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On the group of automorphisms of a quasi-affine variety

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Abstract Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. We show that if the automorphisms group of a quasi-affine variety X over $\mathbb{K}$ is infinite, then X is uniruled.

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1 Introduction

Automorphism groups of open varieties have always attracted a lot of attention, but the nature of these groups is still not well-known. For example the group of automorphisms of $\mathbb{K}^n$ is understood only in the case $n = 2$ (and $n = 1$, of course). Let Y be an open variety. It is natural to ask when the group $\mathrm{Aut}(Y)$ of automorphisms of Y is finite. A partial answer to this question is given in our papers [7–9] and [10]. In [6], Iitaka proved that $\mathrm{Aut}(Y)$ is finite if Y has a maximal logarithmic Kodaira dimension. Here we focus on the group of automorphisms of an affine or, more generally, quasi-affine variety over an algebraically closed field of characteristic zero. Let us recall that a quasi-affine variety is an open subvariety of some affine variety. We prove the following:

Theorem 1.1 *Let X be a quasi-affine (in particular affine) variety over an algebraically closed field of characteristic zero. If the automorphism group* $\mathrm{Aut}(X)$ *is infinite, then X is uniruled, i.e., X is covered by rational curves.*

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This generalizes our old results from [9] and [10]. Our proof uses in a significant way a recent progress in the Minimal Model Program (see [1,2,14]) and is based on our old ideas from [7–9] and [10].

In particular, if X is a quasi-affine non-uniruled variety, then the automorphism group $\mathrm{Aut}(X)$ of X is finite. We show (cf. Proposition 7.2) that conversely, for every $k \geq 1$ and every finite group G there is a k-dimensional affine (smooth) non-uniruled variety X^k_G such that $\mathrm{Aut}(X^k_G) = G$. Hence in this version our result is optimal.

If a variety X is uniruled it may happened that the group $\mathrm{Aut}(X)$ is infinite and discrete. Indeed, M. H, El-Huti [3] showed the following interesting fact:

Example 1.2 Take the cubic surface $H_c \subset \mathbb{C}^3$ defined by $x^2+y^2+z^2-xyz = c,\ c \in \mathbb{C}$. Then the group $\mathrm{Aut}(H_c)$ is generated by a subgroup G isomorphic to $\mathbb{Z}/2\star\mathbb{Z}/2\star\mathbb{Z}/2$ and a finite subgroup V induced by affine linear mappings that preserves H_c. In fact

$$\mathrm{Aut}(H_c) = (\mathbb{Z}/2 \star \mathbb{Z}/2 \star \mathbb{Z}/2) \rtimes S_4,$$

where S_4 is the permutation group in 4 elements, see [11]. □

However, it is possible that if $\mathrm{Aut}(X)$ is non-discrete, we can obtain a more precise information on X than merely non-uniruledness. In particular Hanspeter Kraft and Mikhail Zaidenberg proposed the following:

Kraft-Zaidenberg Conjecture. *Assume that X is a quasi-affine variety with non-discrete group of automorphisms. Than on X acts effectively either the group $\mathbb{G}_a = \{\mathbb{K}, 0, +\}$ or the group $\mathbb{G}_m = \{\mathbb{K}^*, 1, \cdot\}$.*

2 Terminology

We assume that the ground field $\mathbb{K}$ is algebraically closed of characteristic zero. For an algebraic variety X (variety is here always irreducible) we denote by $\mathrm{Aut}(X)$ the group of all regular automorphisms of X and by $\mathrm{Bir}(X)$ the group of all birational transformations of X. By $\mathrm{Aut}_1(X)$ we mean the group of all birational transformation which are regular in codimension one, i.e., which are regular isomorphisms outside subsets of codimension at least two. If $X \subset \mathbb{P}^n(\mathbb{K})$ then we put $\mathrm{Lin}(X) = \{f \in \mathrm{Aut}(X) : f = \mathrm{res}_X T, T \in \mathrm{Aut}(\mathbb{P}^n(\mathbb{K}))\}$. Of course, the group $\mathrm{Lin}(X)$ is always an affine group.

Let $f : X - \to Y$ be a rational mapping between projective normal varieties. Then f is determined outside some (minimal) closed subset F of codimension at least two. If $S \subset X$ and $S \not\subset F$ then by $f(S)$ we mean the set $f(S \setminus F)$. Similarly for $R \subset Y$ we will denote the set $\{x \in X \setminus F : f(x) \in R\}$ by $f^{-1}(R)$.

If $f : X - \to Y$ is a birational mapping and the mapping f^{-1} does not contract any divisor, we say that f is a birational contraction.

An algebraic variety X of dimension $n > 0$ is called uniruled if there exists a variety W of dimension $n-1$ and a rational dominant mapping $\phi : W \times \mathbb{P}^1(\mathbb{K}) - \to X$. Equivalently, an algebraic variety X is uniruled if and only if for every point $x \in X$, there exists a rational curve Γ_x in X through this point.

We say that a divisor D is $\mathbb{Q}$-Cartier if for some non-zero integer $m \in \mathbb{Z}$ the divisor mD is Cartier. If every divisor on X is $\mathbb{Q}$-Cartier, then we say that X is $\mathbb{Q}$-factorial.

In this paper we treat a hypersurface $H = \bigcup_{i=1}^{r} H_i \subset X$ as a reduced divisor $\sum_{i=1}^{r} H_i$, and conversly a reduced divisor will be treated as a hyperserface.

3 Weil divisors on a normal variety

In this section we recall (with suitable modifications) some basic results about divisors on a normal variety (see e.g., [5]).

Definition 3.1 Let X be a normal complete variety. We will denote by $\mathrm{Div}(X)$ the group of all Weil divisors on X. For $D \in \mathrm{Div}(X)$ the set of all effective Weil divisors linearly equivalent to D is called a complete linear system given by D and denoted by $|D|$. Moreover, we set $L(D) := \{f \in \mathbb{K}(X) : f = 0 \text{ or } D + (f) \geq 0\}$.

We have the following (e.g., [5], 2.16, p.126)

Proposition 3.2 *If D is an effective divisor on a normal complete variety X, then $L(D)$ is a finite-dimensional vector space (over $\mathbb{K}$).*

Remark 3.3 The set $|D|$ (if non-empty) has a natural structure of projective space of dimension $\dim L(D) - 1$. By a basis of $|D|$ we mean any subset $\{D_0, ..., D_n\} \subset |D|$ such that $D_i = D + (\phi_i)$ and $\{\phi_0, ..., \phi_n\}$ is a basis of $L(D)$.

Let us recall the next

Definition 3.4 If D is an effective Weil divisor on a normal complete variety X, then by a canonical mapping given by $|D|$ and a basis ϕ we mean the mapping $i_{(D,\phi)} = (\phi_0 : ... : \phi_n) : X \to \mathbb{P}^n(\mathbb{K})$, where $\phi = \{\phi_0, ..., \phi_n\} \subset L(D)$ is a basis of $L(D)$.

Let X be a normal variety and Z a closed subvariety of X. Put $X' = X \setminus Z$. We would like to compare the groups $\mathrm{Div}(X)$ and $\mathrm{Div}(X')$. It can be easily checked that the following proposition is true (compare [4], 6.5., p. 133):

Proposition 3.5 *Let $j_{X'} : \mathrm{Div}(X) \ni \sum_{i=1}^{r} n_i D_i \to \sum_{i=1}^{r} n_i (D_i \cap X') \in \mathrm{Div}(X')$. Then $j_{X'}$ is an epimorphism that preserves linear equivalence. If additionally* $\operatorname{codim} Z \geq 2$, *then $j_{X'}$ is an isomorphism.*

Now we define the pull-back of a divisor under a rational map $f : X - \to Y$. Recall that a Cartier divisor can be given by a system $\{U_\alpha, \phi_\alpha\}$, where $\{U_\alpha\}$ is some open covering of X, $\phi_\alpha \in \mathcal{O}(U_\alpha)$ and $\phi_\alpha/\phi_\beta \in \mathcal{O}^*(U_\alpha \cap U_\beta)$.

Definition 3.6 Let $f : X \to Y$ be a dominant morphism between complete varieties. Let D be a Cartier divisor on Y given by a system $\{U_\alpha, \phi_\alpha\}$. By the pullback of the divisor D by f we mean the divisor f^*D given by the system $\{f^{-1}(U_\alpha), \phi_\alpha \circ f\}$. More generally if X, Y are complete and let f be a rational map. If X_f denotes the domain of f, we put $f^*(D) := (j_{X_f})^{-1}(\mathrm{res}_{X_f} f)^* D$. Finally let f be as above and let D be an arbitrary Weil divisor on Y. Let us assume additionally that $\operatorname{codim} f^{-1}(\mathrm{Sing}(Y)) \geq 2$. Then we have a regular map $f : X_f \setminus W \to Y_{\mathrm{reg}}$ (where $W := f^{-1}(\mathrm{Sing}(Y))$ and we put $f^*D := (j_{X_f \setminus W})^{-1} f^*(j_{Y_{\mathrm{reg}}}(D))$.

By a simple verification we have:

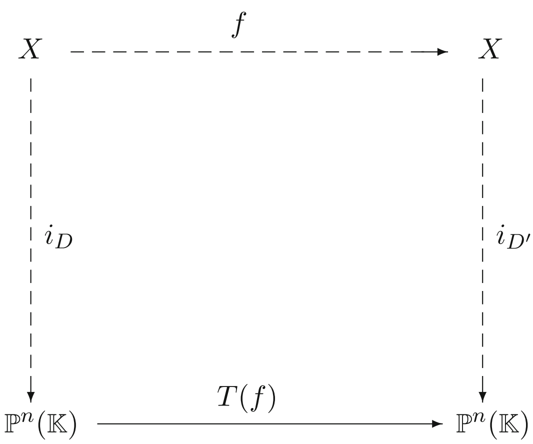
Proposition 3.7 *Let $f : X - \to Y$ be a dominant rational mapping between complete normal varieties, such that $f^{-1}(\mathrm{Sing}(Y))$ has codimension at least two. Then $f^* : \mathrm{Div}(Y) \ni D \to f^*D \in \mathrm{Div}(X)$ is a well-defined homomorphism preserving linear equivalence. Moreover,* $\mathrm{Supp}(f^*(D))$ *coincides with the* $\dim X - 1$*-dimensional part of the set $\overline{f^{-1}(\mathrm{Supp}(D))}$. In particular if D is an effective Cartier divisor, we have* $\mathrm{Supp}(f^*(D)) = \overline{f^{-1}(\mathrm{Supp}(D))}$.

Proof Let $W := \overline{f^{-1}(\mathrm{Sing}(Y))}$. By the assumption we have codim $W \geq 2$. Take $X' := X_f \setminus W$. Since X and X' differ by subsets of codimension at least two it is enough to prove our statement for regular mapping $f' : X' \to Y$ and for Weil divisors with support outside $Sing(Y)$, i.e., for Cartier divisors on a smooth variety. But now the statement is obvious. □

Corollary 3.8 *Let f be as in Proposition 3.7. Let us assume additionally that f is an isomorphism in codimension one. Then $f^* : \mathrm{Div}(Y) \to \mathrm{Div}(X)$ is an isomorphism preserving linear equivalence.* □

Finally we have the following important result:

Proposition 3.9 *Let X be a normal complete variety and $f \in \mathrm{Aut}_1(X)$. Let D be an effective divisor on X and $f^*D' = D$. Then $\dim|D| = \dim |D'| := n$ and there exists a unique automorphism $T(f) \in \mathrm{Aut}(\mathbb{P}^n(\mathbb{K}))$ such that the folowing diagram commutes*



Proof First of all let us note that $T(f)$, if it exists, is unique. Further, by Corollary 3.8, we have $f^*(|D'|) = |D|$ and f^* transforms any basis of $|D'|$ onto a basis of $|D|$. Let ϕ and ψ be suitable bases such that $i_D = i_{(D,\phi)}$ and $i_{D'} = i_{(D',\psi)}$.

We have $i_{D'} \circ f = (\psi_0, ..., \psi_n) \circ f$. But $f^*(D' + (\psi_i)) = f^*(D') + f^*(\psi_i) = D + (\psi_i \circ f)$. It means that rational functions $(\psi_i \circ f), i = 0, ..., n$ form a basis of $L(D)$. Hence there exists a non-singular matrix $[a_{ij}]$ such that $\psi_i \circ f = \sum_{j=0}^{n} a_{ij}\phi_j$. Now it is clear that it is enough to take as $T(f)$ the projective automorphism of $\mathbb{P}^n(\mathbb{K})$ given by the matrix $[a_{ij}]$. □

Corollary 3.10 *Let G be a subgroup of* $\mathrm{Aut}_1(X)$ *such that $G^*D = D$ for some effective divisor D. Let us denote $\overline{i_D(X)} = X' \subset \mathbb{P}^n(\mathbb{K})$, $n = \dim |D|$. Then there is a natural homomorphism $T : G \to \mathrm{Lin}(X')$. Moreover, if D is very big (i.e., the mapping i_D is a birational embedding), then T is a monomorphism.*

Proof It is enough to take above $D' = D$ and $\phi = \psi$. The last statement is obvious. □

Remark 3.11 In our application we deal only with normal $\mathbb{Q}-$factorial varieties. Hence we could restrict our attention only to $\mathbb{Q}$-Cartier divisors. However, the author thinks that the language of Weil divisors is more natural here.

4 Varieties with good covers

We begin this section by recalling the definition of a big divisor (see [12], p. 67):

Definition 4.1 Let X be a projective n-dimensional variety and D a Cartier divisor on X. The divisor D is called big if $\dim H^0(X, \mathcal{O}_X(kD)) > ck^n$ for some $c > 0$ and $k >> 1$.

If $f : X \to Y$ is a birational morphisms and D is a big (Cartier) divisor, then its pullback $f^*(D)$ is also big. Indeed, the line bundle $\mathcal{O}_X(mf^*(D)) = f^*\mathcal{O}_Y(mD)$ has at least as many sections as the bundle $\mathcal{O}_Y(mD)$. We show later that it is also true for suitable birational mappings (see Lemma 4.5). We have the following characterization of big divisors (see [12], Lemma 2.60, p. 67):

Proposition 4.2 *Let X be a projective n-dimensional variety and D a Cartier divisor on X. Then the following are equivalent:*

1. *D is big,*
2. *for some $m \geq 1$ we have $mD \sim A + E$, where A is ample and E is effective Cartier divisor,*
3. *for $m >> 0$ the rational map ι_{mD} associated with the system $|mD|$ is a birational embedding,*
4. *the image of ι_{mD} has dimension n for $m >> 0$.*

In the sequel we need the following observation:

Lemma 4.3 *Let X be a smooth projective variety and let $D = \sum_{i=1}^r a_i D_i$ be a big divisor on X. Then* $\mathrm{Supp}(D) = \sum_{i=1}^r D_i$ *is also a big divisor on X.*

Proof Let $a = \max_{i=1,\ldots,r}\{a_i\}$ and $b_i = a - a_i$. The divisor $E = \sum_{i=1}^r b_i D_i$ is effective. By condition 2) of Proposition 4.2 the divisor $D + E = a\mathrm{Supp}(D)$ is also big. Hence we conclude by 3) of Proposition 4.2. □

Definition 4.4 Let X be a normal projective variety and let D be a Weil divisor on X. We say that D is very big if the rational map ι_D associated with the system $|D|$ is a birational embedding. We say that D is big if for some $m \geq 1$ the divisor mD is very big.

It is easy to see that for Cartier divisors this definition coincide with the previous one. We have the following simple lemma:

Lemma 4.5 *Let X, Y be normal projective varieties and let $\phi : X - \to Y$ be a birational mapping such that* codim $\phi^{-1}(\mathrm{Sing}(Y)) \geq 2$. *If D is an effective big divisor on Y, then the divisor $\phi^*(D)$ on X is also big.*

Proof It is enough to assume that D is very big and prove that then $\phi^*(D)$ is also very big. Take $f_0 = 1$ and let divisors $\{D + (f_0), D + (f_1), ..., D + (f_s)\}$, where the $f_i \in \mathbb{K}(Y)$, form a basis of the system $|D|$. By the assumption, the regular mapping $\Psi : Y \backslash \mathrm{Supp}(D) \ni x \mapsto (f_1(x), ..., f_s(x)) \in \mathbb{K}^s$ is a birational morphism. The system $|\phi^*(D)|$ contains divisors $\{\phi^*(D), \phi^*(D) + (f_1 \circ \phi), ..., \phi^*(D) + (f_s \circ \phi)\}$. Since the collection of rational functions $1, f_1 \circ \phi, ..., f_s \circ \phi$ is linearly independent, we can extend it to some basis B of $L(\phi^*(D))$. Let $\Psi' : X \setminus |\mathrm{Supp}(\phi^*(D))| \to \mathbb{K}^N$ be a mapping given by a system $|\phi^*(D)|$ and the basis B. The mapping Ψ' composed with a suitable projection $\mathbb{K}^N \to \mathbb{K}^s$ is equal to $\Psi \circ \phi$. Since the latter mapping is birational, the mapping Ψ' is also birational. □

We shall use:

Definition 4.6 Let X be an (open) variety. We say that X has a good cover Y, if there exists a completion $\overline{X}$ of X and a smooth projective variety Y with a birational morphism $g : Y \to \overline{X}$ such that:

1. $D := g^{-1}(\overline{X} \setminus X)$ is a big hypersurface in Y,
2. $\mathrm{Aut}(X) \subset \mathrm{Aut}(Y \setminus D)$, i.e., every automorphism of X can be lifted to an automorphism of $Y \setminus D$.

Our next aim is to show that quasi-affine varieties have good covers.

Proposition 4.7 *Any quasi-affine variety X has a good cover.*

Proof By the assumption, there is an affine variety X_1 such that $X \subset X_1$ is an open dense subset. Since X_1 is affine, we can assume that it is a closed subvariety of some $\mathbb{K}^N$. Denote by $\overline{X}$ the projective closure of X_1 in $\mathbb{P}^N$. Let π_∞ be the hyperplane at infinity in $\mathbb{P}^N$ and $V := \overline{X} \,.\, \pi_\infty$ be a divisor at infinity on $\overline{X}$. Of course V is a big (even very ample) Cartier divisor.

Let $h : Y \to \overline{X}$ be a canonical desingularization of $\overline{X}$ (see e.g., [13,16]). Then $h_{|h-1(X)} : h^{-1}(X) \to X$ is a canonical desingularization of X. In particular every automorphism of X has a lift to an automorphism of $h^{-1}(X)$, i.e., $\mathrm{Aut}(X) \subset \mathrm{Aut}(h^{-1}(X)) = \mathrm{Aut}(Y \setminus h^{-1}(V))$. Since V is a big divisor, so is its pullback $h^*(V)$.

Note that $Z := Y \backslash h^{-1}(X)$ is a closed subvariety of Y. Let J_Z be the ideal sheaf of Z and let $f : Y' \to Y$ be a canonical principalization of J_Z (see e.g., [13,16]). Thus $D := f^{-1}(Z)$ is a hypersurface, which contains a big hypersurface $V' = \mathrm{Supp}(f^*h^*(V))$. Since $D = V' + E$, where E is an effective divisor, the hypersurface D is also big by Proposition 4.2.

Finally if we take $g = f \circ h : Y' \to \overline{X}$, then conditions 1) and 2) of Definition 4.6 are satisfied. □

5 The Quasi minimal model

In this section, following [14], we introduce the notion of quasi-minimal models (for details see [14]). This is a weaker analog of the usual notion of minimal model, which has an advantage that to prove its existence we do not need the full strength of the Minimal Model Program.

Definition 5.1 (See [14]) An effective $\mathbb{Q}$-divisor M on a variety X is said to be $\mathbb{Q}$-movable if for some $n > 0$ the divisor nM is integral and generates a linear system without fixed components. Let X be a projective variety with $\mathbb{Q}$-factorial terminal singularities. We say that X is a quasi-minimal model if there exists a sequence of $\mathbb{Q}$-movable $\mathbb{Q}$-divisors M_j whose limit in the Neron-Severi space $NSW_{\mathbb{Q}}(X) = NSW(X) \otimes \mathbb{Q}$ is K_X.

By the recent progress in the minimal model program (see [1,2,14]), every non-uniruled smooth variety has a quasi-minimal model. In fact, if we ran MMP on X and we do all possible divisorial contractions (and all necessary flips) we achieve a quasi-minimal model Y, together with a mapping $\phi : X - \to Y$ that is a composition of divisorial contractions and flips. In particular ϕ is a birational contraction, i.e., the mapping ϕ^{-1} does not contract any divisor (cf. [14], section 4, Corollary 4.5). Thus we get:

Theorem 5.2 *Let X be a smooth projective non-uniruled variety. Then there is a quasi-minimal model Y and a birational contraction $\phi : X - \to Y$.* ◻

Quasi minimal models have the following very important property (cf. [14], section 4, Proposition 4.6):

Theorem 5.3 *Let X be a quasi-minimal model. Then* $\mathrm{Bir}(X) = \mathrm{Aut}_1(X)$. ◻

6 Main result

Now we can start our proof. The first step is

Proposition 6.1 *Let X be a normal complete non-uniruled variety and let H be a big hypersurface in X. Then the group* $\mathrm{Stab}_X(H) = \{f \in \mathrm{Aut}_1(X) : f^*H = H\}$ *is finite.*

Proof For some $m \in \mathbb{N}$ the divisor mH is very big. We have $\mathrm{Stab}_X(H) = \{f \in \mathrm{Aut}_1(X) : f^*H = H\} = \mathrm{Stab}_X(mH) = \{f \in \mathrm{Aut}_1(X) : f^*(mH) = mH\}$. By the assumption, the variety $X' = \overline{i_{mH}(X)}$ is birationally equivalent to X. In view of Corollary 3.10 it is enough to prove that the group $\mathrm{Lin}(X')$ is finite. Since X is non-uniruled, the variety X' is non-uniruled too. But the group $\mathrm{Lin}(X')$ is an affine group and if it is infinite, then by Rosenlicht Theorem (see [15]), we have that X' is ruled - which is impossible. ◻

Now we can prove our main result:

Theorem 6.2 *Let X be an open variety with a good cover. If the group* $\mathrm{Aut}(X)$ *is infinite, then X is uniruled.*

Proof Assume that $\mathrm{Aut}(X)$ is infinite. Let $f : \overline{Y} \to \overline{X}$ be a good cover of X and take $Y = f^{-1}(X)$. Then $\mathrm{Aut}(Y)$ is also infinite. We have to prove that X is uniruled. To do this it suffices to prove that Y is uniruled.

Assume that Y is not uniruled. By Theorem 5.3 there exists a quasi-minimal model Z and a birational contraction $\phi : \overline{Y} - \to Z$. Take $\psi = \phi^{-1}$. The mapping ψ is a regular mapping outside some closed subset F of codimension ≥ 2. By the Zariski Main Theorem the mapping ψ restricted to $Z \setminus F$ is an embedding.

Take a mapping $G \in \mathrm{Aut}(Y)$, in fact $G \in \mathrm{Bir}(\overline{Y})$. The mapping G induces a birational mapping $g \in \mathrm{Bir}(Z)$. Since $\mathrm{Bir}(Z) = \mathrm{Aut}_1(Z)$ we have $g \in \mathrm{Aut}_1(Z)$. The mapping g is a morphisms outside a closed subset R of codimension ≥ 2. Since the mapping g is an automorphism in codimension one we have codim $\overline{g^{-1}(F)} \geq 2$. Denote $V := Z \setminus (F \cup \overline{g^{-1}(F)} \cup R)$ and $U = g(V)$. The mapping g restricted to V is an embedding by the Zariski Main theorem. In particular the set U is open and $g : V \to U$ is an isomorphism. The mapping ψ embeds sets V and U into Y. Denote $V' := \psi(V)$ and $U' = \psi(U)$. Under this identification, the mapping $g : V \to U$ corresponds to the mapping $G : V' \to U'$. Let $D = \overline{Y} \setminus Y$ be a big hypersurface, as in the definition of a good cover. The hypersurface $D' := \psi^*(D)$ is also big (see Lemma 4.5) and $D' \cap V$ corresponds to $D \cap V'$. Since $G(V' \setminus D) = U' \setminus D$, we have that g transforms irreducible components of $D' \cap V$ onto irreducible components of $D' \cap U$. In particular $g^*(D') = D'$. This means that $\mathrm{Aut}(Y) \subset \mathrm{Stab}_Z(D') \subset \mathrm{Aut}_1(Z)$. By Proposition 6.1 this contradicts our assumption. ◻

Corollary 6.3 *Let X be a quasi-affine (in particular affine) variety. If the group* $\mathrm{Aut}(X)$ *is infinite, then X is uniruled.*

7 Automorphisms of affine non-uniruled varieties

As we know, if X is a quasi-affine non-uniruled variety, then it has a finite automorphism group. We show now that conversely, for every $k \geq 1$ and every finite group G, there is a k-dimensional affine (smooth) non-uniruled variety X_G^k such that $\mathrm{Aut}(X_G^k) = G$. We start with:

Lemma 7.1 *Let $\Gamma_1, ..., \Gamma_k$ be affine curves with $0 < g(\Gamma_1) < g(\Gamma_2) < \ldots g(\Gamma_k)$ (here $g(X)$ denotes the genus of a curve X). Then*

$$\mathrm{Aut}\left(\prod_{i=1}^{k} \Gamma_i\right) = \prod_{i=1}^{k} \mathrm{Aut}(\Gamma_i).$$

Proof We proceed by induction. The case $k = 1$ is trivial. Assume $k > 1$. Let $\Phi \in \mathrm{Aut}(\prod_{i=1}^{k} \Gamma_i)$. For a point $a \in \prod_{i=2}^{k} \Gamma_i$ let $\Gamma_a := \Gamma_1 \times \{a\}$ and let $\Gamma'_a := \Phi(\Gamma_a)$. Since the curve Γ_a cannot dominate any curve Γ_i for $i > 1$ we have that $\Gamma'_a = \Gamma_1 \times \phi(a)$ where $\phi(a) \in H := \prod_{i=2}^{k} \Gamma_i$. Hence $\Phi : \Gamma_1 \times H \ni (x, a) \mapsto (\psi(x, a), \phi(a)) \in \Gamma_1 \times H$. For a fixed $a \in H$, the mapping $\psi(x, a) : \Gamma_1 \ni x \mapsto \psi(x, a) \in \Gamma_1$ is an automorphism of Γ_1. Since the group $\mathrm{Aut}(\Gamma_1)$ is finite, we have that $\psi(x, H)$ consists of one point, i.e., the mapping ψ does not depend on $a \in H$. In particular,

$\psi \in \mathrm{Aut}(\Gamma_1)$. The mapping $\phi : H \to H$ is an automorphism and we conclude the proof by induction. □

Now we prove:

Proposition 7.2 *For every $k \geq 1$ and every finite group G, there is a k-dimensional affine (smooth) non-uniruled variety X_G^k such that $\mathrm{Aut}(X_G^k) = G$.*

Proof First we assume $k = 1$ and we construct a non-rational curve Γ_1 with $\mathrm{Aut}(\Gamma_1) = G$. Since G is a finite group there is a number n such that G is a subgroup of the permutation group S_n. Consider a mapping

$$F : \mathbb{K}^n \ni x \mapsto (s_1(x), \dots, s_n(x)) \in \mathbb{K}^n,$$

where $s_1, \dots, s_n$ are all elementary symmetric polynomials of n variables. The group S_n acts effectively on general fibers of F. By (a variant of) the Bertini Theorem the inverse image of a general hyperplane is again a smooth irreducible hypersurface (we are in characteristic zero!). If we repeat this argument several times we see that the inverse image $F^{-1}(H)$ of a general plane $H \subset \mathbb{K}^n$ is a smooth irreducible surface. Now let Λ be a general curve on H of fixed degree $d > 2$. Again by the Bertini Theorem the inverse image Γ of Λ is a smooth irreducible curve. Of course Γ is non-rational, in particular it has finite automorphism group and by the construction $S_n \subset \mathrm{Aut}(\Gamma)$. Let $x \in \Gamma$ be a general point such that $\#\mathrm{Aut}(\Gamma).x = \#\mathrm{Aut}(\Gamma)$. Put $\Gamma_1 = \Gamma \setminus G.x$. It is easy to see that $\mathrm{Aut}(\Gamma_1) = G$ and we take $X_G^1 := \Gamma_1$. If $k > 1$, then we choose curves $\Gamma_2, \dots, \Gamma_k$ such that:

1. $\mathrm{Aut}(\Gamma_i) = \{identity\}$,
2. $g(\Gamma_1) < g(\Gamma_2) < \cdots < g(\Gamma_k)$. Now put $X_G^k := \prod_{i=1}^k \Gamma_i$ and apply Lemma 7.1.

□

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