



Asymptotic and Exact Expansions of Heat Traces

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Abstract We study heat traces associated with positive unbounded operators with compact inverses. With the help of the inverse Mellin transform we derive necessary conditions for the existence of a short time asymptotic expansion. The conditions are formulated in terms of the meromorphic extension of the associated spectral zeta-functions and proven to be verified for a large class of operators. We also address the problem of convergence of the obtained asymptotic expansions. General results are illustrated with a number of explicit examples.

Keywords Heat traces · Asymptotic expansions · Spectral theory · Zeta-functions · General Dirichlet series

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1 Introduction

Given a positive, possibly unbounded, operator P with a compact resolvent, acting on a separable infinite-dimensional Hilbert space \mathcal{H} one can define the associated heat operator e^{-tP} for $t > 0$. The latter, under some mild conditions on P , is trace class for any $t > 0$. It turns out that a close inspection of the function $t \mapsto \text{Tr } e^{-tP}$, reveals a lot of information of geometrical nature. For instance, if P is a differential operator of Laplace type defined on a closed Riemannian manifold then the classical results [40] show that there exists an asymptotic expansion of the form

$$\text{Tr } e^{-tP} \underset{t \downarrow 0}{\sim} \sum_{k \geq 0} a_k(P) t^{(k-d)/2}, \tag{1}$$

where d is the dimension of the manifold and a_k 's – called Seeley–DeWitt coefficients – are given by the integrals over the manifold of some geometrical invariants. Moreover, the coefficients a_k can be expressed as

$$a_k(P) = \text{Res}_{s=(d-k)/2} \Gamma(s) \zeta_P(s),$$

where $\zeta_P(s) := \text{Tr } P^{-s}$ is the (meromorphic extension of the) spectral zeta-function associated with P .

The existence of an asymptotic expansion of $\text{Tr } e^{-tP}$ was proven for P being a classical positive elliptic pseudodifferential operator of order $m \in \mathbb{N}$ (see [39] and references therein). In this case,

$$\text{Tr } e^{-tP} \underset{t \downarrow 0}{\sim} \sum_{k=0}^{\infty} a_k(P) t^{(-d+k)/m} + \sum_{l=1}^{\infty} b_l(P) t^l \log t$$

and

$$a_k(P) = \text{Res}_{s=(d-k)/m} \Gamma(s) \zeta_P(s), \quad b_l(P) = -\text{Res}_{s=-l} (s+l) \Gamma(s) \zeta_P(s).$$

In fact, this result can be extended to certain classes of *non-classical* pseudodifferential operators. In [53] for instance, the heat traces of pseudodifferential operators with log-polyhomogeneous symbols have been studied.

With the development of noncommutative geometry [16], the need came to investigate heat traces associated with positive functions of Dirac operators in the framework of spectral triples. Unfortunately, there is no analogue of the formula (1) for a general spectral triple [72]. In fact, the existence of an asymptotic expansion of the heat trace is assumed whenever needed in applications (see for instance [19, Section 11], [21, Section 2.1]) and has been proven rigorously only for a few specific examples [29, 37, 71]. Whence the results of [37, 71] essentially used the methods of pseudodifferential calculus, the casus of the standard Podleś sphere [29] required completely different tools (see Section 4.2).

We note that the interplay between heat traces and spectral zeta-functions has been investigated in a very general framework of von Neumann algebras by a number of authors [7, 16, 54, 67]. However, the efforts of the latter focused on the leading behaviour of heat traces and its consequences for measurability.

The purpose of this paper is to study heat traces associated with general densely defined positive operators P with compact inverses. In particular, we formulate sufficient conditions for the existence of a small t asymptotic expansion of $\text{Tr} e^{-tP}$. Having an asymptotic expansion at hand, a natural question one may pose is that of its convergence. This issue has not been studied in full generality, even in the context of P being a differential operator (see however [29, 46]). We show how the conditions on P shall be refined in order to get an exact formula for the heat trace valid on some open interval $(0, T)$. The motivation behind our work comes from noncommutative geometry, but the framework of the studies is even wider.

The heat trace methods have multifarious applications in theoretical physics (see [70] for a review). They are in common use in quantum field theory [2, 5, 30, 36], also in its noncommutative version [38]. In general, one only disposes of an asymptotic expansion of the heat trace as $t \downarrow 0$. This implies that the field-theoretic calculations performed with the help of this method are only perturbative. Needless to say that a control on the convergence of a perturbative expansion is of crucial importance.

In noncommutative geometry, the heat trace is the cornerstone of bosonic spectral action computations [9, 19, 46–48]. The large energies expansion of the latter is based on the asymptotic expansion of the heat trace associated with the relevant Dirac operator. Recently, also the exact computations of the spectral action gained interest [10, 28, 29, 57, 58, 61, 68] due to their possible application to the study of cosmic topology.

The plan of the paper presents itself as follows: In Section 2 we recollect some basic notions on spectral functions associated with positive operators. Then, in Section 3 we discuss in details the interplay between the meromorphic extension of the spectral zeta-function ζ_P and the asymptotic expansion of the associated heat trace $\text{Tr} e^{-tP}$, by gathering results on general Dirichlet series [43, 44] and the Mellin transform [35, 63]. Moreover, we present a set of sufficient assumptions on P so that the associated heat trace is controlled for t in some non-empty open interval. Section 4 illustrates the general theorems with various special cases and examples coming from Dirac-type operators on both classical manifolds and noncommutative spaces. We end with an outlook on the possible generalisations and applications of our results. We also discuss the limitations of the method and compare its usefulness with the Tauberian theorems commonly used in this domain.

2 Preliminaries

2.1 Notations

Let us first fix some notations:

- \mathbb{N} denotes the non-negative integers, \mathbb{N}^+ the positive ones, \mathbb{Z}^* stands for the non-zero integers and \mathbb{R}^+ for positive reals.
- Unless stated otherwise, t will always denote a positive parameter and s a complex one.
- $f(s) \approx g(s)$ means that $\lim_{|s| \rightarrow \infty} f(s)/g(s) = 1$.

- $f(x) = \mathcal{O}_{x \rightarrow x_0}(g(x))$ means that $\limsup_{x \rightarrow x_0} |f(x)/g(x)| < \infty$, for $x, x_0 \in \mathbb{R} \cup \{\pm\infty\}$. The notation $f(x) = \mathcal{O}_{x_0}(g(x))$ will be used when the variable is obvious.
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- $f(t) \underset{t \downarrow 0}{\sim} \sum_n \phi_n(t)$ denotes an asymptotic expansion (see Definition 4) of f as t tends to 0 from above.

2.2 Heat Traces

Let P denote an unbounded operator on a Hilbert space. Throughout the paper we will assume the following on P :

General Assumption P is a positive densely defined operator with a compact inverse, acting on a separable infinite-dimensional Hilbert space.

Such a strong assumption allows us, among other things, to use freely the spectral theorem for P . As a consequence, the spectrum $\sigma(P)$ is a discrete subset of \mathbb{R} , which can be ordered into a sequence increasing to infinity

$$\sigma(P) = (\lambda_n)_{n=0}^\infty, \quad 0 < \lambda_0 < \lambda_1 < \dots, \quad \lim_{n \rightarrow \infty} \lambda_n = \infty. \tag{2}$$

We will denote the multiplicity of the eigenvalue $\lambda_n \in \sigma(P)$ by M_n . For further purposes, we also define the spectral growth function as

$$N(\lambda) := \sum_{\{n: \lambda_n \leq \lambda\}} M_n. \tag{3}$$

Now we define the main object of our interest:

Definition 1 The heat trace of the operator P is the function $\text{htr}_P : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, defined as

$$\text{htr}_P(t) := \text{Tr} e^{-tP}.$$

We say that the heat trace is *well-defined* if e^{-tP} is a trace class operator for any $t > 0$.

Using (2) the heat trace can be written as

$$\text{htr}_P(t) = \sum_{n=0}^\infty M_n e^{-t\lambda_n}. \tag{4}$$

The sum is of the form of a general Dirichlet series, which is defined as

$$\sum_{n=0}^\infty a_n e^{-s b_n}, \tag{5}$$

for s in some (possibly empty) subset of \mathbb{C} , $a_n \in \mathbb{C}$ and $(b_n)_{n=0}^\infty$ a sequence of real numbers increasing to infinity. Throughout this paper we will meet only examples

with a_n real and non-negative for $n \in \mathbb{N}$ (usually we will simply have $a_n = M_n$). In such a case the convergence of (5) is governed by the following theorem:

Theorem 1 ([44], **Theorem 7 with the footnote**) *If $a_n > 0$ for $n \in \mathbb{N}$ and $\sum_{n=0}^\infty a_n = \infty$ then the general Dirichlet series (5) converges for $\Re(s) > L$ and diverges for $\Re(s) < L$, where L is given by*

$$L = \limsup_{n \rightarrow \infty} b_n^{-1} \log(a_0 + \dots + a_n),$$

and $L \geq 0$.

We will call such L the *abscissa of convergence* of the general Dirichlet series (5). The inequality $L \geq 0$ follows from the fact that for $s = 0$ the series (5) is equal to $\sum_{n=0}^\infty a_n$, which is divergent. Note that the abscissa can be infinite, what means that the series is nowhere convergent.

Proposition 1 *The heat trace of the operator P is well-defined:*

1. *if and only if $N(\lambda_n) = \mathcal{O}_{n \rightarrow \infty}(e^{\epsilon \lambda_n})$ for any positive ϵ ;*
2. *if there exist $\alpha \geq 0$ such that $M_n = \mathcal{O}_\infty(n^\alpha)$ and $\log n = \mathcal{O}_\infty(\lambda_n)$ (i.e. λ_n grow faster than $\log n$).*

Proof In order to have htr_P well-defined, we need the abscissa of convergence L of the series (4) to be 0. Taking $a_n = M_n, b_n = \lambda_n$ for all $n \in \mathbb{N}$ we see that $\sum_{n=0}^\infty a_n = \infty$ as P has an infinite number of eigenvalues. Now, by Theorem 1 we get that htr_P is well-defined iff $L = \limsup_{n \rightarrow \infty} \lambda_n^{-1} \log N(\lambda_n)$ equals to 0. This is equivalent to the statement that for any $\epsilon > 0$ there exists n_0 such that for any $n \geq n_0$ we have $\lambda_n^{-1} \log N(\lambda_n) \leq \epsilon$ or, equivalently, $N(\lambda_n) \leq e^{\epsilon \lambda_n}$. This in turn is equivalent to $N(\lambda_n) = \mathcal{O}_\infty(e^{\epsilon \lambda_n})$ for any $\epsilon > 0$ and thus 1 follows.

For statement 2 let us first note that since $M_n = \mathcal{O}_\infty(n^\alpha)$ for some $\alpha \geq 0$, then $N(\lambda_n) = \mathcal{O}_\infty(n^{\alpha+1})$. Take a positive constant C such that $N(\lambda_n) \leq Cn^{\alpha+1}$ for $n \geq N$ for some $N \in \mathbb{N}$. Then,

$$L = \limsup_{n \rightarrow \infty} \lambda_n^{-1} \log(N(\lambda_n)) \leq \limsup_{n \rightarrow \infty} \lambda_n^{-1} \log(Cn^{\alpha+1}) = \limsup_{n \rightarrow \infty} \lambda_n^{-1} (\log C + (\alpha + 1) \log n),$$

what tends to 0 under the hypothesis of 2. □

Note that the condition in 2 is only sufficient: consider e.g. $\lambda_n = n^2$, and $M_n = 2^n$ which satisfies the hypothesis of 1 but not the one of 2.

To conclude this subsection, we remark that heat traces can be defined for operators bounded from below, possibly with a non-trivial kernel. However, the zeta-functions described below apply only to positive invertible operators (see, however, Section 3.3).

2.3 Spectral Zeta-functions

Definition 2 The *zeta-function* associated with the operator P is a complex function

$$\mathbb{C} \supset \text{Dom}(\zeta_P) \ni s \mapsto \zeta_P(s) = \text{Tr } P^{-s}.$$

We say that ζ_P is *well-defined* if $\text{Dom}(\zeta_P)$ is non-empty.

Using the spectral theorem this can be written as

$$\zeta_P(s) = \sum_{n=0}^{\infty} M_n \lambda_n^{-s}, \tag{6}$$

which is again a general Dirichlet series (5) with $a_n = M_n$ and $b_n = \log \lambda_n$. Using Theorem 1 we get

Proposition 2 *The abscissa of convergence of the zeta-series (6) is given by*

$$L = \inf \{ \alpha \in \mathbb{R} : N(\lambda_n) = \mathcal{O}_{n \rightarrow \infty} (\lambda_n^\alpha) \}. \tag{7}$$

Note that it may happen that $L = +\infty$, which means that the zeta-function is not well-defined (e.g. when $\lambda_n = \log n$, $M_n = 1$ for $n \geq 2$, then $N(\lambda) \approx \exp \lambda$).

Proof Take any α such that there exists a constant c satisfying $N(\lambda_n) \leq c \lambda_n^\alpha$. Then by setting $a_n = M_n$, $b_n = \log \lambda_n$ we have by Theorem 1

$$L = \limsup_{n \rightarrow \infty} \frac{\log N(\lambda_n)}{\log \lambda_n} \leq \limsup_{n \rightarrow \infty} \frac{\log(c \lambda_n^\alpha)}{\log \lambda_n} = \alpha. \tag{8}$$

Thus $L \leq K$, where K denotes the RHS of (7).

On the other hand, if we suppose that $L < K$ then there exists α such that $L < \alpha < K$. Using (8) we can find such $n_0 \in \mathbb{N}$ that $\log N(\lambda_n) (\log \lambda_n)^{-1} < \alpha$ for any $n \geq n_0$. Then we get that $N(\lambda_n) < \lambda_n^\alpha$ for $n \geq n_0$, which contradicts the assumption $\alpha < K$. Thus $L = K$ and the proposition is proved. \square

Proposition 3 *If an operator P is such that its zeta-function is well-defined with finite abscissa of convergence L , then its heat trace is also well-defined and*

$$\text{htr}_P(t) = \mathcal{O}_0(t^{-\alpha}), \text{ for all } \alpha > L.$$

Proof Comparing Proposition 1 assertion 1. with Proposition 2 we get that htr_P is well-defined.

Now, let us take any $\alpha > L \geq 0$. Then, there exists a positive constant $C(\alpha)$ such that

$$x^\alpha e^{-x} \leq C(\alpha),$$

for any $x > 0$, as the function $x \mapsto x^\alpha e^{-x}$ is bounded on $\mathbb{R}^+ \cup \{0\}$ for any $\alpha \geq 0$. Therefore, for any $\alpha > L$ we have

$$0 \leq t^\alpha \text{htr}_P(t) = \sum_{n=0}^{\infty} M_n t^\alpha e^{-t \lambda_n} \leq C(\alpha) \sum_{n=0}^{\infty} M_n \lambda_n^{-\alpha} = C(\alpha) \zeta_P(\alpha) < \infty.$$

Hence, $\text{htr}_P(t) = \mathcal{O}_0(t^{-\alpha})$. □

2.4 Mellin Transform

Definition 3 (see [63] for instance) The Mellin transform of a locally Lebesgue integrable function f defined over \mathbb{R}^+ is a complex function $\mathcal{M}[f]$ given by

$$\mathcal{M}[f](s) = \int_0^\infty f(t) t^{s-1} dt.$$

The inverse Mellin transform of a meromorphic function g , denoted by $\mathcal{M}^{-1}[g]$, reads

$$\mathcal{M}^{-1}[g](t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} g(s) t^{-s} ds,$$

for some real c such that the integral converges for all $t > 0$.

In general, the Mellin transform is defined only in some region of the complex plane. This region turns out to be a strip, called the fundamental strip (see [35, Definition 1]). If $f(t) = \mathcal{O}_0(t^\alpha)$ and $f(t) = \mathcal{O}_\infty(t^\beta)$, then $\mathcal{M}[f](s)$ exists at least in the strip $-\alpha < \Re(s) < -\beta$ (cf. [35], Lemma 1). The invertibility of Mellin transform is addressed by the following theorem.

Theorem 2 ([35], Theorem 2) *Let f be a continuous function. If c is a real number belonging to the fundamental strip of $\mathcal{M}[f]$ and $\mathbb{R} \ni y \mapsto \mathcal{M}[f](c+iy)$ is Lebesgue integrable, then for any $t \in \mathbb{R}^+$*

$$f(t) = \mathcal{M}^{-1}[\mathcal{M}[f]](t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{M}[f](s) t^{-s} ds.$$

We have established the framework and now we are ready to formulate and prove the main results.

3 General Results

The Mellin transform has a direct application to the study of the asymptotic expansions of heat traces. Recall that P is, by General Assumption, a positive densely defined operator with compact inverse. Let us start with the following lemma.

Lemma 1 *Let P be such that its zeta-function is well-defined with abscissa of convergence L . Then for any $s \in \mathbb{C}$ with $\Re(s) > L$,*

$$\mathcal{M}[\text{htr}_P](s) = \Gamma(s) \zeta_P(s). \tag{9}$$

Proof For s with $\Re(s) > L$ we pick any α such that $\Re(s) > \alpha > L$. From Corollary 3 we know that $\text{htr}_P(t) = \mathcal{O}_0(t^{-\alpha})$ and the integral

$$\mathcal{M}[\text{htr}_P](s) = \int_0^\infty \text{htr}_P(t) t^{s-1} dt$$

converges (absolutely) at 0. It also converges absolutely at ∞ for any s , since

$$\text{Tr } e^{-tP} = \sum_{n=0}^\infty M_n e^{-t\lambda_n} = e^{-t\lambda_0} \sum_{n=0}^\infty M_n e^{-t(\lambda_n - \lambda_0)} \leq e^{-t\lambda_0} c \quad \text{for } t \geq 1, \quad (10)$$

where $c = \sum_{n=0}^\infty M_n e^{-(\lambda_n - \lambda_0)} = e^{\lambda_0} \text{Tr } e^{-P} < \infty$.

Since $M_n e^{-t\lambda_n} t^{s-1} > 0$ for all $n \in \mathbb{N}$, $t > 0$ and s with $\Re(s) > L$, we can use the Tonelli Theorem to exchange the sum with integral in the following calculation:

$$\begin{aligned} \mathcal{M}[\text{htr}_P](s) &= \int_0^\infty \sum_{n=0}^\infty M_n e^{-t\lambda_n} t^{s-1} dt = \sum_{n=0}^\infty M_n \int_0^\infty e^{-t\lambda_n} t^{s-1} dt, \\ &= \sum_{n=0}^\infty M_n \lambda_n^{-s} \int_0^\infty e^{-y} y^{s-1} dy = \zeta_P(s) \Gamma(s). \end{aligned} \quad (11)$$

□

For further convenience we adopt the notation

$$\mathcal{Z}(s) := \Gamma(s)\zeta_P(s),$$

for $\Re(s) > L$. If moreover, ζ_P extends to a meromorphic function on some larger region $D \subset \mathbb{C}$ then \mathcal{Z} also has a meromorphic extension to D , since Γ is meromorphic on \mathbb{C} . For a meromorphic function f we also denote by $\mathfrak{P}_f(D)$ the set of its poles contained in the region $D \subset \mathbb{C}$.

The inverse of the relation (9) (compare [35, Theorem 5]) produces an expansion of htr_P :

Theorem 3 *Let P be an operator satisfying the General Assumption and such that:*

1. ζ_P is well-defined with abscissa of convergence $L \geq 0$.
2. ζ_P has a meromorphic continuation to the half-plane $\Re(s) > L'$ for some real $L' < L$.
3. There exist real numbers c, R , such that $L' < -R < L < c$, and \mathcal{Z} is regular and Lebesgue integrable on lines $\Re(s) = -R$ and $\Re(s) = c$.
4. There exists an increasing sequence $(y_k)_{k \in \mathbb{Z}}$, with $y_0 = 0$ and $y_k \rightarrow \pm\infty$ as $k \rightarrow \pm\infty$, such that

$$\sup_{x \in [-R, c]} |\mathcal{Z}(x + iy_k)| \rightarrow 0, \quad (12)$$

as $k \rightarrow \pm\infty$ and the suprema for all $k \in \mathbb{Z}^*$ are finite.

Let D_k denote a rectangle $\{x + iy : -R \leq x \leq c, y_{-k} \leq y \leq y_k\} \subset \mathbb{C}$ for $k \in \mathbb{N}^+$ and $D_0 = \emptyset$ and let $S_k := \mathfrak{P}_{\mathcal{Z}}(D_k \setminus D_{k-1})$ (see Fig. 1).

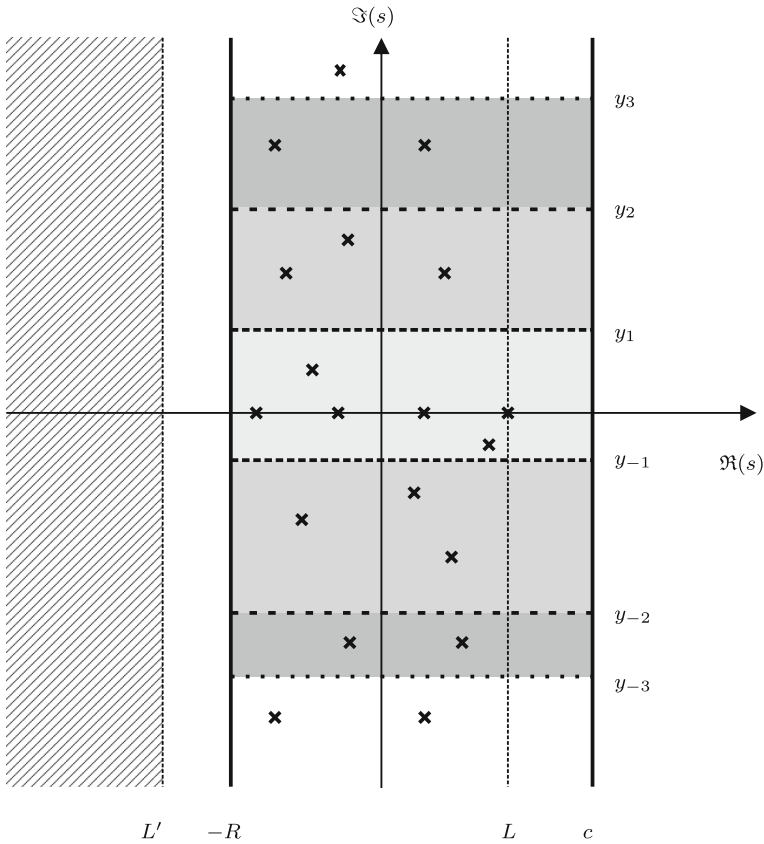


Fig. 1 Illustration for Theorem 3. The crosses stand for poles of the function \mathcal{Z}

Then, for $t > 0$, we have

$$\text{htr}_P(t) = \sum_{k=1}^{\infty} \sum_{s \in S_k} r_s(t) + F_R(t), \tag{13}$$

where the, possibly infinite, series over k is convergent with

$$r_s(t) := \text{Res}_{s'=s} \left(\Gamma(s') \zeta_P(s') t^{-s'} \right),$$

$$F_R(t) := \frac{1}{2\pi i} \int_{-R-i\infty}^{-R+i\infty} \Gamma(s) \zeta_P(s) t^{-s} ds.$$

Moreover, $F_R(t) = \mathcal{O}_0(t^R)$.

Proof On the strength of Assumptions 3 and 4 the function \mathcal{Z} is regular at the boundary of D_k . Thus, by the residue theorem,

$$\frac{1}{2\pi i} \int_{\partial D_k} \mathcal{Z}(s) t^{-s} ds = \sum_{s \in \mathfrak{P}_{\mathcal{Z}}(D_k)} r_s(t) = \sum_{m=1}^k \sum_{s \in S_m} r_s(t), \tag{14}$$

where the contour ∂D_k is oriented counter-clockwise. In the sum above only a finite number of residues is taken into account, as the region D_k is bounded and the set of poles $\mathfrak{P}_{\mathcal{Z}}(D_k)$ has no accumulation points. Let us decompose the boundary of the rectangle into four sides

$$I_{-R}(k) := \int_{-R+iy_{-k}}^{-R+iy_k} \mathcal{Z}(s) t^{-s} ds, \quad I_c(k) := \int_{c+iy_{-k}}^{c+iy_k} \mathcal{Z}(s) t^{-s} ds, \\ I_H^{\pm}(k) := \int_{-R+iy_{\pm k}}^{c+iy_{\pm k}} \mathcal{Z}(s) t^{-s} ds,$$

so that

$$\int_{\partial D_k} \mathcal{Z}(s) t^{-s} ds = I_c(k) - I_H^+(k) - I_{-R}(k) + I_H^-(k).$$

We now estimate

$$|I_H^{\pm}(k)| = \left| \int_{-R}^c \mathcal{Z}(x + iy_{\pm k}) t^{-x-iy_{\pm k}} dx \right| \leq \sup_{x \in [-R, c]} |\mathcal{Z}(x + iy_{\pm k})| \int_{-R}^c t^{-x} dx \xrightarrow{k \rightarrow \infty} 0. \tag{15}$$

To analyse the integrals I_{-R}, I_c we will use the Assumption 3. Integrability over the line $\Re(s) = c$ allows us to apply Theorem 2, which, together with Lemma 1, gives us the following limit

$$\frac{1}{2\pi i} \lim_{k \rightarrow \infty} I_c(k) = \text{htr}_P(t). \tag{16}$$

On the other hand, integrability of \mathcal{Z} along the line $\Re(s) = -R$ allows us to write

$$\frac{1}{2\pi i} \lim_{k \rightarrow \infty} I_{-R}(k) = F_R(t).$$

Thus, by taking the limit $k \rightarrow \infty$ of (14) we get

$$\sum_{m=0}^{\infty} \sum_{s \in S_m} r_s(t) = \lim_{k \rightarrow \infty} \frac{1}{2\pi i} \int_{\partial D_k} \mathcal{Z}(s) t^{-s} ds = \text{htr}_P(t) - F_R(t).$$

To finish the proof we observe that

$$F_R(t) t^{-R} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{Z}(-R + iy) t^{-iy} dy = \frac{1}{2\pi} \mathcal{F}[y \mapsto \mathcal{Z}(-R + iy)] \left(-\frac{\log t}{2\pi} \right) \xrightarrow{t \rightarrow 0} 0,$$

where \mathcal{F} denotes the Fourier transform, and the limit is a consequence of the Riemann–Lebesgue lemma. □

The series in (13) is just the sum of residues of the function $\mathcal{Z}(s)t^{-s}$ over the poles contained in the strip $-R < \Re(s) < c$ and one could be tempted to write it down as

$$\sum_{s \in S} r_s(t), \tag{17}$$

where summation goes over $S = \mathfrak{P}_{\mathcal{Z}}(\{s \in \mathbb{C} : -R < \Re(s) < c\})$. Whilst this second form looks simpler, it ignores the information about grouping and arrangement of terms, which may be significant. Indeed, Theorem 3 states that the series over k in formula (13) is convergent, and in general this may only be a conditional convergence. That is why residues have to be grouped into (finite) sums over S_k , and then summed in the order given by index k . The grouping and order are consequences of the choice of the sequence (y_k) and the assumption (12) may fail for a different choice of sequence.

However, if the function \mathcal{Z} has only a finite number of residues in the strip $-R < \Re(s) < c$ or the sum (17) is absolutely convergent then rearrangements of terms are allowed and one can safely write

$$\text{htr}_P(t) = \sum_{s \in S} r_s(t) + F_R(t), \tag{18}$$

instead of (13). One clue about the absolute convergence is given by the following result:

Proposition 4 *If the hypothesis of Theorem 3 is fulfilled with the assumption (12) altered for a stronger one:*

$$\sum_{k \in \mathbb{Z}^*} \sup_{x \in [-R, c]} |\mathcal{Z}(x + iy_k)| < \infty, \tag{19}$$

then for any $t > 0$

$$\sum_{k=1}^{\infty} \left| \sum_{s \in S_k} r_s(t) \right| < \infty.$$

Proof First note that for any $t > 0$

$$\sum_{k=1}^{\infty} \left| \sum_{s \in S_k} r_s(t) \right| = \frac{1}{2\pi i} \sum_{k=1}^{\infty} \left| \int_{\partial(D_k \setminus D_{k-1})} \mathcal{Z}(s) t^{-s} ds \right|.$$

Now, for any $k \geq 1$ we decompose the boundary $\partial(D_k \setminus D_{k-1})$ as in the proof of Theorem 3 and estimate

$$\begin{aligned} \left| \int_{\partial(D_k \setminus D_{k-1})} \mathcal{Z}(s) t^{-s} ds \right| &\leq |I_{-R}(k) - I_{-R}(k-1)| + |I_c(k) - I_c(k-1)| + \\ &\quad + |I_H^+(k-1)| + |I_H^-(k-1)| + |I_H^+(k)| + |I_H^-(k)|, \end{aligned}$$

with the convention $I_H^{\pm}(0) := 0$ to comply with $D_0 = \emptyset$. So the contributions of the horizontal integrals $I_H^{\pm}(k), I_H^{\pm}(k+1)$ add together instead of canceling out as it happened in the proof of Theorem 3.

Therefore,

$$\sum_{k=1}^{\infty} \left| \int_{\partial(D_k \setminus D_{k-1})} \mathcal{Z}(s) t^{-s} ds \right| \leq \int_{-R-i\infty}^{-R+i\infty} |\mathcal{Z}(s) t^{-s}| ds + \int_{c-i\infty}^{c+i\infty} |\mathcal{Z}(s) t^{-s}| ds + 2 \sum_{k=1}^{\infty} (|I_H^+(k)| + |I_H^-(k)|).$$

The convergence of the integrals along the vertical lines follows since \mathcal{Z} is Lebesgue integrable (Assumption 3 of Theorem 3).

On the other hand, the sum over horizontal contributions can be estimated as in (15):

$$\sum_{k=1}^{\infty} (|I_H^+(k)| + |I_H^-(k)|) \leq \sum_{k \in \mathbb{Z}^*} \sup_{x \in [-R, c]} |\mathcal{Z}(x + iy_k)| \int_{-R}^c t^{-x} dx,$$

which is finite by assumption (19). □

In most cases (see Section 4), one can also avoid the grouping of the residues into S_k 's, by finding some denser sequence $(y_k)_{k \in \mathbb{Z}}$ such that each S_k contains only one pole of \mathcal{Z} . Also, if there are poles of \mathcal{Z} lying on a common line $\Re(s) = \text{const.}$, one can resort to the more general Theorem 4. However, for a denser sequence the assumptions (12) or (19) of the Theorem 3 may not be fulfilled.

Unfortunately, we were not able to tailor an example, where the analytic structure of ζ_P is such that the grouping or arrangement of the terms are important. It might turn out that the operatorial aspect of heat traces, which leads to a specific subclass of general Dirichlet series, implies that one can always replace formula (13) with (18). However, as the problem is open, we emphasise once again that in general formulae the series in (13) is conditionally convergent only.

It is instructive to write down explicitly an individual term $r_s(t)$. If the function \mathcal{Z} has a pole of order n at s_0 , then it has a Laurent expansion $\mathcal{Z}(s) = \sum_{k=-n}^{\infty} b_k(s_0) (s - s_0)^k$ in some open punctured disc with the center at s_0 . On the other hand,

$$t^{-s} = e^{-s_0 \log t} e^{-(s-s_0) \log t} = t^{-s_0} \sum_{k=0}^{\infty} \frac{(-\log t)^k}{k!} (s - s_0)^k, \quad \forall s_0, s \in \mathbb{C}, t > 0.$$

Therefore, the residue $r_{s_0}(t)$ being the coefficient in front of $(s - s_0)^{-1}$ in the Laurent expansion of $\mathcal{Z}(s)t^{-s}$ at $s = s_0$ reads

$$r_{s_0}(t) = t^{-s_0} \sum_{k=0}^{n-1} \frac{b_{-k-1}(s_0)}{k!} (-\log t)^k. \tag{20}$$

Note that $r_{s_0}(t) = \mathcal{O}_0(t^{-\Re(s_0)}(\log t)^{n-1})$, what also means that $r_{s_0}(t) = \mathcal{O}_0(t^{-\Re(s_0)+\delta})$ for arbitrarily small $\delta > 0$.

Remark 1 One could in principle allow the function \mathcal{Z} to have *essential singularities* as long as they are isolated (see [8, last point on p. 453] for a motivation). For such functions the residue (i.e. the -1^{st} term of the Laurent expansion) is well-defined. Since Γ is a meromorphic function and $s \mapsto t^{-s}$ is an entire one for all $t > 0$, the function

$$s \mapsto \mathcal{Z}(s)t^{-s}, \tag{21}$$

has isolated singularities only. At an essential singularity, the formula (20) yields an infinite series. This series is absolutely convergent for every $t > 0$ in the punctured disc of convergence of the Laurent expansion of \mathcal{Z} at s_0 . Indeed, in the interior of this punctured disc the Laurent series of the function (21) is a product of two absolutely convergent series, and as such it is absolutely convergent.

3.1 Asymptotic Expansions

Theorem 3 gives us information about the behavior of $\text{htr}_P(t)$ at $t = 0$ up to the order t^R . If ζ_P can be meromorphically continued to the whole complex plane and satisfies suitable growth conditions, then Theorem 3 can give us the behavior of htr_P at 0 up to an arbitrary finite order, i.e. an asymptotic expansion. Recall the definition [22, 31, 43]:

Definition 4 Let $(\phi_n)_{n \in \mathbb{N}}$ be a sequence of functions from \mathbb{R}^+ to \mathbb{C} . We call this sequence an asymptotic scale at $t = 0$ if for any $n \in \mathbb{N}$ we have $\phi_{n+1}(t) = \mathcal{O}_0(\phi_n(t))$.

For a function $f : \mathbb{R}^+ \rightarrow \mathbb{C}$ the formal series $\sum_{n=0}^\infty \phi_n(t)$, with $(\phi_n)_{n \in \mathbb{N}}$ being an asymptotic scale, is called an *asymptotic expansion* (or *asymptotic series*) of f at $t = 0$ if for any $N \in \mathbb{N}$

$$f(t) - \sum_{n=0}^N \phi_n(t) = \mathcal{O}_0(\phi_{N+1}(t)).$$

In this case, we write

$$f(t) \underset{t \downarrow 0}{\sim} \sum_{n=0}^\infty \phi_n(t).$$

Now, by an iterative argument exploiting Theorem 3 we obtain the asymptotic expansion of htr_P , which is the main result of the paper:

Theorem 4 Let P be a an operator satisfying the General Assumption and such that:

1. ζ_P is well-defined with abscissa of convergence $L \geq 0$.
2. ζ_P has a meromorphic continuation to the whole complex plane.
3. There exists a sequence $(R_n)_{n \in \mathbb{N}}$ of real numbers strictly increasing to infinity, such that $-R_0 > L$, $-R_n < L$ for $n \geq 1$, and for each $n \in \mathbb{N}$ function $\mathcal{Z}(s) = \Gamma(s)\zeta_P(s)$ is regular and Lebesgue integrable over the vertical line $\Re(s) = -R_n$.
4. For each $n \geq 1$ there exists a strictly increasing sequence $(y_k^{(n)})_{k \in \mathbb{Z}}$ with $y_0^{(n)} = 0$ and $y_k^{(n)} \rightarrow \pm\infty$ as $k \rightarrow \pm\infty$ such that

$$\sup_{x \in [-R_n, -R_{n-1}]} \left| \mathcal{Z} \left(x + iy_k^{(n)} \right) \right| \rightarrow 0,$$

as $k \rightarrow \pm\infty$ and the suprema for all $k \in \mathbb{Z}^*$ are finite.

For $n, k \geq 1$ let D_k^n denote a rectangle $\{x + iy : -R_n \leq x \leq -R_{n-1}, y_{-k}^{(n)} \leq y \leq y_k^{(n)}\} \subset \mathbb{C}$ and $D_0^n = \emptyset$ for $n \geq 1$. Set $S_k^n = \mathfrak{P}_{\mathcal{Z}}(D_k^n \setminus D_{k-1}^n)$ (see Figs. 1 and 2). Then, for $t > 0$

$$\text{htr}_P(t) \underset{t \downarrow 0}{\sim} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sum_{s \in S_k^n} r_s(t), \tag{22}$$

where

$$r_s(t) = \text{Res}_{s'=s} \left(\mathcal{Z}(s') t^{-s'} \right).$$

The RHS of (22) is to be understood as the asymptotic (formal) series $\sum_{n=1}^{\infty} \phi_n(t)$, with $\phi_n : \mathbb{R}^+ \rightarrow \mathbb{C}$ being an asymptotic scale defined by the convergent series

$$\phi_n(t) := \sum_{k=0}^{\infty} \sum_{s \in S_k^n} r_s(t), \quad \text{for } n \geq 1. \tag{23}$$

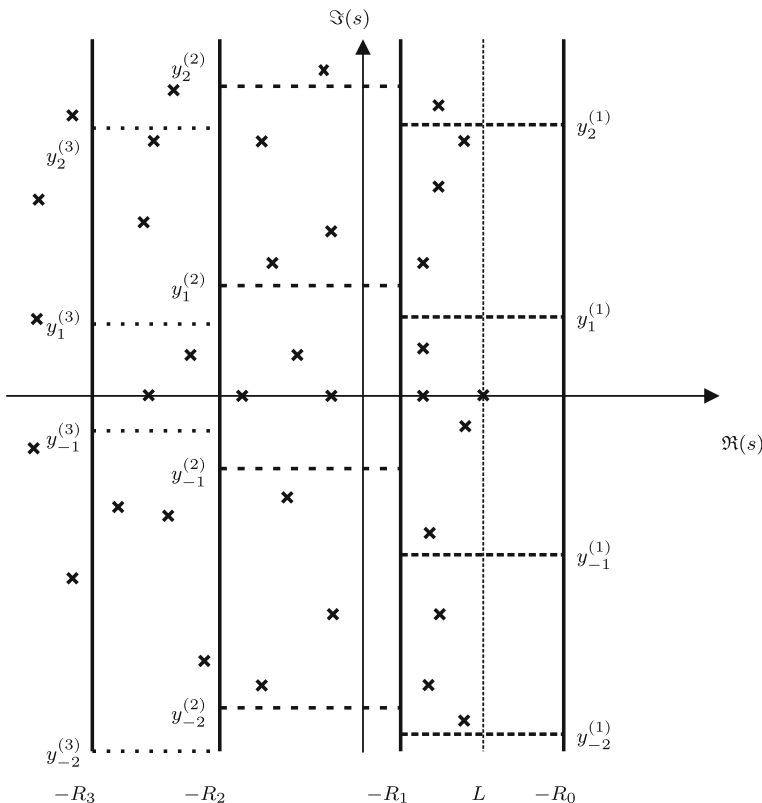


Fig. 2 Illustration for Theorem 4. See also Fig. 1

Proof Let us first apply Theorem 3 to ζ_P with $c = -R_0$, $R = R_1$ and $y_k = y_k^{(1)}$ for $k \in \mathbb{Z}$. All of the assumptions are readily fulfilled and we obtain

$$\text{htr}_P(t) = \phi_1(t) + F_{R_1}(t), \tag{24}$$

where ϕ_1 is given by (23).

Then, we use Theorem 3 again with $c = -R_1$, $R = R_2$ and $y_k = y_k^{(2)}$ for $k \in \mathbb{Z}$. Strictly speaking in this case c is not greater than L as required by Assumption 3, but as long as \mathcal{Z} is regular and Lebesgue integrable on the lines $\Re(s) = -R_1$ and $\Re(s) = -R_0$ the same arguments apply. The contour integral, now being $2\pi i(F_{R_1}(t) - F_{R_2}(t))$, is again equal to the sum of residues, giving

$$F_{R_1}(t) = \phi_2(t) + F_{R_2}(t),$$

instead of (24). Repeating this argument for the function ζ_P in each strip $-R_{n+1} \leq \Re(s) \leq -R_n$ we obtain a recurrence relation:

$$F_{R_n}(t) = \phi_{n+1}(t) + F_{R_{n+1}}(t). \tag{25}$$

Thus, for any $N \in \mathbb{N}^+$ we have

$$\text{htr}_P(t) = \sum_{n=1}^N \phi_n(t) + F_{R_N}(t),$$

where $F_{R_N}(t) = \mathcal{O}_0(t^{R_N})$ and by Definition 4 of an asymptotic expansion we conclude that

$$\text{htr}_P(t) \underset{t \downarrow 0}{\sim} \sum_{n=1}^{\infty} \phi_n(t).$$

□

The choice of the sequence (R_n) determines the asymptotic scale (ϕ_n) of the expansion (22). Thus, in our method the residues are first summed in each vertical strip $\{s \in \mathbb{C} : -R_{n+1} < \Re(s) < -R_n\}$ yielding $\phi_n(t)$, and then the contributions from subsequent strips form an asymptotic series.

3.2 Exact and Almost Exact Expansions

Having an asymptotic series for $\text{htr}_P(t)$, it is natural to ask whether this series converges for some t . This can be checked by analysing the limit of $F_{R_N}(t)$ as $N \rightarrow \infty$. If this limit exists and $F_{R_N} \rightarrow F_\infty < \infty$ locally uniformly on the interval $(0, T)$ for some $T > 0$, then the series in (22) converges locally uniformly on $(0, T)$ to $\text{htr}_P(t) - F_\infty(t)$. Note that F_∞ is $\mathcal{O}_0(t^R)$ for every $R \in \mathbb{R}$. Let us adopt the following definition:

Definition 5 If the series on the RHS of (22) converges (to $\text{htr}_P(t) - F_\infty(t)$) locally uniformly on $(0, T)$ for some $T > 0$ then we say that htr_P has an *almost exact expansion* on $(0, T)$. If moreover $F_\infty = 0$ identically, then htr_P has an *exact expansion* on $(0, T)$.

In Section 4 we will meet various examples of almost exact and exact expansions of heat traces, with $T < \infty$ and $T = \infty$ as well. Let us now give a general sufficient condition for the existence of an exact expansion.

Theorem 5 *Let the assumptions of Theorem 4 be fulfilled and let the estimate*

$$|\mathcal{Z}(-R_n + iy)| \leq C_n e^{-\epsilon_n |y|} \tag{26}$$

hold for every $y \in \mathbb{R}$ and $n \in \mathbb{N}$, where C_n, ϵ_n are some positive constants for $n \in \mathbb{N}$. Assume moreover that the sequence $\sqrt[n]{C_n/\epsilon_n}$ is bounded for $n \in \mathbb{N}$.

Then,

$$\text{htr}_P(t) = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \sum_{s \in S_k^n} r_s(t), \tag{27}$$

for $t \in (0, T)$, where

$$T := \left(\limsup_{n \rightarrow \infty} \sqrt[n]{\frac{C_n}{\epsilon_n}} \right)^{-1}$$

and the series over n is locally uniformly convergent on $(0, T)$.

Proof Let us estimate the reminder $F_{R_n}(t)$ as follows

$$\begin{aligned} |F_{R_n}(t)| &= \frac{1}{2\pi} \left| \int_{-\infty}^{\infty} \mathcal{Z}(-R_n + iy) t^{R_n - iy} dy \right| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} C_n e^{-\epsilon_n |y|} t^{R_n} dy \\ &= \frac{C_n t^{R_n}}{2\pi} 2 \int_0^{\infty} e^{-\epsilon_n y} dy = \frac{C_n t^{R_n}}{2\pi} \frac{2}{\epsilon_n}. \end{aligned}$$

Let $0 < T' < T$. Then, for any $t \in (0, T']$

$$\limsup_{n \rightarrow \infty} \left(t \sqrt[n]{\frac{C_n}{\epsilon_n}} \right) = \frac{t}{T} \leq \frac{T'}{T}.$$

Hence, for sufficiently large n we have $t \sqrt[n]{\frac{C_n}{\epsilon_n}} < a$, where $a \in (T'/T, 1)$ is some constant independent of t . Then

$$|F_{R_n}(t)| \leq \frac{C_n t^{R_n}}{\epsilon_n \pi} < \frac{a^{R_n}}{\pi} \rightarrow 0, \tag{28}$$

so $F_{R_n}(t)$ tends to 0 uniformly for $t \in (0, T']$ as $n \rightarrow \infty$. Since T' can be any number in $(0, T)$, the theorem is proven. □

Note that again the order of summation in (27) is important and the convergence may be only conditional. As in the case of the vertical sum over k (see (17)), the convergence properties depend on the choice of the (R_n) sequence. And as in Proposition 4 we can refine the assumptions of Theorem 5 to obtain an absolute convergence of the series over n in formula (27).

Let us first adopt the following definition:

Definition 6 Let the operator P be such that

$$\text{htr}_P(t) = \sum_{n=1}^{\infty} \phi_n(t), \quad \text{for } t \in (0, T), \tag{29}$$

with some $0 < T \leq +\infty$, where $(\phi_n)_{n \in \mathbb{N}}$ is the asymptotic scale given by (23). We say that the heat trace associated with P has an *absolutely exact* expansion on $(0, \tilde{T})$ if the series

$$\sum_{n=1}^{\infty} |\phi_n(t)|$$

is locally uniformly convergent on $(0, \tilde{T})$ with some $0 < \tilde{T} \leq T$.

Note, that since the RHS of (29) is not in general a Taylor series, its domain of convergence does not necessarily coincide with that of absolute convergence, hence $\tilde{T} \leq T$. In particular if $0 = \tilde{T} < T$, then the expansion of the heat trace associated with P will be exact on $(0, T)$, but nowhere absolutely exact.

By using a similar reasoning as the one used in the proof of Proposition 4, we can refine the assumptions of Theorem 5 to obtain an absolutely exact heat trace expansion.

Proposition 5 *Let the assumptions of Theorem 5 be fulfilled and moreover let $\log n = \mathcal{O}_{\infty}(R_n)$ (i.e. R_n grow faster than $\log n$). Then the expansion is absolutely exact on the whole domain $(0, T)$.*

Proof As announced, we proceed similarly to the proof of Proposition 4, but now we shall add together the contributions of subsequent vertical integrals. From (25) we have

$$|\phi_n(t)| = |F_{R_n}(t) - F_{R_{n-1}}(t)| \leq |F_{R_n}(t)| + |F_{R_{n-1}}(t)|.$$

Now, (28) implies

$$\sum_{n=1}^{\infty} |\phi_n(t)| \leq 2 \sum_{n=1}^{\infty} |F_{R_n}(t)| \leq 2 \sum_{n=1}^{\infty} \frac{C_n t^{R_n}}{\epsilon_n \pi} < \frac{2}{\pi} \sum_{n=1}^{\infty} a^{R_n}.$$

with $a \in (T'/T, 1)$ for any $T' \in (0, T)$. Therefore, it suffices to show that the last series is convergent for any $a < 1$. Taking $x = -\log a$ we see that it is again a general Dirichlet series (5) with $a_n = 1, b_n = R_n$ for $n \in \mathbb{N}$. By Theorem 1 we conclude that its abscissa of convergence is

$$\limsup_{n \rightarrow \infty} \frac{\log n}{R_n},$$

which equals 0 by hypothesis. Thus, the series is convergent for $x > 0$, i.e $a < 1$. \square

The characterisation of an absolutely exact expansion through Proposition 5 is particularly useful. If we have an exact expansion of a heat trace htr_P on $(0, T)$, it is sufficient to check whether the sequence (R_n) of our choice grows faster than $\log n$.

In fact, all of the examples of exact expansions presented in Section 4 are actually absolutely exact in the same domain.

Let us also note, that if we have an exact expansion of the heat trace for an open interval $(0, T)$, then htr_P actually provides an analytic continuation of the RHS of (27) to the whole half line \mathbb{R}^+ .

Let us now turn to the case of an almost exact expansion of heat traces. The situation is somewhat different than that of an exact expansion, as one would need precise formulae rather than estimates to guarantee that the limit of F_{R_N} as $N \rightarrow \infty$ is finite, but non-zero. It may be seen as a kind of “critical” case, in the sense that a slight perturbation of the zeta-function renders the expansion divergent (see Section 4.1.4). One of these specific cases is captured by the following proposition.

Proposition 6 *If the operator P fulfilling the assumptions of Theorem 4 is such that $\mathfrak{P}_{\mathcal{Z}}(\mathbb{C})$ is a finite set, then the expansion*

$$\text{htr}_P(t) = \sum_{s \in \mathfrak{P}_{\mathcal{Z}}(\mathbb{C})} r_s(t) + F_{\infty}(t) \tag{30}$$

is almost exact for all $t > 0$, but not exact.

Proof Let us note that $\mathfrak{P}_{\mathcal{Z}}(\mathbb{C})$ being finite requires $\mathfrak{P}_{\zeta_P}(\mathbb{C})$ to be finite, but also that the zeros of ζ_P cancel all but a finite number of poles of Γ , i.e. $\zeta_P(-n) = 0$ for all but a finite number of $n \in \mathbb{N}$. In this case, Theorem 4 yields an asymptotic expansion, which has only a finite number of terms, hence it converges for all $t > 0$. As t tends to infinity we have from (10) that $\text{htr}_P(t) = \mathcal{O}_{t \rightarrow \infty}(e^{-t\lambda_0})$. But this is not compatible with the behaviour $t^{-s}(\log t)^k$ (compare (20)) of the summands of the first term on the RHS of (30). Hence, $F_{\infty}(t)$ cannot be 0 and the expansion (30) is almost exact, but not exact. □

3.3 Truncated Zeta-function

For further purposes it is convenient to define also the following spectral function.

Definition 7 For any $N \in \mathbb{N}^+$, a *truncated zeta-function* associated with the operator P is a complex function

$$\mathbb{C} \supset \text{Dom}(\zeta_P) \ni s \mapsto \hat{\zeta}_P^N(s) = \sum_{n=N}^{\infty} M_n \lambda_n^{-s}. \tag{31}$$

The two zeta-functions are related by

$$\zeta_P(s) = \sum_{n=0}^{N-1} M_n \lambda_n^{-s} + \hat{\zeta}_P^N(s). \tag{32}$$

The first term of the RHS of the above formula is an entire function of s , hence the analytic properties of ζ_P and $\hat{\zeta}_P^N$ are identical.

We can actually slightly relax the growth rate Assumptions 3 and 4 of Theorem 4 by considering the truncated zeta-function (31) at the place of ζ_P .

Proposition 7 *If the assumptions of Theorem 3 hold for $\hat{\zeta}_P^N$ and $\hat{\mathcal{Z}} : s \mapsto \Gamma(s)\hat{\zeta}_P^N(s)$ with some finite $N \in \mathbb{N}^+$, then for $t > 0$ we have*

$$\text{htr}_P(t) = \sum_{n=0}^{N-1} M_n e^{-t\lambda_n} + \sum_{n=0}^{\infty} \sum_{s \in S_n} \hat{r}_s(t) + \hat{F}_R(t),$$

where

$$\begin{aligned} \hat{r}_s(t) &:= \text{Res}_{s'=s} \left(\hat{\mathcal{Z}}(s') t^{-s'} \right), \\ \hat{F}_R(t) &:= \frac{1}{2\pi i} \int_{-R-i\infty}^{-R+i\infty} \hat{\mathcal{Z}}(s) t^{-s} ds = \mathcal{O}_0(t^R). \end{aligned}$$

Proof The proof of Theorem 3 will work equally well for this case with the only difference at line (16) which will now read

$$\frac{1}{2\pi i} \lim_{n \rightarrow \infty} \hat{I}_c(n) = \mathcal{M}^{-1}[\hat{\mathcal{Z}}](t).$$

Applying the inverse Mellin transform to relation (32) multiplied by $\Gamma(s)$ we get (using Lemma 1)

$$\text{htr}_P(t) = \sum_{n=0}^{N-1} M_n \mathcal{M}^{-1}[\lambda_n^{-s} \Gamma(s)] + \mathcal{M}^{-1}[\hat{\mathcal{Z}}](t).$$

Note that for any $n \in \mathbb{N}$ the function $\lambda_n^{-s} \Gamma(s)$ is integrable over any vertical line $\Re(s) = c \notin \mathbb{N}$ (as Γ decays exponentially on verticals, compare (52)) and that it equals to $\mathcal{M}[\exp(-t\lambda_n)](s)$ (cf. (11)). The claim then follows from Theorem 2. \square

As ζ_P and $\hat{\zeta}_P^N$ differ by an entire function, the assumptions (i) and (ii) of Theorem 3 are fulfilled by $\hat{\zeta}_P^N$ if and only if they are fulfilled by ζ_P . In view of Proposition 7 it is convenient to introduce the notion of a truncated heat trace:

$$\hat{\text{htr}}_P^N(t) := \text{htr}_P(t) - \sum_{n=0}^{N-1} M_n e^{-t\lambda_n}. \tag{33}$$

Then all of the considerations about the truncated zeta-function can be expressed in one simple remark:

Remark 2 All of the results concerning the asymptotic and exact expansions of heat traces hold with ζ_P altered for the truncated zeta-function $\hat{\zeta}_P^N$ (31). One proceeds as in Proposition 7 and substitutes $\hat{\text{htr}}_P^N$ for htr_P and $\hat{\mathcal{Z}}$ for \mathcal{Z} in all of the assertions.

The truncation procedure proves very useful when some of the assumptions on the zeta-function are not met, because of a finite number of terms in its definition (or

because of a finite number of eigenvalues of P). We will encounter one such situation when discussing operators with polynomial spectrum in the next section. Zeta-functions of such operators do not generally satisfy the assumption of integrability on vertical lines (Assumption 3 of Theorems 3 and 4), however they can always be truncated to get an asymptotic expansion of the truncated heat trace (see Theorem 7).

Another situation when truncation is beneficial is when the operator P under consideration is not positive or invertible, but there exists such a real constant c , that $P + c$ satisfies the General Assumption (P still has to be self-adjoint and have compact resolvent). Such a P has a finite number of non-positive eigenvalues and we can truncate them. The heat trace htr_P still may be defined, although the definition of zeta-function ζ_P is ambiguous at best. However, $\hat{\zeta}_P^N$ makes sense for N such that for every $n \geq N$ one has $\lambda_n > 0$, if only its abscissa of convergence is finite. Then one can obtain the asymptotic expansion of htr_P from Proposition 7. An important special case is the following:

Remark 3 Let P be a positive densely defined operator with compact resolvent and non-trivial kernel, acting on a separable infinite-dimensional Hilbert space. Let $(\lambda_n)_{n=0}^\infty$ denote the strictly increasing sequence of its eigenvalues, that tends to infinity. Then $\lambda_0 = 0$ with $M_0 = \dim \ker P < \infty$ and every other eigenvalue is positive. If htr_P is well-defined then from (33) one obtains

$$\text{htr}_P(t) = \dim \ker P + \hat{\text{htr}}_P^1(t)$$

and one can apply the Mellin transform techniques to obtain the asymptotic expansion of $\hat{\text{htr}}_P^1(t)$ if only $\hat{\zeta}_P^1$ is well-defined and possesses the required properties.

4 Examples

4.1 Operators of Polynomial Spectrum

In this section we investigate heat traces associated with operators, the eigenvalues and multiplicities of which are given by polynomials. For brevity we shall call such operators – the *operators of polynomial spectrum*. They appear naturally in the context of Dirac and Laplace operators on spheres [6, 69] and their isospectral deformations [17, 18, 24–26, 62].

Moreover, the results presented in this section apply almost directly when the spectra of the relevant operators can be written as

$$\sigma(P) = \bigcup_{k=1}^K \left\{ \lambda_n^k : n \in \mathbb{N} \right\}, \tag{34}$$

for some finite $K \in \mathbb{N}^+$, and for each $k \in \{1, \dots, K\}$ the eigenvalues λ_n^k and their respective multiplicities M_n^k are given by polynomials in n . Since the (inverse) Mellin transform is linear, one can apply the general theory to each sequence $(\lambda_n^k)_{n \in \mathbb{N}}$ separately. One can find examples of such spectra in the framework of Dirac operators on some homogeneous spaces [3, 57, 68], like the

Poincaré sphere or lens spaces. Indeed, [57, (6.3), (6.11)] give the decomposition of Dirac spectrum into four¹ ($K = 4$) polynomial sequences for quaternionic space $SU(2)/Q8$. The same paper describes the spectrum of Poincaré sphere with $K = 120$ [57, Prop. 7.2] and of four-dimensional lens space $SU(2)/\mathbb{Z}_4$ with $K = 4$ [57, Lemma 11.20].

By combining the results of [59] with general theorems presented in the preceding section we are able to prove the existence of an asymptotic expansion of the heat traces associated with operators of polynomial spectra (see Theorem 7). We also derive sufficient conditions for the convergence of the expansion.

4.1.1 Asymptotic Expansions

Let us start with the following theorem summarising the behaviour of zeta-functions associated with operators in the considered class (compare [59, Theorems A and B]).

Theorem 6 *Let P be an operator satisfying the General Assumption with eigenvalues $\lambda_n = A(n)$ and multiplicities $M_n = B(n)$ for $n \in \mathbb{N}$, where A and B are polynomials.*

Then:

1. *Roots of A are not in \mathbb{N} ;*
2. *ζ_P is well-defined, with the abscissa of convergence $L = (1 + \deg B)/\deg A$;*
3. *ζ_P admits a meromorphic extension to the whole complex plane;*
4. *$\Re_{\zeta_P}(\mathbb{C}) \subset \frac{1}{\deg A}(1 + \deg B - \mathbb{N}) \setminus (-\mathbb{N})$ and all of the poles are of first order;*
5. *For any $\epsilon > 0$ there exists $N > 0$ such that the truncated zeta-function (see (31)), $\hat{\zeta}_P^N$, obeys the following growth rate along the vertical lines*

$$\left| \hat{\zeta}_P^N(x + iy) \right| = \mathcal{O}_\infty \left(e^{\epsilon|y|} \right),$$

for any $x \in \mathbb{R}$.

Proof The point 1 is a direct consequence of the fact, that P is invertible, and hence all its eigenvalues are non-zero. Now, the zeta-function associated with the operator P for $\Re(s) > L$ can be written as

$$\zeta_P(s) = \sum_{n=0}^\infty \frac{B(n)}{A(n)^s}, \tag{35}$$

since the roots of A are not in \mathbb{N} . The corresponding spectral growth function reads

$$N(\lambda_n) = \sum_{k=0}^n B(k) =: \tilde{B}(n),$$

¹To be precise, eigenvalues and multiplicities are in [57] indexed by $n \in \mathbb{Z}$, and hence they need only $K/2$ polynomials. In our convention with $n \in \mathbb{N}$ we need separate polynomials for positive and negative part of spectrum, what doubles the number of polynomials given in [57].

and it is a classical result (see e.g. the Faulhaber’s formula) that \tilde{B} is a polynomial and $\deg \tilde{B} = \deg B + 1$. Hence, by Theorem 1 the abscissa of convergence of ζ_P equals (cf. (8))

$$L = \limsup_{n \rightarrow \infty} \frac{\log N(\lambda_n)}{\log \lambda_n} = \limsup_{n \rightarrow \infty} \frac{\log \tilde{B}(n)}{\log A(n)} = \frac{\deg B + 1}{\deg A}. \tag{36}$$

Point 3 is the content of Theorem B in [59] (see also Remark 1 therein).

To prove points 4 and 5 we introduce for $r \in \mathbb{N}^+$ the following multi-variable series (see [59, Section 1, Formula (2)])

$$\zeta_r(s_1, \dots, s_r; \alpha_1, \dots, \alpha_r) := \sum_{n=0}^{\infty} (n + \alpha_1)^{-s_1} (n + \alpha_2)^{-s_2} \dots (n + \alpha_r)^{-s_r}, \tag{37}$$

for $s_1, \dots, s_r \in \mathbb{C}$ with $\Re(s_1 + \dots + s_r) > 1$ and $\alpha_1, \dots, \alpha_r \in \mathbb{C} \setminus (-\mathbb{N})$. The branch of logarithm in $(n + \alpha_j)^{-s_j} = \exp(-s_j \log(n + \alpha_j))$ is chosen to be $-\pi < \arg(n + \alpha_j) \leq \pi$.

If we take $r = \deg A$ and α_i for $i = 1, \dots, \deg A$ to be the roots of A , then we can write the polynomials A and B as

$$A(n) = a \prod_{i=1}^{\deg A} (n + \alpha_i), \quad B(n) = \sum_{j=0}^{\deg B} \tilde{b}_j (n + \alpha_1)^j.$$

Plugging this representation into (35) and using the definition (37) we get (compare [59, Section 1, Formula (3)])

$$\zeta_P(s) = a^{-s} \sum_{j=0}^{\deg B} \tilde{b}_j \zeta_{\deg A}(s - j, s, \dots, s; \alpha_1, \dots, \alpha_{\deg A}). \tag{38}$$

In [59] the analytic properties of ζ_r functions are studied, what allows to draw conclusions about the analytic properties of ζ_P on the strength of formula (38). Formulae (10) and (12) in [59, Section 1] provide an explicit meromorphic continuation of ζ_r to \mathbb{C}^r . The proof is based on the induction on r with the Riemann zeta-function as the starting point $r = 1$.

The claim $\mathfrak{P}_{\zeta_P}(\mathbb{C}) \subset \frac{1}{\deg A}(1 + \deg B - \mathbb{N})$ then follows from [59, Theorem A]. On the other hand, [59, Theorem C] implies that $-\mathbb{N} \not\subset \mathfrak{P}_{\zeta_P}(\mathbb{C})$. The fact that the poles ζ_P are at most of first order is a consequence of [59, Lemma 5] and the formula [59, Section 1, Formula (12)] (see also [59, Section 1.2 and p. 242]).

Let us now pass on to the last point of the claim – the estimate of $\hat{\zeta}_P^N$ on vertical lines. We first quote the result [59, Proposition 1 (iii)] translated to our notation:

$$|\zeta_r(x_1 + iy_1, \dots, x_r + iy_r; \alpha_1, \dots, \alpha_{r-1}, 0)| = \mathcal{O}_{\infty} \left(C(y_1, \dots, y_r) \cdot e^{\rho_1|y_1| + \dots + \rho_{r-1}|y_{r-1}|} \right),$$

with $\rho_i = |\arg \alpha_i|$ and C – a polynomial. Now, we repeat the proof of Proposition 1 (iii) [59, p. 240] with the formula (12) instead of (10) therein. The reasoning goes

along the same lines, [59, Lemma 1] still applies with $y = \Im(z)$, $A = -\frac{\pi}{2}$, $\alpha = y_r$, $B = \rho_r - \frac{\pi}{2}$, $\beta = 0$ and we conclude that

$$|\zeta_r(x_1 + iy_1, \dots, x_r + iy_r; \alpha_1, \dots, \alpha_{r-1}, \alpha_r)| = \mathcal{O}_\infty \left(C(y_1, \dots, y_r) \cdot e^{\rho_1|y_1| + \dots + \rho_{r-1}|y_{r-1}| + \rho_r|y_r|} \right). \tag{39}$$

Hence, by (38) we obtain

$$|\zeta_P(x + iy)| = \mathcal{O}_\infty \left(C(y)^{\deg A} \cdot e^{|y| \sum_{i=1}^{\deg A} |\arg \alpha_i|} \right) = \mathcal{O}_\infty \left(e^{|y| \left(\sum_{i=1}^{\deg A} |\arg \alpha_i| + \delta \right)} \right),$$

for any $\delta > 0$.

Here comes the advantage of using the truncated zeta-function $\hat{\zeta}_P^N$, since by starting the zeta-series at $n = N$ instead of $n = 0$ we effectively shift $\alpha_i \rightarrow \alpha_i + N$ (compare [59, Remark 1]). This means that by taking N large enough one can make $\sum_{i=1}^{\deg A} |\arg \alpha_i|$ arbitrarily small and assertion 5 is proven. \square

Typically (i.e. for generic polynomials A and B) the zeta-functions ζ_P will have an infinite number of poles on the negative part of the real axis. This is a consequence of the [59, formulae (10) and (12)]. Let us illustrate this property with the following simple example.

Example 1 Let P be an operator with eigenvalues $\lambda_n = A(n) = n(n + \alpha)$ for $n \in \mathbb{N}^+$ with some $\alpha \in \mathbb{R}^+$ and no degeneracies (i.e. $B(n) = 1$), then $\mathfrak{P}_{\zeta_P}(\mathbb{C}) = \frac{1}{2} - \mathbb{N}$.

Indeed, from the formulae (38) and [59, (10)] we have for $\Re(s) > -M/2$ with any $M \in \mathbb{N}$:

$$\zeta_P(s) = \sum_{j=0}^M \binom{-s}{j} \zeta(2s + j) \alpha^j + h_M(s),$$

where h_M is a remainder term holomorphic for $\Re(s) > -M/2$. So for any $n \in \mathbb{N}$ we have

$$\operatorname{Res}_{s=\frac{1}{2}-n} \zeta_P(s) = \operatorname{Res}_{s=\frac{1}{2}-n} \left(\sum_{j=0}^{2n} \binom{-s}{j} \zeta(2s + j) \alpha^j + h_{2n}(s) \right) = \binom{n - \frac{1}{2}}{2n} \alpha^{2n} \neq 0.$$

There exist however special operators P with fine-tuned polynomials A and B for which the zeta-function ζ_P will only have a finite number of poles. It happens for instance in the following case:

Proposition 8 *If all of the roots of the polynomial A are equal then the set $\mathfrak{P}_{\zeta_P}(\mathbb{C})$ is finite.*

Proof Since $A(n) = a(n + \alpha)^{\deg A}$, we have for $\Re(s) > (1 + \deg B)/\deg A$

$$\zeta_P(s) = \sum_{n=0}^{\infty} \sum_{j=0}^{\deg B} \tilde{b}_j a^{-s} (n + \alpha)^{-(\deg A)s+j} = a^{-s} \sum_{j=0}^{\deg B} \tilde{b}_j \zeta_H((\deg A)s - j, \alpha),$$

where ζ_H is the Hurwitz zeta-function. Since ζ_H is meromorphic on \mathbb{C} with a single simple pole at 1 and the sum over j for ζ_P is finite, the assertion follows. \square

The equality of all roots of A is a sufficient condition for $\mathfrak{P}_{\zeta_P}(\mathbb{C})$ to be finite, but not a necessary one. Consider for instance the operator P with eigenvalues $\lambda_n = A(n) = n^3 + 1$ and degeneracies $B(n) = n^2$. Then, by using

$$(1 + x)^{-s} = \sum_{k=0}^{\infty} \binom{-s}{k} x^k, \quad \text{for } |x| < 1,$$

we obtain

$$\zeta_P(s) = \sum_{n=0}^{\infty} \frac{n^2}{(n^3 + 1)^s} = \sum_{n=0}^{\infty} n^{-3s+2} (1 + n^{-3})^{-s} = \sum_{j=0}^{\infty} \binom{-s}{j} \zeta(3s + 3j - 2),$$

which gives a meromorphic extension of ζ_P to the whole complex plane. On the other hand, the Theorem 6 assertion 4 implies that $\mathfrak{P}_{\zeta_P}(\mathbb{C}) \subset \frac{1}{3}(3 - \mathbb{N}) \setminus (-\mathbb{N})$. Moreover, the poles of ζ_P come only from the poles of $s \mapsto \zeta(3s + 3j - 2)$. But at $s = \frac{k}{3} - n$, $\zeta(k - 3n + 3j - 2)$ is finite for all $n \in \mathbb{N}$, $j \in \mathbb{N}$ and $k \in \{1, 2\}$ since the only pole of the Riemann zeta-function is at 1. Hence, ζ_P has only one simple pole $\mathfrak{P}_{\zeta_P}(\mathbb{C}) = \{1\}$.

It is interesting to compare this result with [40, Lemma 1.10.1] and [39, p. 2]. The former tells us that zeta-functions associated with classical positive elliptic differential operators on (finite-dimensional) compact manifolds have a finite number of poles only. On the other hand, the latter says that positive elliptic classical pseudodifferential operators generically do have an infinite number of simple poles, including at $-\mathbb{N}$.

Clearly, elliptic pseudodifferential operators on compact manifolds need not be of polynomial spectrum — for instance, the scalar Laplacian on \mathbb{T}^2 [32]. On the other hand, one might ask whether any operator of polynomial spectrum can be realised as a classical elliptic pseudodifferential operator on some compact manifold. We consider it as an interesting open problem, the solution to which may shed more light on the geometrical meaning of the operators in this class.

Finally, let us turn to the heat traces of operators with eigenvalues and multiplicities given by polynomials. The following Theorem establishes the existence of an asymptotic expansion of heat trace for any operator in this class.

Theorem 7 *Let P be an operator with eigenvalues $\lambda_n = A(n)$ and multiplicities $M_n = B(n)$, where A and B are polynomials of degree $\deg A$ and $\deg B$ respectively.*

Then, there exists $N \in \mathbb{N}$ such that

$$\widehat{\text{htr}}_P^N(t) = \text{htr}_P(t) - \sum_{n=0}^{N-1} B(n)e^{-A(n)t} \underset{t \downarrow 0}{\sim} \sum_{k=0}^{\infty} \underset{s=\ell-k/(\deg A)}{\text{Res}} \left(\Gamma(s)\widehat{\zeta}_P^N(s) \right) t^{\ell-k/(\deg A)}, \tag{40}$$

where $\ell = (1 + \deg B)/(\deg A)$.

Proof The claim follows from a direct application of Theorem 4 together with Remark 2. The assumptions are met by Theorem 6. □

Remark 4 Let us remark that one could actually extend Theorem 6 to operators, the eigenvalues and multiplicites of which can be written as

$$\lambda_n = an^{\gamma_0} \prod_{i=1}^{\deg A} (n^{\gamma_i} + \alpha_i), \quad M_n = \sum_{j=0}^{\deg B} b_j n^{\beta_j},$$

with $\gamma_i > 0$ for $i = 1, \dots, \deg A$, $\sum_{i=0}^{\deg A} \gamma_i > 0$ and $b_j > 0$ for at least one $j \in \{0, \dots, \deg B\}$. Then, instead of (37), one would need to seek a meromorphic continuation to \mathbb{C}^r of the functions

$$\widetilde{\zeta}_r(s_1, \dots, s_r; \gamma_0, \dots, \gamma_r; \alpha_1, \dots, \alpha_r) := \sum_{n=0}^{\infty} n^{-s_0\gamma_0} (n^{\gamma_1} + \alpha_1)^{-s_1} \dots (n^{\gamma_r} + \alpha_r)^{-s_r}.$$

The latter could again be accomplished with the help of the Mellin–Barnes formula as in [59]. It is plausible that ζ_P functions obtained in this way will only have first order isolated poles and some bound similar to (39) can be established. On the other hand, the regularity of ζ_P at $-\mathbb{N}$ is not to be expected in general.

We have chosen to formulate Theorem 6 for operators of polynomial spectrum, as considering more general ones described above would add to the complexity without being strongly motivated. In fact, we were not able to find any reasonable geometric example where the operator falls into this larger class, but not the one of operators of polynomial spectrum.

Generically, the expansion (40) will only be an asymptotic one. There exist, however, operators for which the formula (40) will be exact or almost exact on $(0, T)$ for some $0 < T \leq +\infty$.

4.1.2 Exact Expansions

In this subsection we consider a class of operators, which have an exact expansion of the associated heat traces. In particular, it will serve as an illustration for Theorem 5.

Proposition 9 *Let P be an operator of polynomial spectrum with eigenvalues $\lambda_n = A(n)$ and multiplicities $M_n = B(n)$. If $\deg A = 1$, i.e. if $A(n) = a(n + \alpha)$ for some $a > 0$, $\alpha \geq 0$, then the asymptotic expansion (40) is absolutely exact on $(0, 2\pi/a)$.*

Proof To start, let us consider an operator P_0 with $a = 1, \alpha = 1$ and $B(n) = 1$. We can calculate directly:

$$\zeta_P(s) = \sum_{n=1}^{\infty} n^{-s} = \zeta(s), \text{ for } \Re(s) > 1.$$

The Riemann zeta-function ζ extends meromorphically to \mathbb{C} with a single simple pole at 1, whereas Γ has simple poles at non-positive integer numbers. The values of the residues read

$$\text{Res}_{s=1} \zeta(s) = 1, \quad \text{Res}_{s=-n} \Gamma(s) = \frac{(-1)^n}{n!} \quad \forall n \in \mathbb{N}.$$

Moreover, the values of ζ at non-positive integers are given by

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}, \quad \forall n \in \mathbb{N},$$

where B_n denote the Bernoulli numbers with the convention $B_1 = +1/2$. As a consequence, ζ vanishes at negative even integers, so the function $\mathcal{Z}: s \mapsto \Gamma(s)\zeta(s)$ is in fact regular at these points, and the set of its poles is $\mathfrak{P}_{\mathcal{Z}}(\mathbb{C}) = \{1, 0, -1, -3, -5, \dots\}$. Theorem 7 therefore yields

$$\text{htr}_{P_0}(t) \underset{t \downarrow 0}{\sim} t^{-1} - \frac{1}{2} + \sum_{n=0}^{\infty} \frac{B_{2n+2}}{(2n+2)!} t^{2n+1}. \tag{41}$$

To check if this expansion is exact, by Theorem 5, we need to find an explicit bound of the form (26) for a suitable sequence (R_n) tending monotonically to $+\infty$. We can, for instance, choose

$$R_0 = -\frac{3}{2}, \quad R_1 = -\frac{1}{2}, \quad R_2 = \frac{1}{2}, \quad R_n = 2(n-2), \quad \text{for } n \geq 3, \tag{42}$$

as \mathcal{Z} is regular at negative even integers. Now, recall the Riemann functional equation [1, Formula (23.2.6)]:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s).$$

By changing s to $(1-s)$ we get

$$\mathcal{Z}(s) = \Gamma(s)\zeta(s) = \frac{2^{s-1} \pi^s \zeta(1-s)}{\sin(\pi(1-s)/2)}.$$

For $s = -R_n + iy$ with $n \geq 3$ the denominator of the above expression equals

$$\sin\left(\pi\left(\frac{1}{2} + n - i\frac{y}{2}\right)\right) = \cos\left(\pi\left(n - i\frac{y}{2}\right)\right) = (-1)^n \cos\left(-i\frac{\pi y}{2}\right) = (-1)^n \cosh\left(\frac{\pi y}{2}\right).$$

Thus, we have

$$|\mathcal{Z}(-R_n + iy)| = \frac{2^{-2n-1} \pi^{-2n} |\zeta(2n+1-iy)|}{\cosh\left(\frac{\pi y}{2}\right)}.$$

For any $y \in \mathbb{R}$, $2 \cosh(y) > e^{|y|}$ and if $x > 1$ then

$$|\zeta(x + iy)| \leq \sum_{k=1}^{\infty} |k^{-x-iy}| = \sum_{k=1}^{\infty} k^{-x} = \zeta(x).$$

Hence,

$$|\mathcal{Z}(-2n + iy)| \leq (2\pi)^{-2n} \zeta(2n + 1) e^{-\frac{\pi|y|}{2}},$$

what means that \mathcal{Z} satisfies the assumptions of Theorem 4 with $C_n := (2\pi)^{-2n} \zeta(2n + 1)$ and a constant $\epsilon_n := \pi/2$. Now,

$$\limsup_{n \rightarrow \infty} \sqrt[n]{C_n/\epsilon_n} = \frac{1}{2\pi} < \infty,$$

as $\zeta(x) \rightarrow 1$ for $x \rightarrow \infty$. Thus, on the strength of Theorem 5, we see that the expansion (41) is exact on $(0, 2\pi)$. Moreover, with the choice (42) of the sequence (R_n) , Proposition 5 applies and the expansion (41) is in fact absolutely exact on $(0, 2\pi)$.

Let us now turn to the general case of an operator P with $A(n) = a(n + \alpha)$ and $B(n) = \sum_{j=0}^{\deg B} b_j n^j$. For any $t > 0$ we have

$$\text{htr}_P(t) = \sum_{n=0}^{\infty} \sum_{j=0}^{\deg B} b_j n^j e^{-ta(n+\alpha)} = e^{-ta\alpha} \sum_{j=0}^{\deg B} b_j \sum_{n=0}^{\infty} n^j e^{-tan} = e^{-ta\alpha} \sum_{j=0}^{\deg B} b_j \frac{d^j}{dt^j} \text{htr}_{P_0}(at).$$

Since $\mathbb{R}^+ \ni t \mapsto \text{htr}_{P_0}(t)$ is an analytic function with an absolutely exact expansion for $t \in (0, 2\pi)$, we conclude that the expansion of htr_P is absolutely exact for $(0, 2\pi/a)$. □

Remark 5 Since the eigenvalues of the operator at hand grow linearly, the heat trace

$$\text{htr}_P(t) = \sum_{n=0}^{\infty} B(n) e^{-tA(n)}$$

associated with P can be summed explicitly – for instance, $\text{htr}_{P_0}(t) = \frac{1}{2} (\coth \frac{t}{2} - 1)$ for any $t > 0$. Hence, htr_P , as a linear combination of derivatives of htr_{P_0} , is actually a complex analytic function of t around 0. In particular, it admits a Laurent expansion around $t = 0$ with the radius of convergence precisely equal to $2\pi/a$ (recall that $\coth \frac{t}{2}$ is singular at $t = 2\pi i$).

We conclude that in this particular subclass the asymptotic expansion of $\text{htr}_P(t)$ as $t \downarrow 0$ obtained from the Mellin transform is equal to the Laurent expansion of htr_P and the value of abscissa of convergence of the former precisely coincides with the value of the radius of convergence of the latter. In particular, it shows that with the estimates adopted in the statement of Theorem 5 one can obtain the *maximal* region of convergence of the expansion of a heat trace.

Proposition 9 has a direct geometrical application:

Corollary 1 *Let \mathcal{D} be the Dirac operator acting on a spinor bundle of S^d — the d -dimensional sphere with round metric. Then, the asymptotic expansion of the heat trace associated with $|\mathcal{D}|$ is exact on $(0, 2\pi)$.*

Proof Recall first that on S^1 there are two possible spin structures, whereas for S^d with $d \geq 2$ there is only one available [4]. In the case of the trivial spin structure on S^1 one has

$$\lambda_{\pm n}(\mathcal{D}) = \pm n, \quad M_{\pm n}(\mathcal{D}) = 1, \quad \text{for } n \in \mathbb{N},$$

hence

$$\lambda_n(|\mathcal{D}|) = n, \quad M_n(|\mathcal{D}|) = 2 - \delta_{n,0}, \quad \text{for } n \in \mathbb{N}.$$

So, for S^1 with the trivial spin structure the operator $|\mathcal{D}|$ is not invertible and we have to use truncated zeta-function. From Remark 3 we get

$$\text{htr}_{|\mathcal{D}|} = 1 + \hat{\text{htr}}^1_{|\mathcal{D}|} = 1 + 2 \text{htr}_{P_0},$$

where P_0 is as in the proof of Proposition 9.

In the case of the non-trivial spin structure, the spectrum of the Dirac operator \mathcal{D} fits into the general pattern for S^d [4, 6, 69]:

$$\lambda_{\pm n}(\mathcal{D}) = \pm \left(n + \frac{d}{2}\right), \quad M_{\pm n}(\mathcal{D}) = 2^{\lfloor \frac{d}{2} \rfloor} \binom{n+d-1}{d-1}, \quad \text{for } n \in \mathbb{N}. \quad (43)$$

Hence, the operator $|\mathcal{D}|$ with

$$\lambda_n(|\mathcal{D}|) = n + \frac{d}{2}, \quad M_n(|\mathcal{D}|) = 2^{\lfloor \frac{d}{2} \rfloor + 1} \binom{n+d-1}{d-1}, \quad \text{for } n \in \mathbb{N},$$

meets the conditions of Proposition 9 with $a = 1$, $\alpha = \frac{d}{2}$ and $B(n) = M_n$ and the assertion follows. □

4.1.3 Almost Exact Expansions

In this subsection we turn to the case of almost exact (but not exact) expansions of heat traces associated with operators of polynomial spectrum.

Proposition 10 *Let P be an operator with eigenvalues $\lambda_n = A(n)$ and multiplicities $M_n = B(n)$ for $n \in \mathbb{N}$. If $A(n) = a(n + \alpha)^{2k} + \beta$ for some $a \in \mathbb{R}^+$, $\alpha \in \frac{1}{2}\mathbb{N}$, $\beta \in \mathbb{R}$, $k \in \mathbb{N}^+$ and $B(n) = \sum_{j=0}^{(\deg B)/2} \tilde{b}_j (n + \alpha)^{2j}$ with $\deg B$ even, then the asymptotic expansion (40) is almost exact for all $t > 0$, but not exact.*

Proof At first let us note that with $P = P' + \beta$ we have

$$\text{htr}_P(t) = e^{-t\beta} \text{htr}_{P'}(t).$$

Thus, it is sufficient to consider the case $\beta = 0$.

Moreover, we can assume that $\alpha = \frac{1}{2}$ or $\alpha = 0$. Indeed, let us denote by $\tilde{\alpha} = \alpha - [\alpha]$ the fractional part of α , then we have

$$\begin{aligned} \text{htr}_{P'}(t) &= \sum_{n=0}^{\infty} B(n)e^{-tA(n)} = \sum_{j=0}^{(\deg B)/2} \tilde{b}_j \sum_{n=0}^{\infty} (n + \alpha)^{2j} e^{-ta(n+\alpha)^{2k}} = \\ &= \sum_{j=0}^{(\deg B)/2} \tilde{b}_j \sum_{n=0}^{\infty} (n + [\alpha] + \tilde{\alpha})^{2j} e^{-ta(n+[\alpha]+\tilde{\alpha})^{2k}} = \\ &= \sum_{j=0}^{(\deg B)/2} \tilde{b}_j \sum_{n=[\alpha]}^{\infty} (n + \tilde{\alpha})^{2j} e^{-ta(n+\tilde{\alpha})^{2k}} = \\ &= \sum_{j=0}^{(\deg B)/2} \tilde{b}_j \sum_{n=0}^{\infty} (n+\tilde{\alpha})^{2j} e^{-ta(n+\tilde{\alpha})^{2k}} - \sum_{j=0}^{(\deg B)/2} \tilde{b}_j \sum_{n=0}^{[\alpha]-1} (n+\tilde{\alpha})^{2j} e^{-ta(n+\tilde{\alpha})^{2k}}, \end{aligned}$$

and the second term is regular at $t = 0$.

Let us denote by P'' the operator with $\lambda_n(P'') = a(n + \tilde{\alpha})^{2k}$ and $M_n(P'') = M_n(P)$. For $\Re(s) > (1 + \deg B)/(\deg A)$ we have

$$\zeta_{P''}(s) = \sum_{n=0}^{\infty} \sum_{j=0}^{(\deg B)/2} \tilde{b}_j a^{-s} (n + \tilde{\alpha})^{2j-2ks} = a^{-s} \sum_{j=0}^{(\deg B)/2} \tilde{b}_j \zeta_H(2ks - 2j, \tilde{\alpha}),$$

where ζ_H is the Hurwitz zeta-function. This formula provides a meromorphic extension of $\zeta_{P''}$ to the whole complex plane. Since the Hurwitz zeta-function has a single simple pole at 1 for any $\tilde{\alpha} \in \mathbb{C} \setminus -\mathbb{N}$, we have $\mathfrak{P}_{\zeta_{P''}}(\mathbb{C}) = \frac{1}{2k} \{1, 3, \dots, \deg B + 1\}$. What is more, for $s = -n$ with $n \in \mathbb{N}$ we have

$$\zeta_{P''}(-n) = a^n \sum_{j=0}^{(\deg B)/2} \tilde{b}_j \zeta_H(-2kn - 2j, \tilde{\alpha}) = -a^n \sum_{j=0}^{(\deg B)/2} \tilde{b}_j \frac{B_{2kn+2j+1}(\tilde{\alpha})}{2kn + 2j + 1},$$

where B_n denotes the Bernoulli polynomial of degree n . But if $\tilde{\alpha} = 0$ or $\tilde{\alpha} = \frac{1}{2}$ then $B_{2n+1}(\tilde{\alpha}) = 0$ [1, Section 23], hence for any $n \in \mathbb{N}^+$ $\zeta_{P''}(-n) = 0$. So if $\mathcal{Z}(s) = \Gamma(s)\zeta_{P''}(s)$ as usually, we have $\mathfrak{P}_{\mathcal{Z}}(\mathbb{C}) = \frac{1}{2k} \{0, 1, 3, \dots, \deg B + 1\}$. Thus, on the strength of Proposition 6 and formula (40) we finally conclude that for all $t > 0$

$$\begin{aligned} \text{htr}_P(t) &= e^{-t\beta} \sum_{j=0}^{(\deg B)/2} \tilde{b}_j \Gamma\left(\frac{1+2j}{2k}\right) (at)^{-\frac{1+2j}{2k}} - e^{-t\beta} \tilde{b}_0 \left(\tilde{\alpha} - \frac{1}{2}\right) + \\ &\quad - e^{-t\beta} \sum_{j=0}^{(\deg B)/2} \tilde{b}_j \sum_{n=0}^{[\alpha]-1} (n + \tilde{\alpha})^{2j} e^{-ta(n+\tilde{\alpha})^{2k}} + F_{\infty}(t), \end{aligned} \tag{44}$$

where $F_{\infty}(t) \neq 0$ has a null Taylor expansion at $t = 0$. □

In some cases, one could be able to compute the remainder function F_∞ explicitly.

Example 2 Let P be an operator with $\lambda_n(P) = n^2$ and $M_n = 1$ for $n \in \mathbb{N}$, then

$$\text{htr}_P(t) = \sum_{n=0}^{\infty} e^{-tn^2} = \frac{1}{2}(\theta_3(0; e^{-t}) - 1),$$

with θ_3 being the Jacobi theta function defined as

$$\theta_3(z; q) := \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niz}.$$

For such an operator one can compute explicitly (see [30] and also [28]):

$$F_\infty(t) = \sqrt{\frac{\pi}{t}} \frac{1}{2} \left(-1 + \theta_3\left(0; e^{-\pi^2/t}\right) \right).$$

Hence, formula (44) turns out to be the famous Jacobi identity [30, (3.13)]. This example has been analysed in details in [30] and has direct applications to quantum physics.

Therefore, one can regard the formula (44) as a generalisation of the Jacobi identity for special functions defined by $\text{htr}_P(t)$ with P as in Proposition 10.

Proposition 10 again has a direct geometrical application.

Corollary 2 *Let \mathcal{D} be the Dirac operator acting on a spinor bundle of S^d – the d -dimensional sphere with round metric. If d is odd, then the asymptotic expansion of the heat trace associated with \mathcal{D}^{2k} with $k \in \mathbb{N}^+$ is almost exact for all $t > 0$, but not exact.*

Proof To apply Proposition 10 we need to show that $M_n(\mathcal{D}^{2k})$ can be written as a polynomial in $\lambda_n(\mathcal{D}^2)$. For trivial spin structure on S^1 it is obvious, however we need to resort again to the truncated zeta-function, as in Corollary 1.

In all other cases, by formula (43) we have $\lambda_n(\mathcal{D}^{2k}) = \left(n + \frac{d}{2}\right)^{2k}$. Note that

$$\begin{aligned} M_n(\mathcal{D}^{2k}) &= M_n(|\mathcal{D}|) && (45) \\ &= 2^{\lfloor \frac{d}{2} \rfloor + 1} \left(\lambda_n(|\mathcal{D}|) + \frac{d}{2} - 1\right) \left(\lambda_n(|\mathcal{D}|) + \frac{d}{2} - 2\right) \cdots \left(\lambda_n(|\mathcal{D}|) - \frac{d}{2} + 2\right) \left(\lambda_n(|\mathcal{D}|) - \frac{d}{2} + 1\right). \end{aligned}$$

So if d is odd, then

$$M_n(\mathcal{D}^{2k}) = 2^{\lfloor \frac{d}{2} \rfloor + 1} \left(\lambda_n(|\mathcal{D}|)^2 - \left(\frac{d}{2} - 1\right)^2\right) \left(\lambda_n(|\mathcal{D}|)^2 - \left(\frac{d}{2} - 2\right)^2\right) \cdots \left(\lambda_n(|\mathcal{D}|)^2 - \left(\frac{1}{2}\right)^2\right).$$

Hence, $M_n(\mathcal{D}^{2k})$ can indeed be written as a polynomial in $\lambda_n(|\mathcal{D}|)^2 = \lambda_n(\mathcal{D}^2)$. \square

The comparison of Corollaries 1 and 2 shows a sharp contrast between the heat traces associated with the operators $|\mathcal{D}|$ and \mathcal{D}^2 for *odd-dimensional* spheres. For the even-dimensional ones the discrepancy is even more dramatic, as we shall now see.

4.1.4 Divergent Expansions

For a generic operator of polynomial spectrum with eigenvalues growing at least quadratically one expects the associated heat trace expansion to be only an asymptotic one. In particular, we have the following result:

Proposition 11 *Let \mathcal{D} be the Dirac operator acting on a spinor bundle of S^d – the d -dimensional sphere with round metric. If d is even, then the asymptotic expansion of the heat trace associated with \mathcal{D}^2 is only asymptotic.*

Proof If d is even, formula (46) yields

$$\begin{aligned}
 M_n(\mathcal{D}^2) &= 2^{1+\frac{d}{2}} \left(\lambda_n(|\mathcal{D}|)^2 - \left(\frac{d}{2} - 1\right)^2 \right) \left(\lambda_n(|\mathcal{D}|)^2 - \left(\frac{d}{2} - 2\right)^2 \right) \cdots \left(\lambda_n(|\mathcal{D}|)^2 - 1^2 \right) \lambda_n(|\mathcal{D}|) \\
 &=: \sum_{m=0}^{d-1} c_m \lambda_n(|\mathcal{D}|)^m,
 \end{aligned}$$

with $c_m = 0$ for m even. Moreover,

$$\text{sign } c_{d-1-2q} = (-1)^q, \quad \forall q = 0, \dots, \frac{d}{2} - 1. \tag{46}$$

The zeta-function associated with the operator $P = \mathcal{D}^2$ on S^d reads

$$\zeta_P(s) = \sum_{n=0}^{\infty} M_n(\mathcal{D}^k) \lambda_n(|\mathcal{D}|)^{-2s} = \sum_{m=0}^{d-1} c_m \sum_{n=0}^{\infty} \left(n + \frac{d}{2}\right)^{-2s+m} = \sum_{m=0}^{d-1} c_m \zeta_H\left(2s - m, \frac{d}{2}\right),$$

where ζ_H is the Hurwitz zeta-function. It turns out, that the latter can actually be replaced by the Riemann zeta-function. Indeed, let us note that

$$\zeta_H\left(s, \frac{d}{2}\right) = \sum_{n=0}^{\infty} \left(n + \frac{d}{2}\right)^{-s} = \sum_{n=1}^{\infty} n^{-s} - \sum_{n=1}^{d/2-1} n^{-s} = \zeta(s) - F_d(s),$$

with $F_d(s) := \sum_{n=1}^{d/2-1} n^{-s}$ for $d \geq 4$ and $F_2(s) = 0$. Hence,

$$\zeta_P(s) = \sum_{m=0}^{d-1} c_m \zeta(2s - m) - \sum_{m=0}^{d-1} c_m F_d(2s - m).$$

Let us investigate the second term on the RHS of the above equality. For any $s \in \mathbb{C}$,

$$\sum_{m=0}^{d-1} c_m F_d(2s - m) = \sum_{n=1}^{d/2-1} n^{-2s} \sum_{m=0}^{d-1} c_m n^m = \sum_{n=1}^{d/2-1} n^{-2s} B\left(n - \frac{d}{2}\right),$$

where B is the polynomial defining the multiplicities of \mathcal{D}^2 , i.e. $M_n(\mathcal{D}^2) = B(n)$ for all $n \in \mathbb{N}$. But, since $B(n) = 2^{\lfloor \frac{d}{2} \rfloor + 1} \binom{n+d-1}{d-1}$ by (43) and d is even, we have $B(n - \frac{d}{2}) = 0$ for every $n = 1, \dots, \frac{d}{2} - 1$. Thus,

$$\zeta_P(s) = \sum_{m=0}^{d-1} c_m \zeta(2s - m).$$

Now, Theorem 7 yields the following asymptotic expansion:

$$\begin{aligned} \text{htr}_P(t) &\underset{t \downarrow 0}{\sim} \sum_{k=0}^{\infty} \text{Res}_{s=(d-k)/2} \Gamma(s) \zeta_P(s) t^{(k-d)/2} \\ &= \sum_{k=0}^{d-1} \Gamma\left(\frac{d-k}{2}\right) c_{d-k-1} t^{(k-d)/2} + \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \sum_{m=0}^{d-1} c_m \zeta(-2p - m) t^p. \end{aligned} \tag{47}$$

Let us investigate the convergence of the series

$$\sum_{p=0}^{\infty} d_p t^p, \quad \text{with } d_p = \frac{(-1)^p}{p!} \sum_{m=0}^{d-1} c_m \zeta(-2p - m). \tag{48}$$

Recall that [1, (23.2.15)]

$$\zeta(-2p - m) = -\frac{B_{2p+m+1}}{2p + m + 1},$$

where B_n are the Bernoulli numbers.

Since $c_m = 0$ for even m , therefore in the definition of d_p the only non-vanishing terms are those with $2p + m + 1$ even. Moreover, we have

$$\text{sign } B_{2n} = (-1)^{n+1},$$

and by (46)

$$\text{sign } c_m = (-1)^{\frac{d-1-m}{2}},$$

thus

$$\text{sign } c_m B_{2p+m+1} = (-1)^{\frac{d-1-m}{2}} (-1)^{\frac{2p+m+1}{2}+1} = (-1)^{\frac{d}{2}+p+1}.$$

Therefore,

$$\text{sign } d_p = (-1)^{\frac{d}{2}}$$

so all of the terms in series (48) over p have the same sign. Moreover, since $\text{sign } c_m B_{2p+m+1}$ does not depend on m it is sufficient to study the behaviour of $c_m \zeta(-2p - m)/(p!)$ as p grows, for a fixed m .

We have [1, (23.1.15)]

$$(-1)^{n+1} B_{2n} > \frac{2(2n)!}{(2\pi)^n},$$

so that

$$c_m \frac{\zeta(-2p - m)}{p!} = c_m \frac{B_{2p+m+1}}{p!(2p + m + 1)} > \frac{2 |c_m|}{(2\pi)^{(m+1)/2}} \frac{(2p + m)!}{(2\pi)^p p!} \xrightarrow{p \rightarrow \infty} \infty.$$

Hence, the asymptotic series in (47) diverges for any $t > 0$. □

The lesson from the example of spheres is that if one is interested in the convergence properties of heat trace expansion one should work with $|\mathcal{D}|$, which is of the first order, rather than with \mathcal{D}^2 , despite the fact that the former is not a differential operator, but only a pseudodifferential one.

4.2 Operators of Exponential Spectrum

In this section we consider the class of *operators of exponential spectrum*, i.e. the ones with the spectrum $\sigma(P) = (q^{-n})_{n \in \mathbb{N}}$ for some $0 < q < 1$. We shall also assume that the multiplicities are given by a polynomial. This type of operators appears naturally in the context of quantum groups [50, 51] and their homogeneous spaces [27, 29, 60] (see also [52] and [35, Example 12]), as well as in the framework of fractal spaces [11–13, 41, 42].

Proposition 12 *Let P be an operator with $\lambda_n = q^{-n}$ for some $0 < q < 1$ and $M_n = p(n)$ for some polynomial p of degree m . Then, the asymptotic expansion of the heat trace htr_P is absolutely exact for all $t > 0$ and can be expressed as*

$$\text{htr}_P(t) = \frac{\tilde{p}(1)}{(m+1)!} (-\log_q t)^{m+1} + \sum_{j=0}^m (r_j + G_j(\log_q t)) (\log_q t)^j + \sum_{n=1}^{\infty} \frac{\tilde{p}(q^{-n})}{(1-q^{-n})^{m+1}} \frac{(-t)^n}{n!}. \tag{49}$$

$\log_q t = \frac{\log t}{\log q}$, \tilde{p} is a polynomial of degree m , r_i are constants (with respect to t) and G_i are Fourier series completely determined by q and the polynomial p .

Proof We start with the analysis of the zeta-function. For $\Re(s) > 0$ we have

$$\zeta_P(s) = \sum_{n=0}^{\infty} p(n)q^{ns} = \sum_{j=0}^m p_j \sum_{n=0}^{\infty} n^j q^{ns} =: \frac{\tilde{p}(q^s)}{(1 - q^s)^{m+1}}.$$

The polynomial \tilde{p} is completely determined by p via the formula

$$\sum_{n=0}^{\infty} n^j q^{ns} = \text{Li}_{-j}(q^s) = \frac{1}{(1 - q^s)^{j+1}} \sum_{k=0}^{j-1} \left\langle \begin{matrix} j \\ k \end{matrix} \right\rangle q^{s(j-k)}, \tag{50}$$

where Li is the polylogarithm (Jonquière’s) function [49] and $\langle j \rangle_k$ stand for Eulerian numbers of the first kind [14]. From the formula (50) we also deduce that $\tilde{p}(1) = p_m \cdot m! \neq 0$. Indeed,

$$\tilde{p}(q^s) = \sum_{j=0}^m p_j (1 - q^s)^{m-j} \sum_{k=0}^{j-1} \langle j \rangle_k q^{s(j-k)}$$

and, using the summation formula for Eulerian numbers [14, p. 242],

$$\tilde{p}(1) = p_m \sum_{k=0}^{m-1} \langle m \rangle_k = p_m \cdot m!.$$

Therefore, we conclude that the function $\mathcal{Z} : s \mapsto \Gamma(s)\zeta_P(s)$ has a meromorphic extension to the whole complex plane with:

- first order poles at $s \in -\mathbb{N}^+$,
- $m + 1$ order poles at $s \in \frac{2\pi i}{\log q} \mathbb{Z}^*$,
- $m + 2$ order pole at $s = 0$.

To see that the assumptions of Theorem 4 are met let us choose $R_n = n + \frac{1}{2}$ and $y_k^{(n)} = \frac{2\pi}{\log q} (k + \frac{1}{2})$. Let us also denote by \hat{p} the polynomial $\hat{p}(x) = \sum_{i=0}^m |\tilde{p}_i| x^i$.

On the horizontal lines of the contour integration we have

$$\left| \zeta_P \left(x + iy_k^{(n)} \right) \right| = \frac{\left| \tilde{p} \left(q^{x+iy_k^{(n)}} \right) \right|}{\left| 1 - q^{x+iy_k^{(n)}} \right|^{m+1}} = \frac{\left| \tilde{p} \left(q^{x+iy_k^{(n)}} \right) \right|}{(1 + q^x)^{m+1}} \leq \frac{\hat{p}(q^x)}{(1 + q^x)^{m+1}}$$

and Γ decays exponentially on verticals [63, (2.1.21)], hence Assumption 4 of Theorem 4 is fulfilled. Similarly, on the vertical lines of integration we have

$$\left| \zeta_P(-R_n + iy) \right| = \frac{\left| \tilde{p}(q^{-R_n+iy}) \right|}{\left| 1 - q^{-R_n+iy} \right|^{m+1}} \leq \frac{\hat{p}(q^{-R_n})}{\left| 1 - q^{-R_n} \right|^{m+1}}. \tag{51}$$

To show that we have an exact expansion of htr_P valid for any $t > 0$ we need to estimate the Gamma function more precisely. The Euler reflection formula [63, (2.1.20)] together with the estimate (see [29, Formula (43)])

$$|\Gamma(x + iy)|^{-1} \leq (2\pi)^{-1/2} e^{x+|y| |\arg(x+iy)|} (x^2 + y^2)^{-x/2+1/4}, \quad \text{for } x > 0, y \in \mathbb{R}$$

gives

$$\begin{aligned} |\Gamma(-R_n + iy)| &\leq \sqrt{\frac{\pi}{2}} \frac{1}{\cosh(\pi y)} e^{1+R_n+|y||1+R-iy|} \left((1 + R_n)^2 + y^2 \right)^{-R_n/2-1/4}, \\ &\leq \sqrt{\frac{\pi}{2}} e^{1+R_n} R_n^{-R_n-1/2} \frac{1}{\cosh(\pi y)} e^{|y||1+R-iy|}, \\ &\leq \sqrt{2\pi} e^{1+R_n} R_n^{-R_n-1/2} \frac{e^{\frac{\pi}{2}|y|}}{e^{\pi y} + e^{-\pi y}}, \\ &\leq \sqrt{2\pi} e^{1+R_n} R_n^{-R_n-1/2} e^{-\frac{\pi}{2}|y|}. \end{aligned} \tag{52}$$

The above result together with (51) yields the following constants in Assumption 3 of Theorem 5:

$$\epsilon_n = \frac{\pi}{2}, \quad C_n = \sqrt{2\pi} e^{3/2} \frac{e^n \hat{p}(q^{-n-1/2})}{|1 - q^{-n-1/2}|} \left(n + \frac{1}{2}\right)^{-n}.$$

Therefore, Theorem 5 applies and

$$\limsup_{n \rightarrow \infty} \sqrt[n]{C_n/\epsilon_n} = 0, \quad \text{for any } 0 < q < 1,$$

yielding $T = \infty$. Moreover, since $R_n = n + \frac{1}{2}$, Proposition 5 applies and the expansion is absolutely exact.

The formula (49) results from a direct calculation of the residues. □

This result is somewhat surprising at the first sight. Clearly, $\text{htr}_P(t)$ decays exponentially as t grows to infinity, whereas the RHS of (49) seems to diverge to infinity because of the $\log t$ terms. However, it turns out that the sum over n in the RHS of (49) compensates for large t the divergent terms $(\log t)^j$, the constant terms and the oscillatory part to yield an exponential decay (compare [35, Example 12] and [29, Section 4.1]).

It is interesting to note that if P is an operator of exponential spectrum, then so is P^r for any $r \in \mathbb{R}_+$ (compare also [29, Section 4.4]). Moreover, given the exact expansion of the heat trace (49), one immediately obtains that for htr_{P^r} simply by changing $q \rightsquigarrow q^r$.

5 Outlook

We start the concluding section of this paper with an exploration of the limitations of our general theorems. Having in mind the exactness result of the previous section one could naively expect that the faster the eigenvalues of a positive operator grow, the better the convergence properties of the associated heat trace expansion are. However, as we shall show below the exponential growth of eigenvalues establishes in fact a limit of applicability of the inverse Mellin transform.

Proposition 13 *Let P be such that $M_n = \mathcal{O}_\infty(n^b)$ for some $b \in \mathbb{R}^+$. If*

$$\lim_{n \rightarrow \infty} \frac{\lambda_{n+1}}{\lambda_n} = +\infty, \tag{53}$$

then the function ζ_P is holomorphic for $\Re(s) > 0$, but does not admit a meromorphic continuation through $\Re(s) = 0$.

Proof The general Dirichlet series defining the zeta-function associated with P reads

$$\zeta_P(s) = \sum_{n=0}^{\infty} M_n \lambda_n^{-s} = \sum_{n=0}^{\infty} M_n e^{-s \log \lambda_n}. \tag{54}$$

We note that (53) implies that $\lambda_n = \mathcal{O}_\infty(e^{g(n)})$ with $g(n)/n \rightarrow +\infty$. Moreover, the assumption on the power-like growth of multiplicities assures that $\zeta_P(s)$ is convergent for $\Re(s) \geq 0$ – see [23, 45] and references therein.

Under the assumption (53), the series (54) is a *lacunary Dirichlet series* [45]:

$$\zeta_P(s) = \sum_{n=0}^\infty a_n z^{\mu_n},$$

with $z = e^{-s}$, $\mu_n = \log \lambda_n$ and $a_n = M_n$. Then, classical results [55] (see also [33], [56, Theorem 1], [45] and [23]) show that the vertical line $\Re(s) = 0$ is a natural boundary of analyticity for $\zeta_P(s)$. The latter means that the poles of ζ_P are dense on the imaginary axis and therefore, ζ_P cannot be extended to the left complex half-plane. □

We shall call the operators satisfying the assumptions of Proposition 13 *lacunary operators*. Proposition 13 does not imply directly that $\text{htr}_P(t)$ does not have an asymptotic expansion as $t \downarrow 0$ for lacunary operators – in particular, both htr_P and ζ_P are well-defined. It just states that the technique of the inverse Mellin transform developed in this paper does not apply in this case. For an example of a lacunary operator in the realm of non-classical pseudodifferential operators see [64].²

Can anything be said about the small t behaviour of heat traces associated with lacunary operators? The answer is positive and can be deduced from the following Tauberian Theorem due to Hardy and Littlewood (known also under the name of Karamata Theorem).

Theorem 8 ([34] **Theorem XIII.5.2**) *Let $G : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a function of bounded variation (see [73, Chapter 1]) and such that the following Riemann–Stieltjes integral*

$$\omega(t) = \int_0^\infty e^{-t\lambda} dG(\lambda),$$

converges for $t > 0$. Then, the following are equivalent:

$$G(\lambda) \approx \lambda^L F(\lambda), \quad \text{as } \lambda \rightarrow \infty,$$

and

$$\omega(t) \approx t^{-L} \Gamma(L + 1) F(t^{-1}), \quad \text{as } t \downarrow 0,$$

where F is a slowly varying function, i.e $F(\tau x)/F(\tau) \rightarrow 1$ as $\tau \rightarrow \infty$ for every $x > 0$.

As a direct application of Theorem 8 we obtain the following result:

Corollary 3 *Let P be a positive operator such that ζ_P has a finite abscissa of convergence L . If*

$$N(\lambda) \approx \lambda^L F(\lambda),$$

²We thank Bruno Iochum for pointing out this reference to us.

with F slowly varying, then

$$\text{htr}_P(t) \approx \Gamma(L + 1)t^{-L}F(t^{-1}), \quad \text{as } t \downarrow 0.$$

Proof The function N , being a step function, is of bounded variation [73, Chapter 1]. Moreover, since ζ_P has a finite abscissa of convergence, htr_P is well-defined and thus

$$\text{htr}_P(t) = \sum_{n=0}^{\infty} M_n e^{-t\lambda_n} = \int_0^{\infty} e^{-t\lambda} dN(\lambda),$$

for all $t > 0$. Proposition 2 implies that for large λ we have $N(\lambda) \approx \lambda^L F(\lambda)$, with $F(\lambda) = \mathcal{O}_{\infty}(\lambda^{\delta})$ for every $\delta > 0$. If moreover, F is slowly varying, then Theorem 8 applies and the conclusion follows. \square

Corollary 3 applies also to lacunary operators and one can use it to detect the leading behaviour of htr_P as $t \downarrow 0$.

Example 3 Let us consider a lacunary operator P with $\lambda_n = e^{n^2}$ and no degeneracy (i.e. $M_n \equiv 1$). Its zeta-function reads

$$\zeta_P(s) = \sum_{n=0}^{\infty} e^{-sn^2} = \frac{1}{2}(\theta_3(0; e^{-s}) + 1), \quad \text{for } \Re(s) > 0,$$

where θ_3 is the Jacobi theta function we met in Example 2, but now playing the role of the zeta-function. It is a lacunary function and does not admit a meromorphic continuation to the left complex half-plane.

On the other hand, since

$$N(\lambda) = \sum_{\{n:e^{n^2} \leq \lambda\}} 1 \approx \sqrt{\log \lambda}$$

and the function $\sqrt{\log}$ is slowly varying, Theorem 8 implies

$$\text{htr}_P(t) = \sum_{n=0}^{\infty} e^{-te^{n^2}} \approx \sqrt{-\log t}, \quad \text{as } t \downarrow 0.$$

However, the assumption of N being a regularly varying function is a non-trivial one and puts limitations on the usefulness of Theorem 8.

Example 4 Let P be an operator with $\lambda_n(P) = 2^n$ and $M_n(P) = 2^n$. Then,

$$N(\lambda) = \sum_{n:2^n \leq \lambda} 2^n = \sum_{n=0}^{\lfloor \log_2 \lambda \rfloor} 2^n = 2^{\lfloor \log_2 \lambda \rfloor + 1} - 1 \approx \lambda \cdot F(\lambda),$$

but $F(\lambda) := 2^{\lfloor \log_2 \lambda \rfloor + 1} \lambda^{-1}$ is *not* slowly varying. Indeed, the limit $\lim_{\lambda \rightarrow \infty} F(x\lambda)/F(\lambda)$ exists (and is equal to 1) only if $x = 2^m$ with $m \in \mathbb{N}$.

On the other hand,

$$\zeta_P(s) = \sum_{n=0}^{\infty} 2^n 2^{-sn} = \frac{1}{1 - 2^{s-1}}.$$

The results of Section 4.2 can be easily adapted to this setting yielding the following *absolutely exact* expansion

$$\text{htr}_P(t) = -\frac{t^{-1}}{\log 2} \sum_{k \in \mathbb{Z}} \Gamma\left(1 - \frac{2\pi ik}{\log 2}\right) t^{\frac{2\pi ik}{\log 2}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{t^n}{1 - 2^{-n-1}},$$

which is valid for all $t > 0$. Note that the leading term of htr_P is of the form $t^{-1}G(t)$, where G is oscillating and thus not slowly varying.

We have seen that the spectral growth function (3) of an operator is a primary quantity that allows us to determine whether an asymptotic expansion of the associated heat trace can be obtained via the inverse Mellin transform. Proposition 1 establishes an upper bound on the spectral growth of P , whereas Proposition 13 gives a lower one. One might therefore expect that for the intermediate values of growth rates, the inverse Mellin transform technique guarantees the existence of an asymptotic expansion of heat traces. However, this is not the case as we shall show below.

Proposition 14 *Let P be an operator such that its spectral growth function satisfies*

$$N(\lambda) \approx (\log \lambda)^a, \quad \text{as } \lambda \rightarrow \infty, \tag{55}$$

for some $a \in \mathbb{R}^+$. If $a \notin \mathbb{N}$, then ζ_P has an abscissa of convergence $L = 0$, but is not meromorphic around $s = 0$.

Proof Proposition 2 implies that ζ_P is well-defined with an abscissa of convergence $L = 0$. To see that ζ_P is not meromorphic around $s = 0$ we invoke the Hardy–Littlewood Tauberian Theorem 8 once again.

With $M \in \mathbb{N}$ such that $\lambda_M \leq 1$ and $\lambda_{M+1} > 1$, we write the zeta-function ζ_P on \mathbb{R}^+ as Riemann–Stieltjes integral (compare [65, Section 13.3])

$$\zeta_P(s) = \int_0^{\infty} \lambda^{-s} dN(\lambda), \quad \hat{\zeta}_P^M(s) = \int_1^{\infty} \lambda^{-s} dN(\lambda).$$

On the strength of [73, Theorem 11a] we can change variables in the Riemann–Stieltjes integral and rewrite

$$\hat{\zeta}_P^M = \int_1^{\infty} e^{-s \log \lambda} dN(\lambda) = \int_0^{\infty} e^{-s\mu} d\tilde{N}(\mu),$$

for $s > 0$, with $\tilde{N}(\mu) = \sum_{n: \log \lambda_n \leq \mu} M_n$, which is of bounded variation on \mathbb{R}_+ . Assumption (55) implies

$$\tilde{N}(\mu) \approx \mu^a, \quad \text{as } \mu \rightarrow \infty,$$

thus, by Theorem 8, we have

$$\hat{\zeta}_P^M(s) \approx s^{-a}\Gamma(a + 1), \quad \text{as } s \downarrow 0.$$

Therefore, if an extension of $\hat{\zeta}_P^M$ to $\Re(s) \leq 0$ exists at all, then the point $s = 0$ is not a pole unless $a \in \mathbb{N}^+$. Since $\hat{\zeta}_P^M$ and ζ_P differ by an entire function, the same conclusion holds for ζ_P . □

Let us stress that Proposition 14 has a “no-go” character only. Having $a \in \mathbb{N}^+$ does not imply that ζ_P can be extended to the left complex half-plane in a meromorphic way.

Proposition 14 is in accordance with Proposition 13, but it also provides examples of pathological operators, which are not lacunary.

Example 5 Consider an operator P with $\lambda_n = e^{n^{2/3}}$ and no degeneracy (i.e. $M_n \equiv 1$). Although its eigenvalues grow slower than exponentially, the zeta-function ζ_P cannot be meromorphic around $s = 0$, as

$$\zeta_P(s) \approx \Gamma(-1/2)s^{-3/2}, \quad \text{as } s \downarrow 0 \text{ along } \Im(s) = 0.$$

On the other hand, by Theorem 8,

$$\text{htr}_P(t) = \sum_{n=0}^{\infty} e^{-te^{n^{2/3}}} \approx (-\log t)^{3/2}, \quad \text{as } t \downarrow 0.$$

The above examples show that Tauberian theorems and the inverse Mellin transform have different domains of applicability and can be considered as complementary tools in the study of the asymptotic behaviour of heat traces.

There is also a third method, namely the Poisson summation, not discussed in this paper but widely used in the field of spectral geometry. The Poisson summation formula can be written as

$$\sum_{n=-\infty}^{\infty} f(tn) = t^{-1} \sum_{k=-\infty}^{\infty} \mathcal{F}[f]\left(\frac{k}{t}\right), \tag{56}$$

where \mathcal{F} is the Fourier transform and f is a real function satisfying suitable conditions (as usual, the more we assume on f , the more we know on the convergence and properties of the Fourier series on the RHS – see [66, 74]). The main advantage of the Poisson summation method is that it allows to obtain the asymptotic expansion of $\text{Tr}(f(tP))$ for quite a general class of functions f , whereas the Mellin transform technique in its basic form described in this paper applies only to $f(x) = e^{-x}$. Hence, Poisson summation method is better suited to spectral action calculations. On the other hand, the Poisson summation may be used only for operators with polynomial spectra (see Section 4.1), and only for those with eigenvalues given by first order

polynomial.³ This is quite restrictive, however many symmetric sphere-like manifolds and their deformations provide Dirac operators with spectra of the desired form [57]. Poisson summation also generalises easily to the multi-index case, which can be applied to obtain spectral action on tori. This cannot be easily done by direct application of Mellin transform technique.⁴ These properties make Poisson summation a convenient method for calculations of spectral action on some specific spaces, but make it less suitable for more general considerations, which are the aim of this paper.

The second distinguishing property of Poisson summation method is the fact, that it provides only the principal part of the spectral action asymptotic expansion. This principal part depends solely on $\mathcal{F}[f](0)$ term in (56). If one is interested in the non-singular part of the expansion, then one needs to analyse the whole RHS of (56). This would arguably neither help in finding the asymptotic expansion, nor say anything about its exactness. We found the Mellin transform technique more useful in achieving the aims of present paper.

Let us now sum up the results of the paper. In Section 3 we have presented general theorems on the existence and convergence of heat traces associated with positive unbounded operators with compact inverses. The necessary conditions were formulated in terms of spectral zeta-functions. The non-existence of meromorphic extensions of the latter sets a natural limitation of applicability of the inverse Mellin transform. However, lacunary operators seem to be pathological anyway from the viewpoint of noncommutative geometry. For instance, if a spectral triple would have a lacunary Dirac operator, then it would not have a dimension spectrum [20]. On the other hand, the assumption (12), even in its more refined version (19), seems to be a mild one. In fact, a similar one was adopted in [20, p. 206] “on the technical side”.

In Section 4 devoted to examples we always worked with operators, the spectrum of which is known explicitly. Therefore, the operatorial aspect of the problem was somewhat hidden. In practice, one is rarely granted the comfort of knowing the full spectrum of a given operator. Even if this is the case, one would like study the behaviour of the heat trace when a fixed operator P is perturbed to $P + A$, with some bounded A . Clearly, a bounded perturbation of P would not change the leading behaviour of $\text{htr}_P(t)$ at small t , but it can, at least in principle, spoil the asymptotic expansion.

Indeed, perturbations may drastically change the analytic properties of the associated zeta-functions. For instance, the modulus of the Dirac operator on the standard Podleś sphere has (up to a multiplicative constant) the following eigenvalues $\lambda_n(|\mathcal{D}_q|) = q^{-n} - q^n$ (see [27]). It can thus be considered as a sum of an operator of exponential spectrum P and a trace class perturbation Q . It turns out that the poles of ζ_{P+Q} form a regular lattice on the left complex half-plane [29], whereas the poles of ζ_P are located only on the imaginary axis (see Section 4.2). Although, the convergence properties of the small t expansion of $\text{htr}_{P+Q}(t)$ are not altered by

³One needs to remember, that in Poisson formula eigenvalues are indexed by \mathbb{Z} , while in Mellin transform approach the index set is \mathbb{N} . Usually this difference can be circumvented, e. g. by using the decomposition (34) which works for both methods.

⁴This is because the spectrum of Dirac operators on tori cannot be easily ordered into an increasing sequence and because the zeta-function would involve multi-index summation.

the perturbation Q , but the estimates of contour integrals are much more subtle and tedious to control (see [29, Proposition 4.3]).

We regard the investigation of the impact of perturbations on the asymptotic expansion of heat traces an important and natural next step in the study initiated in this paper. We hope that a combination of our results with the techniques developed in [7, 54, 67] can lead to a better understanding of heat traces outside the realm of classical pseudodifferential operators.

A promising application of the results presented in the paper concerns the spectral action [15]. By using the semi-group property of heat operators and (distributional) Laplace transform techniques one can attempt to extend both the existence and exactness results to functionals of the form $\text{Tr } f(tP)$ for suitable cut-off functions f — see [29] for an explicit example and [28] for a more detailed study. Let us note that the non-perturbative calculations carried out in [10, 57, 58, 61, 68] using Poisson summation formula are, according to the nomenclature adopted in Definition 5, only *almost exact*. However, with the help of Theorem 5 (compare also [28, Proposition 3.2.2] and [29, Theorem 5.2]) one is actually able to get exact formulae for the spectral action for an explicit class of cut-off functions. This technique may prove potentially useful in the study of cosmic topology [57].

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