. We denote

$$v_\psi := |J_\psi|^{\frac{n-2}{2n}} v \circ \psi.$$

We know that $\lambda(A^{v_\psi}(y)) = \lambda(A^v(\psi(y)))$. Namely, $A^{v_\psi}(y)$ and $A^v(\psi(y))$ differ only by an orthogonal conjugation.

Introduce

$$w^{(x)} := (v_{x, \bar{\lambda}(x)})_\psi, \quad x \in \mathbb{R}^n.$$

We deduce from the above properties of v that for every $x \in \mathbb{R}^n$, there exists some $\delta(x) > 0$ such that

$$\begin{aligned} v_\psi &\geq w^{(x)} \text{ in } B_{\delta(x)}(0) \setminus \{0\}, \\ w^{(x)}(0) &= \alpha = \liminf_{y \rightarrow 0} v_\psi(y), \\ \Delta v_\psi &\leq 0, \text{ in } B_{\delta(x)}(0) \setminus \{0\}, \end{aligned}$$

Let $D = \{x \in \mathbb{R}^n \mid v \text{ is differentiable at } x\}$. Since $v \in C_{loc}^{0,1}(\mathbb{R}^n)$, the Lebesgue measure of $\mathbb{R}^n \setminus D$ is 0. It is clear that $w^{(x)}(y)$ is differentiable at $y = 0$ if v is differentiable at x .

By [8, Lemma 4.1],

$$\nabla w^{(x)}(0) = \nabla w^{(\tilde{x})}(0), \quad \forall x, \tilde{x} \in D.$$

Namely, for some $V \in \mathbb{R}^n$,

$$\nabla w^{(x)}(0) = V, \quad \forall x \in D.$$

A calculation yields

$$\nabla w^{(x)}(0) = (n - 2)\alpha x + \alpha^{\frac{n}{n-2}} v(x)^{\frac{n}{n-2}} \nabla v(x).$$

Thus

$$\nabla_x \left(\frac{n - 2}{2} \alpha^{\frac{n}{n-2}} v(x)^{\frac{n}{n-2}} - \frac{n - 2}{2} \alpha |x|^2 + V \cdot x \right) = 0, \quad \forall x \in D.$$

Consequently, for some $\bar{x} \in \mathbb{R}^n$ and $d \in \mathbb{R}$,

$$v(x)^{-\frac{2}{n-2}} \equiv \alpha^{-\frac{2}{n-2}} |x - \bar{x}|^2 + d\alpha^{-\frac{2}{n-2}}.$$

Since $v > 0$, we must have $d > 0$, so

$$v(x) \equiv \left(\frac{\alpha^{\frac{2}{n-2}}}{d + |x - \bar{x}|^2} \right)^{\frac{n-2}{2}}.$$

We have proved that v is of the form (11) for some $\bar{x} \in \mathbb{R}^n$ and some positive constants a and b .

To finish the proof, we show that $f(2b^2a^{-2}, \dots, 2b^2a^{-2}) = 1$ when (18) and (19) are in effect. For $\delta > 0$, let

$$v_\delta(x) = v(x) - \delta|x|^2.$$

Since $v_k \rightarrow v$ in $C^0(\bar{B}_\delta(0))$, there exists $\beta_k \rightarrow 0$ and $x_k \rightarrow 0$ such that $\hat{v}_k := v_\delta + \beta_k$ satisfies

$$(v_k - \hat{v}_k)(x_k) = 0 \text{ and } v_k - \hat{v}_k \geq 0 \text{ near } x_k.$$

We have $A^{\hat{v}_k}(x_k) \geq A^{v_k}(x_k)$. Therefore, by (19), $\lambda(A^{\hat{v}_k}(x_k)) \in \Gamma$, and by (15),

$$f(\lambda(A^{\hat{v}_k}(x_k))) \geq f(\lambda(A^{v_k}(x_k))) = 1. \tag{44}$$

Noting that $A^{\hat{v}_k}(x_k) \rightarrow 2b^2a^{-2}I$ as $\delta \rightarrow 0, k \rightarrow \infty$, we infer that $2b^2a^{-2} > t_0$. (Indeed, if $2b^2a^{-2} \leq t_0$, then, for small $\rho > 0$, we have $A^{\hat{v}_k}(x_k) < (t_0 + \rho)I$ for small δ and large k , which implies, by (44), (14) and (15), that $1 \leq f(\lambda(A^{\hat{v}_k}(x_k))) < f(t_0 + \rho, \dots, t_0 + \rho)$, which contradicts (18).) In view of (19), this implies that $(2b^2a^{-2}, \dots, 2b^2a^{-2}) \in \Gamma$. We can now send $k \rightarrow \infty$ and then $\delta \rightarrow 0$ in (44) to obtain

$$f(\lambda(A^v(0))) \geq 1, \text{ i.e. } f(2b^2a^{-2}, \dots, 2b^2a^{-2}) \geq 1.$$

Using $v^\delta(x) = v(x) + \delta|x|^2$ instead of v_δ and the fact that $(2b^2a^{-2}, \dots, 2b^2a^{-2}) \in \Gamma$, one can easily derive

$$f(2b^2a^{-2}, \dots, 2b^2a^{-2}) \leq 1.$$

Theorem 1.1 is established. □

2.3 Proof of Theorem 1.2

Proof of Theorem 1.2 We start with some preparation as in the proof of Theorem 1.1. We may assume that $R_k \geq 5$ for all k .

By hypotheses, v_k is super-harmonic and positive on $\mathbb{R}^n \setminus \{0\}$. Therefore, v_* is super-harmonic and non-negative on $\mathbb{R}^n \setminus \{0\}$. Hence either $v_* \equiv 0$ or $v_* > 0$ in $\mathbb{R}^n \setminus \{0\}$. In the former case we are done. We assume henceforth that the latter holds.

Now, for every $\beta > 2$, there exists some positive constant $C(\beta)$, independent of k , such that $C(\beta)^{-1} \leq M_k v_k \leq C(\beta)$ in $B_\beta(0) \setminus B_{1/\beta}(0)$. It follows from Theorem 2.1 that $|\nabla \ln v_k| \leq C'(\beta)$ in $B_{\beta/2}(0) \setminus B_{2/\beta}(0)$. It follows, after passing to a subsequence, that for every $0 < \alpha < 1$, $M_k v_k \rightarrow v_*$ in $C_{loc}^\alpha(\mathbb{R}^n \setminus \{0\})$, $v_* \in C_{loc}^{0,1}(\mathbb{R}^n \setminus \{0\})$ and

$$|\nabla \ln v_*| \leq C'(\beta) \text{ in } B_{\beta/2}(0) \setminus B_{2/\beta}(0). \tag{45}$$

By the super-harmonicity and the positivity of v_* , we can find $c_0 > 0$ such that

$$v_*(y) \geq 2c_0|y|^{2-n}, \quad \forall |y| \geq 1. \tag{46}$$

Hence, passing to a subsequence and shrinking R_k and $c_0 > 0$ if necessary, we can assume without loss of generality that, for all k ,

$$|M_k v_k(y) - v_*(y)| \leq (R_k)^{-n}, \quad \forall R_k^{-1} \leq |y| \leq R_k \tag{47}$$

and

$$M_k v_k(y) \geq c_0(1 + |y|)^{2-n} \quad \forall 0 < |y| < R_k. \tag{48}$$

Denote

$$(v_k)_{x,\lambda}(y) := \left(\frac{\lambda}{|y-x|} \right)^{n-2} v_k \left(x + \frac{\lambda^2(y-x)}{|y-x|^2} \right),$$

the Kelvin transformation of v_k . We use $(v_k)_\lambda$ to denote $(v_k)_{0,\lambda}$.

Lemma 2.5 *Under the hypotheses of Theorem 1.2, there exists a function $\lambda^{(0)} : \mathbb{R}^n \setminus \{0\} \rightarrow (0, \infty)$ such that $\lambda^{(0)}(x) \leq |x|$ and, for all k ,*

$$(v_k)_{x,\lambda} \leq v_k \text{ in } B_{R_k}(0) \setminus (B_\lambda(x) \cup \{0\}), \quad \forall 0 < \lambda < \lambda^{(0)}(x) \text{ and } |x| \leq \frac{R_k}{5}.$$

Proof We adapt the proof of Lemma 2.2. For $0 < |x| < \frac{R_k}{5}$, we have, by (47) and (48), for all k that

$$\frac{1}{c_1(x)} \leq M_k v_k \leq c_1(x) \text{ in } B_{4r_1(x)}(x) \subset B_{R_k}(0), \tag{49}$$

where

$$r_1(x) = \frac{1}{8}|x| \text{ and } c_1(x) = \max \left\{ 1 + \sup_{B_{4r_1(x)}(x)} v_*, \frac{1}{c_0}(1 + 2|x|)^{n-2} \right\}.$$

By Theorem 2.1 and (49), there exists $c_2(x) > 0$, independent of k , such that

$$|\nabla \ln v_k| \leq c_2(x) \text{ in } B_{2r_1(x)}(x).$$

Thus, by Lemma 2.1, we can find $0 < \lambda_1(x) < r_1(x)$ independent of k such that

$$(v_k)_{x,\lambda} \leq v_k \text{ in } B_{\lambda_1(x)}(x) \setminus (B_\lambda(x) \cup \{0\}) \text{ for all } 0 < \lambda < \lambda_1(x). \tag{50}$$

For $0 < \lambda < \lambda_1(x)$, we have, using (49), that

$$(v_k)_{x,\lambda}(y) \leq \frac{\lambda^{n-2}c_1(x)}{|y-x|^{n-2}} \text{ for } y \in B_{R_k}(0) \setminus (B_\lambda(x) \cup \{0\}). \tag{51}$$

For $y \in B_{R_k}(0) \setminus (B_{1+4|x|}(x) \cup \{0\})$, we have $\frac{1}{2}(1 + |y|) \leq |y-x|$ and we obtain, using (51) and (48), that

$$(v_k)_{x,\lambda}(y) \leq \frac{(2\lambda)^{n-2}c_1(x)}{c_0} v_k(y). \tag{52}$$

For $y \in B_{1+4|x|}(x) \setminus (B_{\lambda_1(x)}(x) \cup \{0\})$, we have $1 + |y| \leq 2(1 + 3|x|)$, $|y-x| \geq \lambda_1(x)$ and we obtain, using (51) and (48), that

$$(v_k)_{x,\lambda}(y) \leq \frac{(2\lambda)^{n-2}c_1(x)(1 + 3|x|)^{n-2}}{c_0\lambda_1(x)^{n-2}} v_k(y). \tag{53}$$

Letting

$$\lambda^{(0)}(x) = \min \left\{ \lambda_1(x), \frac{\lambda_1(x)}{2(1 + 3|x|)} \left[\frac{c_1(x)}{c_0} \right]^{\frac{1}{n-2}} \right\} \leq \lambda_1(x),$$

we see that the conclusion of Lemma 2.5 follows from (50), (52) and (53). □

Define, for $0 < |x| \leq R_k/5$, that

$$\bar{\lambda}_k(x) = \sup \left\{ 0 < \mu \leq \min(|x|, \frac{R_k}{5}) \mid (v_k)_{x,\lambda} \leq v_k \text{ in } B_{R_k}(0) \setminus (B_\lambda(x) \cup \{0\}), \forall 0 < \lambda < \mu \right\}.$$

By Lemma 2.5,

$$\bar{\lambda}(x) := \liminf_{k \rightarrow \infty} \bar{\lambda}_k(x) \in [\lambda^{(0)}(x), |x|], \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Clearly,

$$(v_*)_{x,\bar{\lambda}(x)} \leq v_* \text{ in } \mathbb{R}^n \setminus (B_{\bar{\lambda}(x)}(x) \cup \{0\}) \text{ for all } x \in \mathbb{R}^n \setminus \{0\}.$$

We have a dichotomy:

$$\text{either } \bar{\lambda}(x) = |x| \text{ for all } x \in \mathbb{R}^n \setminus \{0\}, \tag{54}$$

$$\text{or } \bar{\lambda}(x_0) < |x_0| \text{ for some } x_0 \in \mathbb{R}^n \setminus \{0\}. \tag{55}$$

In case (54), we obtain that v_* is radially symmetric about the origin thanks to Lemma C.1. To finish the proof, we assume in the rest of the argument that (55) holds and derive a contradiction.

We first collect some properties of $\bar{\lambda}(x)$. We start with an analogue of Lemma 2.3. By (46), let

$$\alpha := \liminf_{|y| \rightarrow \infty} |y|^{n-2} v_*(y) \in (0, \infty].$$

Lemma 2.6 *Under the hypotheses of Theorem 1.2, if $\bar{\lambda}(x) < |x|$ for some $x \in \mathbb{R}^n \setminus \{0\}$, then*

$$\alpha = \lim_{|y| \rightarrow \infty} |y|^{n-2} (v_*)_{x,\bar{\lambda}(x)}(y) = \bar{\lambda}(x)^{n-2} v_*(x) < \infty.$$

Proof We adapt Step 1 in the proof of Lemma 2.3. Assume that $\bar{\lambda}(x) < |x|$ and (without loss of generality) that $\bar{\lambda}_k(x) \rightarrow \bar{\lambda}(x)$. Arguing as before but using the strong maximum principle for solutions with isolated singularities [10, Theorem 1.6] instead of the standard strong maximum principle, this leads to the existence of some $y_k \in \partial B_{R_k}(0)$ such that

$$(v_k)_{x,\bar{\lambda}_k(x)}(y_k) = v_k(y_k).$$

It follows that

$$\lim_{k \rightarrow \infty} |y_k|^{n-2} M_k v_k(y_k) = \lim_{k \rightarrow \infty} |y_k|^{n-2} (M_k v_k)_{x,\bar{\lambda}_k(x)}(y_k) = (\bar{\lambda}(x))^{n-2} v_*(x).$$

This implies, in view of (47), that

$$\alpha \leq \lim_{k \rightarrow \infty} |y_k|^{n-2} v_*(y_k) = \bar{\lambda}(x)^{n-2} v_*(x) = \lim_{|y| \rightarrow \infty} |y|^{n-2} (v_*)_{x,\bar{\lambda}(x)}(y) < \infty.$$

On the other hand, as in the proof of Lemma 2.3, we can use $(v_*)_{x,\bar{\lambda}(x)} \leq v_*$ in $\mathbb{R}^n \setminus (B_{\bar{\lambda}(x)}(x) \cup \{0\})$ to show that

$$\alpha \geq \bar{\lambda}(x)^{n-2} v_*(x).$$

The conclusion is readily seen. □

Lemma 2.7 *Under the hypotheses of Theorem 1.2, if $\bar{\lambda}(x_0) < |x_0|$ for some $x_0 \in \mathbb{R}^n \setminus \{0\}$, then*

$$\limsup_{x \rightarrow x_0} \bar{\lambda}(x) \leq \bar{\lambda}(x_0).$$

Proof Along a subsequence, we have $\bar{\lambda}_k(x_0) \rightarrow \bar{\lambda}(x_0)$.

As in the proof of Lemma 2.6, there exists $y_k \in \partial B_{R_k}(0)$ such that

$$(v_k)_{x_0, \bar{\lambda}_k(x_0)}(y_k) = v_k(y_k). \tag{56}$$

We know that

$$\eta_k := \sup_{B_{|x_0|/2}(x_0)} |M_k v_k - v_*| \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Let m denote the modulus of continuity of v_* in $B_{|x_0|/2}(x_0)$, i.e.

$$m(r) = \sup \left\{ |v_*(x) - v_*(y)| : x, y \in B_{|x_0|/2}(x_0), |x - y| < r \right\}.$$

In the computation below, we use $o(1)$ to denote quantities such that

$$\lim_{k \rightarrow \infty} o(1) = 0.$$

Fix some $\delta > 0$ and consider $|x - x_0| < |x_0|/2$. We note that

$$\left| \left(x + \frac{(\bar{\lambda}_k(x_0) + \delta)^2(y_k - x)}{|y_k - x|^2} \right) - \left(x_0 + \frac{\bar{\lambda}_k(x_0)^2(y_k - x_0)}{|y_k - x_0|^2} \right) \right| = |x - x_0| + o(1).$$

Thus,

$$\left| v_* \left(x + \frac{(\bar{\lambda}_k(x_0) + \delta)^2(y_k - x)}{|y_k - x|^2} \right) - v_* \left(x_0 + \frac{\bar{\lambda}_k(x_0)^2(y_k - x_0)}{|y_k - x_0|^2} \right) \right| \leq m(|x - x_0| + o(1)).$$

It follows that

$$\begin{aligned} M_k(v_k)_{x, \bar{\lambda}_k(x_0) + \delta}(y_k) &= \left(\frac{\bar{\lambda}_k(x_0) + \delta}{|y_k - x|} \right)^{n-2} (M_k v_k) \left(x + \frac{(\bar{\lambda}_k(x_0) + \delta)^2(y_k - x)}{|y_k - x|^2} \right) \\ &\geq \left(\frac{\bar{\lambda}_k(x_0) + \delta}{|y_k - x|} \right)^{n-2} \left[(M_k v_k) \left(x_0 + \frac{\bar{\lambda}_k(x_0)^2(y_k - x_0)}{|y_k - x_0|^2} \right) \right. \\ &\quad \left. - 2\eta_k - m(|x - x_0| + o(1)) \right] \\ &= \left(1 + \frac{\delta}{\bar{\lambda}_k(x_0)} \right)^{n-2} M_k(v_k)_{x_0, \bar{\lambda}_k(x_0)}(y_k) \\ &\quad - \left(\frac{\bar{\lambda}_k(x_0) + \delta}{|y_k - x|} \right)^{n-2} [2\eta_k + m(|x - x_0| + o(1))]. \end{aligned}$$

Recalling (56), we arrive at

$$\begin{aligned} M_k(v_k)_{x, \bar{\lambda}_k(x_0) + \delta}(y_k) &\geq \left(1 + \frac{\delta}{\bar{\lambda}_k(x_0)} \right)^{n-2} M_k v_k(y_k) \\ &\quad - \left(\frac{\bar{\lambda}_k(x_0) + \delta}{|y_k|} \right)^{n-2} [o(1) + m(|x - x_0| + o(1))]. \end{aligned}$$

Thus, in view of (48), we can find small $\bar{\epsilon} > 0$ depending only on $\delta, c, \bar{\lambda}(x_0)$ and the function $m(\cdot)$ such that, for all $|x - x_0| < \bar{\epsilon}$ and for large k ,

$$M_k(v_k)_{x, \bar{\lambda}_k(x_0) + \delta}(y_k) \geq \left(1 + \frac{\delta}{4\bar{\lambda}_k(x_0)}\right)^{n-2} M_k v_k(y_k).$$

This implies that (cf. (56)), that

$$\bar{\lambda}_k(x) \leq \bar{\lambda}_k(x_0) + \delta \text{ for all } |x - x_0| < \bar{\epsilon} \text{ and large } k.$$

The conclusion follows. □

We now return to drawing a contradiction from (55). By Lemma 2.7, we infer from (55) that there exists some $r_0 > 0$ such that $\bar{\lambda}(x) < |x|$ for all $x \in B_{r_0}(x_0)$. We can then argue as in the proof of Theorem 1.1, using Lemma 2.6 instead of Lemma 2.3 to obtain

$$v_*(x) = \left(\frac{a}{1 + b^2|x - \bar{x}|^2}\right)^{\frac{n-2}{2}} \quad x \in B_{r_0}(x_0).$$

for some $\bar{x} \in \mathbb{R}^n$ and some $a, b > 0$. For small $\delta > 0$, let

$$v_*^\delta(x) := v_*(x) + \delta|x - x_0|^2.$$

Since $M_k v_* \rightarrow v_*$ in $C^0(\bar{B}_\delta(x_0))$, there exists $\beta_k \rightarrow 0$ and $x_k \rightarrow x_0$ such that the function $\xi_{k,\delta} := v_*^\delta + \beta_k$ satisfies

$$(M_k v_k - \xi_{k,\delta})(x_k) = 0 \text{ and } M_k v_k - \xi_{k,\delta} \leq 0 \text{ near } x_k.$$

It follows that

$$A^{v_k}(x_k) \geq A^{\frac{1}{M_k} \xi_{k,\delta}}(x_k) = M_k^{\frac{4}{n-2}} A^{\xi_{k,\delta}}(x_k).$$

On the other hand, by hypothesis, there is some $\lambda_* \in \Gamma$ such that $f(\lambda_*) = 1$ (e.g. $\lambda_* = \lambda(A^{v_1}(0))$). By (15), we can find $\hat{\lambda}_* \in \Gamma$ such that $f(\hat{\lambda}_*) > 1$. As $M_k \rightarrow \infty$ and $A^{\xi_{k,\delta}}(x_k) = 2b^2 a^{-2} I + O(\delta)$, we can find k sufficiently large such that $M_k^{\frac{4}{n-2}} A^{\xi_{k,\delta}}(x_k) > \text{diag}(\hat{\lambda}_*)$. We are thus led to

$$A^{v_k}(x_k) > \text{diag}(\hat{\lambda}_*).$$

As $f(\lambda(A^{v_k})) = 1$ and $f(\hat{\lambda}_*) > 1$, the above contradicts (14) and (15). □

3 Local gradient estimates

In this section, we adapt the argument in [12] to prove Theorem 1.6.

For a locally Lipschitz function w in $B_2(0)$, $0 < \alpha < 1$, $x \in B_2(0)$ and $0 < \delta < 2 - |x|$, define

$$[w]_{\alpha,\delta}(x) = \sup_{0 < |y-x| < \delta} \frac{|w(y) - w(x)|}{|y - x|^\alpha},$$

$$\delta(w, x, \alpha) = \begin{cases} \infty & \text{if } (2 - |x|)^\alpha [w]_{\alpha, 2-|x|}(x) < 1, \\ \mu & \text{where } 0 < \mu \leq 2 - |x| \text{ and } \mu^\alpha [w]_{\alpha,\mu}(x) = 1 \\ & \text{if } (2 - |x|)^\alpha [w]_{\alpha, 2-|x|}(x) \geq 1. \end{cases}$$

Note that $\delta(w, x, \alpha)$ is well defined as $[w]_{\alpha,\delta}(x)$ is continuous and non-decreasing in δ . The object $\delta(w, x, \alpha)$ was introduced in [12]. Its reciprocal $\delta(w, x, \alpha)^{-1}$ plays a role similar to

that of $|\nabla w(x)|$ in performing a rescaling argument for a sequence of functions blowing up in C^α -norms. For example, when $\delta = \delta(w, x, \alpha) < \infty$, the rescaled function $\hat{w}(y) := w(x + \delta y) - w(x)$ satisfies

$$\hat{w}(0) = 0 \text{ and } [\hat{w}]_{\alpha,1}(0) = \delta^\alpha [\hat{w}]_{\alpha,\delta}(x) = 1.$$

Proof of Theorem 1.6 By the conformal invariance (7), it suffices to show bound $|\nabla \ln v|$ in $B_{1/4}(0)$.

We first claim that

$$\sup_{x \neq y \in B_{1/2}(0)} \frac{|\ln v(x) - \ln v(y)|}{|x - y|^\alpha} \leq C(\Gamma, \alpha) \text{ for any } 0 < \alpha < 1. \tag{57}$$

Assume otherwise that (57) fails for some $0 < \alpha < 1$. Then there exist $0 < v_i \in C^2(B_2(0))$ such that $f(\lambda(A^{v_i})) = 1$ and $v_i \leq b$ in $B_2(0)$ but

$$\sup_{x \neq y \in B_{1/2}(0)} \frac{|\ln v_i(x) - \ln v_i(y)|}{|x - y|^\alpha} \rightarrow \infty.$$

This implies that, for any fixed $0 < r < 1/2$,

$$\sup_{x \in B_{1/2}(0)} [\ln v_i]_{\alpha,r}(x) \rightarrow \infty \text{ and } \inf_{x \in B_{1/2}(0)} \delta(\ln v_i, x, \alpha) \rightarrow 0.$$

Therefore, there exists $x_i \in B_1(0)$,

$$\frac{1 - |x_i|}{\delta(\ln v_i, x_i, \alpha)} = \sup_{x \in B_1(0)} \frac{1 - |x|}{\delta(\ln v_i, x, \alpha)} \rightarrow \infty.$$

Let $\sigma_i = \frac{1 - |x_i|}{2}$ and $\epsilon_i = \delta(\ln v_i, x_i, \alpha)$. Then

$$\frac{\sigma_i}{\epsilon_i} \rightarrow \infty, \epsilon_i \rightarrow 0, \text{ and } \epsilon_i \leq 2 \delta(\ln v_i, z, \alpha) \text{ for any } |z - x_i| \leq \sigma_i. \tag{58}$$

We now define

$$\hat{v}_i(y) = \frac{1}{v_i(x_i)} v_i(x_i + \epsilon_i y) \text{ for } |y| \leq \frac{\sigma_i}{\epsilon_i}.$$

Then

$$[\ln \hat{v}_i]_{\alpha,1}(0) = \epsilon_i^\alpha [\ln v_i]_{\alpha,\epsilon_i}(x_i) = 1. \tag{59}$$

Also, by (58), for any fixed $\beta > 1$ and $|y| < \beta$, there holds

$$\begin{aligned} [\ln \hat{v}_i]_{\alpha,1}(y) &= \epsilon_i^\alpha [\ln v_i]_{\alpha,\epsilon_i}(x_i + \epsilon_i y) \\ &\leq 2^{-\alpha} \epsilon_i^\alpha \left\{ \sup_{|z - (x_i + \epsilon_i y)| \leq \epsilon_i} [\ln v_i]_{\alpha,\epsilon_i/4}(z) + [\ln v_i]_{\alpha,\epsilon_i/4}(x_i + \epsilon_i y) \right\} \\ &\leq \sup_{|z - (x_i + \epsilon_i y)| \leq \epsilon_i} \delta(\ln v_i, z, \alpha)^\alpha [\ln v_i]_{\alpha,\delta(\ln v_i, z, \alpha)}(z) \\ &\quad + \delta(\ln v_i, x_i + \epsilon_i y, \alpha)^\alpha [\ln v_i]_{\alpha,\delta(\ln v_i, x_i + \epsilon_i y, \alpha)}(x_i + \epsilon_i y) \\ &= 2 \end{aligned} \tag{60}$$

for all sufficiently large i . Since $\hat{v}_i(0) = 1$ by definition, we deduce from (59) and (60) that

$$\frac{1}{C(\beta)} \leq \hat{v}_i(y) \leq C(\beta) \text{ for } |y| \leq \beta \text{ and all sufficiently large } i. \tag{61}$$

We can now apply Theorem 2.1 to obtain

$$|\nabla \ln \hat{v}_i| \leq C(\beta) \text{ in } B_{\beta/2}(0) \text{ for all sufficiently large } i. \tag{62}$$

Passing to a subsequence and recalling (58) and (61), we see that \hat{v}_i converges in $C^{0,\alpha'}$ ($\alpha < \alpha' < 1$) on compact subsets of \mathbb{R}^n to some positive, locally Lipschitz function v_* .

On the other hand, if we define

$$\bar{v}_i(y) = \epsilon_i^{-\frac{n-2}{2}} v_i(x_i + \epsilon_i y) \text{ for } |y| \leq \frac{\sigma_i}{\epsilon_i},$$

then by the conformal invariance (7), we have

$$f(\lambda(A\bar{v}_i)) = 1 \text{ in } B_{\sigma_i/\epsilon_i}(0).$$

Since $\frac{\sigma_i}{\epsilon_i} \rightarrow \infty$, $\hat{v}_i = M_i \bar{v}_i$ where $M_i = v_i(x_i)^{-1} \epsilon_i^{-\frac{n-2}{2}} \rightarrow \infty$ (thanks to the bound $v_i \leq b$), we then conclude from Theorem 1.2 that v_* is constant, namely

$$v_* \equiv v_*(0) = \lim_{i \rightarrow \infty} \hat{v}_i(0) = 1.$$

This contradicts (59), in view of (62) and the convergence of \hat{v}_i to v_* . We have proved (57).

From (57), we can find some universal constant $C > 1$ such that

$$\frac{u(0)}{C} \leq u \leq C u(0) \text{ in } B_{1/2}(0).$$

Applying Theorem 2.1 again we obtain the required gradient estimate in $B_{1/4}(0)$. □

4 Fine blow-up analysis

4.1 A quantitative centered Liouville-type result

In this subsection, we establish:

Proposition 4.1 *Let (f, Γ) satisfy (13)–(15), (18)–(19), (5) and the normalization condition (21). Assume that for a sequence $R_k \rightarrow \infty$, $0 < v_k \in C^2(B_{R_k})$ satisfy*

$$f(\lambda(A^{v_k}))(y) = 1, \quad 0 < v_k(y) \leq v_k(0) = 1, \quad |y| \leq R_k. \tag{63}$$

Then for every $\epsilon > 0$, there exists a constant $\delta_0 > 0$, depending only on (f, Γ) and ϵ , such that, for all sufficiently large k ,

$$|v_k(y) - U(y)| \leq 2\epsilon U(y), \quad \forall |y| \leq \delta_0 R_k. \tag{64}$$

Recall that $U = (1 + |x|^2)^{-\frac{n-2}{2}}$, $A^U \equiv 2I$ and $f(\lambda(A^U)) = 1$ on \mathbb{R}^n .

Proposition 4.1 is equivalent to the following proposition.

Proposition 4.2 *Let (f, Γ) satisfy (13)–(15), (18)–(19), (5) and the normalization condition (21). For any $\epsilon > 0$ there exist $\delta_0, C_0 > 0$ depending only on (f, Γ) and ϵ such that if $0 < u \in C^2(B_R(0))$, $R > 0$, satisfies*

$$f(\lambda(A^u)) = 1 \text{ in } B_R(0) \text{ and } u(0) = \sup_{B_R(0)} u \geq C_0 R^{-\frac{n-2}{2}},$$

then

$$|u(x) - U^{0,u(0)}(x)| \leq 2\epsilon U^{0,u(0)}(x) \text{ for all } x \in B_{\delta_0 R}(0).$$

Proof of the equivalence between Propositions 4.1 and 4.2 It is clear that Proposition 4.2 implies Proposition 4.1.

Consider the converse. Let $\delta_0 = \delta_0(\epsilon)$ be as in Proposition 4.1. Arguing by contradiction, we assume that there are some $\epsilon > 0$ and a sequence of R_k and $u_k \in C^2(B_{R_k}(0))$ such that

$$f(\lambda(A^{u_k})) = 1 \text{ in } B_{R_k}(0) \text{ and } u_k(0) = \sup_{B_{R_k}(0)} u_k \geq k R_k^{-\frac{n-2}{2}}$$

but the last estimate in Proposition 4.2 fails for each k .

Define

$$\bar{u}_k(y) = \frac{1}{u_k(0)} u_k\left(\frac{y}{u_k(0)^{\frac{2}{n-2}}}\right) \text{ for } |y| \leq R_k u_k(0)^{\frac{2}{n-2}} =: \bar{R}_k.$$

Then $f(\lambda(A^{\bar{u}_k})) = 1$ in $B_{\bar{R}_k}(0)$, $\sup_{B_{\bar{R}_k}(0)} \bar{u}_k = \bar{u}_k(0) = 1$, and $\bar{R}_k \geq k^{\frac{2}{n-2}} \rightarrow \infty$. By Proposition 4.1,

$$|\bar{u}_k(y) - U(y)| \leq 2\epsilon U(y) \text{ in } B_{\delta_0 \bar{R}_k}(0) \text{ for all sufficiently large } k.$$

Returning to the original sequence u_k we arrive at a contradiction. □

Lemma 4.1 *Under the hypotheses of Proposition 4.1 except for (5), we have*

$$v_k \rightarrow U, \quad \text{in } C_{loc}^\beta(\mathbb{R}^n), \quad \forall 0 < \beta < 1. \tag{65}$$

Moreover, for every $\epsilon > 0$, there exists $k_0 \geq 1$ such that

$$\min_{|y|=r} v_k(y) \leq (1 + \epsilon)U(r), \quad \forall 0 < r < R_k/5, k \geq k_0. \tag{66}$$

Proof We first prove (65). Since v_k satisfies (63), we deduce from Theorem 1.6 that

$$|\nabla \ln v_k| \leq C \text{ in } B_{R_k-1}, \quad \forall k,$$

where C is independent of k . This yields (65) in view of Theorem 1.1.

We now prove (66). Suppose the contrary, then there exists some $\epsilon > 0$ and sequences of $k_i \rightarrow \infty, 0 < r_i < R_{k_i}/5$ such that

$$v_{k_i} > (1 + \epsilon)U \text{ on } \partial B_{r_i}. \tag{67}$$

Because of (65), $r_i \rightarrow \infty$.

As in the proof of Lemma 2.2, there exists $\lambda_i^{(0)} > 0$ such that

$$(v_{k_i})_\lambda \leq v_{k_i} \text{ in } B_{r_i} \setminus B_\lambda, \forall 0 < \lambda < \lambda_i^{(0)} \text{ and } |x| \leq r_i. \tag{68}$$

By the explicit expression of U , there exists some small $\delta > 0$ independent of i such that, for large i ,

$$U_\lambda(y) \leq (1 + \frac{\epsilon}{4})U(y), \quad \forall y \in \partial B_{r_i}, \lambda_i^{(0)} \leq \lambda \leq 1 + \delta,$$

By the uniform convergence of v_{k_i} to U on compact subsets of \mathbb{R}^n , we have, for large i ,

$$(v_{k_i})_\lambda \leq (1 + \frac{\epsilon}{2})U(y), \quad \forall y \in \partial B_{r_i}, \lambda_i^{(0)} \leq \lambda \leq 1 + \delta,$$

As in the proof of Lemma 2.2, the moving sphere procedure does not stop before reaching $\lambda = 1 + \delta$, namely we have, for large i ,

$$(v_{k_i})_\lambda \leq v_{k_i} \text{ in } B_{r_i} \setminus B_\lambda, \forall 0 < \lambda < 1 + \delta \text{ and } |x| \leq r_i.$$

Sending i to ∞ leads to

$$U_{1+\delta}(y) \leq U(y), \quad \forall 1 + \delta \leq |y| \leq 2.$$

A contradiction—since we see from the explicit expression of U that $U_{1+\delta}(y) > U(y)$ for all $1 < 1 + \delta < |y| \leq 2$. □

Lemma 4.2 *Under the hypotheses of Proposition 4.1, for any $\epsilon > 0$, there exist a small $\delta_1 > 0$ and a large $r_1 > 1$, depending only on (f, Γ) and ϵ , such that, for all sufficiently large k ,*

$$v_k(y) \geq (1 - \epsilon)U(y), \quad \forall |y| \leq \delta_1 R_k, \tag{69}$$

$$\text{and } \int_{r_1 \leq |y| \leq \delta_1 R_k} v_k^{\frac{n+2}{n-2}} \leq \epsilon. \tag{70}$$

Proof Assume without loss of generality that $\epsilon \in (0, 1/2)$. Since $v_k \rightarrow U$ in $C_{loc}^0(\mathbb{R}^n)$, there exist $r_2 > 1$ and k_1 , depending on ϵ , such that for all $k \geq k_1$

$$v_k(y) \geq (1 - \epsilon^2)U(y), \quad \forall |y| \leq r_2, \tag{71}$$

$$v_k(y) \geq (1 - \epsilon^2)U(r_2) \geq (1 - 2\epsilon^2)r_2^{2-n}, \quad \forall |y| = r_2. \tag{72}$$

By (5),

$$\text{Trace}(A^{v_k}) \geq \delta > 0,$$

and therefore

$$-\Delta v_k(y) \geq \frac{n-2}{2} \delta v_k(y)^{\frac{n+2}{n-2}} \quad \text{in } r_2 \leq |y| \leq R_k. \tag{73}$$

Using the superharmonicity of v_k and the maximum principle, we obtain

$$v_k(y) \geq (1 - \epsilon^2) \left(|y|^{2-n} - R_k^{2-n} \right), \quad r_2 \leq |y| \leq R_k.$$

Thus, for any $\delta_2 \in (0, \epsilon^{\frac{2}{n-2}})$, we have for all sufficiently large k that

$$v_k(y) \geq (1 - \epsilon^2)(1 - \delta_2^{n-2})|y|^{2-n} \geq (1 - 2\epsilon^2)|y|^{2-n}, \quad r_2 \leq |y| \leq \delta_2 R_k. \tag{74}$$

Now if $\delta_1 < \delta_2$, (69) is readily seen from (71) and (74).

Let

$$\hat{v}_k(y) := v_k(y) - (1 - 2\epsilon^2)|y|^{2-n}.$$

Then

$$-\Delta \hat{v}_k(y) \geq \hat{f}(y) := \frac{n-2}{2} \delta v_k(y)^{\frac{n+2}{n-2}} \quad \text{in } r_2 \leq |y| \leq \delta_2 R_k,$$

and

$$\hat{v}_k(y) \geq 0, \quad \text{for } y \in \partial(B_{\delta_2 R_k} \setminus B_{r_2}).$$

Let $R'_k = \frac{\delta_2 R_k}{2}$. Enlarging k_1 if necessary, we can apply Corollary A.3 in ‘‘Appendix A’’ to get

$$\min_{|x|=R'_k} \hat{v}_k(x) \geq C^{-1}(\delta_2 R_k)^{2-n} \int_{2r_2 \leq |y| \leq \delta_2 R_k/8} \delta v_k(y)^{\frac{n+2}{n-2}} dy, \quad \forall k \geq k_1, \tag{75}$$

where here and below C is some positive constant depending only on n . On the other hand, by Lemma 4.1, we have (after enlarging k_1 if necessary)

$$\min_{|x|=R'_k} v_k(x) \leq (1 + \epsilon^2)U(R'_k) \leq (1 + 2\epsilon^2)(R'_k)^{2-n}, \quad \forall k \geq k_1,$$

which implies that

$$\min_{|x|=R'_k} \hat{v}_k(x) \leq C \epsilon^2 (\delta_2 R_k)^{2-n}, \quad \forall k \geq k_1. \tag{76}$$

It now follows from (75) and (76) that

$$\int_{2r_2 \leq |y| \leq \delta_2 R_k/8} \delta v_k(y)^{\frac{n+2}{n-2}} dy \leq c_1 \epsilon^2$$

where c_1 depends only on n . (70) is then established for $\epsilon \leq \frac{1}{c_1}$ with $r_1 = 2r_2$ and $\delta_1 = \delta_2/8$. The conclusion for $\epsilon > 1/c_1$ also follows. \square

Lemma 4.3 *Let (f, Γ) satisfy (13)–(15). Then there exist $\delta_3 > 0$ and $C_3 > 1$, depending only on (f, Γ) , such that if $u \in C^2(B_2(0))$ satisfies*

$$f(\lambda(A^u)) = 1, u > 0, \quad \text{in } B_2(0),$$

and

$$\int_{B_2(0)} u^{\frac{2n}{n-2}} \leq \delta_3,$$

then

$$u \leq C_3 \quad \text{in } B_1(0).$$

If (f, Γ) satisfies in addition the conditions (18), (19) and the normalization condition (21), then δ_3 can be chosen to be any constant smaller than $\int_{\mathbb{R}^n} U^{\frac{2n}{n-2}} dx$.

Proof We adapt the proof of [7, Lemma 6.4]. Arguing by contradiction, we can find a sequence of $0 < u_j \in C^2(B_2)$ such that $f(\lambda(A^{u_j})) = 1$ in $B_2(0)$,

$$\int_{B_2(0)} u_j^{\frac{2n}{n-2}} \rightarrow 0$$

but

$$d(y_j)^{\frac{n-2}{2}} u_j(y_j) = \max_{\bar{B}_{3/2}(0)} d(y)^{\frac{n-2}{2}} u_j(y) \rightarrow \infty,$$

where $y_j \in B_{3/2}(0)$ and $d(y) = 3/2 - |y|$.

Let $\sigma_j = \frac{1}{2}d(y_j) > 0$,

$$v_j(z) = \frac{1}{u_j(y_j)} u_j\left(y_j + \frac{1}{u_j(y_j)^{\frac{2}{n-2}}} z\right) \text{ for } |z| < r_j := u_j(y_j)^{\frac{2}{n-2}} \sigma_j \rightarrow \infty.$$

Then by the conformal invariance property (7), $f(\lambda(A^{v_j})) = 1$ in $B_{r_j}(0)$, $v_j(0) = 1$, $v_j \leq 2^{\frac{n-2}{2}}$ in $B_{r_j}(0)$ and

$$\int_{B_{r_j}(0)} v_j^{\frac{2n}{n-2}} \rightarrow 0. \tag{77}$$

By Theorem 1.6, there is a constant C independent of j such that

$$|\nabla \ln v_j| \leq C \text{ in } B_{r_j/2}(0).$$

Thus, after passing to a subsequence, we can assume that v_j converges in $C_{loc}^0(\mathbb{R}^n)$ to some positive function v (as $v_j(0) = 1$). This contradicts (77).

The above argument can be adapted to prove the last assertion of the lemma: Eq. (77) is replaced by

$$\int_{B_{r_j}(0)} v_j^{\frac{2n}{n-2}} \leq \delta_3 < \int_{\mathbb{R}^n} U^{\frac{2n}{n-2}} dx.$$

On the other hand, by Theorem 1.1, we have $v_j \rightarrow U$ in $C_{loc}^0(\mathbb{R}^n)$. This gives a contradiction. \square

Lemma 4.4 *Let (f, Γ) satisfy (13)–(15) and let δ_3, C_3 be as in Lemma 4.3. If $u \in C^2(B_{2R}(0))$ satisfies*

$$f(\lambda(A^u)) = 1, u > 0, \quad \text{in } B_{2R}(0),$$

and

$$\int_{B_{2R}(0)} u^{\frac{2n}{n-2}} \leq \delta_3,$$

then

$$u \leq C_3 R^{-\frac{n-2}{2}} \text{ in } B_R(0).$$

Proof This follows from Lemma 4.3 and a change of variables, $\tilde{u}(y) = R^{\frac{n-2}{2}} u(Ry)$ for $|y| \leq 2$. \square

Lemma 4.5 *Under the hypotheses of Proposition 4.1, there exist positive constants $\delta_4 > 0$ and $C_4 > 1$, depending only on (f, Γ) , such that, for all sufficiently large k ,*

$$v_k(y) \leq C_4 U(y), \quad \forall |y| \leq \delta_4 R_k. \tag{78}$$

Proof Let δ_3 be as in Lemma 4.3. Since $v_k \leq 1$, we deduce from Lemma 4.2, there is $r_1 > 1$ and $\delta_1 > 0$ such that

$$\int_{r_1 \leq |y| \leq \delta_1 R_k} v_k^{\frac{2n}{n-2}} \leq \epsilon. \tag{79}$$

For any $2r_1 < r < \delta_1 R_k/2$, consider

$$\tilde{v}_k(z) = r^{\frac{n-2}{2}} v_k(rz), \quad \frac{1}{2} < |z| < 2.$$

By (79), we have, for large k ,

$$\int_{\frac{1}{2} < |z| < 2} \tilde{v}_k(z)^{\frac{2n}{n-2}} = \int_{\frac{r}{2} < |\eta| < 2r} v_k(\eta)^{\frac{2n}{n-2}} \leq \delta_3.$$

It follows from Lemma 4.4 that

$$\tilde{v}_k(z) \leq C \quad \forall \frac{2}{3} < |z| < \frac{5}{4},$$

for some universal constant C . Since \tilde{v}_k also satisfies $f(\lambda(A^{\tilde{v}_k})) = 1$, we can apply Theorem 1.6 to obtain

$$|\nabla \ln \tilde{v}_k(z)| \leq C \quad \forall |z| = 1,$$

which implies that $\max_{|z|=1} \tilde{v}_k \leq C \min_{\partial B_1} \tilde{v}_k$. Returning to v_k , we obtain

$$\max_{\partial B_r} v_k \leq C \min_{\partial B_r} v_k$$

where C is universal. The conclusion then follows from Lemma 4.1. □

Proof of Proposition 4.1 Fix $\epsilon > 0$. In view of Lemma 4.2 (cf. (69)), we only need to prove that there exist $\delta_0 > 0$ such that, for all sufficiently large k ,

$$v_k(y) \leq (1 + 2\epsilon)U(y), \quad \forall |y| \leq \delta_0 R_k. \tag{80}$$

Suppose the contrary of the above, then, after passing to a subsequence and renaming the subsequence still as $\{v_k\}$ and $\{R_k\}$, there exist $|y_k| = \delta_k R_k, \delta_k \rightarrow 0^+$, such that

$$v_k(y_k) = \max_{|y|=\delta_k R_k} v_k(y) > (1 + 2\epsilon)U(y_k). \tag{81}$$

In view of the convergence of v_k to $U, |y_k| \rightarrow \infty$ as $k \rightarrow \infty$.

Consider the following two rescalings of v_k :

$$\hat{v}_k(z) := |y_k|^{n-2} v_k(|y_k|z) \text{ and } \bar{v}_k(z) = |y_k|^{\frac{n-2}{2}} v_k(|y_k|z), \quad |z| < \frac{R_k}{|y_k|} \rightarrow \infty. \tag{82}$$

By Lemma 4.5, we have

$$\hat{v}_k(z) \leq C|z|^{2-n} \text{ and } \bar{v}_k(z) \leq C|y_k|^{-\frac{n-2}{2}}|z|^{2-n} \tag{83}$$

for some constant C independent of k .

In view of the conformal invariance (7) and (63),

$$f(\lambda(A^{\bar{v}_k}(z))) = 1 \text{ for } |z| < \frac{R_k}{|y_k|}.$$

Recalling (83), we can apply Theorem 1.6 to obtain that for all $0 < \alpha < \beta < \infty$, there exists positive constant $C(\alpha, \beta)$ such that for large k ,

$$|\nabla \ln \bar{v}_k(z)| \leq C(\alpha, \beta), \quad \forall \alpha < |z| < \beta, \tag{84}$$

which implies that

$$|\nabla \ln \hat{v}_k(z)| \leq C(\alpha, \beta), \quad \forall \alpha < |z| < \beta. \tag{85}$$

We know from (82), (81) and Lemma 4.1 that

$$\min_{|z|=1} \hat{v}_k(z) \leq (1 + \epsilon) \frac{|y_k|^{n-2}}{U(y_k)}, \tag{86}$$

and

$$\max_{|z|=1} \hat{v}_k(z) \geq (1 + 2\epsilon) \frac{|y_k|^{n-2}}{U(y_k)}. \tag{87}$$

We deduce from (85), (86) and (87), after passing to a subsequence, that for some positive function \hat{v}^* in $C_{loc}^{0,1}(\mathbb{R}^n \setminus \{0\})$,

$$\hat{v}_k \rightarrow \hat{v}^* \quad \text{in } C_{loc}^\alpha(\mathbb{R}^n \setminus \{0\}), \quad \forall 0 < \alpha < 1. \tag{88}$$

By Theorem 1.2, \hat{v}^* is radially symmetric. On the other hand, we deduce from (86) and (87) after passing to limit that

$$\min_{|z|=1} \hat{v}^*(z) \leq 1 + \epsilon, \quad \text{and} \quad \max_{|z|=1} \hat{v}^*(z) \geq 1 + 2\epsilon. \tag{89}$$

The above violates the radial symmetry of \hat{v}^* . Proposition 4.1 is established. □

4.2 Detailed blow-up landscape

The proof of Theorem 1.3 uses the following consequence of the Harnack-type inequality for conformally invariant equations, see [4, 7, 16].

Lemma 4.6 *Let (f, Γ) satisfy (13)–(15) and (5). There exists a constant C_6 , depending only on (f, Γ) , such that if $u \in C^2(B_3(0))$ is a positive solution of*

$$f(\lambda(A^u)) = 1 \text{ in } B_3(0)$$

then

$$\int_{B_1(0)} |u|^{\frac{2n}{n-2}} dx \leq C_6.$$

Proof We give the proof here for completeness. By (5),

$$-\Delta u \geq \frac{n-2}{2} \delta u^{\frac{n+2}{n-2}} > 0 \text{ in } B_2(0).$$

Thus, by Corollary A.2 in ‘‘Appendix A’’ as well as the maximum principle,

$$\inf_{B_2(0)} u = \inf_{B_2(0) \setminus B_{3/2}(0)} u \geq \frac{1}{C} \int_{B_1(0)} u^{\frac{n+2}{n-2}} dx.$$

It follows that

$$\int_{B_1(0)} u^{\frac{2n}{n-2}} dx \leq C \sup_{B_1(0)} u \inf_{B_2(0)} u.$$

The conclusion follows from the above estimate and the Harnack-type inequality [8, Theorem 1.2]. (Note that (5) is used again here.) □

Proof of Theorem 1.3 In view of Proposition 4.2 and (vi), it suffices to establish the theorem for $\epsilon = \epsilon_0 := 1/2$.

By Lemma 4.6,

$$\int_{B_2(0)} u^{\frac{2n}{n-2}} dx \leq C_6. \tag{90}$$

The constant \bar{m} in the result can be selected to be the least integer satisfying

$$\bar{m} \geq 2C_6 \left(\int_{B_1} U^{\frac{2n}{n-2}} dx \right)^{-1}.$$

(Clearly, this is an obvious upper bound for m if the x^i ’s satisfies (iii).)

Let δ_3 and C_3 be the constants in Lemma 4.4. Fix some $N_0 > \frac{C_1}{\delta_3}$. Then there is some $r_0 \in (3/2, 2)$ such that

$$\int_{r_0 < |x| < r_0 + \frac{1}{2N_0}} u^{\frac{2n}{n-2}} dx \leq \delta_3.$$

By Lemma 4.4, this implies that

$$u(x) \leq C_3 (8N_0)^{\frac{n-2}{2}} =: C_7 \text{ for all } r_0 + \frac{1}{8N_0} < |x| < r_0 + \frac{3}{8N_0}. \tag{91}$$

Let C_0 and δ_0 be as in Proposition 4.2 (corresponding to $\epsilon = \epsilon_0$). We can assume without loss of generality that

$$C_0 > 2 \text{ and } \delta_0 < 1. \tag{92}$$

We now declare

$$C_* = \max \left(2C_7, C_0(2\delta_0^{-1})^{\frac{\bar{m}(n-2)}{2}} (4N_0)^{-\frac{n-2}{2}} \right). \tag{93}$$

This choice of C_* will become clear momentarily.

Let $U_1 = B_{r_0 + \frac{3}{8N_0}}(0)$ and $V_1 = B_{r_0 + \frac{1}{8N_0}}(0) \subset U_0$. By (93), $C_* \geq 2C_7$, and so, by (91), there is some $x^1 \in V_1 \subset B_2(0)$ such that

$$u(x^1) = \sup_{U_1} u \geq C_*.$$

Let $R_1 = \frac{1}{4N_0}$, then (93) gives

$$C_* \geq C_0 R_1^{-\frac{n-2}{2}}.$$

Hence, an application of Proposition 4.2 to u on the ball $B_{R_1}(x^1)$ leads to

$$|u(x) - U^{x^1, u(x^1)}(x)| \leq \epsilon_0 U^{x^1, u(x^1)}(x), \quad \forall x \in B_{\delta_0 R_1}(x^1).$$

In particular, for $\frac{\delta_0 R_1}{2} \leq |x - x^1| \leq \delta_0 R_1$,

$$u(x) \leq 2U^{x^1, u(x^1)}(x) \leq \frac{2}{u(x^1)|x - x^1|^{n-2}} \leq \frac{2^{n-1}}{C_*(\delta_0 R_1)^{n-2}} \leq \frac{C_*}{2}, \tag{94}$$

where we have used (93) in the last estimate.

Let $U_2 = U_1 \setminus B_{\delta_0 R_1/2}(x^1)$ and $V_2 = V_1 \setminus B_{\delta_0 R_1}(x^1) \subset U_1$. If

$$\sup_{U_2} u \leq C_*,$$

we stop. Otherwise, in view of (94), there is some $x^2 \in V_2$ such that

$$u(x^2) = \sup_{U_2} u \geq C_*.$$

We then let $R_2 = \frac{\delta_0 R_1}{2}$ so that (93) implies

$$C_* \geq C_0 R_2^{-\frac{n-2}{2}}.$$

Hence, by Proposition 4.2,

$$|u(x) - U^{x^2, u(x^2)}(x)| \leq \epsilon_0 U^{x^2, u(x^2)}(x), \quad \forall x \in B_{\delta_0 R_2}(x^2).$$

We then repeat the above process to define U_3, V_3 , and to decide if a local max x^3 can be selected in V_3 , etc. As explain above, the number m of times this process can be repeated cannot exceed \bar{m} .

We have obtained the set of local maximum points $\{x^1, \dots, x^m\}$ of u and have verified (i) and (iv) for

$$\delta_* = \left(\frac{\delta_0}{2}\right)^{\bar{m}} \frac{1}{2N_0} \leq \delta_0 R_m.$$

(vi) is readily seen as

$$dist(x^i, \partial U_i) \geq R_i \geq \delta_*.$$

(ii) is also clear for

$$K \geq \left(\frac{2}{\delta_0}\right)^{\bar{m}} 4N_0 \geq \frac{2}{\delta_0 R_{m-1}}.$$

From construction, we have

$$\sup_{U_{m+1}} u \leq C_*.$$

By Theorem 1.6, this implies that

$$|\nabla \ln u(x)| \leq C_8 \text{ for all } x \in V_m = V_0 \setminus \cup_{i=1}^m B_{\delta_0 R_i}(x^i). \tag{95}$$

Also, note that, for $\delta_* < |x - x^i| < \delta_0 R_i$, we have

$$\frac{1}{u(x^i)} \left(\frac{1}{(C_*)^{-\frac{4}{n-2}} + (\delta_0 R_i)^2} \right)^{\frac{n-2}{2}} \leq U^{x^i, u(x^i)}(x) \leq \frac{1}{u(x^i)} (\delta_*)^{-(n-2)},$$

and so

$$\frac{1}{C_9 u(x^i)} \leq u(x) \leq \frac{C_9}{u(x^i)} (C_*)^{-2} (\delta_*)^{-(n-2)} \tag{96}$$

It is now clear that (iii) and (v) hold for K sufficiently large. The proof is complete. □

5 A quantitative Liouville theorem

Proof of Theorem 1.5 Assume by contradiction that, for some $\epsilon \in (0, 1/2]$, there exist $v_k \in C^2(B_{3R_k}(0), R_k \rightarrow \infty$, such that $f(\lambda(A^{v_k})) = 1$ in $B_{3R_k}(0)$ and $v_k \geq \gamma$ in $B_{r_1}(0)$ but, for each k ,

$$(24) \text{ and } (25) \text{ cannot simultaneously hold for any } \bar{x}. \tag{97}$$

Define

$$u_k(y) = R_k^{\frac{n-2}{2}} v_k(R_k y) \text{ for } |y| \leq 3.$$

Then $f(\lambda(A^{u_k})) = 1$ in $B_3(0)$ and

$$u_k \geq R_k^{\frac{n-2}{2}} \gamma \text{ in } B_{r_1/R_k}(0). \tag{98}$$

Thus, by applying Theorem 1.4 and after passing to a subsequence, we can select sets of local maximum points $\{x_k^1, \dots, x_k^m\}$ of u_k such that assertions (i)–(vi) in Theorem 1.4 hold. We can also assume that $x_k^i \rightarrow x_*^i, 1 \leq i \leq m$.

By assertions (i), (iv) and (v) of Theorem 1.4, u_k converges locally uniformly to zero in $B_1 \setminus \{x_*^1, \dots, x_*^m\}$. Thus, in view of (98), we must have $x_*^{i_0} = 0$ for some (unique) $1 \leq i_0 \leq m$. Clearly, $B_{r_1/R_k}(0) \subset B_{\delta_*}(x_k^{i_0})$ for large k .

Recalling assertions (iv), (vi) and returning to the original sequence v_k we get, for $\bar{x}_k = R_k x_k^{i_0}$ and $\bar{\mu}_k = v_k(\bar{x}_k) = \sup_{B_{\delta_* R_k}(\bar{x}_k)} v_k$, that $B_{r_1}(0) \subset B_{\delta_* R_k}(\bar{x}_k), \bar{\mu}_k \geq \gamma$ and

$$(1 - \epsilon)U^{\bar{x}_k, \bar{\mu}_k} \leq v_k \leq (1 + \epsilon)U^{\bar{x}_k, \bar{\mu}_k} \text{ in } B_{\delta_* R_k}(\bar{x}_k).$$

We then have

$$\gamma \leq v_k(0) = 2 \left(\frac{\bar{\mu}_k^{\frac{2}{n-2}}}{1 + \bar{\mu}_k^{\frac{4}{n-2}} |\bar{x}_k|^2} \right)^{\frac{n-2}{2}} \leq \frac{2}{\bar{\mu}_k |\bar{x}_k|^{n-2}} \leq \frac{2}{\gamma |\bar{x}_k|^{n-2}}$$

This implies

$$|\bar{x}_k| \leq 2^{\frac{1}{n-2}} \gamma^{-\frac{2}{n-2}}.$$

On the other hand, since $B_{r_1}(0) \setminus B_{r_1/2}(\bar{x}_k) \neq \emptyset$, we can select some $y_k \in B_{r_1}(0)$ such that

$$|\bar{x}_k - y_k| \geq \frac{r_1}{2}.$$

This implies that

$$\gamma \leq v(y_k) \leq 2U^{\bar{x}_k, \bar{\mu}_k}(y_k) \leq \frac{2}{\bar{\mu}_k |y_k - \bar{x}_k|^{n-2}} \leq \frac{2^{n-1}}{\bar{\mu}_k r_1^{n-2}},$$

and so

$$\bar{\mu}_k \leq \frac{2^{n-1}}{\gamma r_1^{n-2}}.$$

We have thus shown that $\bar{x} = \bar{x}_k$ satisfies both (24) and (25), which contradicts (97). □

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Appendix A: A remark on positive superharmonic functions

Let $G(x, y)$ be the Green’s function of $-\Delta$ in $B_1 \setminus B_\rho \subset \mathbb{R}^n, 0 < \rho < 1/2$:

$$G(x, y) := \frac{1}{n(n-2)\alpha(n)} \left(\frac{1}{|x-y|^{n-2}} - h(x, y) \right), \quad x, y \in B_1 \setminus B_\rho,$$

where $\alpha(n)$ denotes the volume of the unit ball in \mathbb{R}^n , and $h(x, y)$ satisfies, for $x \in B_1 \setminus B_\rho$,

$$\begin{cases} -\Delta_y h(x, y) = 0, & y \in B_1 \setminus B_\rho, \\ h(x, y) = \frac{1}{|x-y|^{n-2}}, & y \in \partial(B_1 \setminus B_\rho). \end{cases}$$

Lemma A.1 For any $0 < 2\rho < \rho_0 < \rho_1 < \rho_2 < 1$, there exists some constants $C, C' > 1$, depending only on $n, \rho_0, \rho_1, \rho_2$, such that the Green's function G for $B_1 \setminus B_\rho$ satisfies

$$G(x, y) \geq \frac{1}{C} \left(1 - \frac{\rho^{n-2}}{|y|^{n-2}} \right), \quad \forall \rho \leq |y| \leq \rho_0, \quad \rho_1 \leq |x| \leq \rho_2.$$

Consequently,

$$G(x, y) \geq 1/C', \quad \forall 2\rho \leq |y| \leq \rho_0, \quad \rho_1 \leq |x| \leq \rho_2.$$

Proof In the following we use C_1, C_2, \dots to denote positive constants depending only on ρ_0, ρ_1, ρ_2 and n . For a fixed x satisfying $\rho_1 \leq |x| \leq \rho_2$, it follows from the maximum principle that for some positive constant C_1 ,

$$0 < h(x, y) \leq C_1, \quad \forall y \in B_1 \setminus B_\rho.$$

It follows that for some positive constants C_2 and C_3 ,

$$G(x, y) \geq |x - y|^{2-n} - C_1 \geq 1/C_2, \quad \forall y \in B_{2/C_3}(x) \setminus \{x\} \subset B_1 \setminus B_{\rho_0}.$$

Since $G(x, y)$ is a positive harmonic function of y in $(B_1 \setminus B_\rho) \setminus \{x\}$. We can apply the Harnack inequality to obtain, for some C_4 ,

$$G(x, y) \geq 1/C_4, \quad \forall |y| = \rho_0.$$

By the maximum principle,

$$G(x, y) \geq \frac{1}{C_4} \left(1 - \frac{\rho^{n-2}}{|y|^{n-2}} \right), \quad \rho < |y| \leq \rho_0.$$

It follows that for some K ,

$$G(x, y) \geq 1/K, \quad 2\rho \leq |y| \leq \rho_0.$$

Lemma A.1 is established. \square

Corollary A.1 For any $0 < 2\rho < \rho_0 < \rho_1 < \rho_2 < 1$, let

$$\begin{cases} -\Delta \tilde{v} = \tilde{f} \geq 0, & B_1 \setminus B_\rho, \\ \tilde{v} \geq 0, & \partial(B_1 \setminus B_\rho). \end{cases}$$

Then, for some constants $C, C' > 1$ depending only on $n, \rho_0, \rho_1, \rho_2$,

$$\inf_{\rho_1 \leq |x| \leq \rho_2} \tilde{v}(x) \geq \frac{1}{C} \int_{\rho \leq |y| \leq \rho_0} \left(1 - \frac{\rho^{n-2}}{|y|^{n-2}} \right) \tilde{f}(y) dy \geq \frac{1}{C'} \int_{2\rho \leq |y| \leq \rho_0} \tilde{f}(y) dy.$$

Proof For $\rho_1 \leq |x| \leq \rho_2$, we use the Green's formula to obtain

$$\tilde{v}(x) = \int_{B_1 \setminus B_\rho} G(x, y) \tilde{f}(y) dy.$$

Corollary A.1 follows from Lemma A.1. \square

Corollary A.2 For any $0 < \rho_0 < \rho_1 < \rho_2 < 1$, let

$$\begin{cases} -\Delta \tilde{v} = \tilde{f} \geq 0, & B_1, \\ \tilde{v} \geq 0, & \partial B_1. \end{cases}$$

Then, for some constants C depending only on $n, \rho_0, \rho_1, \rho_2$,

$$\inf_{\rho_1 \leq |x| \leq \rho_2} \tilde{v}(x) \geq \frac{1}{C} \int_{|y| \leq \rho_0} \tilde{f}(y) dy.$$

Proof This follows from Corollary A.1 by sending $\rho \rightarrow 0$. □

Corollary A.3 For $0 < r < R/2$, let

$$\begin{cases} -\Delta v = f \geq 0, & B_R \setminus B_r, \\ v \geq 0, & \partial(B_R \setminus B_r). \end{cases}$$

Then, for any $\frac{2r}{R} < \rho_0 < \rho_2 < \rho_2 < 1$, there exist some constants $C, C' > 1$ depending only on $n, \rho_0, \rho_1, \rho_2$,

$$\begin{aligned} \inf_{\rho_1 R \leq |x| \leq \rho_2 R} v(x) &\geq \frac{1}{C R^{n-2}} \int_{r \leq |y| \leq \rho_0 R} \left(1 - \frac{r^{n-2}}{|y|^{n-2}}\right) f(y) dy \\ &\geq \frac{1}{C' R^{n-2}} \int_{2r \leq |y| \leq \rho_0 R} f(y) dy. \end{aligned}$$

Proof Performing a change of variables

$$\tilde{v}(x) := R^{n-2} v(Rx), \quad \tilde{f}(x) := R^n f(Rx), \quad \rho = r/R,$$

we obtain Corollary A.3 from Corollary A.1 □

Appendix B: A remark on viscosity solutions

In this section we consider the convergence of viscosity solutions in a slightly more general context. Let $\mathbb{R}^{n \times n}, \text{Sym}^{n \times n}, \text{Sym}_+^{n \times n}$ denote the set of $n \times n$ matrices, symmetric matrices, and positive definite symmetric matrices, respectively. Let $\mathcal{M} = \mathbb{R}^{n \times n}$ or $\mathcal{M} = \text{Sym}^{n \times n}$. Let $\Omega \subset \mathbb{R}^n, U \subset \mathcal{M}$ be open, $F \in C(U), A \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n \times \text{Sym}^{n \times n}; \mathcal{M})$ and consider partial differential equations for the form

$$F(A(x, u, \nabla u, \nabla^2 u)) = 0.$$

To keep the notation simple, we will abbreviate $A[u] = A(\cdot, u, \nabla u, \nabla^2 u)$, and whenever we write $F(M)$, we implicitly assume that $M \in U$.

In applications, it is frequently assumed that

$$M + N \in U \text{ for } M \in U \text{ and } N \in \text{Sym}_+^{n \times n}, \tag{99}$$

$$F(M + N) \geq F(M) \text{ for } M \in U \text{ and } N \in \text{Sym}_+^{n \times n}, \tag{100}$$

$$A(x, z, p, M) - A(x, z, p, M + N) \text{ is non-negative definite for } N \in \text{Sym}_+^{n \times n}. \tag{101}$$

In the main body of the paper,

$$\begin{aligned} \mathcal{M} &= \text{Sym}^{n \times n}, U \text{ is the set of symmetric matrices such that } \lambda(U) \in \Gamma, \\ F(M) &= f(\lambda(M)) - 1, \text{ and } A[v] = A^v. \end{aligned} \tag{102}$$

The following definition is “consistent” with the assumptions (99), (100) and (101) and with Definition 1.1.

Definition B.1 A continuous function v is an open set $\Omega \subset \mathbb{R}^n$ is a viscosity supersolution (respectively, subsolution) of

$$F(A[v]) = 0, \text{ in } \Omega,$$

when the following holds: if $x_0 \in \Omega$, $\varphi \in C^2(\Omega)$, $(v - \varphi)(x_0) = 0$, and $v - \varphi \geq 0$ near x_0 , then

$$F(A[\varphi](x_0)) \geq 0.$$

(respectively, if $(v - \varphi)(x_0) = 0$, and $v - \varphi \leq 0$ near x_0 , then either $A[\varphi](x_0) \notin U$ or $F(A[\varphi](x_0)) \leq 0$). Equivalently, we write $F(A[v]) \geq 0$ (respectively, $F(A[v]) \leq 0$) in Ω in the viscosity sense.

We say that v is a viscosity solution if it is both a viscosity supersolution and a viscosity subsolution.

Proposition B.1 Let $\Omega \subset \mathbb{R}^n$, $U \subset \mathcal{M}$ be open, $F \in C(U)$, $A \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n \times \text{Sym}^{n \times n}; \mathcal{M})$ and assume that the structural conditions (99), (100) and (101) are in effect.

- (a) If v_k satisfies $F(A[v_k]) \leq 0$ in Ω in the viscosity sense and if v_k converges in $C^0_{loc}(\Omega)$ to v , then v also satisfies $F(A[v]) \leq 0$ in Ω in the viscosity sense.
- (b) Assume further that the closure of $F^{-1}([0, \infty))$ in \mathcal{M} is a subset of U , i.e.

$$\text{if } A_k \in U, F(A_k) \geq 0 \text{ and } A_k \rightarrow A \text{ as } k \rightarrow \infty, \text{ then } A \in U. \tag{103}$$

If v_k satisfies $F(A[v_k]) \geq 0$ in Ω in the viscosity sense and if v_k converges in $C^0_{loc}(\Omega)$ to v , then v also satisfies $F(A[v]) \geq 0$ in Ω in the viscosity sense.

Remark B.1 In general, condition (103) cannot be dropped in (b).

As a first example, consider

$$F(M) = \text{tr}(M), \quad U = \{M \in \text{Sym}^{n \times n} : \text{tr}(M) > 0\},$$

$$A[v] = -\nabla^2 v.$$

Clearly, (99), (100) and (101) are satisfied. Now, if $v_k = -\frac{1}{k}|x|^2$, then $A[v_k] \in U$ but $v = \lim_{k \rightarrow \infty} v_k = 0$ does not satisfy $A[v] \in U$.

The situation does not improve even if one imposes that v_k is a solution and that U is a maximal set where the ellipticity condition (100) holds. See Remark 2.1.

Remark B.2 If (f, Γ) satisfies (1)–(4) and if (F, U) are given by (102), then all hypotheses of Proposition B.1 are met. In particular if $f(\lambda(A^{v_k})) = 1$ in some open set $\Omega \subset \mathbb{R}^n$ in the viscosity sense and v_k converges in $C^0_{loc}(\Omega)$ to v , then $f(\lambda(A^v)) = 1$ in Ω in the viscosity sense.

Proof The proof is standard and we include it here for readers’ convenience. We will only show (b). The proof of (a) is similar. Fix $x_0 \in \Omega$ and assume that $\varphi \in C^2(\Omega)$ such that $(v - \varphi)(x_0) = 0$ and $v - \varphi \geq 0$ in some small ball $B_\rho(x_0)$. We need to show that $F(A[\varphi](x_0)) \geq 0$.

For $\delta \in (0, \rho)$, let $\varphi_\delta(x) = \varphi(x) - \delta|x - x_0|^2$. Since $\varphi_\delta(x) = \varphi(x) - \delta^3 \leq v(x) - \delta^3$ for $|x - x_0| = \delta$, the convergence of v_k to v implies that

$$v_k(x) \geq \varphi_\delta(x) + \frac{1}{2}\delta^3 \text{ for all } |x - x_0| = \delta \text{ and all sufficiently large } k.$$

Thus, as $\min_{B_\delta(x_0)}(v - \varphi_\delta) = 0$, there is some $x_k \in B_\delta(x_0)$ such that

$$\beta_k := (v_k - \varphi_\delta)(x_k) = \min_{B_\delta(x_0)} (v_k - \varphi_\delta) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Also, since

$$\beta_k = (v_k - \varphi)(x_k) + \delta|x_k - x_0|^2 \geq (v_k - v)(x_k) + \delta|x_k - x_0|^2,$$

we have that $x_k \rightarrow x_0$ as $k \rightarrow \infty$.

Now since $\hat{\varphi}_k := \varphi_\delta + \beta_k$ satisfies $(v_k - \hat{\varphi}_k)(x_k) = 0$, $v_k - \hat{\varphi}_k \geq 0$ in $B_\delta(x_0)$, and since v_k is a solution of $F(A[v_k]) = 0$, we infer that

$$F(A[\hat{\varphi}_k](x_k)) \geq 0.$$

Sending $k \rightarrow \infty$ and then $\delta \rightarrow 0$ and using (103), we deduce that $A[\varphi](x_0) \in U$ and

$$F(A[\varphi](x_0)) \geq 0,$$

as desired. (Note that (103) is not required to show that (a).) □

Appendix C: A calculus lemma

We collect here a couple calculus statements which was used when we applied the method of moving spheres in the body of the paper.

Lemma C.1 *Let $f \in C^0(\mathbb{R}^n \setminus \{0\})$, $n \geq 1$, $\nu > 0$. Assume that, for all $x \in \mathbb{R}^n$ and $0 < \lambda \leq |x|$, there holds*

$$\left(\frac{\lambda}{|y-x|}\right)^\nu f\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) \leq f(y) \text{ for } y \in \mathbb{R}^n \setminus (B_\lambda(x) \cup \{0\}).$$

Then f is radially symmetric about the origin and $f : (0, \infty) \rightarrow (0, \infty)$ is non-increasing.

Proof This was established in [10]. See the argument following equation (110) therein. □

Corollary C.1 *Let $f \in C^0(\mathbb{R}^n)$, $n \geq 1$, $\nu > 0$. Assume that*

$$\left(\frac{\lambda}{|y-x|}\right)^\nu f\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) \leq f(y), \quad \forall \lambda > 0, x \in \mathbb{R}^n, |y-x| \geq \lambda.$$

Then $f \equiv \text{constant}$.

References

1. Caffarelli, L., Gidas, B., Spruck, J.: Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. *Commun. Pure Appl. Math.* **42**, 271–297 (1989)
2. Caffarelli, L., Nirenberg, L., Spruck, J.: The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian. *Acta Math.* **155**, 261–301 (1985)
3. Chang, S.-Y.A., Gursky, M.J., Yang, P.: Entire solutions of a fully nonlinear equation. In: *Lectures on Partial Differential Equations*, Vol. 2 of *New Studies in Advanced Mathematics*, pp. 43–60. International Press, Somerville (2003)
4. Chen, C.-C., Lin, C.-S.: Estimates of the conformal scalar curvature equation via the method of moving planes. *Commun. Pure Appl. Math.* **50**, 971–1017 (1997)

5. Gidas, B., Ni, W.M., Nirenberg, L.: Symmetry of positive solutions of nonlinear elliptic equations in \mathbf{R}^n . In: *Mathematical Analysis and Applications, Part A, Vol. 7 of Advanced in Mathematical Supply Studies*, pp. 369–402. Academic Press, New York (1981)
6. Guan, P., Wang, G.: Local estimates for a class of fully nonlinear equations arising from conformal geometry. *Int. Math. Res. Not.* **26**, 1413–1432 (2003)
7. Li, A., Li, Y.Y.: On some conformally invariant fully nonlinear equations. *Commun. Pure Appl. Math.* **56**, 1416–1464 (2003)
8. Li, A., Li, Y.Y.: On some conformally invariant fully nonlinear equations. II. Liouville, Harnack and Yamabe. *Acta Math.* **195**, 117–154 (2005)
9. Li, Y.Y.: Prescribing scalar curvature on S^n and related problems. I. *J. Differ. Equ.* **120**, 319–410 (1995)
10. Li, Y.Y.: Conformally invariant fully nonlinear elliptic equations and isolated singularities. *J. Funct. Anal.* **233**, 380–425 (2006)
11. Li, Y.Y.: Degenerate conformally invariant fully nonlinear elliptic equations. *Arch. Ration. Mech. Anal.* **186**, 25–51 (2007)
12. Li, Y.Y.: Local gradient estimates of solutions to some conformally invariant fully nonlinear equations. *Commun. Pure Appl. Math.* **62**, 1293–1326 (2009). (*C. R. Math. Acad. Sci. Paris* 343 (2006), no. 4, 249–252)
13. Li, Y.Y., Nguyen, L.: Harnack inequalities and Bôcher-type theorems for conformally invariant, fully nonlinear degenerate elliptic equations. *Commun. Pure Appl. Math.* **67**, 1843–1876 (2014)
14. Li, Y.Y., Zhang, L.: Liouville-type theorems and Harnack-type inequalities for semilinear elliptic equations. *J. Anal. Math.* **90**, 27–87 (2003)
15. Li, Y.Y., Zhu, M.: Uniqueness theorems through the method of moving spheres. *Duke Math. J.* **80**, 383–417 (1995)
16. Schoen, R.: *Courses at Stanford University, 1988, and New York University (1989)*