Is the dynamical system stable?

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Dedicated to Professor János Aczél on 90th anniversary of his birth.

Abstract. In this paper we consider stability in the Ulam–Hyers sense, and in other similar senses, for the five equivalent definitions of one-dimensional dynamical systems.

Mathematics Subject Classification. Primary 39B12, 39B82; Secondary 37E05, 26A18.

Keywords. Translation equation, dynamical system, Ulam–Hyers stability, b-stability, uniform b-stability, inverse stability, superstability, inverse superstability, hiperstability, inverse hiperstability.

1. Introduction

In the theory of dynamical systems there exist several notions of their stability (see e.g. [9]). We consider here a rendering of Ulam–Hyers stability and other similar stabilities to one-dimensional dynamical systems.

The theory of stability of functional equations started with the question posed by Ulam: if a function satisfies Cauchy's equation for the additive function up to some degree of accuracy, does there exist an additive function close to this function? Hyers investigated in [3] (the first paper on the stability of functional equations) this question of Ulam's.

More precisely, we say that *a functional equation is Ulam–Hyers stable* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every function H which satisfies this functional equation approximately, with δ-accuracy, there exists a solution F of this functional equation which is in the ε-neighborhood of H. In this way we can also consider the stability of systems of functional equations. We can also consider the stability of a functional equation, or a system of functional equations, in some class of functions demanding that H and F appearing in the definition above are from this class of functions.

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There are also various modifications of the notion above known, e.g. b-stability, uniform b-stability, superstability, inverse stability and so on. The precise definitions of these concepts will be given later.

In this paper we deal with stability (in these various senses) of systems of functional equations, or one functional equation, in some classes of functions, which define equivalently one-dimensional dynamical systems. In the whole paper let I be an interval in $\mathbb{R}$ with nonempty interior.

The classic definition of dynamical system reads as follows:

Definition 1. The continuous function $F\colon \mathbb{R} \times I \to I$, is called *a dynamical system* if *the translation equation*:

$$F(t, F(s, x)) = F(t + s, x) \quad \textit{for } t, s \in \mathbb{R},\ x \in I \qquad (1.1)$$

as well as *the identity condition*:

$$F(0, x) = x \quad \textit{for } x \in I \qquad (1.2)$$

are satisfied.

It has been proved in [7] that if $I = \mathbb{R}$, then the system (1.1) & (1.2) is Ulam–Hyers stable, that is for every $\varepsilon > 0$ there exists a $\delta = \varepsilon/10$ such that for every continuous function $H\colon \mathbb{R} \times I \to I$ for which

$$|H(t, H(s, x)) - H(t + s, x)| \leq \delta \quad \textit{for } t, s \in \mathbb{R}\ \textit{and } x \in I \qquad (1.3)$$

and

$$|H(0, x) - x| \leq \delta \quad \textit{for } x \in I \qquad (1.4)$$

there exists a dynamical system F such that

$$|H(t, x) - F(t, x)| \leq \varepsilon \quad \textit{for } t \in \mathbb{R},\ x \in I. \qquad (1.5)$$

If $I \neq \mathbb{R}$, the system (1.1) and (1.2) is not stable (see Remark 3.3, too). In [7] these results were presented in the section colloquially named as "stability of dynamical systems". Colloquially, but incorrectly[1]. In the theory of stability in the sense of Ulam–Hyers, it is the functional equation that can be stable or not. And the dynamical system is a function which is a solution of some system of equations. Thus the correct question is: is the system of functional equations, which defines dynamical systems, stable? For the system (1.1) and (1.2) the answer is yes, if $I = \mathbb{R}$, and no, if $I \neq \mathbb{R}$.

But there are also other systems of equations, which are equivalent to the system (1.1) and (1.2). For example, dynamical systems may also be defined equivalently in the following way:

Definition 2. The continuous function $F\colon \mathbb{R} \times I \to I$ is called a dynamical system if F is a solution of the translation equation such that

$$F'(0, x) = 1 \quad \textit{for } x \in I \qquad (1.6)$$

[hereinafter $F'(0, x)$ means the derivative of $F(0, \cdot)\colon I \to I$ at the point x].

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[1]In the title of this paper, too.

We will prove in this paper that the system (1.1) and (1.6) [equivalent to the system (1.1) and (1.2)] is Ulam–Hyers stable for every interval I, i.e., for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every continuous function $H \colon \mathbb{R} \times I \to I$ which satisfies (1.3) and

$$|H'(0, x) - 1| \leq \delta \quad \text{for } x \in I$$

there exists a dynamical system F satisfying (1.5).

Though, for convenience, we will write that the dynamical system is stable (or not), having in mind the system of equations defining this dynamical system.

Remark 1.1. The situation described above—that from two equivalent functional equations one may be stable and the other not, occurs also in the case of the equation of homomorphism. For the equivalent equations

$$f(xy) = f(x) + f(y)$$

and

$$f(x) + f(y) - f(xy) = 0$$

for $f \colon \mathbb{R} \to \mathbb{R}$, where $\rho(a, b) = |e^a - e^b|$ is a metric in $\mathbb{R}$, the second equation is stable and the first is not (see [1]).

In the example above, the metric is not natural. (With the natural metric $\rho(a, b) = |a - b|$ one should consider the equations

$$\exp(f(xy)) = \exp(f(x) + f(y))$$

and

$$\exp(f(x) + f(y) - f(xy)) = 1$$

to obtain this phenomenon).

In our case the metric is natural. So there have to be other reasons why dynamical systems in the sense of different, however equivalent, definitions are stable or not.

It is interesting that in the class of continuous functions the translation equation is stable [7]. The identity equation, that is Eq. (1.2) is stable too (for $F(t, x) = H(t, x) - H(0, x) + x$ we have $|F(t, x) - H(t, x)| \leq \delta$ if $|H(0, x) - x| \leq \delta$). However, the system (1.1) and (1.2) is not stable if $I \neq \mathbb{R}$. For other similar cases in the theory of stability see also [4].

2. Other definitions of dynamical system and other definitions of stability

We have already mentioned in the introduction that a one-dimensional dynamical system can be defined equivalently by definitions 1 and 2. But there are also other equivalent definitions.

Let $\mathcal{K}_1$ be the class of all continuous functions $F\colon \mathbb{R} \times I \to I$ such that $F(0,\cdot)$ is strictly increasing,

let $\mathcal{K}_2$ be the class of all continuous functions $F\colon \mathbb{R} \times I \to I$ such that $F'(0,\cdot)$ exists,

let $\mathcal{K}_3$ be the class of all continuous functions $F\colon \mathbb{R} \times I \to I$ such that F is a surjection.

Definition 3. The solution of the translation equation $F\colon \mathbb{R} \times I \to I$, such that $F \in \mathcal{K}_1$, is called a dynamical system.

This definition is equivalent to definitions 1 and 2 since $F(0, F(0,x)) = F(0,x)$, thus $F(0,\cdot)$ is the identity function on $F(0,I)$, and as it is strictly increasing, is the identity function on I.

If F is a continuous solution of the translation equation such that $F(0,\cdot)$ is increasing, but not strictly increasing, then F may not be a dynamical system. For example, the function

$$F(t,x) = \begin{cases} 0, & \text{if} \quad x \le 0,\, t \in \mathbb{R}; \\ x, & \text{if} \quad x \in (0,1),\, t \in \mathbb{R}; \\ 1, & \text{if} \quad x \ge 1,\, t \in \mathbb{R} \end{cases}$$

is a continuous solution of the translation equation, $F(0,\cdot)$ is increasing, but not strictly, and it is not a dynamical system.

Definition 4. The non-constant function $F\colon \mathbb{R} \times I \to I$ is called a dynamical system if F is a solution of the translation equation and $F \in \mathcal{K}_2$.

This definition is equivalent to the definition 1 (and, hence, to the definitions 2 and 3) since $F(0, F(s,x)) = F(s,x)$, $F(0,u) = u$ for $u \in F(\mathbb{R}, I)$ (a subinterval of I) and the existence of $F'(0,x)$ implies $F(\mathbb{R}, I) = I$.

Definition 5. $F\colon \mathbb{R} \times I \to I$, $F \in \mathcal{K}_3$, which satisfies the translation equation, is called a dynamical system.

This definition is equivalent to the precedents, since $F(0, F(t,x)) = F(t,x)$, thus, taking into account the surjectivity of F, we have $F(0,x) = x$ for $x \in I$.

The Ulam–Hyers stability has already been made precise, in the introduction, for system (1.1) and (1.2) and for system (1.1) and (1.6). Moreover, we say that the translation equation is Ulam–Hyers stable in the class $\mathcal{K}_i$ for $i = 1, 2, 3$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $H \in \mathcal{K}_i$ such that (1.3) holds, there exists a dynamical system $F\colon \mathbb{R} \times I \to I$ such that (1.5) holds true.

Thus we have explained what we mean by "Ulam–Hyers stability of dynamical systems in the sense of definitions 1–5".

For a given function $H\colon \mathbb{R} \times I \to I$ we put

$$\mathbb{H}(t,s,x) := H(t, H(s,x)) - H(t+s, x), \quad s,t \in \mathbb{R}, x \in I.$$

In the theory of functional equations several notions of stability are considered (see [5] and [6]):

a/**b-stability**

– For dynamical systems in the sense of definition 1 (respectively 2):

for every continuous function $H\colon \mathbb{R} \times I \to I$ if $\mathbb{H}$ and $H(0,\cdot) - \mathrm{Id}_I$ (respectively $\mathbb{H}$ and $H'(0,\cdot) - 1$) are bounded, then there exists a dynamical system $F\colon \mathbb{R} \times I \to I$ such that $H - F$ is bounded,

– For dynamical systems in the sense of definition 3 (respectively 4, 5):

for every function $H \in \mathcal{K}_1$ (respectively $H \in \mathcal{K}_2$, $H \in \mathcal{K}_3$) if $\mathbb{H}$ is bounded, then there exists a dynamical system $F\colon \mathbb{R} \times I \to I$ such that $H - F$ is bounded,

    b/**uniform b-stability**

i.e., b-stability in which the boundedness of $H - F$ does not depend on H, more precisely:

– For dynamical systems in the sense of definition 1 (respectively 2):

for every $\delta > 0$ there exists an $\varepsilon > 0$ such that for every continuous function $H\colon \mathbb{R} \times I \to I$ if $\mathbb{H}$ and $H(0,\cdot) - \mathrm{Id}_I$ (respectively $H'(0,\cdot) - 1$) are bounded by δ, then there exists a dynamical system $F\colon \mathbb{R} \times I \to I$ such that $H - F$ is bounded by ε,

– For dynamical systems in the sense of definition 3 (respectively 4, 5):

for every $\delta > 0$ there exists an $\varepsilon > 0$ such that for every function $H \in \mathcal{K}_1$ (respectively $H \in \mathcal{K}_2$, $H \in \mathcal{K}_3$) if $\mathbb{H}$ is bounded by δ, then there exists a dynamical system $F\colon \mathbb{R} \times I \to I$ such that $H - F$ is bounded by ε,

    c/**inverse stability**

– For dynamical systems in the sense of definition 1 (respectively 2):

for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $H\colon \mathbb{R} \times I \to I$ if there exists a dynamical system $F\colon \mathbb{R} \times I \to I$ such that

$$|H(t,x) - F(t,x)| \leq \delta, \quad for\ t \in \mathbb{R},\ x \in I, \tag{2.1}$$

then

$$|\mathbb{H}(t,s,x)| \leq \varepsilon, \quad for\ s,t \in \mathbb{R},\ x \in I \tag{2.2}$$

and $|H(0,x) - x| \leq \varepsilon$ for $x \in I$ (respectively $|H'(0,x) - 1| \leq \varepsilon$ for $x \in I$),

– For dynamical systems in the sense of definition 3 (respectively 4, 5):

for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $H\colon \mathbb{R} \times I \to I$ such that $H \in \mathcal{K}_1$ (respectively $H \in \mathcal{K}_2$, $H \in \mathcal{K}_3$)) if there exists a dynamical system $F\colon \mathbb{R} \times I \to I$ such that (2.1) is satisfied, then we have (2.2),

    d/**inverse b-stability**

– For dynamical systems in the sense of definition 1 (respectively 2):

for every $H\colon \mathbb{R} \times I \to I$ if there exists a dynamical system $F\colon \mathbb{R} \times I \to I$ such that $H - F$ is bounded, then $\mathbb{H}$ and $H(0,\cdot) - \mathrm{Id}_I$ (respectively $H'(0,\cdot) - 1$) are bounded,

– For dynamical systems in the sense of definition 3 (respectively 4, 5):

for every $H \in \mathcal{K}_1$ (respectively $H \in \mathcal{K}_2$, $H \in \mathcal{K}_3$)) if there exists a dynamical system $F\colon \mathbb{R} \times I \to I$ such that $H - F$ is bounded, then $\mathbb{H}$ is bounded,

    e/**inverse uniform b-stability**

i.e., the inverse b-stability for which the boundedness of the difference/ differences appearing in the consequence of the implication does not depend on H. More precisely:

– For dynamical systems in the sense of definition 1 (respectively 2):
for every $\delta > 0$ there exists an $\varepsilon > 0$ such that for every $H\colon \mathbb{R} \times I \to I$ if there exists a dynamical system $F\colon \mathbb{R} \times I \to I$ such that $H - F$ is bounded by δ, then $\mathbb{H}$ and $H(0,\cdot) - \mathrm{Id}_I$ (respectively $H'(0,\cdot) - 1$) are bounded by ε,
– For dynamical systems in the sense of definition 3 (respectively 4, 5):
for every $\delta > 0$ there exists an $\varepsilon > 0$ such that for every $H \in \mathcal{K}_1$ (respectively $H \in \mathcal{K}_2$, $H \in \mathcal{K}_3$)) if there exists a dynamical system $F\colon \mathbb{R} \times I \to I$ such that $H - F$ is bounded by δ, then $\mathbb{H}$ is bounded by ε,

     f/**superstability**
– For dynamical systems in the sense of definition 1 (respectively 2):
if $\mathbb{H}$ and $H(0,\cdot) - \mathrm{Id}_I$ (respectively $H'(0,\cdot) - 1$) are bounded, then H is bounded or it is a dynamical system,
– For dynamical systems in the sense of definition 3 (respectively 4, 5):
for every $H \in \mathcal{K}_1$ (respectively $H \in \mathcal{K}_2$, $H \in \mathcal{K}_3$)) if $\mathbb{H}$ is bounded, then H is bounded or it is a dynamical system.

     g/**inverse superstability**
– For dynamical systems in the sense of definition 1 (respectively 2):
if H is bounded or it is a dynamical system, then $\mathbb{H}$ and $H(0,\cdot) - \mathrm{Id}_I$ (respectively $H'(0,\cdot) - 1$) are bounded,
– For dynamical systems in the sense of definition 3 (respectively 4, 5):
for every $H \in \mathcal{K}_1$ (respectively $H \in \mathcal{K}_2$, $H \in \mathcal{K}_3$)) if H is bounded or it is a dynamical system, then $\mathbb{H}$ is bounded.

     h/**hiperstability**
– For dynamical systems in the sense of definition 1 (respectively 2):
if $\mathbb{H}$ and $H(0,\cdot) - \mathrm{Id}_I$ (respectively $H'(0,\cdot) - 1$) are bounded, then H is a dynamical system,
– For dynamical systems in the sense of definition 3 (respectively 4, 5):
for every $H \in \mathcal{K}_1$ (respectively $H \in \mathcal{K}_2$, $H \in \mathcal{K}_3$)) if $\mathbb{H}$ is bounded, then H is a dynamical system.

     i/**inverse hiperstability**
– For dynamical systems in the sense of definition 1 (respectively 2):
if H is a dynamical system, then $\mathbb{H}$ and $H(0,\cdot) - \mathrm{Id}_I$ (respectively $H'(0,\cdot) - 1$) are bounded,
– For dynamical systems in the sense of definition 3 (respectively 4, 5):
for every $H \in \mathcal{K}_1$ (respectively $H \in \mathcal{K}_2$, $H \in \mathcal{K}_3$)) if H is a dynamical system, then $\mathbb{H}$ is bounded.

3. Positive results

Theorem 3.1. *Let $H\colon \mathbb{R} \times I \to I$ be a continuous function satisfying*

$$|\mathbb{H}(t,s,x)| \le \delta_1 \quad \text{for } t, s \in \mathbb{R},\ x \in I \text{ and some } \delta_1 > 0, \qquad (3.1)$$

and

$$|H(0,x) - x| \leq \delta_2 \quad \text{for } x \in I, \text{ and some } \delta_2 > 0, \tag{3.2}$$

such that $H(0,\cdot)$ is monotone. Then there exists a dynamical system $F \colon \mathbb{R} \times I \to I$ such that

(a) *if $H(0,\cdot)$ is increasing, then*

$$|H(t,x) - F(x,t)| \leq \max\{6\delta_1 + \delta_2, 9\delta_1\} \quad \text{for } t \in \mathbb{R}, x \in I,$$

(b) *if $H(0,\cdot)$ is decreasing, then*

$$|H(t,x) - F(t,x)| \leq 2\delta_2 \quad \text{for } t \in \mathbb{R}, x \in I. \tag{3.3}$$

Proof. (a) Let us consider the following cases.

i) For every $x \in I$ the interval $H(\mathbb{R}, x)$ has the length not greater than $6\delta_1$. Put $F(t,x) := x$. Then, of course, F is a dynamical system and

$$|F(t,x) - H(t,x)| \leq |x - H(0,x)| + |H(0,x) - H(t,x)| \leq \delta_2 + 6\delta_1, \, t \in \mathbb{R}, \, x \in I.$$

ii) There are some $x \in I$ for which the length of the interval $H(\mathbb{R}, x) =: A_x$ is greater than $6\delta_1$. In this case we use some facts proven in [7] (see the beginning of the proof of Theorem 1.1, section 3 in [7]). Put $\mathcal{L}_{6\delta_1} := \{x \in I : |A_x| > 6\delta_1\}$. The intervals A_x, for $x \in \mathcal{L}_{6\delta_1}$, are either equal or disjoint. Let $(B_n : n \in N)$, where $N \subset \mathbb{N}$ is a set of indices, be the injective sequence of all intervals A_x, $x \in \mathcal{L}_{6\delta_1}$. We have

$$\bigcup_{n \in N} B_n = \bigcup_{x \in \mathcal{L}_{6\delta_1}} A_x = \mathcal{L}_{6\delta_1} \cap H(\mathbb{R}, I). \tag{3.4}$$

It has been shown that B_n are open intervals,

$$H(t, \inf B_n) = \inf B_n, \; H(t, \sup B_n) = \sup B_n \quad \text{for } t \in \mathbb{R} \tag{3.5}$$

provided $\inf B_n$, respectively $\sup B_n$, are in I (see the proof of Lemma 2.3(iii) in [7]), and for any point x of B_n we have $A_x = B_n$. Thus for the intervals B_n the assumptions of the main result from [2] are satisfied and we infer that there exist homeomorphisms $h_n \colon B_n \to \mathbb{R}$ such that

$$|h_n^{-1}(h_n(x) + t) - H(t,x)| \leq 9\delta_1, \quad t \in \mathbb{R}, x \in B_n. \tag{3.6}$$

Let $c := \inf \bigcup_{n \in N} B_n$ and $d := \sup \bigcup_{n \in N} B_n$, particularly, $c = \inf B_n$ and $d = \sup B_m$ for some $n, m \in N$. We will show that for $x < c$ and for $x > d$ we have $|A_x| \leq 6\delta_1$. Fix an $x < c$ (for $x > d$ the proof is analogous). By the monotonicity of $H(0,\cdot)$ and (3.5) we have $H(0,x) \leq H(0,c) = c$. Since $H(0,x) \in H(\mathbb{R}, I)$ we deduce by (3.4) that $|A_{H(0,x)}| \leq 6\delta_1$. Hence $|A_x| \leq 6\delta_1$, as follows from Corollary 2.3(i) from [7]. We have shown that

$$[(-\infty, c) \cup (d, \infty)] \cap I \subset I \setminus \mathcal{L}_{6\delta_1}.$$

But

$$I \setminus H(\mathbb{R}, I) \subset [(-\infty, c) \cup (d, \infty)] \cap I,$$

thus

$$\mathcal{L}_{6\delta_1} \cap (I \setminus H(\mathbb{R}, I)) = \varnothing$$

and we get

$$\mathcal{L}_{6\delta_1} = (\mathcal{L}_{6\delta_1} \cap H(\mathbb{R}, I)) \cup (\mathcal{L}_{6\delta_1} \cap (I \setminus H(\mathbb{R}, I))) = \bigcup_{n \in N} B_n.$$

Put

$$F(t, x) = \begin{cases} h_n^{-1}(h_n(x) + t), & \text{if} \quad x \in B_n, \, t \in \mathbb{R}, \, n \in N; \\ x, & \text{if} \quad x \notin \bigcup_{n \in N} B_n, \, t \in \mathbb{R}. \end{cases}$$

F defined in this way is a dynamical system (see [8]). Moreover, by (3.6) we know that $|F(t, x) - H(t, x)| \leq 9\delta_1$ for $x \in B_n$, $n \in N$, $t \in \mathbb{R}$. If $x \notin \bigcup_{n \in N} B_n = \mathcal{L}_{6\delta_1}$ then $|A_x| \leq 6\delta_1$ and we have

$$|F(t, x) - H(t, x)| \leq |x - H(0, x)| + |H(0, x) - H(t, x)| = \delta_2 + 6\delta_1, \quad t \in \mathbb{R}.$$

(b) If $H(0, \cdot)$ is decreasing, the interval I must be bounded since in the opposite case $\lim_{x \to +\infty} |H(0, x) - x| = +\infty$ or $\lim_{x \to -\infty} |H(0, x) - x| = +\infty$, thus their is a contradiction to (3.2). Put $a := \inf I$ and $b := \sup I$ and $H(0, a) := \lim_{x \to a+0} H(0, x)$ if $a \notin I$, $H(0, b) := \lim_{x \to b-0} H(0, x)$ if $b \notin I$. We have

$$b - a = b - H(0, b) + H(0, b) - H(0, a) + H(0, a) - a \leq$$
$$\delta_2 + [H(0, b) - H(0, a)] + \delta_2 \leq 2\delta_2.$$

The function F given by $F(t, x) = x$, for $x \in I$ and $t \in \mathbb{R}$, is a dynamical system for which (3.3) is satisfied. $\square$

Remark 3.2. The assumption (3.2) is essential in the theorem. Really, for $I = \mathbb{R}$ and $H(t, x) := f(x)$, where

$$f(x) = \begin{cases} -1, & \text{for} \quad x < -1; \\ x, & \text{for} \quad |x| \leq 1; \\ 1, & \text{for} \quad x > 1, \end{cases}$$

we have $f(f(x)) = f(x)$ for $x \in \mathbb{R}$, thus (3.1) is satisfied, $H(0, \cdot)$ is monotone and there does not exist any dynamical system F for which $|H - F|$ is bounded, since $|H(0, x) - F(0, x)| = |1 - x| \to +\infty$ if $x \to +\infty$.

Remark 3.3. The assumption that $H(0, \cdot)$ is monotone is essential in the theorem, too. Let us consider the following example. Suppose that I is an interval bounded from below. Let $a := \inf I$ and $b \in I$, $b > a$. Put $\varepsilon := \frac{b-a}{4}$. Fix any $\delta > 0$. Let $c \in (a, b)$ be such a point that $c - a < \min\{\frac{\delta}{2}, \varepsilon\}$. Let $f \colon I \to \mathbb{R}$ be a differentiable[2] function such that $f(a) \in (c, c + \frac{\delta}{2})$, $f(x) = x$ for $x \geq c$, $f(x) > x$ for $x < c$, f is strictly decreasing on interval (a, d) and strictly increasing on interval (d, c) for some $d \in (a, c)$. Let $c_1 \in (a, d)$ be such a point

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[2] For this example it is sufficient that f is continuous—however, see the end of this remark.

that $f(c_1) = c$. Let $h \colon (c, b) \to \mathbb{R}$ be a strictly increasing homeomorphism. We define $H \colon \mathbb{R} \times I \to I$ by the formula

$$H(t, x) = \begin{cases} h^{-1}(h(f(x)) + t), & \text{if} \quad f(x) \in (c, b), \ t \in \mathbb{R}; \\ f(x), & \text{if} \quad f(x) \in I \setminus (c, b), \ t \in \mathbb{R}. \end{cases}$$

Then $H(0, x) = f(x)$ for $x \in I$. We have

$$|f(x) - x| \le |f(a) - a| = (f(a) - c) + (c - a) \le \delta, \quad x \in I. \qquad (3.7)$$

Thus

$$|H(0, x) - x| \le \delta, \quad x \in I.$$

Now we check that (3.1) is satisfied with $\delta_1 = \delta$. Fix $x \in I$ and $s, t \in \mathbb{R}$. Let us consider the following cases.

   i) $x < c_1$.
In this case $f(x) \in (c, b)$ and $H(s, x) = h^{-1}(h(f(x)) + s) \in (c, b)$, hence $f(H(s, x)) = H(s, x) \in (c, b)$. We have

$$\begin{aligned} H(s + t, x) &= h^{-1}(h(f(x)) + s + t) = h^{-1}(h(h^{-1}(h(f(x)) + s)) + t) \\ &= h^{-1}(h(H(s, x)) + t) = h^{-1}(h(f(H(s, x))) + t) = H(t, H(s, x)). \end{aligned}$$

   ii) $x \in [c_1, c] \cup ([b, \infty) \cap I)$.
In this case $f(x) \in I \setminus (c, b)$ and $f(f(x)) \in I \setminus (c, b)$. Hence $H(s + t, x) = f(x)$ and $H(s, H(t, x)) = H(s, f(x)) = f(f(x))$. From (3.7) we infer that $|f(f(x)) - f(x)| \le \delta$, thus (3.1) is satisfied.
   iii) $x \in (c, b)$.
In this case $f(x) = x \in (c, b)$, hence

$$H(s, H(t, x)) = H(s, h^{-1}(h(f(x)) + t)) = h^{-1}(h(f(x)) + s + t) = H(s + t, x).$$

Suppose that $F \colon \mathbb{R} \times I \to I$ is a dynamical system such that $|F(t, x) - H(t, x)| \le \varepsilon$. Fix $x \in (a, c_1)$. We have

$$\lim_{t \to \infty} H(t, x) = \lim_{t \to \infty} h^{-1}(h(f(x)) + t) = b.$$

Hence there exists a $t_1 \in \mathbb{R}$ such that

$$F(t_1, x) > b - 2\varepsilon > c. \qquad (3.8)$$

We also have $F(0, x) = x < c$. From the continuity of $F(\cdot, x)$ we infer that there exists an $s \in \mathbb{R}$ such that $F(s, x) = c$. Put $t_2 = t_1 - s$. Then

$$|F(t_1, x) - c| = |F(t_2, F(s, x)) - c| = |F(t_2, c) - H(t_2, c)| \le \varepsilon.$$

But $b - c = (b - a) - (c - a) > 4\varepsilon - \varepsilon = 3\varepsilon$, therefore $F(t_1, x) \le c + \varepsilon < b - 3\varepsilon + \varepsilon = b - 2\varepsilon$, which is a contradiction to (3.8).

Moreover this example proves that the dynamical system from the definition 1 is not Ulam–Hyers stable if $I \ne \mathbb{R}$ (even in the class of functions $F \colon \mathbb{R} \times I \to I$ for which $F'(0, x)$ exists).

Remark 3.4. The function $H(0, \cdot)$ is evidently monotone if the function $H'(0, \cdot)$ has the constant sign, in particular if $H'(0, x) \neq 0$ for $x \in I$. The example above shows that if $H'(0, x)$ is zero at least at one point x then the system may be non-stable.

Remark 3.5. The general form of dynamical systems is as follows [8]:

$$F(t, x) = \begin{cases} h^{-1}(h(x) + t), & \text{if} \quad x \in I_n, \, t \in \mathbb{R}, \, n \in N; \\ x, & \text{if} \quad x \in I \setminus \bigcup_{n \in N} I_n, \, t \in \mathbb{R}, \end{cases} \qquad (3.9)$$

where $I_n \subset I$, $n \in N \subset \mathbb{N}$, are open and disjoint intervals, and $h_n \colon I_n \to \mathbb{R}$, $n \in N$, are the homeomorphisms.

Notice that the dynamical system F in the above proof is of this form (with $I_n = B_n$) and, moreover, $\inf_{n \in N} |I_n| = \inf_{n \in N} |B_n| \geq 6\delta > 0$.

Let us call any F of the form (3.9) with the additional assumption $\inf_{n \in N} |I_n| > 0$ a *simple dynamical system*. Every dynamical system F satisfies the conditions (3.1) and (3.2) with arbitrary $\delta_1 > 0$ and $\delta_2 > 0$, hence for every $\varepsilon > 0$ there exists a simple dynamical system F_1 such that $|F(t, x) - F_1(t, x)| \leq \varepsilon$ for $t \in \mathbb{R}$, $x \in I$.

Thus for every dynamical system F there exists a simple dynamical system arbitrarily close to F. However, not every simple dynamical system can be approximated by a dynamical system of the form (3.9) with $\inf_{n \in N} |I_n| = 0$. Indeed, let $F \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a simple dynamical system given by $F(t, x) = t + x$. If F_1 is a dynamical system with $\inf_{n \in N} |I_n| = 0$ then it has at least one fixed-point, e.g. x_0. We have $|F(t, x_0) - F_1(t, x_0)| = |t + x_0 - x_0| = |t| \to +\infty$ as $t \to +\infty$. Thus there does not exist a dynamical system F_1 with $\inf_{n \in N} |I_n| = 0$ such that $|F(t, x) - F_1(t, x)|$ is bounded.

If the dynamical system is stable, it is possible to formulate *the problem of uniqueness*: for H given, is the dynamical system F which approximates H unique or not? The answer is not. Really, if H is a dynamical system which is not simple, then there exist two such dynamical systems: H and, by the above, the simple dynamical system F which approximates H.

Corollary 3.6. *The translation equation is stable in the class of continuous functions $H \colon \mathbb{R} \times I \to I$ for which $H(0, x) = x$ for $x \in I$.*

Remark 3.7. In the above corollary we stated that if a continuous $H \colon \mathbb{R} \times I \to I$ satisfies the identity condition exactly and the translation equation approximately, then there exists $F \colon \mathbb{R} \times I \to I$ which satisfies both the identity condition and the translation equation and is close to H.

Let us consider the "reverse" case in which H satisfies the translation equation exactly and the identity condition approximately. In Theorem 4.2 from [7]

it was proven that[3] there exists an $\varepsilon > 0$ such that for every $\delta > 0$ there exists a continuous $H \colon \mathbb{R} \times I \to I$ which satisfies the translation equation (exactly) and the identity condition up to δ and is such that for every dynamical system $F \colon \mathbb{R} \times I \to I$ we have $|H(t, x) - F(t, x)| > \varepsilon$ for some $(t, x) \in \mathbb{R} \times I$.

Corollary 3.8. *If the continuous function $H \colon \mathbb{R} \times I \to I$ is such that*

$$|\mathbb{H}(t, s, x)| \leq \delta \quad \text{for } t, s \in \mathbb{R}, \ x \in I \ \text{ and some } \delta \in \left(0, \frac{2}{5}\right),$$

and

$$|H'(0, x) - 1| \leq \delta \quad \text{for } x \in I,$$

then there exists a dynamical system $F \colon \mathbb{R} \times I \to I$ such that

$$|H(t, x) - F(t, x)| \leq 10\delta \quad \text{for } t \in \mathbb{R}, \ x \in I.$$

Proof. Put $h(x) = H(0, x)$ for $x \in I$. We have

$$|H(0, H(0, x)) - H(0, x)| \leq \delta, \quad x \in I,$$

thus (3.2) is satisfied for every $x \in H(0, I) = h(I)$. We have $|h'(x) - 1| \leq \delta$, thus:

$$0 < 1 - \delta \leq h'(x) \leq 1 + \delta, \tag{3.10}$$

thus the function h is increasing. Let $y_1 = \inf I$, $y_2 = \sup I$, $x_1 = \inf h(I)$, $x_2 = \sup h(I)$. We will show that (3.2) is satisfied with $\delta_2 = 4\delta$. Let us consider some cases.

1/ For $y_1 > -\infty$ and $y_2 = +\infty$, h is unbounded (since, in the contrary case, we would have

$$\frac{h(n) - h(y_1 + 1)}{n - (y_1 + 1)} = h'(\xi(n)) \to 0 \ \text{ for } n \to +\infty,$$

which is a contradiction to (3.10)).

    a/ If $x_1 = y_1$, then $h(I) = I$ and the condition (3.2) is satisfied.

    b/ If $x_1 > y_1$ and $y_1 \in I$, then we have $h(y_1) = x_1$ and $|h(x_1) - x_1| \leq \delta$, and since $h(x_1) - x_1 = h(x_1) - h(y_1) = h'(\xi)(x_1 - y_1)$ for a ξ

$$(1 - \delta)(x_1 - y_1) \leq h'(\xi)(x_1 - y_1) = |h(x_1) - x_1| \leq \delta,$$

thus $x_1 - y_1 \leq \frac{\delta}{1-\delta}$.

    We have for $x \in [y_1, x_1)$

$$|h(x) - x| \leq |h(x) - x_1| + |x_1 - x| = h'(\xi)(x - y_1) + (x_1 - x)$$

$$\leq \frac{\delta}{1 - \delta}[h'(\xi) + 1] \leq \frac{\delta(\delta + 2)}{1 - \delta} \leq 4\delta.$$

---

[3] Let the second author correct some lapse from [7] on this occasion. Actually, in the proof of this Theorem 4.2 in [7] it was incorrectly stated "Such H belongs to $\mathcal{D}_\delta(I)$, actually even to $\mathcal{D}(I)$". It should be written there: "Such H belongs to $\mathcal{D}_\delta(I)$, actually H even satisfies the translation equation".

Since $|h(x) - x| \leq 4\delta$ for $x \in I$, by the Theorem 3.1 there exists a dynamical system F such that $|H(t, x) - F(t, x)| \leq 10\delta$ for $t \in \mathbb{R}$, $x \in I$.

c/ If $x_1 > y_1$ and $y_1 \notin I$, then we put $h(y_1) = x_1$ and we consider as above.
2/ If $y_1 = -\infty$ and $y_2 = +\infty$, then by (3.10) the function h is unbounded from above and below, thus $h(I) = \mathbb{R}$ and (3.2) is satisfied.

The proof in the other cases is analogous. $\qquad\qquad\square$

The above corollary shows that a dynamical system in the sense of definition 2 is Ulam–Hyers stable as well as uniformly b-stable (hence b-stable).

Let us recall

Theorem 3.9. (Theorem 4.1 in [7]) *If $H \colon \mathbb{R} \times I \to I$ is continuous, surjective, and satisfies (1.3) then there exists a dynamical system $F \colon \mathbb{R} \times I \to I$ such that $|F(t, x) - H(t, x)| \leq 9\delta$.*

This theorem shows that a dynamical system in the sense of definition 5 is Ulam–Hyers stable and uniformly b-stable (hence b-stable).

Remark 3.10. In this case the function $H(0, x)$ may be non-monotone, e.g. the function

$$H(t, x) = f(x) = \begin{cases} x + \delta, & \text{for} \quad x \leq 0; \\ -x + \delta, & \text{for} \quad 0 < x < \delta; \\ x - \delta, & \text{for} \quad x \geq \delta, \end{cases}$$

for which $|H(0, x) - x| = |f(x) - x| \leq \delta$ for $x \in \mathbb{R}$.

Moreover, notice that if $I = \mathbb{R}$ then inequality (1.4) for continuous $H \colon \mathbb{R} \times I \to I$ implies that H is surjective. Thus we have the following

Corollary 3.11. *If $I = \mathbb{R}$ then a dynamical system in the sense of definition 1 is uniformly b-stable.*

The other "positive results" are trivial:

Proposition 3.12. *If I is bounded then a dynamical system in the sense of definitions 1–5 is uniformly b-stable, thus b-stable, superstable and inversely superstable.*

If I is bounded then a dynamical system in the sense of definitions 1, 3–5 is inversely uniformly b-stable, thus inversely b-stable, too.

A dynamical system in the sense of definitions 3–5 is inversely superstable.

A dynamical system in the sense of definitions 1–5 is inversely hiperstable.

4. Negative results

Theorem 4.1. *A dynamical system in the sense of definitions 3 and 4 (even if we suppose that $F'(0, x) \neq 0$ for $x \in I$) is not Ulam–Hyers stable.*

Proof. Let $a, b \in I$, $a < b$, $\varepsilon < \frac{b-a}{2}$, $\delta > 0$. Let $f : I \to [a, \frac{a+b}{2}] \cap [a, a + \delta]$ be a strictly increasing differentiable function such that $f'(x) \neq 0$, $x \in I$. For $H(t, x) = f(x)$ we have $|\mathbb{H}(t, s, x)| \leq \delta$ and

$$|H(0, b) - F(0, b)| = |H(0, b) - b| = b - H(0, b) \geq b - \frac{a + b}{2} = \frac{b - a}{2} > \varepsilon$$

for every dynamical system F. □

Theorem 4.2. *If I is unbounded, a dynamical system in the sense of definitions 3 and 4 (even if we suppose that $F'(0, x) \neq 0$ for $x \in I$) is not b-stable (thus is not uniformly b-stable).*

Proof. Let's say that I is unbounded from above. Let $\varepsilon > 0$, $a, b \in I$, $a < b$, $b - a < \varepsilon$. Let $f : I \to [a, b]$ be a strictly increasing function, differentiable with $f'(x) \neq 0$. For $H(t, x) = f(x)$ we have $|\mathbb{H}(t, s, x)| \leq b - a \leq \varepsilon$. But $|H(0, x) - F(0, x)| = |f(x) - x| \to \infty$ as $x \to \infty$ for every dynamical system F. □

Theorem 4.3. *If unbounded I is bounded from one side then a dynamical system in the sense of definition 1 is not b-stable (thus is not uniformly b-stable).*

Proof. Suppose that I is an interval bounded from below, unbounded from above. Let $a := \inf I$ and $b \in I$, $a < b$. Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by $f(x) = |x - b| + b$. Let $h : (b, \infty) \to \mathbb{R}$ be a strictly increasing homeomorphism. We define $H : \mathbb{R} \times I \to I$ by the formula

$$H(t, x) = \begin{cases} h^{-1}(h(f(x)) + t), & \text{if} \quad f(x) \in (b, \infty),\ t \in \mathbb{R}; \\ f(x), & \text{if} \quad f(x) = b,\ t \in \mathbb{R}. \end{cases}$$

Then $H(0, x) = f(x)$ for $x \in I$. We have

$$|f(x) - x| \leq |f(a) - a| \quad x \in I. \tag{4.1}$$

Thus $|H(0, x) - x|$ is bounded. Moreover, the translation equation is satisfied, since $f(f(x)) = f(x)$.

Suppose that $F : \mathbb{R} \times I \to I$ is a dynamical system such that $|F(t, x) - H(t, x)| \leq M(x)$ for some $M : I \to \mathbb{R}$. Fix $x \in (a, b)$. We have

$$\lim_{t \to \infty} H(t, x) = \lim_{t \to \infty} h^{-1}(h(f(x)) + t) = \infty.$$

Hence

$$\lim_{t \to \infty} F(t, x) = \infty. \tag{4.2}$$

We also have $F(0, x) = x < b$. From the continuity of $F(\cdot, x)$ we infer that there exists an $s \in \mathbb{R}$ such that $F(s, x) = b$. We have $|F(t, b) - H(t, b)| = |F(t, b) - b| \leq M(b)$ for every $t \in \mathbb{R}$, this gives

$$F(t, x) = F(t - s, F(s, x)) = F(t - s, b) \in [b - M(b), b + M(b)], \quad t \in \mathbb{R},$$

which is a contradiction to (4.2). □

Theorem 4.4. *A dynamical system in the sense of definitions 1–5 is not inversely stable.*

Proof. Fix $a, b, c, d \in I$, $c < a < b < d$. Let $\varepsilon < \min\{(b-a)/2, 2(a-c), 1/2\}$. Let $\delta \in (0, \varepsilon/2)$. We define $F \colon \mathbb{R} \times I \to I$ by the formula

$$F(t, x) = \begin{cases} h^{-1}(h(x) + t), & \text{if} \quad x \in (a, b), \, t \in \mathbb{R}; \\ x, & \text{if} \quad x \in I \setminus (a, b), \end{cases}$$

where $h \colon (a, b) \to \mathbb{R}$ is a strictly increasing homeomorphism. Let $g \colon \mathbb{R} \to [0, \delta]$ be a differentiable function such that $g(x) = 0$ for $x \leq a$ and for $x \geq d$, $g(b) > 0$, $|g'(x)| \leq 1/2$ and there exists a $y \in (a, b)$ such that $g'(y) > \varepsilon$.

We define $H \colon \mathbb{R} \times I \to \mathbb{R}$ by $H(t, x) = F(t, x) - g(x)$. To see that $H(\mathbb{R}, I) \subset I$ let us consider the following:
if $x \in (-\infty, a] \cap I$, then $H(t, x) = F(t, x) \in I$;
if $x \in (a, \infty) \cap I$, then $H(t, x) = F(t, x) - g(x) \in [F(t, x) - \delta, F(t, x)] \subset [a - \delta, F(t, x)] \subset [c, F(t, x)] \subset I$.

We have $|H(t, x) - F(t, x)| = |g(x)| \leq \delta$.

Moreover, $H(s + t, b) = F(s + t, b) - g(b) = b - g(b) \geq b - \delta$ and $H(t, b) = F(t, b) - g(b) = b - g(b) \in (a, b)$. Hence $F(s, H(t, b)) \to a$ as $s \to -\infty$. Let s_0 be such that $F(s_0, H(t, b)) < a + \delta$. We have $H(s_0, H(t, b)) = F(s_0, H(t, b)) - g(H(t, b)) < a + \delta$.

Therefore $H(s_0 + t, b) - H(s_0, H(t, b)) > b - \delta - a - \delta > 2\varepsilon - 2\delta > \varepsilon$.

Furthermore, $H(0, x) = F(0, x) - g(x) = x - g(x)$, thus $H(0, \cdot)$ is differentiable, $H'(0, x) = 1 - g'(x) > 0$, so $H(0, \cdot)$ is strictly increasing.

Moreover, H is surjective: for $x \leq c$ or $x \geq d$ we have $x = F(0, x) = F(0, x) - 0 = H(0, x)$, hence $x \in V := H(\mathbb{R}, I)$. But V is an interval, thus $V = I$.

Additionally, $|H'(0, y) - 1| = |g'(y)| > \varepsilon$. $\qquad \square$

Theorem 4.5. *If I is unbounded then a dynamical system in the sense of definitions 1, 3–5 is not inversely b-stable (thus is not inversely uniformly b-stable).*

For every I, a dynamical system in the sense of definition 2 is not inversely b-stable (thus is not inversely uniformly b-stable).

Proof. Let
$a = 0$, $\alpha = 1$ if $I = \mathbb{R}$,
$a = \inf I$, $\alpha = 1$ if I is bounded from below,
$a = \sup I$, $\alpha = -1$ if I is bounded from above.
Put $F(t, x) = (x - a)e^t + a$ for $t \in \mathbb{R}$, $x \in I$, and $H(t, x) = F(t, x) + \alpha \delta$ for some $\delta > 0$. We see that F is a dynamical system, $|H(t, x) - F(t, x)| \leq \delta$ for $t \in \mathbb{R}$, $x \in I$, $H'(0, x) = 1$ and $\mathbb{H}(t, s, x) = \alpha \delta e^t$ is unbounded. Thus a dynamical system in the sense of definitions 1–4 is not inversely stable.

If $I = \mathbb{R}$, then the function H is a surjection, thus the example above proves that a dynamical system in the sense of definition 5 is not inversely stable.

If unbounded I is bounded for example from below, then H is not a surjection. In this case we put:

$$H^*(t,x) = \begin{cases} H(t,x), & \text{if} \quad x > a+1, \, t \in \mathbb{R}; \\ (\delta + e^t)(x-a) + a, & \text{if} \quad x < a+1, \, x \in I, \, t \in \mathbb{R}. \end{cases}$$

This function is a surjection, $|H^*(t,x) - F(t,x)| \leq \delta$ and $\mathbb{H}^*(t,s,x)$ is unbounded. Thus a dynamical system in the sense of definition 5 is not inversely stable in this case either.

Thus we have proven the first part of this Theorem.

Now assume that I is an arbitrary nondegenerate interval. Let $f \colon I \to I$ be a differentiable function such that $f(\cdot) - \mathrm{Id}_I$ is bounded, and with unbounded derivative. Then for $H, F \colon \mathbb{R} \times I \to I$ given by $H(t,x) = f(x)$ for $x \in I$ and $t \in \mathbb{R}$, $F(t,x) = x$ for $t \in \mathbb{R}$ and $x \in I$, we have: $|F - H|$ is bounded, F is a dynamical system, $|H'(0, \cdot) - 1|$ is unbounded. $\qquad \square$

Theorem 4.6. *If I is unbounded then a dynamical system in the sense of definitions 1–5 is not superstable.*

Proof. Let $a, b \in I$, $a < b$. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a differentiable function, with $3/2 \geq f'(x) \geq 1/2$ such that $f(x) = x$ for $x \geq b$ and for $x \leq a$, $f(x) > x$ for $x \in (a,b)$. We have $|f(x) - x| \leq b - a$ for $x \in \mathbb{R}$, in particular $|f(f(x)) - f(x)| \leq b - a$ for $x \in \mathbb{R}$. Let us define $H(t,x) = f(x)$. Then $|\mathbb{H}(t,s,x)| \leq b - a$, $H(0, \cdot)$ is differentiable with $|H'(0,x) - 1| \leq 1/2$. Moreover $H(0, \mathbb{R}) = \mathbb{R}$, hence H is a surjection. But H is not a dynamical system (for $x \in (a,b)$ we have $H(0,x) = f(x) > x$) and H is unbounded.

Thus we have proven that a dynamical system in the sense of definitions 1, 2 and 5 is not superstable. Now we are going to prove that the dynamical system, also in the sense of definitions 3 and 4, is not superstable.

Assume for example that I is an interval unbounded from above, $\inf I < a \in I$. Let $f \colon \mathbb{R} \to [a - 1, \infty)$ be a differentiable function, with $f'(x) > 0$ such that $f(x) = x$ for $x \geq a$, $f(x) > x$ for $x < a$. We have $|f(x) - x| \leq 1$ for $x \in [a-1, a]$, and $f(x) = x$ for $x \geq a$, hence $|f(f(x)) - f(x)| \leq 1$ for $x \in \mathbb{R}$. Let us define $H(t,x) = f(x)$. Then $|\mathbb{H}(t,s,x)| \leq 1$, $H(0, \cdot)$ is differentiable with $H'(0,x) > 0$ (hence strictly increasing). But H is not a dynamical system (for $x < a$ we have $H(0,x) = f(x) > x$) and H is unbounded. $\qquad \square$

Theorem 4.7. *If I is unbounded then a dynamical system in the sense of definition 1 is not inversely superstable.*

Proof. For bounded H the difference $H(0,x) - x$ is unbounded. $\qquad \square$

Theorem 4.8. *A dynamical system in the sense of definition 2 is not inversely superstable.*

Proof. If $H(t,x) = f(x)$, where $f \colon I \to I$ is a bounded differentiable function with the derivative unbounded, then $H'(0,\cdot) - 1$ is unbounded. $\qquad\square$

Theorem 4.9. *A dynamical system in the sense of definitions 1–5 is not hiperstable.*

Proof. A dynamical system in the sense of definitions 1–5 is not superstable for unbounded I, thus, for such I, it is not hiperstable.

Assume that I is bounded. Let $f \colon I \to I$ be a differentiable, not identically equal to Id_I, strictly increasing surjection with bounded derivative (for example

$$f(x) = \frac{(x-a)^2}{b-a} + a, \quad x \in I,$$

where $a = \inf I$, $b = \sup I$). Then H given by $H(t,x) := f(x)$ belongs to $\mathcal{K}_i$ for $i = 1, 2, 3$, the functions $\mathbb{H}$, $H(0,\cdot) - \mathrm{Id}_I$ and $H'(0,\cdot) - 1$ are bounded but H is not a dynamical system since $f \circ f \neq f$. $\qquad\square$

5. Summary

In the table below, the answer is given to the question: is a dynamical system in the sense of the definition given in the first row of the table, stable in the sense given in the first column?

Every stability is considered in the class of continuous functions.

	def.1 ((1.1) & $F(0,x)=x$)	def .2 ((1.1) & $F'(0,x)=1$)	def. 3 ((1.1) & $F(0,\cdot)$ strictly increasing)	def. 4 ((1.1) & $F'(0,x)$ exists)	def.5 ((1.1) & F surjection)
Ulam-Hyers stability	only for $I = \mathbb{R}$	for every I	for no I		for every I
b-stability uniform b-stability	only for I bounded or $I = \mathbb{R}$	for every I	only for I bounded		for every I
inverse stability	for no I				
inverse b-stability inverse uniform b-stability	only for I bounded	for no I	only for I bounded		
superstability	only for I bounded				
inverse superstability	only for I bounded	for no I	for every I		
hiperstability	for no I				
inverse hiperstability	for every I				

The stability of a dynamical system depends thus on the system considered as its definition and often on the interval I (there are only four possibilities: every I, bounded I, $I = \mathbb{R}$, no I).

This table permits to determine the relations between the different types of stabilities for the dynamical system of specific definition. E.g. inverse b-stability, inverse uniform b-stability, superstability and inverse superstability are equivalent for a system in the sense of definition 1 since they are equivalent to the boundedness of I. Moreover only superstability (for bounded I) and the inverse hiperstability (for every I) are true for a system for every definition.

Acknowledgements

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