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New theta-function identities and general theorems for the explicit evaluations of Ramanujan's continued fractions

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Abstract We prove some new theta-function identities for two continued fractions of Ramanujan which are analogous to those of Ramanujan–Göllnitz–Gordon continued fraction. Then these identities are used to prove new general theorems for the explicit evaluations of the continued fractions.

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الملخص

نثبت متطابقات دالة – ثيتا جديدة لكسرَيْ رامانوجان مستمرين مماثلة لكسر رامانوجان – جولنتز – جوردون المستمر. ثم نستخدم تلك المتطابقات لإثبات مبرهنات عامة جديدة لتقييمات صريحة للكسور المستمرة.

1 Introduction

Throughout the paper, we assume $|q| < 1$ and

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n).$$

Ramanujan's general theta-function $f(a,b)$ is given by

$$f(a,b) = \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \quad |ab| < 1. \tag{1.1}$$

Three special cases of $f(a,b)$ are

$$\phi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}, \tag{1.2}$$

$$\psi(q) := f(q,q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.3}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{1.4}$$

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In his notebooks, Ramanujan recorded several q-continued fractions which have beautiful theories. Among them is the celebrated Ramanujan–Göllnitz–Gordon continued fraction $K(q)$ defined by

$$K(q) := \cfrac{q^{1/2}}{1+q+\cfrac{q^2}{1+q^3+\cfrac{q^4}{1+q^5+\cdots}}}, \quad |q|<1. \tag{1.5}$$

On page 299 of his second notebook [11], Ramanujan recorded a product representation of $K(q)$, namely

$$K(q) := q^{1/2}\frac{(q;q^8)_\infty(q^7;q^8)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty}. \tag{1.6}$$

Without the knowledge of Ramanujan's work, Göllnitz [8] and Gordon [9], independently, rediscovered and proved (1.6). Shortly thereafter, Andrews [1] proved (1.6) as a corollary of a more general result. Ramanathan [10] also found an alternative proof of (1.6). In addition to (1.6), Ramanujan offered the following two other identities [11, p. 299] for $K(q)$:

$$\frac{1}{K(q)} - K(q) = \frac{\phi(q^2)}{q^{1/2}\psi(q^4)} \tag{1.7}$$

and

$$\frac{1}{K(q)} + K(q) = \frac{\phi(q)}{q^{1/2}\psi(q^4)}. \tag{1.8}$$

Proofs of (1.7) and (1.8) can be found in [7] and [12]. Yuttanan [19] also proved that

$$\frac{1}{K(q)} + 2 - K(q) = \frac{\phi(q^{1/2})}{q^{1/2}\psi(q^4)} \tag{1.9}$$

and

$$\frac{1}{K(q)} - 2 - K(q) = \frac{\phi(-q^{1/2})}{q^{1/2}\psi(q^4)}. \tag{1.10}$$

For further references on $K(q)$ see Chan and Huang [7], Vasuki and Kumar [18], and Baruah and Saikia [3].

In this paper, we prove some theta-function identities analogous to (1.7)–(1.10) for the continued fractions $T(q)$ and $W(q)$ which are defined, respectively, as

$$T(q) := \cfrac{q}{1-q^2+\cfrac{q^4}{1-q^6+\cfrac{q^8}{1-q^{10}+\cdots}}}, \quad |q|<1. \tag{1.11}$$

and

$$W(q) := \cfrac{q}{1-q^2+\cfrac{q^2(1+q^2)^2}{1-q^6+\cfrac{q^4(1+q^4)^2}{1-q^{10}+\cdots}}}, \quad |q|<1. \tag{1.12}$$

and use them to prove new general theorems for the explicit evaluations of $T(q)$ and $W(q)$. The continued fractions $T(q)$ and $W(q)$ are introduced and studied by Saikia in [14] and [15], respectively. Saikia [14, p. 4, Theorem 3.1] proved that

$$T(q) = \frac{f(q)-f(-q)}{f(q)+f(-q)}. \tag{1.13}$$

The identity analogous to (1.13) and satisfied by the continued fraction $W(q)$ is [15, Theorem 3.1]:

$$2W(q) = \frac{f^2(q)-f^2(-q)}{f^2(q)+f^2(-q)}. \tag{1.14}$$

Saikia also established some modular relations and explicit values for $T(q)$ and $W(q)$ in [14] and [15], respectively.

In Sects. 3 and 4, we prove new theta-function identities for the continued fractions $T(q)$ and $W(q)$, respectively. In Sects. 5 and 6, we prove new general theorems for the explicit evaluations of $T(q)$ and $W(q)$ by using theta-function identities established in Sects. 3 and 4, respectively, and give examples of explicit evaluations. Section 2 is devoted to record some preliminary results for ready references in this paper.

To end this introduction, we define some parameters of theta-functions which will be used in the explicit evaluations of $T(q)$ and $W(q)$. For any positive real numbers k and n, define

$$A_{k,n} = \frac{\phi(-q)}{2\,k^{1/4}q^{k/4}\psi(q^{2k})}; \qquad q = e^{-\pi\sqrt{n/k}}, \tag{1.15}$$

$$s_{4,n} = \frac{f(q)}{\sqrt{2}q^{1/8}f(-q^4)}; \qquad q = e^{-\pi\sqrt{n}/2}, \tag{1.16}$$

$$J_n = \frac{f(-q)}{\sqrt{2}q^{1/8}f(-q^4)}; \qquad q = e^{-\pi\sqrt{n}}. \tag{1.17}$$

The parameter $A_{k,n}$ is introduced by Saikia [13, p. 107, (1.7)]. The parameter $s_{4,n}$ is the particular case $k = 4$ of the general parameter $s_{k,n}$ defined by

$$s_{k,n} = \frac{f(q)}{k^{1/4}q^{(k-1)/24}f(-(-1)^k q^k)}; \qquad q = e^{-\pi\sqrt{n/k}} \tag{1.18}$$

and is due to Berndt [6, p. 9, (4.7)]. The parameter J_n is the particular case $k = 4$ of the general parameter $r_{k,n}$, introduced by Yi [20, p. 11, (2.1.1)] (also see [6, p. 9, (4.6)]) and defined by

$$r_{k,n} := \frac{f(-q)}{k^{1/4}q^{(k-1)/24}f(-q^k)}, \quad q = e^{-2\pi\sqrt{n/k}}. \tag{1.19}$$

Yi [20] evaluated several explicit values of the parameter $r_{k,n}$.

2 Preliminary results

This section is devoted to record some transformation formulas and P-Q theta-function identities which will be used in the succeeding sections. The P-Q identities presented in Lemmas 2.6–2.8 are new. Since modular equations are key in the proofs of P-Q theta-function identities, first we define Ramanujan's modular equation.

The ordinary or Gaussian hypergeometric function ${}_2F_1(a, b; c; x)$ is defined by

$${}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n,$$

where $(a)_0 = 1$ and $(a)_n = a(a+1)(a+2)\cdots a(a+n-1)$ for $n \geq 1$ and $|x| < 1$.

Let, for $0 < \alpha < 1$,

$$z_1 = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right) \quad \text{and} \quad q = \exp\left(-\pi\csc(\pi/2)\frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)}\right). \tag{2.1}$$

Assume that for some integer n

$$n\frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)}. \tag{2.2}$$

Then a modular equation of degree n is a relation between α and β induced by (2.2). We often say β has degree n over α. The multiplier m connecting α and β is defined by $m = z_1/z_n$, where $z_n = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)$.

Lemma 2.1 [13, p. 111, Theorem 4.1] *We have*

$$A_{k,1} = 1 \quad and \quad A_{k,1/n} = 1/A_{k,n}.$$

Lemma 2.2 [2, p. 9, Theorem 6.1] *We have*

$$J_{1/n} = 1/J_n.$$

Lemma 2.3 [5, p. 40, Entry 25] *We have*

$$\phi(q) + \phi(-q) = 2\phi(q^4), \tag{2.3}$$

$$\phi(q) - \phi(-q) = 4q\psi(q^8), \tag{2.4}$$

$$\phi^2(q) - \phi^2(-q) = 8q\psi^2(q^4), \tag{2.5}$$

$$\phi^2(q) + \phi^2(-q) = 2\phi^2(q^2), \tag{2.6}$$

$$\phi(q)\phi(-q) = \phi^2(-q^2). \tag{2.7}$$

Lemma 2.4 [4, Lemmas 3.10 and 3.11] *We have*

$$\phi(q) = \frac{f^5(-q^2)}{f^2(-q)f^2(-q^4)}, \quad \phi(-q) = \frac{f^2(-q)}{f(-q^2)},$$

$$\psi(q) = \frac{f^2(-q^2)}{f(-q)}, \quad f(q) = \frac{f^3(-q^2)}{f(-q)f(-q^4)}.$$

Lemma 2.5 [5, p. 122–123, Entry 10(ii) and Entry 11(v)] *If z_1, q, and α are related by (2.1), then*

$$\phi(-q) = \sqrt{z_1}(1-\alpha)^{1/4} \quad and \quad \psi(q^8) = \frac{1}{4}\sqrt{z_1}\left\{1 - (1-\alpha)^{1/4}\right\}/q.$$

Lemma 2.6 *If* $P = \dfrac{\phi(-q)}{q\psi(q^8)}$ *and* $Q = \dfrac{\phi(-q^2)}{q^2\psi(q^{16})}$,

$$then \quad Q^2 - P^2Q - 4PQ - 2P^2 - 8P = 0. \tag{2.8}$$

Proof Transcribing using Lemma 2.5, we find that

$$(1-\alpha)^{1/4} = \frac{P}{4+P} \quad \text{and} \quad \beta = 1 - \left(\frac{Q}{4+Q}\right)^4, \tag{2.9}$$

where β has degree 2 over α. From [5, p. 214, Entry 24(iii)] we note that, if β has degree 2 over α, then

$$m\sqrt{1-\alpha} + \sqrt{\beta} = 1 \tag{2.10}$$

and

$$m^2\sqrt{1-\alpha} + \beta = 1. \tag{2.11}$$

Eliminating m between (2.10) and (2.11) and then simplifying, we deduce that

$$\left(1 + \beta + (\beta - 1)\sqrt{1-\alpha}\right)^2 - 4\beta = 0. \tag{2.12}$$

Employing (2.9) in (2.12) and factorizing using *Mathematica*, we obtain

$$\left(Q^2 - P^2Q - 4PQ - 2P^2 - 8P\right)$$

$$\times \left(32P + 8P^2 + 16PQ + 4P^2Q + 4Q^2 + 4PQ^2 + P^2Q^2\right) = 0. \tag{2.13}$$

Since the second factor is non-zero, we arrive at the desired result. □

Lemma 2.7 *If* $P = \dfrac{\phi(-q)}{q\psi(q^8)}$ *and* $Q = \dfrac{\phi(-q^3)}{q^3\psi(q^{24})}$,

$$\text{then}\quad P^4 - 64PQ - 48P^2Q - 12P^3Q - 48PQ^2 - 30P^2Q^2 - 6P^3Q^2 - 12PQ^3 - 6P^2Q^3 - P^3Q^3 + Q^4 = 0. \tag{2.14}$$

Proof Transcribing using Lemma 2.5, we find that

$$(1-\alpha)^{1/4} = \frac{P}{4+P} \qquad (1-\beta)^{1/4} = \frac{Q}{4+Q}, \tag{2.15}$$

$$\alpha = 1 - \left(\frac{P}{4+P}\right)^4, \quad \text{and} \quad \beta = 1 - \left(\frac{Q}{4+Q}\right)^4, \tag{2.16}$$

where β has degree 3 over α. From [5, p. 230, Entry 5(ii)] we note that, if β has degree 3 over α, then

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = 1 \tag{2.17}$$

and can also be expressed as

$$\alpha\beta - \left(1 - (1-\alpha)^{1/4}(1-\beta)^{1/4}\right)^4 = 0. \tag{2.18}$$

Employing (2.15) and (2.16) in (2.18) and simplifying with the help of *Mathematica*, we arrive at the desired result. □

Lemma 2.8 *If* $P = \dfrac{\phi(-q)}{q\psi(q^8)}$ *and* $Q = \dfrac{\phi(-q^5)}{q^5\psi(q^{40})}$,

$$\text{then}\quad P^6 - 4096PQ - 5120P^2Q - 2560P^3Q - 640P^4Q - 70P^5Q - 5120PQ^2$$

$$-6400P^2Q^2 - 3200P^3Q^2 - 785P^4Q^2 - 80P^5Q^2 - 2560PQ^3 - 3200P^2Q^3 - 1620P^3Q^3$$

$$-400P^4Q^3 - 40P^5Q^3 - 640PQ^4 - 785P^2Q^4 - 400P^3Q^4 - 100P^4Q^4 - 10P^5Q^4 - 70PQ^5$$

$$-80P^2Q^5 - 40P^3Q^5 - 10P^4Q^5 - P^5Q^5 + Q^6 = 0. \tag{2.19}$$

Proof Transcribing using Lemma 2.5, we find that

$$c := (1-\alpha)^{1/8} = \sqrt{\frac{P}{4+P}} \qquad d := (1-\beta)^{1/8} = \sqrt{\frac{Q}{4+Q}}, \tag{2.20}$$

where β has degree 5 over α. From [5, p. 280–281, Entry 13(v) & (vi)] we note that, if β has degree 5 over α, then

$$m = \frac{1 + \left(\dfrac{(1-\beta)^5}{1-\alpha}\right)^{1/8}}{1 + \left\{(1-\alpha)^3(1-\beta)\right\}^{1/8}}. \tag{2.21}$$

$$\frac{5}{m} = \frac{1 - \left(\dfrac{(1-\alpha)^5}{1-\beta}\right)^{1/8}}{1 - \left\{(1-\alpha)(1-\beta)^3\right\}^{1/8}}. \tag{2.22}$$

Employing (2.20) in (2.21) and (2.22), we find that

$$m = \frac{c + d^5}{c(1 + c^3 d)} \tag{2.23}$$

and

$$\frac{5}{m} = \frac{d - c^5}{d(1 - cd^3)}, \tag{2.24}$$

respectively. Eliminating m between (2.23) and (2.24) and simplifying, we deduce that

$$5cd(1 + c^3 d)(1 - cd^3) - (c + d^5)(d - c^5) = 0. \tag{2.25}$$

Equivalently,

$$4(cd - c^5 d^5) = (5c^2 d^4 - 5c^4 d^2 - c^6 + d^6). \tag{2.26}$$

Squaring (2.26) and substituting for c and d from (2.20) and simplifying with the help of *Mathematica*, we arrive at desired result. □

3 New identities for $T(q)$

In this section we prove theta-function identities for $T(q)$ analogous to (1.7)–(1.10).

Theorem 3.1 *We have*

$$\text{(i)}\ \frac{1}{T(q^{1/4})} + T(q^{1/4}) = \frac{\phi(q)}{q^{1/4}\psi(q^2)} = \frac{f^2(q)}{q^{1/4} f^2(-q^4)},$$

$$\text{(ii)}\ \frac{1}{T(q^{1/2})} - T(q^{1/2}) = \frac{\phi(-q)}{q^{1/2}\psi(q^4)},$$

$$\text{(iii)}\ \frac{1}{T(q)} + 2 + T(q) = \frac{\phi(q)}{q\psi(q^8)},$$

$$\text{(iv)}\ \frac{1}{T(q)} - 2 + T(q) = \frac{\phi(-q)}{q\psi(q^8)}.$$

Proof (i) From (1.13), we note that

$$\frac{1}{T(q)} + T(q) = \frac{f(q) + f(-q)}{f(q) - f(-q)} + \frac{f(q) - f(-q)}{f(q) + f(-q)}$$
$$= \frac{2\left\{f^2(q) + f^2(-q)\right\}}{f^2(q) - f^2(-q)}. \tag{3.1}$$

From Lemma 2.4, we note that

$$f^2(q) = \frac{f^6(-q^2)}{f^2(-q) f^2(-q^4)} = \left(\frac{f^5(-q^2)}{f^2(-q) f^2(-q^4)}\right)\left(\frac{f(-q^2)}{f^2(-q)}\right) f^2(-q)$$
$$= \frac{\phi(q)}{\phi(-q)} f^2(-q). \tag{3.2}$$

Employing (3.2) in (3.1), and simplifying, we deduce that

$$\frac{1}{T(q)} + T(q) = \frac{2\{\phi(q) + \phi(-q)\}}{\phi(q) - \phi(-q)}. \tag{3.3}$$

Employing (2.3) and (2.4) in (3.3) and simplifying, we obtain

$$\frac{1}{T(q)} + T(q) = \frac{\phi(q^4)}{q\psi(q^8)}. \tag{3.4}$$

Replacing q by $q^{1/4}$ in (3.4), we prove the first equality. To prove the second equality, from Lemma 2.4 we note that

$$\frac{\phi(q)}{\psi(q^2)} = \frac{f^2(q)}{f^2(-q^4)}. \tag{3.5}$$

Employing (3.5) in the first equality, we arrive at the desired result.
(ii) From (1.13), we deduce that

$$\frac{1}{T(q)} - T(q) = \frac{4f(q)f(-q)}{f^2(q) - f^2(-q)}. \tag{3.6}$$

Employing (3.2) in (3.6) and simplifying, we find that

$$\frac{1}{T(q)} - T(q) = 4\left(\frac{f(q)\phi(-q)}{f(-q)}\right)\left(\frac{1}{\phi(q) - \phi(-q)}\right). \tag{3.7}$$

From Lemma 2.4, we note that

$$\frac{f(q)\phi(-q)}{f(-q)} = \frac{f^2(-q)}{f(-q^4)} = \phi(-q^2). \tag{3.8}$$

Employing (2.4) and (3.8) in (3.7) and simplifying, we obtain

$$\frac{1}{T(q)} - T(q) = \frac{\phi(-q^2)}{q\psi(q^8)}. \tag{3.9}$$

Replacing q by $q^{1/2}$ in (3.9), we arrive at the desired result.
(iii) From (1.13), we deduce that

$$\begin{aligned}\frac{1}{\sqrt{T(q)}} + \sqrt{T(q)} &= \sqrt{\frac{f(q) + f(-q)}{f(q) - f(-q)}} + \sqrt{\frac{f(q) - f(-q)}{f(q) + f(-q)}}\\ &= \frac{2f(q)}{\sqrt{f^2(q) - f^2(-q)}}.\end{aligned} \tag{3.10}$$

Employing (3.2) in (3.10) and simplifying, we obtain

$$\frac{1}{\sqrt{T(q)}} + \sqrt{T(q)} = \frac{2f(q)\sqrt{\phi(-q)}}{f(-q)\sqrt{\phi(q) - \phi(-q)}}. \tag{3.11}$$

Squaring (3.11), we find that

$$\frac{1}{T(q)} + 2 + T(q) = \frac{4f^2(q)\phi(-q)}{f^2(-q)\{\phi(q) - \phi(-q)\}}. \tag{3.12}$$

Employing (2.4) and (3.2) in (3.12) and simplifying, we arrive at the desired result.
(iv) From (1.13), we deduce that

$$\begin{aligned}\frac{1}{\sqrt{T(q)}} - \sqrt{T(q)} &= \sqrt{\frac{f(q) + f(-q)}{f(q) - f(-q)}} - \sqrt{\frac{f(q) - f(-q)}{f(q) + f(-q)}}\\ &= \frac{2f(-q)}{\sqrt{f^2(q) - f^2(-q)}}.\end{aligned} \tag{3.13}$$

Squaring (3.13) and simplifying by employing (3.2), we obtain

$$\frac{1}{T(q)} - 2 + T(q) = \frac{4\phi(-q)}{\phi(q) - \phi(-q)}. \tag{3.14}$$

Employing (2.4) in (3.14) and simplifying, we complete the proof. □

Corollary 3.2 *We have*

$$\text{(i)}\ \frac{1}{T^2(q)}-2+T^2(q)=\frac{\phi^2(-q^2)}{q^2\psi^2(q^8)},$$
$$\text{(ii)}\ \frac{1}{\sqrt{T(q)}}+\sqrt{T(q)}=\sqrt{\frac{\phi(q)}{q\psi(q^8)}},$$
$$\text{(iii)}\ \frac{1}{\sqrt{T(q)}}-\sqrt{T(q)}=\sqrt{\frac{\phi(-q)}{q\psi(q^8)}}.$$

Proof From Theorem 3.1(iii) and (iv), we have

$$\left(\frac{1}{T(q)}+2+T(q)\right)\left(\frac{1}{T(q)}-2+T(q)\right)=\frac{\phi(q)\phi(-q)}{q^2\psi^2(q^8)}. \tag{3.15}$$

Employing (2.7) in (3.15), we arrive at (i). To prove (ii) and (iii), we employ (3.2) in (3.11) and (3.13), respectively, and simplify. □

4 New identities for $W(q)$

This section is devoted to proving theta-function identities analogous to (1.7)–(1.10) for the continued fraction $W(q)$.

Theorem 4.1 *We have*

$$\text{(i)}\ \frac{1}{W(\sqrt{q})}+4W(\sqrt{q})=\frac{\phi^2(q)}{q^{1/2}\psi^2(q^2)}=\frac{f^4(q)}{q^{1/2}f^4(-q^4)},$$
$$\text{(ii)}\ \frac{1}{W(\sqrt{q})}-4W(\sqrt{q})=\frac{\phi^2(-q)}{q^{1/2}\psi^2(q^2)}=\frac{f^4(-q)}{q^{1/2}f^4(-q^4)},$$
$$\text{(iii)}\ \frac{1}{W^2(\sqrt{q})}-16W^2(\sqrt{q})=\frac{\phi^4(-q^2)}{q\psi^4(q^2)}.$$

Proof (i) From (1.14), we deduce that

$$\frac{1}{2W(q)}+2W(q)=\frac{2\left\{f^4(q)+f^4(-q)\right\}}{f^4(q)-f^4(-q)}. \tag{4.1}$$

Squaring (3.2), then employing in (4.1) and simplifying, we obtain

$$\frac{1}{2W(q)}+2W(q)=\frac{2\left\{\phi^2(q)+\phi^2(-q)\right\}}{\phi^2(q)-\phi^2(-q)}. \tag{4.2}$$

Employing (2.5) and (2.6) in (4.2) and simplifying, we obtain

$$\frac{1}{W(q)}+4W(q)=\frac{\phi^2(q^2)}{q\psi^2(q^4)}. \tag{4.3}$$

Replacing q by $\sqrt{q}$ in (4.3) we prove the first equality. To prove the second equality, from Lemma 2.4 we note that

$$\frac{\phi(q)}{\psi(q^2)}=\frac{f^2(q)}{f^2(-q^4)}. \tag{4.4}$$

Employing (4.4) in the first equality, we arrive at the desired result.

(ii) From (1.14), we note that

$$\frac{1}{2W(q)}-2W(q)=\frac{4f^2(q)f^2(-q)}{f^4(q)-f^4(-q)}. \tag{4.5}$$

Employing (3.2) in (4.5) and simplifying, we obtain

$$\frac{1}{2W(q)}+2W(q)=\frac{4\phi(q)\phi(-q)}{\phi^2(q)-\phi^2(-q)}. \tag{4.6}$$

Employing (2.5) and (2.7) in (4.6) and simplifying, we obtain

$$\frac{1}{W(q)}+4W(q)=\frac{\phi^2(-q^2)}{q\psi^2(q^4)}. \tag{4.7}$$

Replacing q by $\sqrt{q}$ in (4.7), we prove the first equality. To prove the second equality, from Lemma 2.4 we note that

$$\frac{\phi(-q)}{\psi(q^2)}=\frac{f^2(-q)}{f^2(-q^4)}. \tag{4.8}$$

Employing (4.8) in the first equality, we arrive at the desired result.

(iii) From part (i) and (ii), we deduce that

$$\left(\frac{1}{W(\sqrt{q})}+4W(\sqrt{q})\right)\left(\frac{1}{W(\sqrt{q})}-4W(\sqrt{q})\right)=\frac{\phi^2(q)\phi^2(-q)}{q\psi^4(q^2)}. \tag{4.9}$$

Employing (2.7) in (4.9) and simplifying, we complete the proof. □

Theorem 4.2 *We have*

$$\text{(i)}\ \frac{1}{\sqrt{W(q)}}+2\sqrt{W(q)}=\frac{\phi(q)}{q^{1/2}\psi(q^4)},$$

$$\text{(ii)}\ \frac{1}{\sqrt{W(q)}}-2\sqrt{W(q)}=\frac{\phi(-q)}{q^{1/2}\psi(q^4)}.$$

Proof (i) From (1.14), we deduce that

$$\frac{1}{\sqrt{2W(q)}}+\sqrt{2W(q)}=\frac{2f^2(q)}{\sqrt{f^4(q)-f^4(-q)}}. \tag{4.10}$$

Employing (3.2) in (4.10) and simplifying, we obtain

$$\frac{1}{\sqrt{2W(q)}}+\sqrt{2W(q)}=\frac{2\phi(q)}{\sqrt{\phi^2(q)-\phi^2(-q)}}. \tag{4.11}$$

Employing (2.5) in (4.11) and simplifying, we arrive at the desired result.

(ii) From (1.14), we deduce that

$$\frac{1}{\sqrt{2W(q)}}-\sqrt{2W(q)}=\frac{2f^2(-q)}{\sqrt{f^4(q)-f^4(-q)}}. \tag{4.12}$$

Employing (3.2) in (4.12) and simplifying, we obtain

$$\frac{1}{\sqrt{2W(q)}}-\sqrt{2W(q)}=\frac{2\phi(-q)}{\sqrt{\phi^2(q)-\phi^2(-q)}}. \tag{4.13}$$

Employing (2.5) in (4.13) and simplifying, we complete the proof. □

Corollary 4.3 *We have*

$$\text{(i)}\ \frac{1}{W(q)}+4+4W(q)=\frac{\phi^2(q)}{q\psi^2(q^4)},$$

$$\text{(ii)}\ \frac{1}{W(q)}-4+4W(q)=\frac{\phi^2(-q)}{q\psi^2(q^4)}.$$

Proof Squaring Theorem 4.2(i) and (ii) we easily arrive at (i) and (ii), respectively. □

5 General theorems for explicit evaluations of $T(q)$

In this section we prove new general theorems for the explicit evaluations of $T(q)$ and give examples.

Theorem 5.1 *If $s_{4,n}$ is as defined in (1.16), then*

$$\frac{1}{T(e^{-\pi\sqrt{n}/8})}+T(e^{-\pi\sqrt{n}/8})=2s_{4,n}^2.$$

Proof Setting $q=\mathrm{e}^{-\pi\sqrt{n}/2}$ in Theorem 3.1(i) and employing the definition of $s_{4,n}$ we complete the proof. □

Remark 5.2 From Theorem 5.1 it is obvious that if we know explicit values of the parameter $s_{4,n}$ the explicit values of $T(e^{-\pi\sqrt{n}/8})$ can be easily evaluated. Baruah and Saikia [3] evaluated $s_{4,n}$ for $n=$1, 2, 3, 4, 5, 7, 8, 9, 10, 12, 13, 15, 16, 18, 20, 24, 25, 28, 32, 36, 40, 52, 64, 68, 72, 100, 108, 144, 196, 2/3, 1/2, 4/7, 4/5, 4/9, 4/25, 4/49, 2/5, 1/3, 1/5, 1/7, 1/13, 1/15, and 1/25. For example, setting $n=4$, employing the value $s_{4,4}=2^{1/8}$ in Theorem 5.1 and solving the resulting equation, we obtain

$$T(\mathrm{e}^{-\pi/4})=2^{1/4}\pm\sqrt{-1+\sqrt{2}}. \tag{5.1}$$

For $|q|<1$, neglecting q^8 and higher powers of q in (1.11), we find that

$$T(q)\approx q(1-q^6)\left((1-q^2)(1-q^6)+q^4\right)^{-1}$$

and for $q=e^{-\pi/4}$, $T(e^{-\pi/4})\approx 0.545559<1$. So choosing minus sign in (5.1), we obtain

$$T(e^{-\pi/4})=2^{1/4}-\sqrt{-1+\sqrt{2}}.$$

Theorem 5.3 *We have*

$$\text{(i)}\ \frac{1}{T(e^{-\pi\sqrt{n}/(2\sqrt{2})})}-T(e^{-\pi\sqrt{n}/(2\sqrt{2})})=2^{5/4}A_{2,n},$$

$$\text{(ii)}\ \frac{1}{T(e^{-\pi/(2\sqrt{2n})})}-T(e^{-\pi/(2\sqrt{2n})})=\frac{2^{5/4}}{A_{2,n}}.$$

Proof Setting $q=\mathrm{e}^{-\pi\sqrt{n/2}}$ in Theorem 3.1(ii) and employing the definition of $A_{k,n}$ with $k=2$, we arrive at (i). Replacing n by $1/n$ in (i) and simplifying using Lemma 2.1, we complete the proof of (ii). □

Remark 5.4 From Theorem 5.3(i) and (ii) it is clear that if we know explicit values of the parameter $A_{2,n}$ then explicit values of $T(e^{-\pi\sqrt{n}/(2\sqrt{2})})$ and $T(e^{-\pi/(2\sqrt{2n})})$ can easily be evaluated, respectively. Saikia [13,17] evaluated $A_{2,n}$ for $n=1$, 2, 3, 4, 5, 7, 9, 25, and 49. For example, setting $n=3$, employing the value $A_{2,3}=\sqrt{2+\sqrt{2}+\sqrt{9+6\sqrt{2}}}$ from [13, p. 115, Theorem 5.4(i)] in Theorem 5.3(i) and (ii) and solving the resulting equations, we evaluate

$$T(e^{-\pi\sqrt{3}/(2\sqrt{2})})=\sqrt{3+2\sqrt{2}+\sqrt{18+12\sqrt{2}}}-2^{1/4}\sqrt{2+\sqrt{2}+\sqrt{9+6\sqrt{2}}}$$

and

$$T(e^{-\pi/(2\sqrt{6})})=\frac{-2^{1/4}+\sqrt{2+2\sqrt{2}+\sqrt{9+6\sqrt{2}}}}{\sqrt{2+\sqrt{2}+\sqrt{9+6\sqrt{2}}}},$$

respectively.

Theorem 5.5 *We have*

$$\text{(i)}\ \frac{1}{T(e^{-\pi\sqrt{n}/2})} - 2 + T(e^{-\pi\sqrt{n}/2}) = 2^{3/2}A_{4,n},$$
$$\text{(ii)}\ \frac{1}{T(e^{-\pi/(2\sqrt{n})})} - 2 + T(e^{-\pi/(2\sqrt{n})}) = \frac{2^{3/2}}{A_{4,n}}.$$

Proof Setting $q = \mathrm{e}^{-\pi\sqrt{n/4}}$ in Theorem 3.1(iv) and employing the definition of $A_{k,n}$ with $k = 4$, we arrive at (i). Replacing n by $1/n$ in (i) and simplifying using Lemma 2.1, we complete the proof of (ii). □

Remark 5.6 From Theorem 5.5(i) and (ii) it is obvious that if we know explicit values of the parameter $A_{4,n}$ then explicit values of $T(e^{-\pi\sqrt{n}/2})$ and $T(e^{-\pi/2\sqrt{n}})$ can easily be evaluated, respectively. For example, setting $n = 1$ in Theorem 5.3(i), noting $A_{4,1} = 1$ from Lemma 2.1 and solving the resulting equation, we evaluate

$$T(e^{-\pi/2}) = 1 + \sqrt{2} - \sqrt{2 + 2\sqrt{2}}.$$

A systematic study of the parameter $A_{k,n}$ for $k = 4$ has not been undertaken and no other value of the parameter $A_{4,n}$ is evaluated in literature. So we devote the remainder of this section to evaluate some new explicit values of $A_{4,n}$ by using P-Q theta-function identities established in Sect. 2.

Theorem 5.7 *We have*

$$(i)\ A_{4,2} = 1 + \sqrt{1 + \sqrt{2}},$$
$$(ii)\ A_{4,4} = 2 + \sqrt{2} + \sqrt{2(4 + 3\sqrt{2})}.$$

Proof (i) Setting $q = \mathrm{e}^{-\pi\sqrt{n/4}}$ in Lemma 2.6 and employing the definition of $A_{k,n}$ with $k = 4$ from (1.15), we obtain

$$P = 2^{3/2}A_{4,n} \quad \text{and} \quad Q = 2^{3/2}A_{4,4n}. \tag{5.2}$$

Setting $n = 1/2$ in (5.2) and simplifying using Lemma 2.1, we obtain

$$P = \frac{2^{3/2}}{A_{4,2}} \quad \text{and} \quad Q = 2^{3/2}A_{4,2}. \tag{5.3}$$

Employing (5.3) in (2.8) and simplifying, we deduce that

$$A_{4,2}^4 - 4A_{4,2}^2 - 4\sqrt{2}A_{4,2} - 2 = 0. \tag{5.4}$$

Solving (5.4) using *Mathematica* and noting $A_{4,n}$ has positive real value greater than unity, we arrive at (i).

(ii) Setting $n = 1$ in (5.2), employing (2.8) and simplifying, we obtain

$$A_{4,4}^2 - (4 + 2\sqrt{2})A_{4,4} - (2 + 2\sqrt{2}) = 0. \tag{5.5}$$

Solving (5.5) and choosing the appropriate root, we complete the proof of (ii). □

Theorem 5.8 *We have*

$$\text{(i)}\ A_{4,3} = \frac{1}{2}\left(2 + \sqrt{2} + \sqrt{18 + 12\sqrt{2})}\right),$$
$$\text{(ii)}\ A_{4,9} = 5 + 3\sqrt{2} + \sqrt{3(17 + 12\sqrt{2})}$$
$$+ \sqrt{93 + 66\sqrt{2} + 314\sqrt{3(17 - 12\sqrt{2})} + 222\sqrt{6(17 - 12\sqrt{2})}}.$$

Proof (i) Setting $q = e^{-\pi\sqrt{n/4}}$ in Lemma 2.7 and employing the definition of $A_{k,n}$ with $k = 4$ from (1.15), we obtain

$$P = 2^{3/2}A_{4,n} \quad \text{and} \quad Q = 2^{3/2}A_{4,9n}. \tag{5.6}$$

Setting $n = 1/3$ in (5.6) and simplifying using Lemma 2.1, we obtain

$$P = \frac{2^{3/2}}{A_{4,3}} \quad \text{and} \quad Q = 2^{3/2}A_{4,3}. \tag{5.7}$$

Employing (5.7) in (2.14) and simplifying with the help of *Mathematica*, we deduce that

$$A_{4,3}^8 - 12A_{4,3}^6 - 24\sqrt{2}A_{4,3}^5 - 46A_{4,3}^4 - 24\sqrt{2}A_{4,3}^3 - 12A_{4,3}^2 + 1 = 0. \tag{5.8}$$

Solving (5.8) using *Mathematica* and choosing the appropriate root, we arrive at (i).

(ii) Setting $n = 1$ in (5.6), employing in (2.14) and simplifying, we obtain

$$A_{4,9}^4 - (20 + 12\sqrt{2})A_{4,9}^3 - (30 + 24\sqrt{2})A_{4,9}^2 - (20 + 12\sqrt{2})A_{4,9} + 1 = 0. \tag{5.9}$$

Solving (5.9) using *Mathematica* and choosing the appropriate root, we complete the proof of (ii). □

Theorem 5.9 *We have*

$$\text{(i)}\ A_{4,5} = \frac{1}{2}\left(3 + \sqrt{2} + \sqrt{5(3 + 2\sqrt{2})}\right) + \frac{1}{\sqrt{2}}\sqrt{\left(19 + 14\sqrt{2} + 2\sqrt{5(3 + 2\sqrt{2})} + 3\sqrt{10(3 + 2\sqrt{2})}\right)},$$

$$\text{(ii)}\ A_{4,25} = 114 + 80\sqrt{2} + 3\sqrt{5(577 + 408\sqrt{2})} + 2\sqrt{12940 + 9150\sqrt{2} + 171\sqrt{5(577 + 408\sqrt{2})} + 120\sqrt{10(577 + 408\sqrt{2})}}.$$

Proof (i) Setting $q = e^{-\pi\sqrt{n/4}}$ in Lemma 2.8 and employing the definition of $A_{k,n}$ with $k = 4$ from (1.15), we obtain

$$P = 2^{3/2}A_{4,n} \quad \text{and} \quad Q = 2^{3/2}A_{4,25n}. \tag{5.10}$$

Setting $n = 1/5$ in (5.10) and simplifying using Lemma 2.1, we obtain

$$P = \frac{2^{3/2}}{A_{4,5}} \quad \text{and} \quad Q = 2^{3/2}A_{4,5}. \tag{5.11}$$

Employing (5.11) in (2.19) and simplifying with the help of *Mathematica*, we deduce that

$$A_{4,5}^{12} - 70A_{4,5}^{10} - 320\sqrt{2}A_{4,5}^9 - 1425A_{4,5}^8 - 1920\sqrt{2}A_{4,5}^7 - 3348A_{4,5}^6 - 1920\sqrt{2}A_{4,5}^5 - 1425A_{4,5}^4 - 320\sqrt{2}A_{4,5}^3 - 70A_{4,5}^2 + 1 = 0. \tag{5.12}$$

Solving (5.12) using *Mathematica* and choosing the appropriate root, we arrive at (i).

(ii) Setting $n = 1$ in (5.10), employing in (2.19) and simplifying, we obtain

$$(1 + y^2)\left(y^4 + (456 + 320\sqrt{2})y^3 - (674 + 480\sqrt{2})y^2 - (456 + 320\sqrt{2})y + 1\right) = 0, \tag{5.13}$$

where

$$y = A_{4,25}. \tag{5.14}$$

dividing second factor of (5.13) by y^2 and rearranging the terms, we obtain

$$\left(y^2 + \frac{1}{y^2}\right) + (456 + 320\sqrt{2})\left(y + \frac{1}{y}\right) - (674 + 480\sqrt{2}) = 0. \tag{5.15}$$

Equivalently,

$$(z^2-2)-(456+320\sqrt{2})z-(674+480\sqrt{2})=0, \tag{5.16}$$

where

$$y+\frac{1}{y}=z. \tag{5.17}$$

Solving (5.16) and choosing the positive real root greater than unity, we obtain

$$z=2\left(114+80\sqrt{2}+3\sqrt{5(577+408\sqrt{2})}\right). \tag{5.18}$$

Employing (5.18) in (5.17), solving for $y=A_{4,25}$, and choosing the appropriate root, we complete the proof of (ii). □

6 General theorems for the explicit evaluations of $W(q)$

In this section we prove general theorems for the explicit evaluations of the continued fraction $W(q)$.

Theorem 6.1 *If $s_{4,n}$ is as defined (1.16), then*

$$\frac{1}{W(e^{-\pi\sqrt{n}/4})}+4W(e^{-\pi\sqrt{n}/4})=4s_{4,n}^4.$$

Proof Setting $q:=\mathrm{e}^{-\pi\sqrt{n}/2}$ in Theorem 4.1(i) and employing the definition of $s_{4,n}$, we complete the proof. □

Remark 6.2 From Theorem 6.1 it is clear that if we know the explicit values of $s_{4,n}$ then explicit values of $W(e^{-\pi\sqrt{n}/4})$ can easily be evaluated. For example, setting $n=8$ in Theorem 6.1, employing the value $s_{4,8}=(1+\sqrt{2})^{1/4}$ from [3, p. 276, Corollary 3.3.(iv)], and solving the resulting equation, we evaluate

$$W(e^{-\pi/\sqrt{2}})=\left(1+\sqrt{2}-\sqrt{2(1+\sqrt{2})}\right)/2.$$

Theorem 6.3 *If J_n is as defined in (1.17), then*

$$\text{(i)}\ \frac{1}{W(\mathrm{e}^{-\pi\sqrt{n}/2})}-4W(\mathrm{e}^{-\pi\sqrt{n}/2})=4J_n^4,$$

$$\text{(ii)}\ \frac{1}{W(\mathrm{e}^{-\pi/(2\sqrt{n})})}-4W(\mathrm{e}^{-\pi/(2\sqrt{n})})=\frac{4}{J_n^4}.$$

Proof Setting $q=\mathrm{e}^{-\pi\sqrt{n}}$ in Theorem 4.1(ii)and employing the definition of J_n, we arrive at (i). To prove (ii), we replace n by $1/n$ in part (i) and use Lemma 2.2. □

Remark 6.4 From Theorem 6.3(i) and (ii) it is clear that if we know explicit values of J_n then explicit values of $W(e^{-\pi\sqrt{n}/2})$ and $W(e^{-\pi/(2\sqrt{n})})$ can easily be calculated, respectively. Baruah and Saikia [2] evaluated J_n for n =1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 15, 16, 17, 18, 19, 23, 25, 31, 36, and 49. Saikia [16] also evaluated J_n for n =15, 5/3, 21, 7/3, 33, and 11/3. For example, setting $n=3$ in Theorem 6.3(i) and (ii), employing the value of J_3 and solving the resulting equations, we evaluate

$$W(e^{-\pi\sqrt{3}/2})=\left(-2-\sqrt{3}+\sqrt{8+4\sqrt{3}}\right)/2$$

and

$$W(e^{-\pi/(2\sqrt{3})})=\left(-1+\sqrt{8+4\sqrt{3}}\right)\left(2-\sqrt{3}\right)/2,$$

respectively.

Theorem 6.5 *We have*

$$\text{(i)}\ \frac{1}{W(e^{-\pi\sqrt{n/2}})}-4+4W(e^{-\pi\sqrt{n/2}})=2^{5/2}A_{2,n}^2,$$
$$\text{(ii)}\ \frac{1}{W(e^{-\pi/\sqrt{2n}})}-4+4W(e^{-\pi/\sqrt{2n}})=\frac{2^{5/2}}{A_{2,n}^2}.$$

Proof Setting $q=\mathrm{e}^{-\pi\sqrt{n/2}}$ in Corollary 4.3(ii) and employing the definition of $A_{k,n}$ with $k=2$, we arrive at (i). To prove (ii), we replace n by $1/n$ in (i) and simplify using Lemma 2.1. □

Remark 6.6 From Theorem 6.5(i) and (ii) it is clear that if we know explicit values of $A_{2,n}$ then explicit values of $W(e^{-\pi\sqrt{n/2}})$ and $W(e^{-\pi/\sqrt{2n}})$ can be evaluated, respectively. For example, setting $n=2$ in Theorem 6.5(i) and (ii), employing the value $A_{2,2}=\sqrt{2+\sqrt{2}}$ from [13, p. 114, Theorem 5.2(i)] and solving the resulting equations, we evaluate

$$W(e^{-\pi})=\left(3+2\sqrt{2}-2\sqrt{4+3\sqrt{2}}\right)/2$$

and

$$W(e^{-\pi/2})=\left(\sqrt{2}-1\right)/2,$$

respectively.

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