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ORIGINAL PAPER

The three-term recursion for Chebyshev polynomials is mixed forward-backward stable

Alicja Smoktunowicz · Agata Smoktunowicz · Ewa Pawelec

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Abstract This paper provides an error analysis of the three-term recurrence relation (TTRR) $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for the evaluation of the Chebyshev polynomial of the first kind $T_N(x)$ in the interval $[-1, 1]$. We prove that the computed value of $T_N(x)$ from this recurrence is very close to the exact value of the Chebyshev polynomial T_N of a slightly perturbed value of x. The lower and upper bounds for the function $C_N(x) = |T_N(x)| + |xT'_N(x)|$ are also derived. Numerical examples that illustrate our theoretical results are given.

Keywords Chebyshev polynomials · Error analysis · Roots of polynomials

Mathematics Subject Classifications (2010) 65G50 · 65D20 · 65L70

1 Introduction

This paper proves the numerical stability of the three-term recurrence relation (TTRR) of Chebyshev polynomials of the first kind $(T_n(x))$:

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \; n = 2, 3, \ldots, \tag{1}$$

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A. Smoktunowicz (✉) · E. Pawelec
Faculty of Mathematics and Information Science, University of Technology,
Koszykowa 75, Warsaw, 00-662 Poland
e-mail: smok@mini.pw.du.pl

E. Pawelec
e-mail: E.Pawelec@mini.pw.edu.pl

A. Smoktunowicz
School of Mathematics, University of Edinburgh, Edinburgh, Scotland EH9 3JZ, UK
e-mail: A.Smoktunowicz@ed.ac.uk

where $T_0(x) = 1,\ T_1(x) = x$.

This formula often appears in practical applications. The TTRR is used in Forsythe's method for evaluating polynomials in Chebyshev form $p_N(x) = \sum_{k=0}^{N} a_k T_k(x)$. There are several algorithms for evaluating Chebyshev series (see [1–5, 13]). Clenshaw's and Forsythe's algorithms are recommended. Chebyshev polynomials of the first kind ($T_n(x)$) are widely used in many applications (see [6, 8, 10, 14]). A desirable property for algorithms is numerical stability (see [12, 17]). The term *stability* is sometimes used to refer to the forward or backward analysis of the algorithm. An error analysis of Clenshaw's algorithm in the general case was first provided by D. Elliott in [9]. See also [2–5, 7, 11, 15], where the authors present the forward error bounds for the evaluation of $p_N(x) = \sum_{k=0}^{N} a_k T_k(x)$.

In this paper we study the mixed forward-backward stability of the TTRR (see [12], Section 1.5). A precise definition of mixed forward-backward stability is now given.

Definition 1 An algorithm W of computing $T_N(x)$ is called mixed forward-backward stable with respect to the data x, if the value $\tilde{T}_N(x)$ computed by W in floating point arithmetic satisfies

$$\tilde{T}_N(x) = (1 + \delta_N) T_N((1 + \Delta_N)x) + \mathcal{O}({\epsilon_M}^2),\ |\delta_N|, |\Delta_N| \le \epsilon_M L, \tag{2}$$

where $L = L(N)$ is a modestly growing function on N and ϵ_M is machine precision.

Simply, the computed value of $T_N(x)$ by mixed forward-backward stable algorithm is very close to the exact value of the Chebyshev polynomial T_N of a slightly perturbed value of x. Throughout this paper, we will ignore the terms of order $\mathcal{O}({\epsilon_M}^2)$. Notice that (see, e.g., Lemma 4.1 in [18]) that the property (2) is equivalent to

$$|\tilde{T}_N(x) - T_N(x)| \le \epsilon_M\, L\, C_N(x) + \mathcal{O}({\epsilon_M}^2), \tag{3}$$

where

$$C_N(x) = |T_N(x)| + |x T_N'(x)| = |T_N(x)| + N |x U_{N-1}(x)|, \tag{4}$$

where $U_{N-1}(x)$ denotes the Chebyshev polynomial of the second kind. These polynomials satisfy the recurrence relations

$$U_n(x) = 2x U_{n-1}(x) - U_{n-2}(x),\ n = 2, 3, \ldots, \tag{5}$$

where $U_0(x) = 1,\ U_1(x) = 2x$.

The paper has been organized as follows. In Section 2, we recall some basic properties of the Chebyshev polynomials. In Section 3, we will use these properties in a derivation of the lower and upper bounds for $C_n(x)$. In Section 4, we present the error analysis for the TTR. We prove that the TTRR is mixed forward-backward stable in the sense of (3) (which is equivalent to (2). Finally, in Section 5 we present some numerical experiments performed in *MATLAB*.

2 Preliminaries

We will need some properties of the Chebyshev polynomials (see [14, 16]). For $-1 \leq x \leq 1$, we have $T_n(x) = \cos(n\Theta)$, where $\Theta = \arccos x$ and $U_{n-1}(x) = \sin(n\Theta)/\sin\Theta$ for $0 < x < 1$.

The following identities hold:

$$U_{n-1}(x) = \frac{T_n{}'(x)}{n},$$

$$T_n(-x) = (-1)^n T_n(x), \quad U_n(-x) = (-1)^n U_n(x).$$

The Chebyshev polynomials of the first kind satisfy the following differential equations;

$$(1 - x^2)T_n''(x) - xT_n'(x) + n^2 T_n(x) = 0 \tag{6}$$

and

$$T_n^2(x) + \frac{1 - x^2}{n^2} T_n'^2(x) = 1. \tag{7}$$

The last equality is a consequence of the trigonometric identity $\cos^2 n\theta + \sin^2 n\theta = 1$.

For $-1 \leq x \leq 1$ and $n = 0, 1, \ldots$ we have the upper bounds

$$|T_n(x)| \leq |T_n(1)| = 1, \quad |U_n(x)| \leq |U_n(1)| = n + 1 \tag{8}$$

and for $-1 < x < 1$

$$|U_n(x)| \leq \frac{1}{\sqrt{1 - x^2}}. \tag{9}$$

The roots (t_i) of $T_n(x)$ are distinct and belong to $(-1, 1)$:

$$t_i = \cos\frac{(2i - 1)\pi}{2n}, \quad i = 1, 2, \ldots, n. \tag{10}$$

The roots (u_i) of $T_n'(x)$ (i.e. the roots of $U_{n-1}(x)$) are:

$$u_i = \cos\frac{i\pi}{n}, \quad i = 1, 2, \ldots, n - 1. \tag{11}$$

Then $-1 < t_n < u_{n-1} < \ldots < u_1 < t_1 < 1$ and

$$T_n(u_i) = (-1)^i \quad i = 1, 2, \ldots, n - 1. \tag{12}$$

For $-1 \leq x \leq 1$ and $m = 0, 1, \ldots$ we get

$$|T_{2m+1}(x)| \leq (2m + 1)|x|, \quad |U_{2m+1}(x)| \leq 2(m + 1)|x|. \tag{13}$$

3 Lower and upper bounds for $C_n(x)$

Since $C_n(-x) = C_n(x)$ for all x, we restrict our considerations to the interval [0, 1]. From (8) it follows that $C_n(x) \leq C_n(1) = n^2 + 1$ for $0 \leq x \leq 1$. By (11)–(12), we have $C_n(u_i) = 1$ for $i = 1, \ldots, n - 1$. If n is odd then $C_n(0) = 0$ (Figs. 1 and 2).

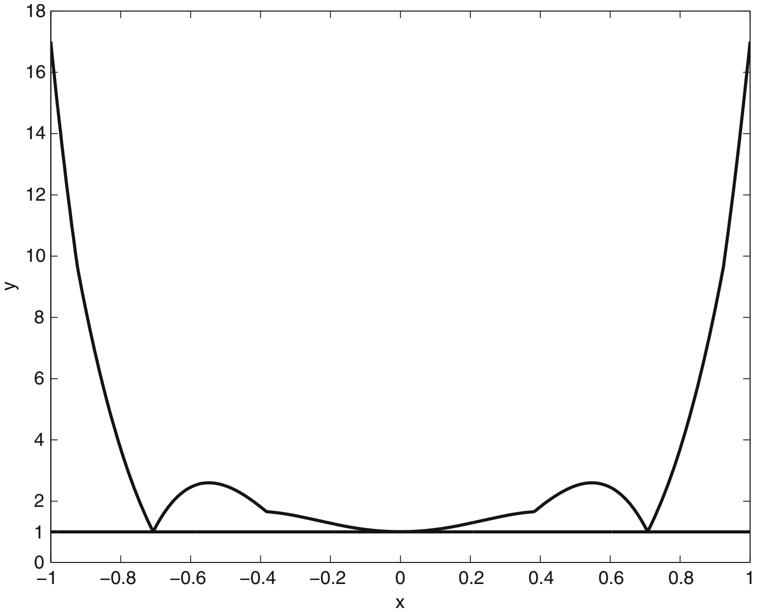

Fig. 1 Plot of $y = C_4(x)$

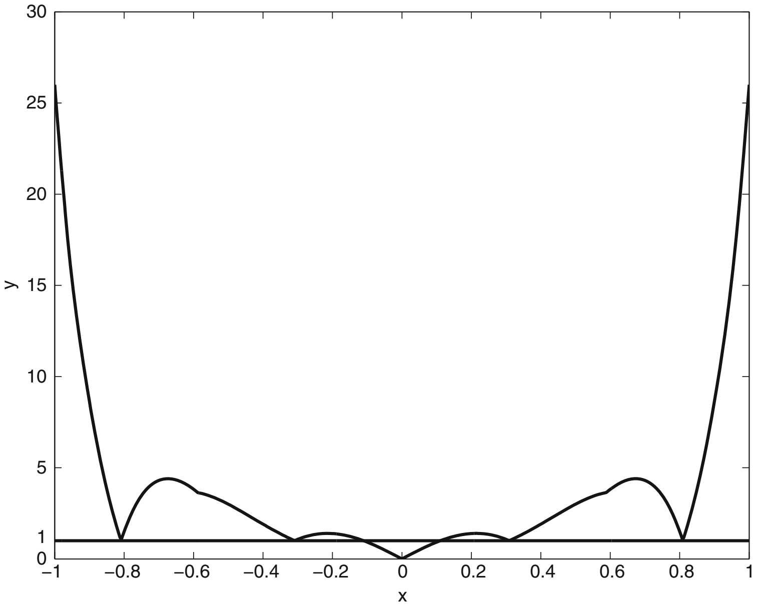

Fig. 2 Plot of $y = C_5(x)$

Theorem 1 *Let n be a natural number. Assume that $s_n \leq x \leq 1$, where*

$$s_n = \frac{1}{\sqrt{n^2+1}}. \tag{14}$$

Then we have

$$C_n(x) = |T_n(x)| + |xT_n'(x)| \geq 1. \tag{15}$$

Proof Notice that the inequality $x^2 \geq s_n^2$ is equivalent to $x^2(n^2+1) \geq 1$, hence $x^2 \geq \frac{1-x^2}{n^2}$. From this and (7) we get

$$C_n^2(x) \geq T_n^2(x) + x^2 T_n'^2(x) \geq T_n^2(x) + \frac{1-x^2}{n^2}\, T_n'^2(x) = 1.$$

The proof is now complete. □

Theorem 2 *Let n be a natural number. Assume that $0 \leq x \leq s_n$, where s_n is defined by (14). Then*

(i) $C_n(x) \geq n|x|$ *for all n,*
(ii) $C_n(x) \geq 1$ *for even n.*

Proof We consider case **(i)**. Clearly, by (14) $1 \geq n^2 x^2$, and since $0 \leq x \leq s_n$, we get

$$C_n^2(x) \geq T_n^2(x) + x^2 T_n'^2(x) \geq n^2x^2T_n^2(x) + x^2 T_n'^2(x) \geq x^2n^2\left(T_n^2(x) + \frac{1}{n^2}\, T_n'^2(x)\right).$$

Therefore,

$$C_n^2(x) \geq x^2n^2\left(T_n^2(x) + \frac{1-x^2}{n^2}\, T_n'^2(x)\right) = x^2n^2,$$

due to (7). Therefore, $C_n(x) \geq n|x|$. This completes the proof of case **(i)**.

Now we consider case **(ii)**. Let $n = 2m$. We first prove that T_{2m} has no roots in $(0, s_{2m})$. By (10), we need to show that

$$t_m = \cos\frac{(2m-1)\pi}{4m} > s_{2m}. \tag{16}$$

Notice that

$$t_m = \cos\left(\frac{\pi}{2} - \frac{\pi}{4m}\right) = \sin\frac{\pi}{4m}.$$

Since $0 < \Theta < \tan\Theta$ for all $0 < \Theta < \frac{\pi}{2}$, we have $\tan^2\Theta > \Theta^2$. From this it follows that $\sin^2\Theta > \frac{\Theta^2}{1+\Theta^2}$. Substituting $\Theta = \pi/4m$ in the above inequality leads to

$$t_m^2 > \frac{\pi^2}{16m^2+\pi^2} > \frac{1}{4m^2+1} = s_{2m}^2,$$

so $t_m > s_{2m}$. This finishes the proof of (16).

We see that T_{2m} has no roots in $(0, s_{2m})$. Moreover, $T_{2m}(0) = (-1)^m$ and $T_{2m}'(0) = 0$. We conclude from (10)–(11) that 0 is the only root of T_{2m}' in the interval $(-s_{2m}, s_{2m})$. Notice that T_{2m} and T_{2m}'' are even, i.e. $T_{2m}(-x) = T_{2m}(x)$ and

$T''_{2m}(-x) = T''_{2m}(x)$ for all x. T'_{2m} is odd, that is, $T'_{2m}(-x) = -T'_{2m}(x)$. Thus we see that the polynomials T_{2m} and T'_{2m} do not change the signs in $(0, s_{2m})$.

More precisely, if m is even, then for all $0 < x < s_{2m}$ we have $T_{2m}(x) > 0$ and $T'_{2m}(x) < 0$, hence $C_{2m}(x) = T_{2m}(x) - xT'_{2m}(x)$. Similarly, if m is odd then $T_{2m}(x) < 0$ and $T'_{2m}(x) > 0$, so $C_{2m}(x) = -T_{2m}(x) + xT'_{2m}(x)$. We see that $C'_{2m}(x) = -T''_{2m}(x)$ if m is even and $C'_{2m}(x) = T''_{2m}(x)$ otherwise.

By (6) for $n = 2m$, we obtain the formula

$$(1 - x^2)T''_{2m}(x) = xT'_{2m}(x) - 2m^2 T_{2m}(x).$$

We see that for all $0 < x < s_{2m}$ we have $T''_{2m}(x) < 0$ if m is even and $T''_{2m}(x) > 0$ if m is odd. We conclude that $C'_{2m}(x) > 0$ for any m, so $C_{2m}(x)$ is increasing in the interval $(0, s_{2m})$. This gives the lower bound $C_{2m}(x) \geq C_{2m}(0) = 1$. The proof of our theorem is now complete. □

4 Error analysis

We analyze the rounding errors in the TTRR. We start with the following lemma.

Lemma 1 *Let $\tilde{T}_n(x)$ denote the quantities computed by the TTRR in floating point arithmetic fl with machine precision ϵ_M. Assume that x is exactly representable in fl ($fl(x) = x$) and $x \in [-1, 1]$. Then*

$$|\tilde{T}_N(x) - T_N(x)| \leq \epsilon_M E_N(x) + \mathcal{O}(\epsilon_M{}^2), \tag{17}$$

where

$$E_N(x) = \sum_{n=2}^{N} (2|x|\,|T_{n-1}(x)| + |T_n(x)|)\,|U_{N-n}(x)|. \tag{18}$$

Proof Note that $\tilde{T}_0(x) = 1$, $\tilde{T}_1(x) = x$ and for $n = 2, 3, \dots$ we have

$$\tilde{T}_n(x) = (2x\,\tilde{T}_{n-1}(x)(1 + \alpha_n) - \tilde{T}_{n-2}(x))(1 + \beta_n), \quad |\alpha_n|, |\beta_n| \leq \epsilon_M.$$

Therefore,

$$\tilde{T}_n(x) = 2x\,\tilde{T}_{n-1}(x) - \tilde{T}_{n-2}(x) + \xi_n, \quad \xi_n = 2x\tilde{T}_{n-1}(x)\alpha_n + \frac{\beta_n}{1 + \beta_n}\tilde{T}_n(x). \tag{19}$$

Let $e_n = \tilde{T}_n(x) - T_n(x)$. We observe that $e_0 = e_1 = 0$ and $e_n = 2xe_{n-1} - e_{n-2} + \xi_n$ for $n = 2, 3, \dots, N$. From this it follows that $e_N = \tilde{T}_N(x) - T_N(x) = \sum_{n=2}^{N} U_{N-n}(x)\xi_n$. Therefore, $|e_N| \leq \sum_{n=2}^{N} |U_{N-n}(x)|\,|\xi_n|$. This, together with (19), leads to

$$|\xi_n| \leq \epsilon_M\,(2|x|\,|T_{n-1}(x)| + |T_n(x)|) + \mathcal{O}(\epsilon_M{}^2), \tag{20}$$

hence

$$|e_N| \leq \epsilon_M \sum_{n=2}^{N} (2|x|\,|T_{n-1}(x)| + |T_n(x)|)\,|U_{N-n}(x)| + \mathcal{O}(\epsilon_M{}^2). \tag{21}$$

The proof of Lemma 1 is now complete. □

Now our task is to bound $E_N(x)$.

Theorem 3 *Let $E_N(x)$ be defined by (18). Then we have*

$$E_N(x) \le E_N(1) = \frac{3N(N-1)}{2} \quad \textit{for } x \in [-1, 1] \tag{22}$$

and

$$E_N(x) \le \frac{3(N-1)}{\sqrt{1-x^2}} \quad \textit{for } x \in (-1, 1). \tag{23}$$

If N is odd and $|x| \le s_N = \frac{1}{\sqrt{N^2+1}}$ then

$$E_N(x) \le \frac{5(N-1)(N+7)}{8}\,|x|. \tag{24}$$

Proof It is obvious that for every $x \in [-1, 1]$ we have

$$E_N(x) \le E_N(1) = 3\sum_{n=2}^{N}(N-n+1) = \frac{3N(N-1)}{2}.$$

This completes the proof of (22).

If $-1 < x < 1$ then by (8)–(9) and (18) we get

$$E_N(x) \le 3\sum_{n=2}^{N}\frac{1}{\sqrt{1-x^2}} = \frac{3(N-1)}{\sqrt{1-x^2}}.$$

This finishes the proof of (23).

Now assume that N is odd and $|x| \le s_N$. We rewrite (18) as follows

$$E_N(x) = A_N(x) + B_N(x)), \tag{25}$$

where

$$A_N(x) = 2|x|\sum_{n=2}^{N}|T_{n-1}(x)|\,|U_{N-n}(x)|, \tag{26}$$

$$B_N(x) = \sum_{n=2}^{N}|T_n(x)|\,|U_{N-n}(x)|. \tag{27}$$

Notice that (9) gives

$$|U_k(x)| \le \frac{1}{\sqrt{1-s_N^2}} \le \frac{3}{2} \quad \text{for } |x| \le s_N, \quad k = 0, 1, \ldots. \tag{28}$$

This, together with the inequality $|T_{n-1}(x)| \le 1$, gives

$$A_N(x) \le 3|x|(N-1). \tag{29}$$

Table 1 The error (33) for the TTRR

N	8	16	32	64	128	256	512	1024
Error (33)	5.25	11.00	21.78	35.00	66.00	165.00	280.75	679.62

To estimate $B_N(x)$ for $N = 2m + 1$, we split it as follows:

$$B_N(x) = \sum_{k=1}^{m} |T_{2k}(x)|\, |U_{N-2k}(x)| + \sum_{k=1}^{m} |T_{2k+1}(x)|\, |U_{N-(2k+1)}(x)|.$$

Note that (13) implies the following upper bounds (for the polynomials of the odd degrees):

$$|U_{N-2k}(x)| \le (N - 2k + 1)\, |x|, \quad |T_{2k+1}(x)| \le (2k + 1)\, |x|.$$

From this and (28) it follows that

$$B_N(x) \le \sum_{k=1}^{m} (N - 2k + 1)\, |x| + \frac{3}{2} \sum_{k=1}^{m} (2k + 1)\, |x|.$$

The last inequality, together with (25) and (29), leads to

$$E_N(x) \le (3(N-1) + m(N-m) + \frac{3}{2} m(m+2))\, |x|.$$

Since $m = (N - 1)/2$, we immediately get (24). □

The bounds (22,23) are not new, see Barrio [2, 5]. To our knowledge, the bound (24) is however new, and allows us to prove the mixed forward-backward stability of the TTRR for all $x \in [-1, 1]$.

By Theorems 1–3 and Lemma 1 we obtain the following theorem.

Theorem 4 *The TTRR is mixed forward-backward stable in* $[-1, 1]$. *For every* $N \ge 2$ *we have*

$$|\tilde{T}_N(x) - T_N(x)| \le \epsilon_M \frac{3N(N-1)}{2} C_N(x) + \mathcal{O}(\epsilon_M{}^2) \quad \textit{for } x \in [-1, 1] \tag{30}$$

and

$$|\tilde{T}_N(x) - T_N(x)| \le \epsilon_M\, 7(N-1)\, C_N(x) + \mathcal{O}(\epsilon_M{}^2) \quad \textit{for } x \in [-0.9, 0.9]. \tag{31}$$

Moreover, if N is odd and $|x| \le s_N = \frac{1}{\sqrt{N^2+1}}$, *then*

$$|\tilde{T}_N(x) - T_N(x)| \le \epsilon_M \frac{5(N-1)(N+7)}{8N} C_N(x) + \mathcal{O}(\epsilon_M{}^2). \tag{32}$$

We see that $L(N)$ in (2) is of order N in $(-1, 1)$ for x not to close to ± 1 and of order N^2 near ± 1.

Table 2 The relative error $R_N(x)$

N	10	10^2	10^3	10^4	10^5	10^6
$R_N(0)$	5	50	500	$5 \cdot 10^3$	$5 \cdot 10^4$	$5 \cdot 10^5$
$R_N(0.5)$	1.63	1.96	1.99	1.99	2.00	2.00
$R_N(0.9)$	1.33	1.61	1.48	1.50	1.48	1.75
$R_N(\cos(2\pi/N))$	14.70	$1.51 \cdot 10^3$	$1.51 \cdot 10^5$	$1.50 \cdot 10^7$	$1.62 \cdot 10^7$	$4.33 \cdot 10^6$
$R_N(\cos(\pi/N))$	29.41	$3.03 \cdot 10^3$	$3.03 \cdot 10^5$	$2.91 \cdot 10^7$	$8.24 \cdot 10^6$	$1.06 \cdot 10^5$
$R_N(1)$	1.33	1.48	1.49	1.49	1.50	1.50

5 Numerical tests

To illustrate our results, we present numerical tests in *MATLAB* with machine precision $\epsilon_M = 2^{-52} \approx 2.2 \cdot 10^{-16}$. We compare the results computed by the TTRR with the exact values of the Chebyshev polynomial $T_N(x)$. They were obtained by implementing the TTRR in high precision using the VPA (Variable Precision Arithmetic) function from *MATLAB*'s Symbolic Math Toolbox, and then rounded to 16th decimal digits. We compute the relative error

$$e_N = \frac{\max_{x \in S} |T_N(x) - \tilde{T}_N(x)|}{\epsilon_M}. \tag{33}$$

Here S consists of the equally spaced checkpoints $t_1, t_2, \ldots, t_{201}$ from the interval $[-1, 1]$, i.e. $t_i = -1 + (i-1)/100$ for $i = 1, 2, \ldots, 201$ (Table 1). In order to show that our theoretical bounds are realistic, we evaluated $R_N(x) = E_N(x)/C_N(x)$ for particular values of x. It is clear that near $x = \pm 1$ the values of R_N are of order N^2 for large N (Table 2). For example, if $\hat{x} = u_1 = \cos(\pi/N)$ (the root of $T_N'(x)$) then $\hat{x}$ is very close to 1, so $E_N(\hat{x}) \approx E_N(1) = \frac{3N(N-1)}{2}$. However, $C_N(\hat{x}) = 1$, so $R_N(\hat{x}) \approx \frac{3N(N-1)}{2}$. Similar conclusion can be found in [2] for a class of parallel algorithms to evaluate Chebyshev series.

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