Almost BPS but still not renormalized

Iosif Bena,[a] **Andrea Puhm,**[a] **Orestis Vasilakis**[b] **and Nicholas P. Warner**[a,b,c]

[a]*Institut de Physique Théorique,*
  CEA Saclay, CNRS-URA 2306, 91191 Gif sur Yvette, France
[b]*Department of Physics and Astronomy, University of Southern California,*
  Los Angeles, CA 90089, U.S.A.
[c]*Institut des Hautes Etudes Scientifiques,*
  Le Bois-Marie, 35 route de Chartres, Bures-sur-Yvette, 91440, France
  E-mail: `iosif.bena@cea.fr`, `andrea.puhm@cea.fr`, `vasilaki@usc.edu`,
  `warner@usc.edu`

ABSTRACT: A key feature of BPS multi-center solutions is that the equations controlling the positions of these centers are not renormalized as one goes from weak to strong coupling. In particular, this means that brane probes can capture the same information as the fully back-reacted supergravity solution. We investigate this non-renormalization property for non-supersymmetric, extremal "almost-BPS" solutions at intermediate coupling when one of the centers is considered as a probe in the background created by the other centers. We find that despite the lack of supersymmetry, the probe action reproduces exactly the equations underlying the fully back-reacted solution, which indicates that these equations also do not receive quantum corrections. In the course of our investigation we uncover the relation between the charge parameters of almost-BPS supergravity solutions and their quantized charges, which solves an old puzzle about the quantization of the charges of almost-BPS solutions.

Contents

1 Introduction

The physics of multi-center four-dimensional BPS solutions and of their five-dimensional counterparts [1–7] could be one of the keys that helps unlock longstanding mysteries of black hole physics, such as the information paradox and the microscopic origin of the black-hole entropy. These solutions yield the easiest-to-construct black-hole microstate geometries [8, 9] and can be used to study the wall-crossing behavior of black-hole partition functions [10, 11]. They also provide the best-known examples of entropy enigmas [10, 12, 13].

    Another key feature of BPS, multi-center solutions is that the equations controlling the positions of these centers, also known as the bubble equations, are not renormalized as one goes from weak to strong effective coupling. At weak coupling, these configurations are

described by a supersymmetric quiver quantum mechanics (QQM) [2], and the equations determining the vevs of the QQM Coulomb-branch fields are the same as those determining the inter-center distances of the fully-back-reacted supergravity solution. This remarkable non-renormalization property has allowed one to compute the symplectic form, quantize the moduli space of supergravity solutions from the QQM perspective [14][1] and has given a clear mapping of some of the microscopic black-hole degrees of freedom to horizonless solutions that exist in the same regime of the moduli space as the classical black hole [15–17].

For BPS solutions one can also test this non-renormalization of the bubble equations by considering multi-center solutions in an intermediate region of the effective coupling, where some of the branes have back-reacted while others are treated as probes. More precisely, one can place a supertube in a multi-center solution and examine its supersymmetric minima. The equations that determine the positions of these minima can then be shown to be identical to the bubble equations [18]. Hence, one can recover the BPS bubble equations both at small effective coupling from the QQM, at intermediate coupling from the supertube DBI action, and at large effective coupling by asking that the fully back-reacted solution has no closed timelike curves.

Since the non-renormalization of the bubble equations is established by invoking quantities protected by supersymmetry, one might naively expect that the beautiful pieces of physics that are protected from renormalization in BPS systems would no longer survive in non-supersymmetric, multi-center solutions or in non-supersymmetric fuzzballs. The purpose of this paper is to demonstrate that, on the contrary, in certain classes of non-supersymmetric multi-center solutions — the so-called almost-BPS solutions [19–21] — the bubble equations are also protected when one goes from intermediate to strong coupling. This suggests that there are no quantum corrections despite the lack of supersymmetry.

This result is quite surprising because there is neither supersymmetry nor any other underlying symmetry that would prevent the bubble equations from receiving corrections and because the bubble equations of almost-BPS solutions contain complicated cubic combinations of the inter-center distances, whose coefficients do not *a priori* appear to be related to anything one can define in a quiver quantum mechanics. However, in retrospect, this result may not look so surprising: Emparan and Horowitz have shown in [22] that the entropy of certain extremal non-supersymmetric black holes can be calculated at weak effective coupling, and does not change as one increases this coupling. Since a certain subclass of almost-BPS solutions can be dualized to the black hole considered in [22], it seems plausible that whatever principle protects the quantum states calculated at weak coupling from being uplifted and disappearing at strong coupling also protects the bubble equations from receiving quantum corrections.[2]

To establish that the almost-BPS bubble equations are not changing with the coupling, we consider an intermediate-coupling configuration where all the centers, except one, have

---

[1]This in turn has confirmed that quantum effects can wipe out macroscopically large regions of certain smooth low-curvature solutions [9, 14].

[2]It is also interesting to note that whatever underlying principle protects the states and bubble equations of almost-BPS system from being uplifted by quantum corrections can be even stronger than supersymmetry, which, for example, does not protect the degeneracy of the D1-D5-P system in the non-Cardy regime [13].

back-reacted into an almost-BPS supergravity solution. One can then probe this solution with a supertube and use the (Dirac-Born-Infeld and Wess-Zumino) action to determine the equilibrium position of the probe. If one considers the back-reaction of the probe then the supertube would become another center of the supergravity solution and its location is fixed by the bubble equations, which arise from requiring that there be no closed time-like curves. The comparison of these two physically distinct conditions, one in field theory on the brane and the other from its gravity dual, yields the test of non-renormalization between the intermediate and strong coupling regimes. At first glance, the equations look nothing like each other: both differ from the BPS bubble equations by extra cubic terms, but these cubic terms are not the same. However, the story is a bit more complicated: Unlike the BPS solution, the quantized electric charge of the supertube DBI action and the electric charge parameters in the supergravity harmonic functions are not the same. One can then ask whether there exists a relation between the DBI and the supergravity electric charges of the supertube that maps one set of bubble equations onto the other. One of the primary results of this paper is to prove that such a relation exists, and write it down for the most general known almost-BPS solution in a single-center Taub-NUT space, involving an arbitrary collection of concentric supertubes and black rings.

The implications of this relation for the physics of almost-BPS solutions are quite significant. First, in a general supergravity solution that contains several supertubes, the quantized charges of these supertubes are not the obvious coefficients in the supergravity harmonic functions. The supergravity parameters are related to the quantized charges by certain shifts that come from dipole-dipole interaction and depend on the location of all the other centers of the solution. An interesting features of these shifts is that they only depend on the dipole charges and positions of the centers that lie between location of the charge in question and the (supersymmetry-breaking) center of the Taub-NUT space.

For almost-BPS black rings the effect of the charge shift is more subtle because the black rings have more than one dipole charge, and hence the formulae that give the dipole-dipole contribution to the charge appear to be degenerate. However one can assemble a black ring by bringing together three supertubes with different kinds of dipole charges, and this will allow us to calculate the shifts between the black-ring supergravity and the quantized charges.

The second significant implication has to do with charge quantization, and solves a longstanding puzzle of the physics of almost-BPS black rings and of multi-center almost-BPS solutions. As one can see from [21] or from [23], the (quantized) asymptotic charge of such solutions is equal to the sum of the supergravity charges of the centers plus an extra dipole-dipole term that depends on the positions of the centers and the moduli of the solution. If the supergravity charges were equal to the quantized ones, this would have implied that the moduli-dependent contribution to the charges are also quantized, and hence multi-center almost-BPS solutions could only exist on special codimension-three slices of the moduli space (where the moduli-dependent contributions are integers). This would have been quite puzzling. The results presented here show that this does not happen. Upon using our formulas that relate the supergravity and the quantized charges, one finds that the asymptotic charge of the almost-BPS black ring is equal to the sum of the quantized

charges of its centers, and hence the almost-BPS black ring as well as other multi-center almost-BPS solutions exists for any values of the moduli. If one keeps the quantized charges fixed and changes the moduli, the quantities that change are simply the supergravity charge parameters of the centers. Our result also implies that the $E_{7(7)}$ quartic invariant of the almost-BPS black ring of [20] needs to be rewritten in terms of the quantized charges, and also yields the relation between the "quantized" angular momentum of this black ring and the angular-momentum parameter that appears in the corresponding almost-BPS harmonic function.

A third, perhaps more unexpected consequence of our result is that if one probes a certain supergravity solution with a supertube and finds, say, two minima, one where the supertube is at the exterior of the other supertubes and one where it is at an intermediate position, these two minima do not correspond to two vacua in the same vacuum manifold of the bubble equations. Indeed, one can relate both minima of the supertube potential to almost-BPS supergravity solutions, and then use our recipe to compute the quantized charges of the centers of these two solutions. Since the shifts of the charges of the centers depend on the position of the tube, the quantized charges of the centers of the two resulting solutions will not be the same, and hence these solutions do not describe different arrangements of the same supertubes, and thus cannot be related by moving in the moduli space of solutions of a certain set of bubble equations. Instead they live in different superselection sectors.

In section 2 we review the structure of BPS and almost-BPS supergravity solutions and in section 3 we calculate and examine the action of supertube probes in these solutions. We show that one can reproduce the supergravity bubble equation of the outermost supertube by considering this supertube as a probe in the fully back-reacted solution formed by the other supertubes and find the equation that relates the supergravity charge parameters and the quantized charges of the probe. We then give a recipe to read off the quantized supertube charges in a general multi-center almost-BPS solution. In section 4 we then show that one can also recover the bubble equations of all the other supertube centers by examining the minima of probe supertubes and relating their supergravity charges to their quantized charges. This demonstrates that all the supergravity data of a multi-center solution can be recovered from the action of supertube probes, and hence this data is not renormalized as one goes from weak to strong effective coupling. We also discuss in more detail the physics behind the shift needed to relate supergravity and quantized charges. Section 5 contains our conclusions and a discussion of further issues arising from this work. Some of the technical details of this paper have been relegated to an appendix.

2 The supergravity solutions

2.1 BPS and almost-BPS equations

The simplest way to describe the solutions of interest is to work in M-theory, with the metric

$$ds_{11}^2 = -\left(Z_1 Z_2 Z_3\right)^{-\frac{2}{3}} (dt + k)^2 + \left(Z_1 Z_2 Z_3\right)^{\frac{1}{3}} ds_4^2 + \left(Z_2 Z_3 Z_1^{-2}\right)^{\frac{1}{3}} (dx_5^2 + dx_6^2) \quad (2.1)$$
$$+ \left(Z_1 Z_3 Z_2^{-2}\right)^{\frac{1}{3}} (dx_7^2 + dx_8^2) + \left(Z_1 Z_2 Z_3^{-2}\right)^{\frac{1}{3}} (dx_9^2 + dx_{10}^2),$$

where ds_4^2 is a four-dimensional hyper-Kähler metric. The three-form potential is given by:

$$\mathcal{A} = A^{(1)} \wedge dx_5 \wedge dx_6 + A^{(2)} \wedge dx_7 \wedge dx_8 + A^{(3)} \wedge dx_9 \wedge dx_{10}, \tag{2.2}$$

where the Maxwell fields are required to obey the "floating brane Ansatz" [24]:

$$A^I = -\varepsilon\, Z_I^{-1}\, (dt + k) + B^{(I)}, \tag{2.3}$$

and where $\varepsilon = \pm 1$ and $B^{(I)}$ is a "magnetic" vector potential on the base, ds_4. We further define the field strengths:

$$\Theta^{(I)} \equiv dB^{(I)}. \tag{2.4}$$

It was shown in [24] that if one chooses the base to be merely Ricci-flat and solves the linear system of equations (for a fixed choice of $\varepsilon = \pm 1$):

$$\Theta^{(I)} = \varepsilon *_4 \Theta^{(I)}, \tag{2.5}$$

$$\widehat{\nabla}^2 Z_I = \frac{1}{2}\, \varepsilon\, C_{IJK} *_4 [\Theta^{(J)} \wedge \Theta^{(K)}], \tag{2.6}$$

$$dk + \varepsilon *_4 dk = \varepsilon\, Z_I \Theta_I, \tag{2.7}$$

then one obtains solutions to the complete equations of motion of M-theory. In particular one obtains a BPS solution [25–27] by requiring that the duality structure of the $\Theta^{(I)}$ matches that of the Riemann tensor of the base. That is, one obtains BPS solutions by taking $\varepsilon = +1$ and choosing the base metric to be self-dual hyper-Kähler, or by taking $\varepsilon = -1$ and choosing the base metric to be anti-self-dual hyper-Kähler. Almost-BPS solutions [19–21] are also obtained by using hyper-Kähler base metrics but supersymmetry is broken by mismatching the duality structure of the $\Theta^{(I)}$ and that of the Riemann tensor of the base.

Here we will work with the Taub-NUT metric:

$$ds_4^2 = V^{-1}(d\psi + A)^2 + V\,(dx^2 + dy^2 + dz^2), \tag{2.8}$$

with

$$V = h + \frac{q}{r}, \tag{2.9}$$

where $r^2 \equiv \vec{y} \cdot \vec{y}$ with $\vec{y} \equiv (x, y, z)$. We will also frequently use polar coordinates: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$. The metric (2.8) is hyper-Kähler if

$$\vec{\nabla}V = \pm\vec{\nabla} \times \vec{A}, \tag{2.10}$$

where ∇ denotes the flat derivative of $\mathbb{R}^3$. For (2.9) we get

$$A = \pm q\,\cos\theta\, d\phi, \tag{2.11}$$

where we chose the integration constant to vanish. We use the orientation with $\epsilon_{1234} = +1$ and then choosing the positive sign in (2.10) results in a the Riemann tensor that is self-dual while choosing the negative sign makes the Riemann tensor anti-self-dual.

Henceforth within this paper, we will follow the conventions of [23] and take

$$\varepsilon = +1 \,, \tag{2.12}$$

and then the BPS solutions and almost-BPS solutions correspond to choosing the $+$ or $-$ sign, respectively, in (2.10) and (2.11).

2.2 BPS and almost-BPS solutions

We now summarize, in some detail, the solutions presented in [23]. Since we are using the conventions (2.12), there will be some sign differences compared to [23].

On a Gibbons-Hawking (GH) base the vector potentials, $B_\pm^{(I)}$ in (2.3) are given by:

$$
\begin{aligned}
B_\pm^{(I)} &= P_\pm^{(I)}(d\psi + A) + \vec{\xi}_\pm^{(I)} \cdot d\vec{y}, & P_+^{(I)} &= V^{-1} K_+^I \,; \\
P_-^{(I)} &= K_-^I \,, & &
\end{aligned}
\tag{2.13}
$$

$$
\nabla^2 K_\pm^I = 0 \,, \qquad\qquad \vec{\nabla} \times \vec{\xi}_+^{(I)} = -\vec{\nabla} K_+^I \,,
$$

$$
\vec{\nabla} \times \vec{\xi}_-^{(I)} = K_-^I \vec{\nabla} V - V \vec{\nabla} K_-^I \,,
\tag{2.14}
$$

where the $\pm$ corresponds to the choice in (2.10) and hence to BPS or almost-BPS respectively.

A "type I" supertube, $I = 1, 2, 3$, has a singular magnetic source for $B^{(I)}$ and singular electric sources for Z_J and Z_K, where I, J, K are all distinct. As in [23], we study an axisymmetric supertube configuration with one supertube of each type on the positive z-axis and thus we take harmonic functions:

$$K_\pm^I = \frac{k_I^\pm}{r_I} \,, \tag{2.15}$$

where

$$r_J \equiv \sqrt{x^2 + y^2 + (z - a_J)^2} \,. \tag{2.16}$$

Without loss of generality we will assume that

$$0 < a_1 \;<\; a_2 \;<\; a_3 \,. \tag{2.17}$$

We then have

$$
\xi_+^{(I)} = -k_I^+ \frac{(r\cos\theta - a_I)}{r_I} \, d\phi \,, \qquad \xi_-^{(I)} = -\frac{k_I^-}{a_I \, r_I} \left[q\,(r - a_I \cos\theta) + h\,a_I\,(r\cos\theta - a_I) \right] d\phi \,,
\tag{2.18}
$$

and we may then write

$$B_+^{(I)} = \frac{K_+^I}{V} \left[d\psi + \frac{q\,a_I}{r} \, d\phi - h\,(r\cos\theta - a_I)\,d\phi \right] \,, \tag{2.19}$$

$$B_-^{(I)} = \frac{k_I^-}{a_I\,r_I} \left[a_I\,d\psi - q\,r\,d\phi - h\,a_I\,(r\cos\theta - a_I)\,d\phi \right] \,. \tag{2.20}$$

One can measure the local dipole strength by taking the integral of $\Theta^{(I)}$ over a small sphere, $S^2_\epsilon \subset \mathbb{R}^3$, around the singular point and one finds:

$$\int_{S^2_\epsilon} \Theta^{(I)} = \begin{cases} -4\,\pi\,k_j^+ & \text{for BPS} \\ -4\,\pi\left(h + \dfrac{q}{a_j}\right)k_j^- & \text{for almost-BPS} \end{cases} \tag{2.21}$$

Hence, the quantized dipole charges are

$$\widehat{k}_j \equiv \left(h + \frac{q}{a_j}\right)k_j^- \tag{2.22}$$

in the almost-BPS solutions.

The electrostatic potentials are relatively simple. For BPS supertubes it is straightforward to apply the results of [5, 6]:

$$Z_1 = L_1 + \frac{K_+^2\,K_+^3}{V} = 1 + \frac{Q_2^{(1)}}{r_2} + \frac{Q_3^{(1)}}{r_3} + \frac{k_2^+\,k_3^+}{(h + \frac{q}{r})\,r_2\,r_3}\,, \tag{2.23}$$

$$Z_2 = L_2 + \frac{K_+^1\,K_+^3}{V} = 1 + \frac{Q_1^{(2)}}{r_1} + \frac{Q_3^{(2)}}{r_3} + \frac{k_1^+\,k_3^+}{(h + \frac{q}{r})\,r_1\,r_3}\,, \tag{2.24}$$

$$Z_3 = L_3 + \frac{K_+^1\,K_+^2}{V} = 1 + \frac{Q_1^{(3)}}{r_1} + \frac{Q_2^{(3)}}{r_2} + \frac{k_1^+\,k_2^+}{(h + \frac{q}{r})\,r_1\,r_2}\,. \tag{2.25}$$

For the almost-BPS supertubes one can use the results in [21]:

$$Z_1 = 1 + \frac{Q_2^{(1)}}{r_2} + \frac{Q_3^{(1)}}{r_3} + \left(h + \frac{q\,r}{a_2\,a_3}\right)\frac{k_2^-\,k_3^-}{r_2\,r_3}\,, \tag{2.26}$$

$$Z_2 = 1 + \frac{Q_1^{(2)}}{r_1} + \frac{Q_3^{(2)}}{r_3} + \left(h + \frac{q\,r}{a_1\,a_3}\right)\frac{k_1^-\,k_3^-}{r_1\,r_3}\,, \tag{2.27}$$

$$Z_3 = 1 + \frac{Q_1^{(3)}}{r_1} + \frac{Q_2^{(3)}}{r_2} + \left(h + \frac{q\,r}{a_1\,a_2}\right)\frac{k_1^-\,k_2^-}{r_1\,r_2}\,. \tag{2.28}$$

Note that the harmonic pieces, L_I, are the same for both the BPS and almost-BPS systems.

The angular momentum vector, k, is decomposed in the usual manner:

$$k = \mu\,(d\psi + A) + \omega\,. \tag{2.29}$$

The details will not be important here but for completeness we give them in appendix A.

2.3 The bubble equations

One defines the usual symplectic inner products for the BPS and almost-BPS charges:

$$\Gamma_{12}^+ = k_1^+\,Q_2^{(1)} - k_2^+\,Q_1^{(2)}\,, \quad \Gamma_{13}^+ = k_1^+\,Q_3^{(1)} - k_3^+\,Q_1^{(3)}\,, \quad \Gamma_{23}^+ = k_2^+\,Q_3^{(2)} - k_3^+\,Q_2^{(3)}\,, \tag{2.30}$$

$$\widehat{\Gamma}_{12} = \widehat{k}_1\,Q_2^{(1)} - \widehat{k}_2\,Q_1^{(2)}\,, \quad \widehat{\Gamma}_{13} = \widehat{k}_1\,Q_3^{(1)} - \widehat{k}_3\,Q_1^{(3)}\,, \quad \widehat{\Gamma}_{23} = \widehat{k}_2\,Q_3^{(2)} - \widehat{k}_3\,Q_2^{(3)}\,. \tag{2.31}$$

Note that we have used the effective, local charges, $\widehat{k}_j$ in (2.31).

It was shown in [23] that supertube regularity requires the locations of the supertubes to be related to the charges via:

$$\frac{\Gamma_{12}^+}{|a_1 - a_2|} + \frac{\Gamma_{13}^+}{|a_1 - a_3|} = \frac{Q_1^{(2)} Q_1^{(3)}}{k_1^+}\left(h + \frac{q}{a_1}\right) - k_1^+ \,, \tag{2.32}$$

$$\frac{\Gamma_{21}^+}{|a_1 - a_2|} + \frac{\Gamma_{23}^+}{|a_2 - a_3|} = \frac{Q_2^{(1)} Q_2^{(3)}}{k_2^+}\left(h + \frac{q}{a_2}\right) - k_2^+ \,, \tag{2.33}$$

$$\frac{\Gamma_{31}^+}{|a_1 - a_3|} + \frac{\Gamma_{32}^+}{|a_2 - a_3|} = \frac{Q_3^{(1)} Q_3^{(2)}}{k_3^+}\left(h + \frac{q}{a_3}\right) - k_3^+ \,. \tag{2.34}$$

for the BPS solutions. For the almost-BPS supertubes one must impose

$$\frac{\widehat{\Gamma}_{12}}{|a_1 - a_2|} + \frac{\widehat{\Gamma}_{13}}{|a_1 - a_3|} = \frac{Q_1^{(2)} Q_1^{(3)}}{\widehat{k}_1}\left(h + \frac{q}{a_1}\right) - \widehat{k}_1 - \epsilon_{123}\, Y \,, \tag{2.35}$$

$$\frac{\widehat{\Gamma}_{21}}{|a_1 - a_2|} + \frac{\widehat{\Gamma}_{23}}{|a_2 - a_3|} = \frac{Q_2^{(1)} Q_2^{(3)}}{\widehat{k}_2}\left(h + \frac{q}{a_2}\right) - \widehat{k}_2 - \epsilon_{213}\, Y \,, \tag{2.36}$$

$$\frac{\widehat{\Gamma}_{31}}{|a_1 - a_3|} + \frac{\widehat{\Gamma}_{32}}{|a_2 - a_3|} = \frac{Q_3^{(1)} Q_3^{(2)}}{\widehat{k}_3}\left(h + \frac{q}{a_3}\right) - \widehat{k}_3 - \epsilon_{312}\, Y \,, \tag{2.37}$$

where

$$Y \equiv \frac{h\, q\, k_1^- k_2^- k_3^-}{a_1 a_2 a_3} = \frac{h\, q\, \widehat{k}_1 \widehat{k}_2 \widehat{k}_3}{(q + h a_1)(q + h a_2)(q + h a_3)} \,. \tag{2.38}$$

and

$$\epsilon_{ijk} \equiv \frac{(a_i - a_j)\,(a_i - a_k)}{|a_i - a_j|\,|a_i - a_k|} \,. \tag{2.39}$$

The only difference between the two sets of equations is the term, Y, which vanishes in the supersymmetric limits of the solution ($h \to 0$ or $q \to 0$), and hence can be thought of as measuring the amount of supersymmetry breaking.

Our goal, in section 3, will be to see how these bubble equations, and particularly the supersymmetry breaking term, can be obtained from a brane-probe calculation.

2.4 The Minkowski-space limit

If one sets $h = 0$ then the metric (2.8) becomes that of flat Minkowski space modded out by $\mathbb{Z}_q$. In this limit, the BPS and almost-BPS solutions must become the same because both solutions are necessarily BPS since the base is flat.

It is evident that the potential functions, (2.23)–(2.25) and (2.26)–(2.28), are identical for $h = 0$. The metric may then be written as

$$ds_4^2 = q\left[\frac{dr^2}{r} + r\left(\left(\frac{d\psi}{q}\right)^2 + d\theta^2 + d\phi^2 \pm \frac{2}{q}\cos\theta\, d\psi\, d\phi\right)\right] \,, \tag{2.40}$$

which takes a more canonical form if one sets $r = \frac{1}{4}\rho^2$. The $\pm$ sign reflects the BPS, or almost-BPS description (2.11) but this is a coordinate artifact.

There are several obvious ways of mapping the BPS description of the metric onto the almost-BPS description but one can most easily see the correct choice by looking at the magnetic fields, $B_{\pm}^{(I)}$, for $h = 0$. Indeed, from (2.20) one sees that the substitution

$$\psi \to -q\,\phi\,, \qquad \phi \to \frac{1}{q}\,\psi\,, \qquad k_I^+ \to \widehat{k}_I \qquad (2.41)$$

maps $B_{+}^{(I)} \to B_{-}^{(I)}$. This transformation also interchanges the choice of signs in (2.40).

In appendix A we also verify that the transformation (2.41), for $h = 0$, also maps the angular momentum vector in the BPS description to that of almost-BPS description.

3 Brane probes in almost-BPS solutions

3.1 Brane probes

We only require a relatively simple result from the brane probe analysis of [18] or the more recent work of [28]. We need the so-called "radius relation" that determines the equilibrium position of a probe in a supergravity background.

One can use the M-theory frame or one can go back to the approach of [18] and work with three-charge solutions in the IIA frame in which the three electric charges, N_1, N_2 and N_3, of the solution correspond to:[3]

$$N_1 : \text{D0} \qquad N_2 : \text{F1}\,(z) \qquad N_3 : \text{D4}\,(5678), \qquad (3.1)$$

where the numbers in the parentheses refer to spatial directions wrapped by the branes and $z \equiv x^{10}$. The magnetic dipole moments of the solutions correspond to:

$$n_1 : \text{D6}\,(y5678z) \qquad n_2 : \text{NS5}\,(y5678) \qquad n_3 : \text{D2}\,(yz), \qquad (3.2)$$

where y denotes the brane profile in the spatial base, $(x^1, \ldots, x^4)$. We will use a D2-brane probe, carrying electric D0 and F1 charges, $\widehat{Q}^{(1)}$ and $\widehat{Q}^{(2)}$, and with a D2-dipole moment, d_3. For the supertube worldsheet coordinates, ζ^μ, we use static gauge and allow the supertube to wind along both the ψ and ϕ angles of the base space according to:

$$x^0 = \zeta^0\,, \ z = \zeta^1\,, \ \psi = \nu_\psi \zeta^2\,, \ \phi = \nu_\phi \zeta^2\,. \qquad (3.3)$$

For such a probe, sitting along the positive z-axis of the base metric, the radius relation is given by:

$$\left[\widehat{Q}^{(1)} + d_3\,\mathcal{B}^{(2)}\right]\left[\widehat{Q}^{(2)} + d_3\,\mathcal{B}^{(1)}\right] = d_3^2\,\frac{Z_3}{V}\,(\nu_\psi + q\nu_\phi)^2\,, \qquad (3.4)$$

where $\mathcal{B}^{(I)}$ is the pullback of the vector field, $B^{(I)}$, onto the profile of the supertube on the spatial base:

$$\mathcal{B}^{(I)} = B_\mu^{(I)}\,\frac{\partial x^\mu}{\partial \zeta_2}\,. \qquad (3.5)$$

---

[3]For later convenience, we have permuted the labels relative to those of [18].

Then, using (2.20), for the BPS solutions:

$$\mathcal{B}^{(I)} = \mathcal{B}_+^{(I)} = \frac{k_I^+}{(h + \frac{q}{r}) \, r \, r_I} \left[\nu_\psi \, r + q \, a_I \, \nu_\phi - h \, r \, (r \cos\theta - a_I) \, \nu_\phi \right], \qquad (3.6)$$

while for non-BPS we have

$$\mathcal{B}^{(I)} = \mathcal{B}_-^{(I)} = \frac{k_I^-}{a_I \, r_I} \left[a_I \, \nu_\psi - q \, r \, \nu_\phi - h \, a_I \, (r \cos\theta - a_I) \, \nu_\phi \right]. \qquad (3.7)$$

In deriving this radius relation we have set the worldvolume field strength $\mathcal{F}_{tz} = +1$, which is the choice one makes when placing supersymmetric probes in supersymmetric solutions, and one may ask whether this choice is justified for a non-BPS probe supertube, especially because this choice does not describe the metastable supertube minima of [28, 29]. There are two ways to see that this choice correctly reproduces the almost-BPS probe supertube minima. The first, and most direct, would be to evaluate the Hamiltonian derived in [28] in an almost-BPS solution, and to find directly that the minima of this Hamiltonian have $\mathcal{F}_{tz} = +1$. While he have not done the former computation, there is a simpler, second argument that shows what the outcome must be. Remember that, unlike non-extremal solutions, almost-BPS solutions have the mass and the charge equal, and the only way a supertube action can yield a minimum with the mass equal to the charge is if its worldvolume field strength satisfies $\mathcal{F}_{tz} = +1$.

3.2 Probing a supergravity solution

We now replace the third supertube in the solution of section 2.2 by a probe: we set $k_3^\pm = Q_3^{(1)} = Q_3^{(2)} = 0$ and consider the action of a probe with charges $\widehat{Q}_{\mathrm{probe}}^{(1)}$, $\widehat{Q}_{\mathrm{probe}}^{(2)}$ and dipole charge d_3 at a location, a_3, on the z-axis and winding once around the ψ-fiber.

We then have:

$$\mathcal{B}_+^{(I)} = \frac{k_I^+}{(h + \frac{q}{a_3}) \, |a_I - a_3|}, \qquad \mathcal{B}_-^{(I)} = \frac{k_I^-}{|a_I - a_3|}. \qquad (3.8)$$

We know from the results of [18] that for the BPS choice the radius relation, (3.4), indeed yields the bubble equation (2.34). We therefore focus on the almost-BPS solution.

Inserting $\mathcal{B}^{(I)} = \mathcal{B}_-^{(I)}$ in (3.4) and using (3.8) one can rearrange the radius relation to give

$$\frac{1}{|a_1 - a_3|} \left[d_3 \, Q_1^{(3)} - \left(h + \frac{q}{a_3} \right) k_1^- \, \widehat{Q}_{\mathrm{probe}}^{(1)} \right] + \frac{1}{|a_2 - a_3|} \left[d_3 \, Q_2^{(3)} - \left(h + \frac{q}{a_3} \right) k_2^- \, \widehat{Q}_{\mathrm{probe}}^{(2)} \right]$$
$$= \left(h + \frac{q}{a_3} \right) \frac{\widehat{Q}_{\mathrm{probe}}^{(1)} \, \widehat{Q}_{\mathrm{probe}}^{(2)}}{d_3} - d_3 + \frac{q \, k_1^- \, k_2^- \, d_3}{|a_1 - a_3| \, |a_2 - a_3|} \left(\frac{a_1 \, a_2 - a_3^2}{a_1 \, a_2 \, a_3} \right). \qquad (3.9)$$

We now explicitly use the ordering (2.17) to write $|a_i - a_j| = a_i - a_j$ for $i > j$ and we use the definition (2.22) to obtain:

$$\frac{1}{|a_1 - a_3|} \left[d_3 \, Q_1^{(3)} - \widehat{k}_1 \, Q_{\mathrm{probe}}^{(1)} \right] + \frac{1}{|a_2 - a_3|} \left[d_3 \, Q_2^{(3)} - \widehat{k}_2 \, Q_{\mathrm{probe}}^{(2)} \right]$$
$$= \left(h + \frac{q}{a_3} \right) \frac{Q_{\mathrm{probe}}^{(1)} \, Q_{\mathrm{probe}}^{(2)}}{d_3} - d_3 - \frac{h \, q \, k_1^- \, k_2^- \, d_3}{a_1 a_2 a_3 \, (h + \frac{q}{a_3})}, \qquad (3.10)$$

– 10 –

where we have introduced

$$Q_{\text{probe}}^{(1)} \equiv \widehat{Q}_{\text{probe}}^{(1)} - \frac{q \, \widehat{k}_2 \, d_3}{a_2 a_3 \, (h + \frac{q}{a_2})(h + \frac{q}{a_3})} \,, \qquad Q_{\text{probe}}^{(2)} \equiv \widehat{Q}_{\text{probe}}^{(2)} - \frac{q \, \widehat{k}_1 \, d_3}{a_1 a_3 \, (h + \frac{q}{a_1})(h + \frac{q}{a_3})} \,.$$
(3.11)

This exactly matches the bubble equation (2.37) provided one makes the identifications:

$$d_3 = \widehat{k}_3 = \left(h + \frac{q}{a_3} \right) k_3^- \,, \qquad Q_{\text{probe}}^{(I)} = Q_3^{(I)} \,.$$
(3.12)

Hence, the action of a probe supertube that is placed at the outermost position of an almost-BPS solution that contains two other supertubes of different species can capture exactly the supergravity information about the location of this supertube. Since all the terms in the supertube bubble equations only contain two- and three-supertube interactions, we will see in section 3.4 that this result can be straightforwardly generalized to a probe supertube placed at the exterior position of an almost-BPS solution containing an arbitrary number of supertubes of arbitrary species.

There is another more compact way to write the formula that gives the shift from the quantized charges to the BPS parameters

$$Q_{\text{probe}}^{(1)} = \widehat{Q}_{\text{probe}}^{(1)} - \frac{q \, \widehat{k}_2 \, \widehat{k}_3}{a_2 \, a_3 V_2 V_3} \,, \qquad Q_{\text{probe}}^{(2)} = \widehat{Q}_{\text{probe}}^{(2)} - \frac{q \, \widehat{k}_1 \, \widehat{k}_3}{a_1 \, a_3 V_1 V_3} \,.$$
(3.13)

3.3 Interpretation of the charge shift

Given that we have matched the probe calculation to the supergravity one by postulating the charge shift above, it is legitimate to ask whether this charge shift has any direct physical interpretation apart from the fact that it maps the probe result onto the supergravity result. We will argue in the remaining part of the paper that this charge shift encodes very non-trivial properties of almost-BPS solutions.

We begin by comparing almost-BPS solutions to BPS multi-center solutions (for which there is no charge shift [18]) in the limits when the two kinds of solutions become identical. We will then explain how this charge shift solves an old puzzle about the relation between quantized charges and moduli in almost-BPS solutions

3.3.1 Mapping BPS to non-BPS solutions

There are two ways to turn an almost-BPS solution into a BPS one. The first is to set the Taub-NUT charge, q, to zero (and obtain black strings and supertubes extended along the S^1 of $\mathbb{R}^3 \times S^1$), and the second is to set the constant, h, in the Taub-NUT harmonic function equal to zero. In the first limit ($q = 0$), the charge shifts automatically vanish, consistent with the result of [18].

However, for $h = 0$, the shift does not vanish and this may seem a little puzzling. To understand the origin of the shift we should remember that, in the $h = 0$ limit, the BPS and the almost-BPS solutions are identical up to the coordinate transformation (2.41). Hence, a supertube wrapping the ψ-fiber once in the almost-BPS solution and a supertube wrapping the ψ-fiber once in the BPS writing of the same solution wrap different circles. Indeed, the

supertube that wraps the ψ-fiber once in the almost-BPS sozflution has $(\nu_\psi, \nu_\phi) = (1, 0)$. Then the transformation (2.41) means that in the BPS writing of the solution this object is a supertube that wraps the ϕ-fiber $\frac{1}{q}$ times, and hence it has $(\nu_\psi, \nu_\phi) = (0, \frac{1}{q})$.

Clearly two supertubes with different windings are not the same object, and the only way their radius relations can be the same is if their charges are different. To be more precise, when $h = 0$, (3.6) reduces to:

$$\mathcal{B}_+^{(I)}(\nu_\psi, \nu_\phi) = \frac{k_I^+}{q(a_3 - a_I)}\left[\nu_\psi\, a_3 + \nu_\phi\, q\, a_I\right] = \frac{k_I^+(\nu_\psi + q\,\nu_\phi)\, a_3}{q(a_3 - a_I)} - k_I^+\,\nu_\phi\,, \qquad (3.14)$$

where we have used (2.17). The first term is identical for either choice of winding numbers: $(\nu_\psi, \nu_\phi) = (1, 0)$ and $(\nu_\psi, \nu_\phi) = (0, \frac{1}{q})$, and this makes sense because, for either choice, the Taub-NUT fiber is being wrapped once. However, the last term is different, which indicates that the almost-BPS supertube also wraps the Dirac string of the background magnetic flux. Hence, if the supertubes with $(\nu_\psi, \nu_\phi) = (1, 0)$ and $(\nu_\psi, \nu_\phi) = (0, \frac{1}{q})$ are to have the same radius their charges must be related by a shift. Using the charge identification $k_I^+ \to \widehat{k}_I$ in (2.41) and using (2.22) for $h = 0$, one sees that this charge shift is simply:

$$\widehat{Q}_{(1,0)}^{(1)} = \widehat{Q}_{(0,\frac{1}{q})}^{(1)} - \frac{d_3\,\widehat{k}_2}{q}\,, \qquad \widehat{Q}_{(1,0)}^{(2)} = \widehat{Q}_{(0,\frac{1}{q})}^{(2)} - \frac{d_3\,\widehat{k}_1}{q}\,. \qquad (3.15)$$

which exactly matches the shift in (3.11) for $h = 0$. Hence the shift between the almost-BPS supergravity charge and the almost-BPS quantized charge is the same as the shift from the BPS quantized charge to the almost-BPS quantized charge. This implies that the BPS quantized and supergravity charges are the same as the almost-BPS supergravity charge.

Another way to understand this result is to remember that one can consider a solution with a probe supertube of a certain charge wrapping a Dirac string, and take away the Dirac string by a large gauge transformation. As explained in [18], this changes the charges of the supertube. If one does this in our situation, and takes the Dirac string away from the location of the probe supertube, both the BPS and the almost-BPS supertubes will have the same charge and wrapping and will become the same object. Clearly, upon back-reaction the supergravity charges of the two will be the same. Hence, the reason why the quantized charge of the almost-BPS supertube is shifted from its supergravity charge is because, when written as a probe in $\mathbb{R}^4$, this supertube wraps a Dirac string non-trivially.

Note that when $h = 0$ this shift involves only integers, but for a generic almost-BPS solution this shift depends also on the moduli. As we will explain in section 4.3 this happens because almost-BPS solutions have an additional magnetic flux on the non-compact cycle extending to infinity.

3.3.2 A puzzle about charges and its resolution

Once one understands how the quantized charges of the centers of an almost-BPS solution are related to the supergravity charge parameters one can re-examine the problem of moduli-dependence of the asymptotic charge of almost-BPS solutions. Recall that for a

BPS solution in Taub-NUT the warp factors go asymptotically like

$$Z^{(1)} \sim \left[Q_2^{(1)} + Q_3^{(1)} \right] \frac{1}{r}, \qquad Z^{(2)} \sim \left[Q_1^{(2)} + Q_3^{(2)} \right] \frac{1}{r},$$

$$Z^{(3)} \sim \left[Q_1^{(3)} + Q_2^{(3)} \right] \frac{1}{r}, \tag{3.16}$$

which confirms the fact that the asymptotic charges are simply the sum of the quantized charges of the individual component supertubes. However, for an almost-BPS solution one has

$$Z^{(1)} \sim \left[Q_2^{(1)} + Q_3^{(1)} + \frac{q\,\widehat{k}_2\,\widehat{k}_3}{a_2\,a_3 V_2 V_3} \right] \frac{1}{r}, \qquad Z^{(2)} \sim \left[Q_1^{(2)} + Q_3^{(2)} + \frac{q\,\widehat{k}_1\,\widehat{k}_3}{a_1\,a_3 V_1 V_3} \right] \frac{1}{r},$$

$$Z^{(3)} \sim \left[Q_1^{(3)} + Q_2^{(3)} + \frac{q\,\widehat{k}_1\,\widehat{k}_2}{a_1\,a_2 V_1 V_2} \right] \frac{1}{r}. \tag{3.17}$$

This appears to imply that if one changes the moduli of a solution continuously, while keeping the charges of the centers fixed, the asymptotic charges will change continuously. If this were true, an almost-BPS solution would have non-quantized asymptotic charges for generic values of the moduli, and would only exist on certain submanifolds of the moduli space where the asymptotic charges are integer. Needless to say, this would be rather peculiar.

Our results show that this does not, in fact, happen. If one expresses the supergravity charge in terms of quantized charges, the moduli-dependent term drops out and the asymptotics become identical to the one in (3.16). Hence, the asymptotic charges of almost-BPS solutions are also equal to the sum of the quantized charges of the centers, and the quantization of charges does not restrict the the moduli space of supertube locations.

To see this we can make the following Gedankenexperiment: we start with a single supertube of species $I = 1$ at position a_1, for which the supergravity charges are the same as the quantized charges

$$Q_1^{(2)} = \widehat{Q}_1^{(2)}, \qquad Q_1^{(3)} = \widehat{Q}_1^{(3)}. \tag{3.18}$$

We then bring in a second supertube of species $I = 2$ at position a_2. The supergravity charge $Q_2^{(3)}$ of this supertubes is shifted from its quantized charge $\widehat{Q}_2^{(3)}$ because of the presence of the first supertube, while its other charge remains the same:

$$Q_2^{(1)} = \widehat{Q}_2^{(1)}, \qquad Q_2^{(3)} = \widehat{Q}_2^{(3)} - \frac{q\widehat{k}_1^{(1)}\widehat{k}_2^{(2)}}{a_1 a_2 V_1 V_2}, \tag{3.19}$$

The asymptotic charges $\widehat{\mathcal{Q}}_{12}^{(I)}$ of the resulting solution are given by the (sum of the) quantized charges, $\widehat{\mathcal{Q}}_{12}^{(1)} = \widehat{Q}_2^{(1)} = Q_2^{(1)}$, $\widehat{\mathcal{Q}}_{12}^{(2)} = \widehat{Q}_1^{(2)} = Q_1^{(2)}$ and

$$\widehat{\mathcal{Q}}_{12}^{(3)} = \widehat{Q}_1^{(3)} + \widehat{Q}_2^{(3)} = Q_1^{(3)} + Q_2^{(3)} + \frac{q\widehat{k}_1^{(1)}\widehat{k}_2^{(2)}}{a_1 a_2 V_1 V_2}, \tag{3.20}$$

which agrees with the coefficient of r^{-1} in $Z^{(3)}$ in (3.17). After back-reacting the second supertube we bring in a third supertube of species $I = 3$ at position a_3 whose supergravity

charges after back-reaction are shifted with respect to their quantized charges

$$Q_3^{(1)} = \widehat{Q}_3^{(1)} - \frac{q\widehat{k}_2^{(2)}\widehat{k}_3^{(3)}}{a_2 a_3 V_2 V_3}\,, \qquad Q_3^{(2)} = \widehat{Q}_3^{(2)} - \frac{q\widehat{k}_1^{(1)}\widehat{k}_3^{(3)}}{a_1 a_2 V_1 V_2}\,, \qquad (3.21)$$

because of the other supertubes. The asymptotic charges $\widehat{\mathcal{Q}}_{123}^{(I)}$ of the solution with all three supertubes back-reacted are $\widehat{\mathcal{Q}}_{123}^{(3)} = \widehat{\mathcal{Q}}_{12}^{(3)} \sim r\, Z^{(3)}$ and

$$\widehat{\mathcal{Q}}_{123}^{(1)} = \widehat{\mathcal{Q}}_{12}^{(1)} + \widehat{Q}_3^{(1)} = Q_2^{(1)} + Q_3^{(1)} + \frac{q\widehat{k}_2^{(2)}\widehat{k}_3^{(3)}}{a_2 a_3 V_2 V_3} \sim r\, Z^{(1)}\,, \qquad (3.22)$$

$$\widehat{\mathcal{Q}}_{123}^{(2)} = \widehat{\mathcal{Q}}_{12}^{(2)} + \widehat{Q}_3^{(2)} = Q_1^{(2)} + Q_3^{(2)} + \frac{q\widehat{k}_1^{(1)}\widehat{k}_3^{(3)}}{a_1 a_3 V_1 V_3} \sim r\, Z^{(2)}\,. \qquad (3.23)$$

As advertised, these charges match the coefficients of r^{-1} in the $Z^{(I)}$. Hence, while the shift between the quantized and supergravity charges might have seemed surprising at first, it represents the missing ingredient necessary to relate the asymptotic charge of a multi-center solution to those of the centers.

One can also use these equations to determine the shift between the quantized and supergravity charges for an almost-BPS black ring in Taub-NUT [20]. Although black rings have no DBI description, one can make a black ring by bringing together three supertubes with three different types of dipole charges. Since in this process both the supergravity and the quantized charges are preserved, the shifts of the black ring charges will be the shifts of the charges of the composing supertubes. For a Taub-NUT almost-BPS black ring with dipole charges $\widehat{k}^{(I)}$ at position a_1 these shifts are:

$$Q^{(I)} = \widehat{Q}^{(I)} - |\epsilon_{IJK}|\frac{q\widehat{k}^{(J)}\widehat{k}^{(K)}}{a_1^2 V_1^2}\,, \qquad (3.24)$$

where capital Latin indices are not summed. Note that there are $n!$ ways of making a black ring starting with n supertubes, corresponding to the different relative orderings of these supertubes, and for each ordering the charge shifts of various supertubes are different, but the sum of all these charge shifts (which gives the black ring charge shift) is always the same.

We are now ready to further use this result and the iterative procedure outlined above in order to unambiguously determine the relation of the supergravity charge parameters and the quantized charges in a solution with an arbitrary number of centers.

3.4 Generalization to many supertubes

We can generalize the calculation in the previous section to an almost-BPS solution containing an arbitrary number of colinear supertubes. Without loss of generality we consider a solution containing $i-1$ supertubes at positions

$$0 < a_1 < a_2 < \ldots < a_{i-1}\,, \qquad (3.25)$$

and we bring a probe supertube of species I (wrapping the tori T_J^2 and T_K^2 in (2.1)) to the point a_i on the z-axis, at the outermost position with respect to the other supertubes

$(a_i > a_{i-1})$. This probe has electric charges $\widehat{Q}_i^{(J)}$, $\widehat{Q}_i^{(K)}$ and dipole charge $d_i^{(I)}$. Starting from the radius relation:

$$\left[\widehat{Q}_i^{(J)} + d_i^{(I)}\mathcal{B}_i^{(K)}\right]\left[\widehat{Q}_i^{(K)} + d_i^{(I)}\mathcal{B}_i^{(J)}\right] = (d_i^{(I)})^2 \frac{Z_i^{(I)}}{V_i} \tag{3.26}$$

with

$$\mathcal{B}_i^{(L)} = \mathcal{B}_-^{(L)}(r = a_i) = \sum_l \frac{k_l^{-(L)}}{|a_l - a_i|} = \sum_l \frac{\widehat{k}_l^{(L)}}{|a_l - a_i|V_l} \quad , \qquad V_i = h + \frac{q}{a_i}\,, \tag{3.27}$$

and

$$Z_i^{(I)} = Z^{(I)}(r = a_i) = 1 + \sum_j \frac{Q_j^{(I)}}{|a_i - a_j|} + |\epsilon_{IJK}|\sum_{j,k}\left(h + \frac{qa_i}{a_j a_k}\right)\frac{k_j^{-(J)}k_k^{-(K)}}{|a_i - a_j||a_i - a_k|}\,, \tag{3.28}$$

we obtain the shift of charges by matching this relation to the bubble equation for the back-reacted probe center (that can be derived from the results of [21]):

$$\left\{\sum_j \frac{1}{|a_i - a_j|}[d_i^{(I)}Q_j^{(I)} - \widehat{k}_j^{(J)}Q_i^{(J)}] + \sum_k \frac{1}{|a_i - a_k|}[d_i^{(I)}Q_k^{(I)} - \widehat{k}_k^{(K)}Q_i^{(K)}]\right\}$$

$$= \frac{Q_i^{(J)}Q_i^{(K)}}{d_i^{(I)}}\left(h + \frac{q}{a_i}\right) - d_i^{(I)} - |\epsilon_{IJK}|\frac{hqd_i^{(I)}}{a_i\left(h + \frac{q}{a_i}\right)}\sum_{j,k}\epsilon_{ijk}\frac{k_j^{-(J)}k_k^{-(K)}}{a_j a_k}\,, \tag{3.29}$$

where there is no summation over capital Latin indices.[4] We find that, upon identifying $d_i^{(I)} = \widehat{k}_i^{(I)} = k_i^{-(I)}V_i$, the shift between the supergravity charge parameters of the supertube and its quantized charges are

$$Q_i^{(J)} = \widehat{Q}_i^{(J)} - |\epsilon_{JKI}|\sum_k \frac{q\widehat{k}_k^{(K)}\widehat{k}_i^{(I)}}{a_k a_i V_k V_i} \tag{3.30}$$

and similarly for $Q_i^{(K)}$, where there is no summation over capital Latin indices. In equations (3.29)–(3.30) the indices j, k run over the positions of supertubes of species J, K respectively.

One can now use this formula and find the charge shifts for all the centers of a certain solution, by constructing it recursively by bringing in probe supertubes. From the previous analysis it is clear that the relation between the supergravity and the quantized charges of a given center in the interior of the solution does not change as one brings more and more supertubes at the outermost positions. Hence, the shifts of the charges of a given supertube center only depend on the locations and dipole charges of the centers that are between this supertube center and the Taub-NUT center, but not on the locations or charges of the centers at its exterior:

$$Q_i^{(J)} = \widehat{Q}_i^{(J)} - |\epsilon_{JKI}|\sum_{k<i}\frac{q\widehat{k}_k^{(K)}\widehat{k}_i^{(I)}}{a_k a_i V_k V_i}. \tag{3.31}$$

---

[4]Note that the terms involving $|\epsilon_{IJK}|$ in $Z^{(I)}$ and the bubble equation (3.29) differ by a factor $1/2$ from the usual form (see, for example, [21]) where *there is* summation over I, J, K.

It is not hard to see that the asymptotic charge of the solution (which can be read off from the asymptotics of $Z^{(I)}$) is now the sum of the quantized charges of all the centers

$$\widehat{\mathcal{Q}}_\infty^{(I)} = \sum_j Q_j^{(I)} + |\epsilon_{IJK}| \sum_{j,k} \frac{q\widehat{k}_j^{(J)}\widehat{k}_k^{(K)}}{a_j a_k V_j V_k} = \sum_i \widehat{Q}_i^{(I)}. \tag{3.32}$$

One can also extend this calculation to describe a configuration containing an arbitrary collection of concentric supertubes and black rings, by constructing the black rings from three different species of supertubes. The general formula relating the supergravity charge parameters to the quantized charges is

$$Q_i^{(J)} = \widehat{Q}_i^{(J)} - |\epsilon_{JKI}| \sum_{k<i} \frac{q\widehat{k}_k^{(K)}\widehat{k}_i^{(I)}}{a_k a_i V_k V_i} - |\epsilon_{JKI}| \frac{q\widehat{k}_i^{(K)}\widehat{k}_i^{(I)}}{a_i^2 V_i^2}, \tag{3.33}$$

where once again the capital latin indices are not being summed.

4 Extracting the complete supergravity data from supertubes

We have seen that if one considers an axially-symmetric brane configuration and brings in a brane probe along the axis from one side of the configuration then the radius relation of that probe exactly reproduces the bubble equation for the charge center that would replace the brane probe in a fully back-reacted supergravity configuration. This correspondence requires the charge shifts described in section 3. Since we are considering only the action of the probe, there is no immediate way in which this action could directly yield the fully back-reacted bubble equations for the other centers in response to the probe. That is, given the n-supertube solution, the probe radius relation for the $(n+1)^{\text{st}}$ supertube yields the exact supergravity bubble equation for that supertube. However, when back-reacted, the probe supertube must introduce modifications to the bubble equations for the other supertubes and these are not given directly by the computations described above.

In contrast, the analysis of brane probes in BPS solutions [18] required no charge shifts and the bubble equations contained only two-body interactions of the form $\Gamma_{ij}|\vec{y}_i - \vec{y}_j|^{-1}$ as in (2.31)–(2.34). In that solution, the action of the probe was used to read off all the two-by-two terms between the probe and the centers, and thus derive the bubble equations for *all* the centers. However, for almost-BPS solutions the bubble equations for a certain center contain complicated three-body terms, which cannot be read off from the action describing the location of another center. However, there is still a (more complicated) way to recover the full supergravity data from DBI actions: given a solution with $(n+1)$ supertubes one can examine all the ways of extracting one supertube and treating it is a probe in the background sourced by the others. As we will show below, this yields all the bubble equations of the solution.

4.1 Reconstructing the bubble equations from probes

For non-BPS supertubes there are three elements for the inductive modification of the bubble equations as one goes from n to $(n+1)$ supertubes:

(i) Compute the charge shifts of all probe charges.

(ii) Write the left-hand sides of the bubble equations with all two-body interactions, $\Gamma_{ij}|a_i - a_j|^{-1}$.

(iii) Compute and include all the supersymmetry breaking terms, Y, as in (2.38) and include them with the correct relative sign on the right-hand sides of the bubble equations as in (2.35)–(2.37).

If one assembles the configuration by bringing in the charges successively along the symmetry axis from the right then the charge shifts are given by (3.30). The two-body interactions are fixed iteratively exactly as in the BPS solution. These terms are also fixed by the consistency condition that the sum of all the bubble equations must be identically zero. The new feature is that we need to specify the algorithm for (iii): We need to generalize the supersymmetry breaking terms and how they are to be included in the bubble equations. It is clear that the bubble equation for a supertube of type I will have an interaction, Y, of the form (2.38) with every pair of supertubes of species J, K where I, J and K are all distinct. The issue is to introduce these terms with the correct signs in each bubble equation. One can do this by making the more formal induction using all the ways of extracting one supertube and treating it is a probe. However, for simplicity, we will start by restricting our attention to configurations in which the probe is always the outermost supertube. As we will see in the next section, the charge shifts are modified when the probe is some other supertube in the configuration and so the discussion is a little more complicated. We will begin by completing the discussion of section 3.4 by giving a recipe for the supersymmetry breaking terms when the probe is the outermost supertube and we will explain its derivation below.

Suppose we have an n-supertube system and we bring in, as a probe, a supertube of species I then the bubble equations for a center of type J at position a_j must include an extra supersymmetry breaking term, $Y^{(J,I)}_{j,(n+1)}$, that is to be subtracted from the right-hand side. These terms are given by:

$$
\begin{aligned}
Y^{(J,I)}_{j,(n+1)} &= |\epsilon_{IJK}| \sum_k \frac{(a_j - a_{n+1})(a_j - a_k)}{|a_j - a_{n+1}||a_j - a_k|} \frac{hq d^{(I)}_{n+1} \hat{k}^{(J)}_j \hat{k}^{(K)}_k}{a_{n+1} a_j a_k V_{n+1} V_j V_k} = \\
&= -|\epsilon_{IJK}| \sum_k \frac{(a_j - a_k)}{|a_j - a_k|} \frac{hq d^{(I)}_{n+1} \hat{k}^{(J)}_j \hat{k}^{(K)}_k}{a_{n+1} a_j a_k V_{n+1} V_j V_k}
\end{aligned}
\tag{4.1}
$$

where, for fixed I and J, the summation over k runs only over supertubes of species K, with I, J, K all distinct. In the last equality we used that $a_{n+1} > a_j$ for $j = 1, 2, \ldots, n$. Thus, as we assemble the system by successively bringing supertubes from the right, for every new supertube added we have to subtract (4.1) from the right hand side of the bubble equations of the previously back-reacted system.

4.2 Assembling colinear supertubes in general

The general bubble equations (3.29) for a system of n colinear supertubes contain supersymmetry breaking terms, Y, that come with a relative sign given by by ϵ_{ijk} defined

– 17 –

in (2.39).[5] In the procedure discussed in section 3, we always had $\epsilon_{ijk} = +1$ since the probe was always being placed in the outermost position of the back-reacted geometry. However, if the probe is of species I and located at a general position, a_i, then we find that its radius relation can be mapped directly onto is supergravity bubble equation if one uses the following more general relation between supergravity and quantized brane charges:

$$Q_i^{(J)} = \widehat{Q}_i^{(J)} - |\epsilon_{JKI}| \sum_k \frac{a_i - a_k}{|a_i - a_k|} \frac{q\widehat{k}_k^{(K)}\widehat{k}_i^{(I)}}{a_k a_i V_k V_i} \,, \tag{4.2}$$

where, once again, the capital latin indices are not being summed and the sum over k runs over supertubes of species K. Thus for $a_i > a_k$ the shift terms get subtracted from the quantized charge as before, but for $a_i < a_k$ the shift terms have to be added! We also find that if one uses the more general shifts in (4.2) then the radius relation also exactly reproduces precisely the correct supersymmetry breaking terms, Y, in the bubble equations.

As a result, by using (4.2), we can recreate the whole set of bubble equations of the system by the following iterative procedure. Consider a $(n + 1)$-supertube system with preassigned order

$$0 < a_1 < a_2 < \ldots < a_{n-1} < a_{n+1} \,. \tag{4.3}$$

Now consider each supertube, in turn, as a probe in the supergravity background of the remaining n and imagine placing the probe in its assigned position, (4.3), to recreate the full $(n + 1)$-supertube system. To be more explicit, start by taking the supertube at a_{n+1} out of the supergravity system and treat it as a probe being placed at position a_{n+1}. This generates the bubble equation for the center at a_{n+1} as described earlier. Next, imagine the supergravity system where the supertube at position a_n has been removed and replaced by a probe placed at position a_n. The radius relation of this probe produces exactly the bubble equation for the center at a_n provided one uses the charge shifts defined in (4.2). One can then repeat this iterative procedure (illustrated in figure 1) for each of the $n + 1$ centers of the system.

As we noted earlier, for a probe of species I, the bubble equations for the center of type J will involve a supersymmetry breaking term, $Y_{j,(n+1)}^{(J,I)}$, that is a sum over all centers of species K (with I, J, K distinct). The importance of this general iterative procedure is that it generates all these terms in the bubble equations in exactly the correct form and yields the formula (4.1) for a general position, a_i, of the probe:

$$Y_{j,i}^{(J,I)} = |\epsilon_{IJK}| \sum_k \frac{(a_j - a_i)(a_j - a_k)}{|a_j - a_i||a_j - a_k|} \frac{hq\hat{d}_{n+1}^{(I)}\hat{k}_j^{(J)}\hat{k}_k^{(K)}}{a_{n+1}a_j a_k V_{n+1} V_j V_k} \,. \tag{4.4}$$

4.3 Topology, charge shifts and back-reacting probes

To understand the charge shifts and their dependence on the location of the supertubes it is important to recall the geometric structure underlying the almost-BPS supergravity

---

[5]This ϵ_{ijk} should not be confused with the factors of $|\epsilon_{IJK}|$ that give the triple intersection number of the dipole charges.
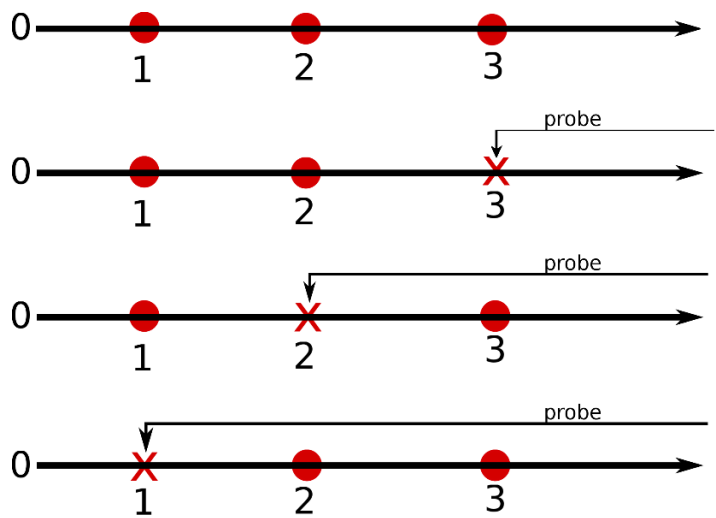
Figure 1. The first graph represents the three-supertube back-reacted geometry while the three below represent the iterative procedure by which one removes each center of the system and treats it as a probe in the background of the remaining two supergravity supertubes. Because supertubes cross over each other we need the formula (4.2) to generate the shift with the correct sign.

solutions. In an appropriate duality frame[6] a solution containing multiple supertubes becomes smooth, and all the charges come from fluxes wrapping topologicaly non-trivial cycles. These cycles can be described in terms of U(1) fibers over paths that end at the locations of the supertubes. The supertube locations are precisely where one of the U(1) fibers pinches off and this, in turn, defines cycles and their intersections. The magnetic dipole charges of the supertube then correspond to cohomological fluxes through these cycles. Hence, in this particular duality frame the back-reaction of a supertube corresponds to blowing up a new cycle and replacing a singular magnetic source with a cohomological flux. If one starts out with a particular supergravity solution and makes a particular choice of homology basis, then introducing a new supertube and back-reacting it will involve blowing up a new cycle and perhaps pinching off other cycles in order to achieve this. Hence, this will involve generically a reshuffling of the homology basis.

The presence of the Chern-Simons term in the electromagnetic action means that the interaction of pairs of magnetic charges can source electric charges and so the change of homology basis arising from the back-reaction of a supertube can lead to shifts of electric charges. Because the magnetic charges on the compact cycles are quantized and the magnetic contributions to electric charges are determined though the intersection form, one would expect all charge shifts to be quantized. This is, indeed, precisely what one finds in all the BPS solutions: the shifts, if they are non-zero, are indeed quantized. What distinguishes the non-BPS solutions is that there is a non-vanishing, normalizable flux on a non-compact cycle that extends to infinity and this flux depends upon the supergravity parameters and moduli. It is the interactions between the fluxes on this non-compact cycle that leads to moduli-dependence of the shifts.

---

[6]This can be realized by performing three generalized spectral flows on almost-BPS solutions [31].

One can recast this geometric picture in more physical terms through a careful examination of Dirac strings. Because a supergravity supertube carries a magnetic charge, the supertube comes with Dirac strings attached (these are, of course, an artifact of trying to write a vector potential for a topologically non-trivial flux). A supertube wraps a U(1) fiber in the background and to define its configuration properly one must specify precisely which Dirac strings are being wrapped by the supertube. This defines the dipole-dipole interaction between the probe and the background[7] and the difference between wrapping and not wrapping will appear as a shift of the electric charges that it contributes to the supergravity solution.

If the supertubes are all colinear then one can choose all the Dirac strings to follow the axis of symmetry out to infinity. One can then set up a configuration in which an outermost probe supertube wraps the Dirac strings of all the other supertubes, and hence its charges get shifted. On the other hand the tubes at the interior of this configuration will not feel the Dirac string of the outermost supertube, and hence the relationship between their supergravity charge and their quantized charge does not shift. In the duality frame where the multi-supertube solution is smooth, this corresponds to blowing up a homology cycle at the outer edge of the original configuration.

If one were to place the probe supertube so that it is the closest to the Taub-NUT center, its Dirac string would generically affect all the other supertubes, and would change the relation between quantized and supergravity charges. This corresponds to blowing up a different new homology element and reshuffling the homology basis. Hence, if one keeps the supergravity charges of the back-reacting supertubes fixed, as one must do in a probe approximation, bringing a supertube to a point that is closer to the center of Taub-NUT than the other supertubes will change the quantized charges of these supertubes. Thus, the resulting configuration will not have the same quantized charges on the centers as when the supertube is at the outermost location, and hence belongs to different sector of the multi-centered solutions. There are, of course, similar consequences to bringing the probe supertube to some point in the middle of the back-reacted supertube centers.

One can also take a more pragmatic perspective and try to understand the charge shifts by compactifying the multi-supertube solutions to obtain a multi-center almost-BPS solution in four dimensions. The five-dimensional, smooth solutions we discuss here are singular at the supertube centers in four dimensions, but this does not impede the calculation of the conserved charges at the singularities. As explained in [36], these conserved charges differ in general from the supergravity charge parameters by dipole-dipole position-dependent shifts similar to the ones found here, and it would be interesting to see whether one can reproduce our results using the four-dimensional KK reduction formulae of [36] (starting, for example, from equation (266) on page 55).

4.4 Quantized charges, supergravity parameters and probes

As we have seen in the previous section, bringing a supertube to some point in the middle of a multi-center supertube solution changes the relation between the quantized charges

---

[7]This wrapping choice determines the new homology element and its intersections with other other elements of homology.

and the supergravity parameters of the supertube centers at its exterior. Hence, all the probe calculations that describe supertubes at interior locations in a multi-center solution are not self-consistent, because the quantized charges of the other centers before bringing in the probe are not the same as the quantized charges with the probe inside. One can think of this as coming from the fact that all probe supertubes come with Dirac strings attached, and when these Dirac strings touch the other centers, the relation between the supergravity and quantized charges of these centers change. Thus, the only probe supertube that one can bring without shifting everybody else's quantized charges is one that lays at the outermost position and whose Dirac string extends away from the supertubes and towards infinity.

An immediate corollary of this is that if one wants to calculate the amplitude for a supertube to tunnel to a vacuum across another center, this calculation cannot be done using the DBI action of that supertube and treating it as a probe in a fixed supergravity background. Instead one would have to change the solution as the probe moves around, and this cannot be done off-shell. One can wonder whether there exists *any* way to compute this tunneling amplitude. Again, the correct probe for this would be a supertube *with Dirac strings attached*, such that one would change the supergravity charges of the back-reacted solution as one moves the probe around, in such a way that the quantized charges stay the same. Unfortunately, there is no known action for such a probe, so probing the interior of a multi-supertube solution with a probe supertube does not correspond to a physical process. Hence, the calculation we performed in section 4.2, where we took the supergravity parameters to be fixed and treated the background of n supertubes merely as one would treat any other supergravity background ignoring the details of how it might have been assembled from other supertubes, should be interpreted as a formal calculation, which does not correspond to a physical process, but which does however reproduce the bubble equations of all the interior supertube centers. It would be clearly interesting to understand the reason for this.

It is also important to stress out that this charge shift subtlety only affects the validity of the probe calculation when the other centers are supertubes or black rings, that interact directly with the Dirac string of the probe. If the other centers are bubbled Gibbons-Hawking centers, where the geometry is smooth, the presence of a Dirac string does not change their four-dimensional charge parameters nor the fluxes wrapping the corresponding cycle in the five-dimensional solution. Hence, one can use the probe supertube action to calculate tunneling probabilities of metastable supertubes in bubbling geometries [28, 29]. Similarly, the description of multi-center non-BPS solutions using supergoop methods [30], being intrinsically non-gravitational, is not affected by this subtlety.

5 Conclusions

We have shown that a brane probe in an almost-BPS background of supertubes can capture the complete supergravity data of the background in which the probe becomes a fully back-reacted source. A similar result was established for BPS solutions in [18] but it is rather surprising that this can also be achieved for almost-BPS solutions in which the supersymmetry is broken, albeit in a rather mild manner. It is surprising because going from a probe to a back-reacted supergravity solution represents going from weak to strong

coupling in the field theory on the brane and when supersymmetry is broken one would expect the parameters of this field theory to be renormalized. Our result therefore suggests that, even though supersymmetry is broken, the couplings of the field theory that govern the probe location are still protected. This does not mean that the theory is unchanged relative to its BPS counterpart: there are terms in the probe action that come from the supersymmetry breaking and these exactly reproduce the corresponding terms found in the supergravity solution.

We suspect that the non-renormalization of the relevant parts of the probe action arises from the rather special form of the supersymmetry breaking: the probe is supersymmetric with respect to every other center in the supergravity solution taken individually and the supersymmetry breaking arises from the fact that these supersymmetries are incompatible between multiple centers. Indeed the supersymmetry breaking terms depend on the product of the charges and dipole charges of three or more centers, and do not have the "two-body" structure of the terms that appears in the BPS bubble equations. It would be interesting to try to extend our analysis to the other known class of non-BPS extremal multi-center solutions, the interacting non-BPS black holes [32, 33], and see if the bubble equations of these solutions are also not renormalized when one goes from weak to strong effective coupling.

One of the new features of our analysis is that the supergravity charge parameters and the quantized charges of the supertubes need to be shifted relative to one another in order to match the supergravity bubble equations and the probe radius relations. By taking a flat-space limit, we saw that part of this difference was related to the choice of how the supertube wraps Dirac strings, or equivalently, how the cycle is blown up after back-reaction. This accounts for a quantized shift but in a Taub-NUT background the shift is no longer obviously quantized because it depends upon moduli. Quantized shifts have also been encountered and understood in the study of black rings [18, 34, 35] but it would be very nice to understand how to extract the quantized charges, and hence the more general moduli-dependent shifts, via a pure supergravity calculation. A good starting point may be to use the four-dimensional reduction formulas of [36] as well as a judicious accounting of the integer charge shifts caused by Dirac string, to try to reproduce the shifts we find.

Our results also suggest further interesting supergravity calculations. For rather mysterious reasons, explicit solutions of the almost-BPS equations still elude us for non-axi-symmetric configurations. Such solutions are extremely simple in the BPS system but for the almost-BPS system even the three-centered, non-axi-symmetric solution is not known. The most one could get is an implicit solution in terms of integrals that one can evaluate asymptotically [36]. Finding the three-centered solution is particularly important because one can then use this solution and the methods of [21] to generate the general, non-axi-symmetric, multi-centered solutions. A two-centered solution is, of course, trivially axi-symmetric and it is easy to introduce a probe in any location. It would therefore be very interesting to see if such a probe could be used to gain insight into the structure of the full supergravity solution.

Given that our results suggest that there are non-renormalization theorems that protect the bubble equations of almost-BPS multi-center solutions, it would be extremely

interesting to investigate whether almost-BPS solutions could be described in the regime of parameters where none of the centers is back-reacted, using a suitable generalization of quiver quantum mechanics of [2]. Clearly, this generalization will have to account for the supersymmetry-breaking terms, which is highly non-trivial: these terms involve three- and four-center interactions, and at first glance no theory of strings on multiple D-branes appears capable of producing such a term. It would be very nice to understand how the charge shifts and supersymmetry breaking terms emerge from the field theory on the branes and whether the effective number of hypermultiplets may be different from the number of hypermultiplets appearing in the Lagrangian. This could be the effect of some of these hypermultiplets becoming massive and no longer contributing to the low energy dynamics. It is thus not too difficult to imagine that our results might be derivable from a field theory analysis and we intend to investigate this further in the future.

Acknowledgments

A Details of the angular momentum vector

The angular momentum vector, k, in the metric (2.1) is decomposed as in (2.29) and here we summarize the results of [23] and give the detailed expressions in our conventions. We also examine the relationship between the BPS and non-BPS descriptions for $h = 0$.

A.1 The function, μ

For BPS solutions, μ is given by the standard expression in [5, 6]:

$$\mu = \frac{1}{6} V^{-2} C_{IJK} K_+^I K_+^J K_+^K + \frac{1}{2} V^{-1} K_+^I L_I + M_+ \,, \tag{A.1}$$

where M_+ is harmonic function that we will take to be

$$M_+ = m_\infty^+ + \frac{m_0^+}{r} + \sum_{j=1}^3 \frac{m_j^+}{r_j} + m_{\text{div}}^+ \, r \cos\theta \,. \tag{A.2}$$

For reasons that will become apparent below, we have added an unphysical harmonic term that diverges at infinity.

For the almost-BPS system one can use the results of [21] to obtain

$$
\begin{aligned}
V\mu =& \frac{1}{2}\sum_{I=1}^{3}K_{-}^{I}(V + h(L_I - 1)) - \frac{q}{2}\sum_{\substack{i,j=1\\j\neq i}}^{3}k_i^- Q_j^{(i)}\frac{r^2 - 2a_i r\cos\theta + a_i a_j}{a_i(a_i - a_j)\,r\,r_i\,r_j}\\
&+ \frac{k_1^- k_2^- k_3^-}{r_1 r_2 r_3}\left[h^2 + \frac{q^2 r\cos\theta}{a_1 a_2 a_3} + \frac{h\,q\left(r^2(a_1 + a_2 + a_3) + a_1 a_2 a_3\right)}{2\,a_1 a_2 a_3\,r}\right] + M_- \,,
\end{aligned}
\tag{A.3}
$$

where M_- is another harmonic function which we will take to be:

$$
M_- = m_\infty^- + \frac{m_0^-}{r} + \sum_{j=1}^{3}\frac{m_j^-}{r_j} + \frac{m_{\mathrm{div}}^-}{r^2}\cos\theta\,.
\tag{A.4}
$$

Again we have added an unphysical harmonic term but this time it diverges at the origin.

Note that regularity at the supertubes requires the following for both classes of solution:

$$
m_1^\pm = \frac{Q_1^{(2)}Q_1^{(3)}}{2\,k_1^\pm}\,,\qquad m_2^\pm = \frac{Q_2^{(1)}Q_2^{(3)}}{2\,k_2^\pm}\,,\qquad m_3^\pm = \frac{Q_3^{(1)}Q_3^{(2)}}{2\,k_3^\pm}\,,
\tag{A.5}
$$

A.2 The one-form, ω

Following [5, 6], for $a_i > a_j$ we define:

$$
\omega_{ij} \equiv -\frac{\left(r^2\sin^2\theta + (r\cos\theta - a_i + r_i)(r\cos\theta - a_j - r_j)\right)}{(a_i - a_j)\,r_i\,r_j}\,,
\tag{A.6}
$$

To define the BPS angular momentum vector, ω, it is also convenient to introduce:

$$
\Delta_1 \equiv 2\left(h + \frac{q}{a_1}\right)m_1^+ - \frac{\Gamma_{12}^+}{(a_2 - a_1)} - \frac{\Gamma_{13}^+}{(a_3 - a_1)} - k_1^+\,,
\tag{A.7}
$$

$$
\Delta_2 \equiv 2\left(h + \frac{q}{a_2}\right)m_2^+ + \frac{\Gamma_{12}^+}{(a_2 - a_1)} - \frac{\Gamma_{23}^+}{(a_3 - a_2)} - k_2^+\,,
\tag{A.8}
$$

$$
\Delta_3 \equiv 2\left(h + \frac{q}{a_3}\right)m_3^+ + \frac{\Gamma_{13}^+}{(a_3 - a_1)} + \frac{\Gamma_{23}^+}{(a_3 - a_2)} - k_3^+\,,
\tag{A.9}
$$

Note that the bubble equations (2.32)–(2.34) and (A.5) imply that $\Delta_j = 0$, $j = 1, 2, 3$. We will not impose these conditions here and we will not remove Dirac strings more generally.

We then find that for the BPS solution, $\omega = \omega_\phi d\phi$, with

$$
\begin{aligned}
\omega_\phi =& \frac{1}{2}\left(\Gamma_{21}^+\,\omega_{21} + \Gamma_{31}^+\,\omega_{31} + \Gamma_{32}^+\,\omega_{32}\right) + q\sum_{i=1}^{3}\frac{m_i^+}{a_i r_i}(r\sin^2\theta + (r\cos\theta - a_i + r_i)(\cos\theta - 1)\\
&+ \frac{1}{2}\sum_{i=1}^{3}\Delta_i\frac{r\cos\theta - a_i}{r_i} + \kappa^+ - \frac{1}{2}\left(\frac{\Gamma_{12}^+}{(a_1 - a_2)} + \frac{\Gamma_{13}^+}{(a_1 - a_3)} + \frac{\Gamma_{23}^+}{(a_2 - a_3)}\right)\\
&+ \left(q\sum_{i=1}^{3}\frac{m_i^+}{a_i} - h\,m_0^+\right)(1 - \cos\theta) - q\,m_\infty^+\cos\theta + m_{\mathrm{div}}^+\left(\frac{1}{2}hr^2 + qr\right)\sin^2\theta\,,
\end{aligned}
\tag{A.10}
$$

and where κ^+ is a constant of integration.

One may use the results of [21] to find ω_ϕ for the almost-BPS solution:

$$
\begin{aligned}
\omega_\phi = &-\frac{1}{2}\sum_{i=1}^{3}\frac{k_i^-}{a_i\,r_i}\big(h\,a_i\,(r\cos\theta - a_i) + q\,(r - a_i\cos\theta)\big) \\
&-\frac{1}{2}\sum_{\substack{i,j=1\\j\neq i}}^{3}\frac{k_i^-\,Q_j^{(i)}}{a_i(a_i - a_j)\,r_i\,r_j}\Big[h\,a_i\,\big(r^2 - (a_i + a_j)r\cos\theta + a_i a_j\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + q\,\big((r^2 + a_i a_j\cos\theta) - r(a_i\cos 2\theta + a_j)\big)\Big] \\
&-\frac{k_1^- k_2^- k_3^-}{a_1 a_2 a_3\,r_1 r_2 r_3}\bigg[q^2\,r^2\sin^2\theta + \frac{1}{2}\,h\,q\,\big(r^3 - (a_1 + a_2 + a_3)r^2\cos\theta \\
&\qquad\qquad\qquad\qquad\qquad + (a_1 a_2 + a_1 a_3 + a_2 a_3)r - a_1 a_2 a_3\cos\theta\big)\bigg] \\
&-\kappa^- - \frac{m_{\mathrm{div}}^-}{r}\sin^2\theta + m_0^-\cos\theta + \sum_{i=1}^{3}\frac{m_i^-}{r_i}(r\cos\theta - a_i)\,,
\end{aligned}
\tag{A.11}
$$

where κ^- is another constant of integration.

A.3   The Minkowski-space limit

For $h = 0$ the BPS and non-BPS solutions must be the same and in section 2.4 it was argued that this should be achieved by the coordinate change (2.41). For the angular momentum vector this means

$$
\begin{aligned}
\mu_-(d\psi_- - q\cos\theta\,d\phi_-) + \omega_{\phi-}d\phi_- &= \mu_+(d\psi_+ + q\cos\theta\,d\phi_+) + \omega_{\phi+}d\phi_+ \\
&= \tilde{\mu}_+(-q\,d\phi_- + \cos\theta\,d\psi_-) + \frac{1}{q}\tilde{\omega}_{\phi+}d\psi_-
\end{aligned}
\tag{A.12}
$$

where the quantities with the tildes have the replacement $k_I^+ \to \widehat{k}_I$. This implies that we must have (for $h = 0$):

$$
\mu_- = \tilde{\mu}_+\cos\theta + \frac{1}{q}\tilde{\omega}_{\phi+}\,, \qquad \omega_{\phi-} = \tilde{\omega}_{\phi+}\cos\theta - q\,\tilde{\mu}_+\sin^2\theta\,.
\tag{A.13}
$$

We have verified this by explicit computation and it works provided that one also makes the substitutions:

$$
m_i^+ \to \frac{a_i}{q}\,m_i^-\,, \quad m_0^+ \to \frac{1}{q}\,m_{\mathrm{div}}^-\,, \quad m_\infty^+ \to \frac{1}{q}\,\kappa^-\,, \quad m_{\mathrm{div}}^+ \to \frac{1}{q}\,m_\infty^-\,, \quad \kappa^+ \to m_0^-\,.
\tag{A.14}
$$

Thus the transformation (2.41) does indeed map the complete BPS description to the almost-BPS description. Note also that we have included the terms $m_{\mathrm{div}}^\pm$ so as to complete this dictionary.

References

[1] F. Denef, *Supergravity flows and D-brane stability*, *JHEP* **08** (2000) 050 [hep-th/0005049] [INSPIRE].

[2] F. Denef, *Quantum quivers and Hall/hole halos*, *JHEP* **10** (2002) 023 [hep-th/0206072] [INSPIRE].

[3] B. Bates and F. Denef, *Exact solutions for supersymmetric stationary black hole composites*, *JHEP* **11** (2011) 127 [hep-th/0304094] [INSPIRE].

[4] J.P. Gauntlett and J.B. Gutowski, *General concentric black rings*, *Phys. Rev.* **D 71** (2005) 045002 [hep-th/0408122] [INSPIRE].

[5] I. Bena and N.P. Warner, *Bubbling supertubes and foaming black holes*, *Phys. Rev.* **D 74** (2006) 066001 [hep-th/0505166] [INSPIRE].

[6] P. Berglund, E.G. Gimon and T.S. Levi, *Supergravity microstates for BPS black holes and black rings*, *JHEP* **06** (2006) 007 [hep-th/0505167] [INSPIRE].

[7] A. Saxena, G. Potvin, S. Giusto and A.W. Peet, *Smooth geometries with four charges in four dimensions*, *JHEP* **04** (2006) 010 [hep-th/0509214] [INSPIRE].

[8] I. Bena, C.-W. Wang and N.P. Warner, *Mergers and typical black hole microstates*, *JHEP* **11** (2006) 042 [hep-th/0608217] [INSPIRE].

[9] I. Bena, C.-W. Wang and N.P. Warner, *Plumbing the abyss: black ring microstates*, *JHEP* **07** (2008) 019 [arXiv:0706.3786] [INSPIRE].

[10] F. Denef and G.W. Moore, *Split states, entropy enigmas, holes and halos*, *JHEP* **11** (2011) 129 [hep-th/0702146] [INSPIRE].

[11] J. Manschot, B. Pioline and A. Sen, *Wall crossing from Boltzmann black hole halos*, *JHEP* **07** (2011) 059 [arXiv:1011.1258] [INSPIRE].

[12] J.P. Gauntlett and J.B. Gutowski, *Concentric black rings*, *Phys. Rev.* **D 71** (2005) 025013 [hep-th/0408010] [INSPIRE].

[13] I. Bena, B.D. Chowdhury, J. de Boer, S. El-Showk and M. Shigemori, *Moulting black holes*, *JHEP* **03** (2012) 094 [arXiv:1108.0411] [INSPIRE].

[14] J. de Boer, S. El-Showk, I. Messamah and D. Van den Bleeken, *Quantizing $N = 2$ multicenter solutions*, *JHEP* **05** (2009) 002 [arXiv:0807.4556] [INSPIRE].

[15] I. Bena, M. Berkooz, J. de Boer, S. El-Showk and D. Van den Bleeken, *Scaling BPS Solutions and pure-Higgs States*, *JHEP* **11** (2012) 171 [arXiv:1205.5023] [INSPIRE].

[16] S.-J. Lee, Z.-L. Wang and P. Yi, *Quiver invariants from intrinsic Higgs states*, *JHEP* **07** (2012) 169 [arXiv:1205.6511] [INSPIRE].

[17] J. Manschot, B. Pioline and A. Sen, *From black holes to quivers*, *JHEP* **11** (2012) 023 [arXiv:1207.2230] [INSPIRE].

[18] I. Bena, N. Bobev, C. Ruef and N.P. Warner, *Supertubes in bubbling backgrounds: Born-Infeld meets supergravity*, *JHEP* **07** (2009) 106 [arXiv:0812.2942] [INSPIRE].

[19] K. Goldstein and S. Katmadas, *Almost BPS black holes*, *JHEP* **05** (2009) 058 [arXiv:0812.4183] [INSPIRE].

[20] I. Bena, G. Dall'Agata, S. Giusto, C. Ruef and N.P. Warner, *Non-BPS black rings and black holes in Taub-NUT*, *JHEP* **06** (2009) 015 [arXiv:0902.4526] [INSPIRE].

[21] I. Bena, S. Giusto, C. Ruef and N.P. Warner, *Multi-center non-BPS black holes: the solution*, *JHEP* **11** (2009) 032 [arXiv:0908.2121] [INSPIRE].

[22] R. Emparan and G.T. Horowitz, *Microstates of a neutral black hole in M-theory*, *Phys. Rev. Lett.* **97** (2006) 141601 [hep-th/0607023] [INSPIRE].

[23] O. Vasilakis and N.P. Warner, *Mind the gap: supersymmetry breaking in scaling, microstate geometries*, *JHEP* **10** (2011) 006 [arXiv:1104.2641] [INSPIRE].

[24] I. Bena, S. Giusto, C. Ruef and N.P. Warner, *Supergravity solutions from floating branes*, *JHEP* **03** (2010) 047 [arXiv:0910.1860] [INSPIRE].

[25] I. Bena and N.P. Warner, *One ring to rule them all . . . and in the darkness bind them?*, *Adv. Theor. Math. Phys.* **9** (2005) 667 [hep-th/0408106] [INSPIRE].

[26] J.B. Gutowski and H.S. Reall, *General supersymmetric AdS$_5$ black holes*, *JHEP* **04** (2004) 048 [hep-th/0401129] [INSPIRE].

[27] J.P. Gauntlett, J.B. Gutowski, C.M. Hull, S. Pakis and H.S. Reall, *All supersymmetric solutions of minimal supergravity in five- dimensions*, *Class. Quant. Grav.* **20** (2003) 4587 [hep-th/0209114] [INSPIRE].

[28] I. Bena, A. Puhm and B. Vercnocke, *Metastable supertubes and non-extremal black hole Microstates*, *JHEP* **04** (2012) 100 [arXiv:1109.5180] [INSPIRE].

[29] I. Bena, A. Puhm and B. Vercnocke, *Non-extremal black hole microstates: fuzzballs of fire or fuzzballs of fuzz?*, *JHEP* **12** (2012) 014 [arXiv:1208.3468] [INSPIRE].

[30] D. Anninos, T. Anous, F. Denef, G. Konstantinidis and E. Shaghoulian, *Supergoop dynamics*, *JHEP* **03** (2013) 081 [arXiv:1205.1060] [INSPIRE].

[31] G. Dall'Agata, S. Giusto and C. Ruef, *U-duality and non-BPS solutions*, *JHEP* **02** (2011) 074 [arXiv:1012.4803] [INSPIRE].

[32] G. Bossard and C. Ruef, *Interacting non-BPS black holes*, *Gen. Rel. Grav.* **44** (2012) 21 [arXiv:1106.5806] [INSPIRE].

[33] G. Bossard and S. Katmadas, *Duality covariant non-BPS first order systems*, *JHEP* **09** (2012) 100 [arXiv:1205.5461] [INSPIRE].

[34] B. de Wit and S. Katmadas, *Near-horizon analysis of $D = 5$ BPS black holes and rings*, *JHEP* **02** (2010) 056 [arXiv:0910.4907] [INSPIRE].

[35] K. Hanaki, K. Ohashi and Y. Tachikawa, *Comments on charges and near-horizon data of black rings*, *JHEP* **12** (2007) 057 [arXiv:0704.1819] [INSPIRE].

[36] G. Bossard, *Octonionic black holes*, *JHEP* **05** (2012) 113 [arXiv:1203.0530] [INSPIRE].